CO 430/630 LECTURE 14 SUMMARY

WINTER 2026

Summary

We looked at some examples of lattices and non-lattices. You checked some simple properties of lattices:

- $\wedge$ and $\vee$ are associative, commutative, and idempotent
- $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ (absorption)
- $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \leq y$

Definition 1. *A lattice L is* distributive *if for all $x, y, z \in L$*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Either of these equations implies the other, which I did for you on the board. We also then looked at examples and non-examples of distributive lattices. Then we stated the *fundamental theorem of finite distributive lattices.*

Theorem 2. *Let L be a finite distributive lattice, then there is a unique (up to isomorphism) finite poset P such that L is isomorphic to the lattice of downsets of P.*

We need a few lemmas to prove this

Lemma 3. *The poset of downsets of a finite poset is a distributive lattice.*

The proof here is just that the union and intersection of two downsets are downsets, and so the rest follows from what we know about union and intersection.

Next we say an element x of a lattice L is *join-irreducible* if we cannot write $x = y \vee z$ with $y < x$ and $z < x$. Then we have the following lemma

Lemma 4. *Let P be a finite poset. A downset D of P is join irreducible in the lattice of downsets of P if and only if it is* principal, *that is $D = \Lambda(x)$ for some $x \in P$.*

This is useful because it gives a bijection between the join irreducibles of the lattice of downsets of P and P itself, and the subposet of the join irreducibles is P as a poset. This gives us our candidate poset for proving the theorem. But first we should prove the lemma.

Proof. If $D = D_1 \cup D_2$ with $D_1 < D$ and $D_2 < D$ then take a maximal element x of $D \setminus D_1$ and a maximal element y of $D \setminus D_2$ which is possible since $D_1 \cap D_2$ is a downset as well. Then if $D = \Lambda(x)$ we'd have $x \geq z$ and $x \geq y$ which is impossible so D is not principal.

If $D = \Lambda(x_1, \ldots, x_k)$ with $x_1, \ldots, x_k$ maximal elements of D and $k > 1$ then let $D_1 = \Lambda(x_1)$, $D_2 = \Lambda(x_2, \ldots, x_k)$ and we have $D = D_1 \cup D_2$ and $D_1 < D$, $D_2 < D$ so D is not join irreducible. $\square$

Next time

Next time we'll prove the fundamental theorem of finite distributive lattices and then get on to Möbius inversion.

References

Stanley's Enumerative Combinatorics chapter 3.