CO 430/630 LECTURE 15 SUMMARY

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Summary

Today we proved the fundamental theorem of finite distributive lattices. You can find the details in Stanley. Here's the argument briefly. Take L a finite distributive lattice. Let P be the subposet of join irreducibles of L. It suffices to show that L is isomorphic to the lattice of downsets of P which we'll call $J(P)$. Given $x \in L$ let $D_x = \{y \in P : y \leq x\}$, then $x \mapsto D_x$ gives a map from L to $J(P)$ which is straightforwardly injective and order preserving. To show this map is surjective, take $D \in J(P)$, then we claim $D = D_x$ where $x = \bigvee_{y \in D} y$. To show the claim, $D \subseteq D_x$ is clear, and for the other inclusion take $z \in D_x$ then we have $\bigvee_{y \in D} y = \bigvee_{y \in D_x} y$ by definition of upper bound. Now take the meet of both sides with z and use the distributive property to get $\bigvee_{y \in D}(z \wedge y) = \bigvee_{y \in D_x}(z \wedge y)$. The right hand side is z since one of the terms is z and the other terms are smaller. For the left hand side to also be z we must have some $y \in D$ with $y \wedge z = z$ (by join irreducibility) but then $y \leq z$ so $y \in D$ since it is a downset.

Furthermore, this gives us an equivalence of categories between the category of finite partial orders with order preserving maps and the category of finite distributive lattices with bounded lattice homomorphisms (a lattice homomorphism is a map of the underlying sets which preserves $\wedge$ and $\vee$ and it is bounded if it also preserves $\hat{0}$ and $\hat{1}$.)

We set up this equivalance of categories as follows. Let $\mathbf{2}$ be the poset of two elements, 0 and 1 with $0 < 1$, and write $J(P)$ for the lattice of downsets of a finite poset P as before. A downset of P is the same information as an order preserving map $P \to \mathbf{2}$, where the downset is just the inverse image of 0 under the map. So $J(P)$ is $\mathrm{Hom}(P, \mathbf{2})$. Further, this plays well with order preserving maps. If $g : Q \to P$ is an order preserving map of finite posets then define $g^* : J(P) \to J(Q)$ the way it has to be, namely for $D \in J(P)$ let r_D be the corresponding map, $r_D(x) = 0$ if $x \in D$ and $r_D(x) = 1$ of $x \neq D$, so $g^*(D) = (r_D \circ g)^{-1}(0)$ in $J(Q)$. This map preserves $\wedge, \vee, \hat{0}, \hat{1}$. So in category language this says that we have $J = \mathrm{Hom}(-, \mathbf{2})$ which is a covariant functor defining an equivalence of categories between the categories described above.

Next we changed to our final poset topic: Möbius inversion.

The ζ-function of a poset P is $\zeta : P \times P \to \mathbb{Z}$, $\zeta(x, y) = 1$ if $x \leq y$, $\zeta(x, y) = 0$ otherwise. It is often convenient to think of ζ as a matrix, so fix a linear extension of P (that is a total order on P so that if $x \leq_P y$ then also $x \leq y$ in the total order) and define the ζ-matrix of P by $\zeta = (\zeta(x_i, x_j))_{i,j}$. This is also called the relation matrix of P. It is the adjacency matrix of P as a digraph (not the Hasse diagram of P, but P itself, with a directed edge for each relation). Like any adjacency matrix we know about the entries of its powers:

$$\zeta^2(x,y) = \sum_{x \le z \le y} 1 = |[x,y]|$$
$$\zeta^k(x,y) = \sum_{x \le x_1 \le \cdots \le x_{k-1} \le y} 1$$

the things we are counting in the second line there are called *multichains*, they are like chains but you can have repetitions. If we want to count honest-to-goodness chains then we need to get rid of the 1s on the diagonal of ζ, that is we need to work with $(\zeta - I)^k$.

Note that ζ is invertible. It will be useful to explicitly understand the inverse, so for P locally finite define $\mu : P \times P \to \mathbb{Z}$ by

$$\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \le z < y} \mu(x,z) & x < y \\ 0 & \text{otherwise} \end{cases}$$

These are equivalent to the equation

$$\sum_{x \le z \le y} \mu(x,z) = \delta(x,y)$$

where $\delta(x,y) = 1$ if $x = y$ and $\delta(x,y) = 0$ if $x \neq y$. Note that μ is well defined inductively. μ is called the *Möbius* function. We can also see it as a matrix which we will also call μ.

Proposition 1. *Let P be a locally finite poset. Then ζ and μ are inverses.*

This is just a computation since we defined things that way

$$(\mu\zeta)(x,y) = \sum_{z \in P} \mu(x,z)\zeta(z,y) = \sum_{x \le z \le y} \mu(x,z)\zeta(z,y) = \sum_{x \le z \le y} \mu(x,z) = \delta(x,y)$$

Then we have Möbius inversion itself in its two forms:

Theorem 2.

(1) *Let P be a poset with all downsets finite and let $f, g : P \to \mathbb{C}$, then*

$$g(x) = \sum_{y \le x} f(y), \forall x \in P \quad \Leftrightarrow \quad f(x) = \sum_{y \le x} \mu(y,x) g(y), \forall x \in P$$

(2) *Let P be a poset with all upsets finite and let $f, g : P \to \mathbb{C}$, then*

$$g(x) = \sum_{y \ge x} f(y), \forall x \in P \quad \Leftrightarrow \quad f(x) = \sum_{y \ge x} \mu(x,y) g(y), \forall x \in P$$

Note that the conditions on downsets and upsets being finite is just so that the sums are finite. If you're working in a valuation ring and your sums converge then that's also fine. The proof is a little linear algebra. Write f and g as column vectors using the same linear extension as for ζ and μ. If I haven't messed up my indexing, the left hand side of the first item says $g = \zeta^t f$ and the right hand side says $f = \mu^t g$ which are equivalent since μ and ζ are inverse, and for the second item we have $g = \zeta f$ and $f = \mu g$ which are likewise equivalent.

Next time

Next time we'll do the three most classic examples and then move on to Hopf algebras.

References

Stanley's Enumerative Combinatorics chapter 3.

For the equivalence of categories, the Wikipedia description is a pretty good stand alone presentation, https://en.wikipedia.org/wiki/Birkhoff%27s_representation_theorem#Functoriality