

# CO 430/630 LECTURE 13 SUMMARY

WINTER 2026

## SUMMARY

Last time we had the midterm. Today we started on posets.

**Definition 1.** A partially ordered set (*poset*)  $P$  is a set (by standard abuse of notation also called  $P$ ) and a binary relation  $\leq$  (or  $\leq_P$  if we need to disambiguate) satisfying

- $x \leq x$  for all  $x \in P$  (*reflexivity*)
- $x \leq y$  and  $y \leq x$  implies  $x = y$  for all  $x, y \in P$  (*antisymmetry*)
- $x \leq y$  and  $y \leq z$  implies  $x \leq z$  for all  $x, y, z \in P$  (*transitivity*)

There's a bunch of not very surprising notation here. We write  $x < y$  to mean  $x \leq y$  and  $x \neq y$ . You can set up the axioms of posets in terms of  $<$  instead of  $\leq$  if you like (you can work out what the axioms on  $<$  need to be). If  $x \leq y$  or  $y \leq x$  we say  $x$  and  $y$  are *comparable*, otherwise we say they are *incomparable*. We say  $y$  covers  $x$  if  $y > x$  and there is no  $z$  such that  $y > z > x$ . I usually write  $y \succ x$  but you also see  $y \triangleright x$  as notation for cover. The *order diagram* or *Hasse diagram* of a poset is the drawing of the graph of the cover relation on the poset where you always draw the edges with the covering element above the element being covered.

We gave a bunch of standard examples including  $B_n$  the boolean lattice of subsets of  $\{1, \dots, n\}$  under containment.

For subposet we want the induced kind, that is a subset with  $\leq$  restricted to that subset.  $f : P \rightarrow Q$  is *order preserving* if  $x \leq_P y$  implies  $f(x) \leq_Q f(y)$ . An isomorphism between posets is an order preserving bijection between them with an order preserving inverse.

A bunch more unsurprising definitions: An interval in a poset  $P$  is  $[x, y] = \{z \in P : x \leq z \leq y\}$ . A poset is *locally finite* if all its intervals are finite. An element  $x \in P$  is *maximal* if there does not exist  $y \in P$  with  $y \geq x$ . Likewise an element  $x \in P$  is *minimal* if there does not exist  $y \in P$  with  $y \leq x$ . A *chain* in a poset is a set of mutually comparable elements. They must, then, be totally ordered. An *antichain* in a poset is a set of mutually incomparable elements.

A *downset* (or order ideal) of a poset  $P$  is a subset  $D \subseteq P$  that is downwards closed, that is, if  $x \in D$  and  $y \in P$  with  $y \leq x$  then  $y \in D$ . An *upset* (or order filter) of a poset  $P$  is a subset  $U \subseteq P$  that is upwards closed, that is, if  $x \in U$  and  $y \in P$  with  $y \geq x$  then  $y \in U$ . The downset generated by  $x_1, \dots, x_k$  is  $\Lambda(x_1, \dots, x_k) = \{y \in P : y \leq x_i \text{ for some } i\}$  and likewise the upset generated by  $x_1, \dots, x_k$  is  $V(x_1, \dots, x_k) = \{y \in P : y \geq x_i \text{ for some } i\}$ . You might hear me call those the inclusive past and the inclusive future of  $x_1, \dots, x_k$ , which is language that comes from the causal set approach to quantum gravity, but most combi people wouldn't use that language.

Notice that the set of downsets of a poset form a poset by inclusion. You drew some examples. In fact, the poset of downsets is not just a poset but is a lattice. However, we need to define lattice before we can say that. Some preliminaries:

Say a poset  $P$  has a *minimum* (not minimal in this case, but actually minimum) element, write it  $\hat{0}$ , if there exists  $\hat{0} \in P$  such that  $x \geq \hat{0}$  for all  $x \in P$ . Likewise say a poset  $P$  has a *maximum* element, write it  $\hat{1}$ , if there exists  $\hat{1} \in P$  such that  $x \leq \hat{1}$  for all  $x \in P$ .

Given  $x, y \in P$  an *upper bound* of  $x$  and  $y$  is an element  $z \in P$  such that  $z \geq x$  and  $z \geq y$ . A *least upper bound* of  $x$  and  $y$  is an upper bound  $z \in P$  such that every upper bound  $w$  of  $x$  and  $y$  satisfies  $w \geq z$ . Write  $z = x \vee y$  and say  $x$  *join*  $y$ .

Likewise given  $x, y \in P$  a *lower bound* of  $x$  and  $y$  is an element  $z \in P$  such that  $z \leq x$  and  $z \leq y$ . A *greatest lower bound* of  $x$  and  $y$  is a lower bound  $z \in P$  such that every lower bound  $w$  of  $x$  and  $y$  satisfies  $w \leq z$ . Write  $z = x \wedge y$  and say  $x$  *meet*  $y$ .

**Definition 2.** A lattice is a poset for which every pair of elements has a least upper bound and a greatest lower bound.

There are other ways to define lattices, including with axioms on the meet and join that don't reference the poset structure directly, but it ends up being equivalent and this way is the most natural for us.

NEXT TIME

Next time we'll talk more about lattices, heading towards the fundamental theorem of distributive lattices.

REFERENCES

Stanley's Enumerative Combinatorics chapter 3.