CO 430/630 LECTURE 10 SUMMARY

WINTER 2026

Summary

Isomorphims travel through natural transformations as follows

Lemma 1. *Let $a \in \mathcal{A}_X$ and $b \in \mathcal{B}_X$ be isomorphic and Φ a natural transformation from $\mathcal{A}$ to $\mathcal{B}$. Then $\Phi_X(a)$ and $\Phi_X(b)$ are also isomorphic.*

The proof follows quickly from the diagram.

If we have a natural transformation Φ from $\mathcal{A}$ to $\mathcal{B}$ such that $\Phi_X : \mathcal{A}_X \to \mathcal{B}_X$ is a bijection for all sets X then we call Φ a *natural equivalence* and say $\mathcal{A}$ and $\mathcal{B}$ are *naturally equivalent* written $\mathcal{A} \equiv \mathcal{B}$. We gave some examples, like perms with directed graphs with in and out degree exactly 1.

We say two species $\mathcal{A}$ and $\mathcal{B}$ are *numerically equivalent* if $A(x) = B(x)$ and write $\mathcal{A} \approx \mathcal{B}$.

Natural equivalence implies numerical equivalence but not the converse. As an example, with linear order $\mathcal{L}$ and permutations $\mathcal{S}$ as defined last time these are clearly numerically equivalent but are not naturally equivalent because all linear orders of the same size are isomorphic (just map the labels appropriate), so the lemma above would imply that all permutations are isomorphic if there were a natural equivalence between $\mathcal{L}$ and $\mathcal{S}$, however this can't be true since different cycle structures can't be mapped between using conjugacy.

We say a species is *connected* of $\mathcal{A}_\emptyset = \emptyset$. This is a useful notion worth having a word for; the choice of word is perhaps a bit inapt in this context, but comes from other contexts.

Species tell the labelled story very nicely: the $\mathcal{A}$-structures on X are the objects labelled by X and transportation is relabelling. The isomorphism classes are the unlabelled objects.

We finished up the class with species operations. These are predominantly the things we've seen before phrased in this language. Let $\mathcal{A}$ and $\mathcal{B}$ be species. We have the following species

- $\mathcal{A}+\mathcal{B}$ defined by $(\mathcal{A}+\mathcal{B})_X = \mathcal{A} \sqcup \mathcal{B}$. The exponential generating series is $A(x)+B(x)$.
- When $\mathcal{B} \subseteq \mathcal{A}$ (i.e. $\mathcal{B}_X \subseteq \mathcal{A}_X$ for all X), then $\mathcal{A}-\mathcal{B}$ is defined by $(\mathcal{A}-\mathcal{B})_X = \mathcal{A}_X \setminus \mathcal{B}_X$. The exponential generating series is $A(x) - B(x)$.
- $\mathcal{A}_{\text{even}}$ defined by $(\mathcal{A}_{\text{even}})_X = \begin{cases} \mathcal{A}_X & \text{if } |X| \text{ is even} \\ \emptyset & \text{otherwise} \end{cases}$. The exponential generating series is $\frac{1}{2}(A(x) + A(-x)$.
- We can restrict in other ways, eg $\mathcal{A}_1$, $\mathcal{A}_{>0}$, likewise. $\mathcal{A}_{\text{odd}}$ has exponential generating series $\frac{1}{2}(A(x) - A(-x))$.
- $\mathcal{A} * \mathcal{B} = \bigsqcup_{S \subseteq X} \mathcal{A}_S \times \mathcal{B}_{X \setminus S}$. The exponential generating series is $A(x)B(x)$.
- Since this *species product* is the best product in this context we define $\mathcal{A}^n = \underbrace{\mathcal{A} * \cdots * \mathcal{A}}_{n \text{ times}}$ and $\mathcal{A}^0 = \mathcal{E}_0 = \{\emptyset\}$ and hence for $\mathcal{A}$ connected define $\mathcal{A}^* = \sum_{n \geq 0} \mathcal{A}^n$, which has exponential generating series $1/(1 - A(x))$.

- If $\mathcal{K}$ is some fixed set then define the species $\mathcal{K} \times \mathcal{A}$ by $(\mathcal{K} \times \mathcal{A})_X = \mathcal{K} \times \mathcal{A}_X$. The exponential generating series is $|K|A(x)$.
- $\mathcal{A}^\bullet$ defined by $(\mathcal{A}^\bullet)_X = \mathcal{A}_X \times X$. The exponential generating series is $\frac{xd}{dx}A(x)$.
- For $\mathcal{B}$ connected define the species $\mathcal{A}[\mathcal{B}]$ by

$$(\mathcal{A}[\mathcal{B}])_X = \bigsqcup_{\{X_1,\dots,X_k\} \in \Pi_X} \mathcal{A}_{X_1,\dots,X_k} \times \mathcal{B}_{X_1} \times \cdots \times \mathcal{B}_{X_k}.$$

  The exponential generating series is $A(B(x))$.

Formally we can define disjoint union by taking a cartesian product of each side with a different singleton. There are some standard drawings you can draw for these, which we drew for species product and composition. We did a few small examples as we went through. We had the generating series in a table as the end, but it's the same information. We finished this off with a slightly more substantial example, permutations with an even number of cycles $\mathcal{E}_{\text{even}}[\mathcal{C}]$ which has exponential generating series $\cosh(\log(1/(1-x)))$ which we can extract coefficients from to get

$$\left[\frac{x^n}{n!}\right] \cosh\left(\log\left(\frac{1}{1-x}\right)\right) = \left[\frac{x^n}{n!}\right]\left(1 + \frac{1}{2}\left(\frac{x^2}{1-x}\right)\right) = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ \frac{n!}{2} & n \geq 2 \end{cases}$$

We finished off with a brief discussion of the format of the midter which will take place on the Wednesday after reading week.

Next time

Next time you'll work on some worksheets again.

References

The standard book on species is *Combinatorial species and tree-like structures* by Bergeron, Labelle, and Leroux.

Another source is the first three videos from the exponential generating series section of one of Kevin's past offerings of this course `https://www.math.uwaterloo.ca/~kpurbhoo/winter2021-co630/co630-videos.html`.

If you want to know more on species you can check out his next few videos in that section as well.