

**CO 430/630 W26 ASSIGNMENT 3 SOLUTIONS**

- (1) (a) This is almost just the usual formula for generating series of sequences, but additionally keeping track of the length of the sequence with the other variable. That is

$$S(x, t) = A(x)t + A(x)^2t^2 + A(x)^3t^3 + \dots = \frac{tA(x)}{1 - tA(x)}$$

- (b) If a sequence is not primitive then it is a concatenation of  $k$  copies of some primitive sequence for some  $k > 1$ .  $k = 1$  gives the primitive sequences themselves. Note that  $P(x^k, t^k)$  counts sequences which are  $k$  copies of a primitive sequence, and so we have

$$S(x, t) = \sum_{k \geq 1} P(x^k, t^k)$$

- (c) The equation from the previous part can be rephrased as follows let  $f(n) = S(x^n, t^n)$  and  $g(n) = P(x^n, t^n)$ , then

$$f(n) = \sum_{n|k} g(k)$$

and the equation from the previous part is the  $n = 1$  instance of this equation. By Möbius inversion on the poset of positive integers by divisibility, that is the usual number theoretic Möbius inversion, we get

$$g(n) = \sum_{n|k} \mu(k/n) f(k)$$

Taking the  $n = 1$  case again and rewriting with our original notation we get

$$P(x, t) = \sum_{k \geq 1} \mu(k) S(x^k, t^k) = \sum_{k \geq 1} \mu(k) \frac{t^k A(x^k)}{1 - t^k A(x^k)}$$

where the last equation is by subbing in the first part.

- (d) By primitivity, each cyclic permutation of a primitive cycle is distinct. Therefore we need to divide the number of primitive sequences of length  $\ell$  by  $\ell$  to get the number of primitive cycles of length  $\ell$ .

Additionally formal integration gives  $\int_0^t u^k \frac{du}{u} = \frac{t^k}{k}$ . Since the  $t$  variable counts the length, this formal integration  $\frac{du}{u}$ , gives what we want, hence the desired formula.

- (e) From the previous part we have  $Q(x, t) = \int_0^t P(x, u) \frac{du}{u}$  and from the part before we have  $P(x, t) = \sum_{k \geq 1} \mu(k) \frac{t^k A(x^k)}{1 - t^k A(x^k)}$ . Putting these together we have

$$\begin{aligned}
Q(x, t) &= \int_0^t \sum_{k \geq 1} \mu(k) \frac{u^k A(x^k)}{1 - u^k A(x^k)} \frac{du}{u} \\
&= \sum_{k \geq 1} \mu(k) \sum_{\ell \geq 1} (u^k A(x^k))^\ell \frac{du}{u} \\
&= \sum_{k \geq 1} \mu(k) \sum_{\ell \geq 1} \frac{(t^k A(x^k))^\ell}{k\ell} \\
&= \sum_{k \geq 1} \frac{\mu(k)}{k} \log \left( \frac{1}{1 - t^k A(x^k)} \right)
\end{aligned}$$

- (f) Similarly to the second part, arbitrary cycles can be obtained through repetitions of primitive cycles and so we have

$$B(x, t) = \sum_{k \geq 1} Q(x^k, t^k)$$

Now using the previous part

$$\begin{aligned}
B(x, t) &= \sum_{\ell \geq 1} \sum_{d \geq 1} \frac{\mu(d)}{d} \log \left( \frac{1}{1 - t^{\ell d} A(x^{\ell d})} \right) \\
&= \sum_{k \geq 1} \sum_{d|k} \frac{\mu(d)}{d} \log \left( \frac{1}{1 - t^k A(x^k)} \right) \\
&= \sum_{k \geq 1} \frac{\phi(k)}{k} \log \left( \frac{1}{1 - t^k A(x^k)} \right)
\end{aligned}$$

(with the substitution  $k = \ell d$  and using the arithmetical identity given in the question) which is what we wanted to prove.

- (2) Before we start it is worth noticing that before we consider  $B$ , a non-attacking rook placement of  $n$  rooks is simply a permutation as the rook placement is permutation matrix with the rooks as the 1s and the other squares as 0s.

So we will use permutation language and elementary permutation counting in the below whenever convenient.

- (a) We'll use inclusion exclusion. Let the set of conditions  $B$ , that is, for each box in  $B$ , the condition of that box is the condition that a rook is in that box.

For  $A \subset B$  let  $f_A$  be the number of non-attacking rook placements of  $n$  rooks with a rook on each position of  $A$ . We can compute  $f_A$  directly. Either the positions of  $A$  are already attacking in which case  $f_A = 0$  or we place rooks in the positions of  $A$ , those use up certain rows and columns, and we're just left with counting arbitrary non-attacking rook placements for the rest, that is just counting permutations of size  $n - |A|$ , so  $f_A = (n - |A|)!$  when  $A$  is non-attacking.

Now by inclusion exclusion we have

$$d_{B,n} = \sum_{A \subseteq B} (-1)^{|A|} f_A$$

Collecting together terms by the size of  $A$  we can rewrite this as

$$\sum_{A \subseteq B} (-1)^{|A|} f_A = \sum_{k=0}^n (-1)^k r_{B,k} (n-k)!$$

We claim that this is the coefficient extraction given in the problem, which we can check by computing

$$[x^n] R_B(-x) \sum_{j \geq 0} j! x^j = [x^n] \sum_{k=0}^n r_{B,k} (-x)^k \sum_{j \geq 0} j! x^j = \sum_{k=0}^n r_{B,k} (-1)^k (n-k)!$$

which agrees, giving the result.

- (b) We want to compute the  $r_{B,k}$  in this case by elementary counting.  $B$  is structured as a zigzag so that if we have a rook in one of the squares then the next square is either the next square to the right or the next square down and in either case, each square can only attack the square immediately before or after it in the zigzag (interpreted cyclically). So we're trying to count ways to put  $k$  non-adjacent things on a rooted cycle of length  $2n$ . One way to count this is to first consider putting  $k$  things on a rooted cycle of length  $2n-k$  and then after each thing put one empty slot so that they are not adjacent. There are  $\binom{2n-k}{k}$  ways to do this. However, this misses some because the last slot will always be empty (either you didn't choose it, or you did in which case you put an empty slot after it). We also count these by putting something in the last slot and now asking the same question of size one smaller in both  $n$  and  $k$ , which there are  $\binom{2n-2-(k-1)}{k-1} = \binom{2n-k-1}{k-1}$  ways to do. This suffices, because we want an empty final slot now so as not to conflict with the one we did put at the end. So in total we have  $r_{B,k} = \binom{2n-k}{k} + \binom{2n-k-1}{k-1} = \frac{2n}{2n-k} \binom{2n-k}{k}$ .

Now apply the previous part to get

$$d_{B,n} = \sum_{k=0}^n \frac{2n}{2n-k} \binom{2n-k}{k} (-1)^k (n-k)!$$

- (3) (a) Consider



The length of the longest chain is 3 and every element is in such a chain (either the one on the left or the one on the right). However the highlighted chain in the middle is also maximal yet of strictly smaller length.

- (b) Suppose not, so suppose there is a maximal chain  $C : x_0 < x_1 < \dots < x_k$  which has length strictly less than  $\ell$ .

Now there is a chain of length  $\ell$  containing both  $x_0$  and  $x_1$ . That chain cannot contain any element strictly less than  $x_0$  or we'd contradict the maximality of

$C$ , so it is a chain starting with  $x_0 < x_1$  and going up from there, let  $E_1$  be the part of this chain from  $x_1$  upwards, which has length  $\ell - 1$ .

There is also a chain of length  $\ell$  containing both  $x_1$  and  $x_2$ . This chain cannot contain more than one element strictly less than  $x_1$  or combining those elements with  $E_1$  we'd get a chain of length greater than  $\ell$ . Letting  $E_2$  be the part of this chain from  $x_2$  upwards we see that  $E_2$  has length  $\ell - 2$  or more.

Continuing likewise, at a given step we have a chain of length  $\ell$  involving  $x_{i-1}$  and  $x_i$  and we let  $E_i$  be the part of this chain from  $x_i$  upwards and we know  $E_i$  has length  $\ell - i$  or more. So knowing that we have a chain of length  $\ell$  involving  $x_i$  and  $x_{i+1}$ , this chain can have at most  $i$  elements below  $x_i$  as otherwise combining with  $E_i$  would give a chain of length greater than  $\ell$  and hence letting  $E_{i+1}$  be the part of this chain from  $x_{i+1}$  upwards we see that  $E_{i+1}$  has length  $\ell - i - 1$  or more.

But now at the final step, we have  $E_k$  which has length  $\ell - k$  or more beginning at  $x_k$ , but this elongates  $C$  and hence contradicts the maximality of  $C$ , proving the result.

- (4) (a) Suppose  $P$  is the intersection of  $d$  linear orderings of  $\{1, 2, \dots, n\}$ , call them  $\leq_1, \dots, \leq_d$ . For  $p \in P$  let  $i_d(p)$  be the index of  $p$  in the  $\leq_i$  order. Associate to  $p$  the vector  $v_p = (i_1(p), i_2(p), \dots, i_d(p)) \in \mathbb{Z}_{\geq 0}^d$ . Then  $p \leq_P q$  iff  $i_i(p) \leq i_i(q)$  for all  $i$  iff  $v_p \leq_{\mathbb{Z}_{\geq 0}^d} v_q$ , so  $P$  is isomorphic to the subset  $\{v_p : p \in P\}$  of  $\mathbb{Z}_{\geq 0}^d$ .

The other direction is a bit more of a nuisance because there may be coordinates that are equal. Suppose  $P$  is a subposet of  $\mathbb{Z}_{\geq 0}^d$  with  $n$  elements. Let  $\leq_i$  be the *preorder* induced on  $P$  by the value of the  $i$ th coordinate of the elements. Note that  $\leq_i$  is not yet a total order as some distinct elements may be at the same position in the order (ie we may have  $a \leq_i b$  and  $b \leq_i a$  without  $a = b$ .)

The next step is to refine each  $\leq_i$  into a total order. So consider  $a_1, a_2, \dots, a_t$  distinct but with  $a_j \leq_i a_k$  and  $a_k \leq_i a_j$  for all  $1 \leq j, k \leq t$ . Comparability in  $P$  divides  $\{a_1, \dots, a_t\}$  into equivalence classes. For each such equivalence class, the elements are totally ordered in  $P$  and hence must be totally ordered the same way in each  $\leq_i$  so refine  $\leq_i$  to totally order them in this way. This gives us some totally ordered blocks of the  $\{a_1, \dots, a_t\}$  where elements of distinct blocks are incomparable. It remains only to totally order these blocks. Now if  $a$  and  $b$  are incomparable in  $P$  then they must be strictly greater in some coordinate and strictly less in some other coordinate, so they are already incompatibly ordered in two of the  $\leq_j$  and hence it does not matter how they are ordered in a  $\leq_i$  where they began in the same position. Therefore we can order these blocks in any way. So choose one such total order and let that be  $\leq_i$ . Doing so for all  $i$  gives  $P$  as an intersection of  $d$  total orders.

Note that in the case either and hence both of these properties hold and  $d$  is minimal for this to be true then we say that  $P$  has *order dimension*  $d$ .

- (b) Suppose  $P$  is the intersection of two total orders  $\leq_1$  and  $\leq_2$ . We can define  $Q$  on the same ground set as  $P$  by taking the intersection of  $\leq_1$  and the reverse of  $\leq_2$ , that is  $x \leq_Q y$  iff  $x \leq_1 y$  and  $y \leq_2 x$ . Then  $x \neq y$  are comparable in  $P$  if they occur in the same order in  $\leq_1$  and  $\leq_2$  and they are comparable in  $Q$  if they

occur in the opposite order in  $\leq_1$  and  $\leq_2$ , giving that  $x \neq y$  are comparable in  $Q$  iff they are incomparable in  $P$ .

In the other direction, suppose we have  $P$  and  $Q$  on the same ground set such that  $x \neq y$  are comparable in  $Q$  iff they are incomparable in  $P$ . Define  $\leq_1$  on that same ground set by  $x \leq_1 y$  if either  $x \leq_P y$  or  $x \leq_Q y$  exactly one of which will occur for  $x \neq y$ . Define  $\leq_2$  by  $x \leq_2 y$  if either  $x \leq_P y$  or  $y \leq_Q x$  and again exactly one of these will occur for  $x \neq y$ , so  $\leq_1$  and  $\leq_2$  are well defined total orders on the ground set of  $P$  and their intersection is  $P$  by construction.

(5) Either way there are four diagrams and they end up being these four diagrams

$$\begin{array}{ccccc}
 B \otimes B \otimes B \otimes B & \xleftarrow{\Delta \otimes \Delta} & B \otimes B & \xrightarrow{m} & B \\
 \downarrow \text{id} \otimes T \otimes \text{id} & & & & \parallel \\
 B \otimes B \otimes B \otimes B & \xrightarrow{m \otimes m} & B \otimes B & \xleftarrow{\Delta} & B \\
 B \otimes B & \xrightarrow{\epsilon \otimes \epsilon} & K \otimes K & & \\
 m \downarrow & & \downarrow m & & \\
 B & \xrightarrow{\epsilon} & K & & \\
 B \otimes B & \xleftarrow{u \otimes u} & K \otimes K & & \\
 \Delta \uparrow & & \uparrow \Delta & & \\
 B & \xleftarrow{u} & K & & \\
 K & \xrightarrow{\text{id}} & K & & \\
 u \downarrow & & \uparrow \epsilon & & \\
 B & \xlongequal{\quad} & B & & 
 \end{array}$$

(6) *Answers will vary.*