CO 430/630 W26 ASSIGNMENT 3

DUE WEDNESDAY APRIL 1 AT 10AM IN CROWDMARK

Do any 4 of the 6 problems!

The point of the assignment is for you to learn by doing the problems. I'm sure you can look up answers to many of these in various ways, but you largely defeat the purpose and undercut your own learning if you look things up instead of thinking. Having said that, you may use any resources, animate or inanimate, **provided you credit them**. In particular credit any use of generative AI, even mundane uses like in generating tikz for figures.

(1) We postponed the proof how the Cyc operator affects the generating series to an assignment, and here it finally is! The goal is to prove that if $\mathcal{B} = \mathsf{Cyc}(\mathcal{A})$ with $\mathcal{A}$ connected (and everything unlabelled) then

$$B(x) = \sum_{k\geq 1} \frac{\phi(k)}{k} \log\left(\frac{1}{1 - A(x^k)}\right)$$

   (a) Let $\mathcal{S} = \mathcal{A} \times \mathcal{A}^*$ be the class of nonempty sequences of elements of $\mathcal{A}$. Give an expression for $S(x,t)$ in terms of $A(x)$ where $S(x,t)$ is the bivariate generating series of $\mathcal{S}$ with x counting the usual weight (summing the weights of the elements of $\mathcal{A}$) and t counting the length of the sequence.

   (b) Call an element $s \in \mathcal{S}$ *primitive* if it is not the concatenation of two or more copies of some shorter sequence. Show that

$$S(x,t) = \sum_{k\geq 1} P(x^k, t^k)$$

   where $P(x,t)$ is the bivariate generating series of primitive sequences.

   (c) Use Möbius inversion and the previous two parts to give an expression for $P(x,t)$ in terms of $A(x)$.

   (d) A cycle is primitive if all of its linear representations are primitive. Let $Q(x,t)$ be the bivariate generating series of primitive cycles. Prove that

$$Q(x,t) = \int_0^t P(x,u)\frac{du}{u}$$

   (e) Integrate term by term using the previous two parts to obtain an expression for $Q(x,t)$ in terms of $A(x)$.

   (f) Using that arbitrary cycles can be obtained by repetitions of primitive cycles and the arithmetical identity $\sum_{d|k} \mu(d)/d = \phi(k)/k$ obtain the desired result.

(2) The rook in chess attacks horizontally or vertically at any distance. A placement of n rooks on an $n \times n$ chess board is *non-attacking* if there is at most one rook in each row and each column.

Let $B \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$. Let $r_{B,k}$ be the number of non-attacking rook placements of k rooks with all rooks on elements of B. Let $d_{B,k}$ be the number of non-attacking rook placements of k rooks with no rooks on elements of B.

(a) Prove that $d_{B,n} = [x^n] R_B(-x) \sum_{j \geq 0} j! x^j$ where $R_B(x) = \sum_{k=0}^{n} r_{B,k} x^k$ is the *rook polynomial.*

(b) Suppose B is $\{(i,j) : j = i, i+1 \mod n\}$. Give an expression for $d_{B,n}$ in terms of finite sums of binomial coefficients. *Hint: use elementary counting to find an expression for the* $r_{B,k}$.

(3) (a) Give an example of a finite poset P such that if ℓ is the length of the longest chain of P, then every $x \in P$ is contained in a chain of length ℓ yet P has a maximal chain of length $< \ell$.

(b) Show that if P is a finite poset with longest chain of length ℓ which has the property of the previous part, that is every $x \in P$ is contained in a chain of length ℓ, and additionally if for every y covering x in P there exists a chain of length ℓ containing both x and y, then every maximal chain of P has length ℓ.

(4) Suppose $\leq_1$ and $\leq_2$ are two total orders on the set P. The *intersection* of $\leq_1$ and $\leq_2$ is the partial order on P where $a \leq b$ in the partial order if and only if $a \leq_1 b$ and $a \leq_2 b$.

(a) Let P be a poset on the vertex set $\{1, 2, \ldots, n\}$. Show that the following two conditions are equivalent.

- P is the intersection of d linear orderings of $\{1, 2, \ldots, n\}$.
- P is isomorphic to a subposet of $\mathbb{Z}_{\geq 0}^d$ with the usual partial order (coordinatewise).

(b) Show further than when $d = 2$ the two conditions of the previous part are also equivalent to the property:

- There exists a poset Q on $\{1, 2, \ldots, n\}$ such that $x \neq y$ are comparable in Q if and only if they are incomparable in P.

(5) Let B be an algebra with maps m and u and a coalgebra with maps Δ and ϵ. Write down the conditions that show that Δ and ϵ are algebra homomorphisms. Write down the conditions that show that m and u are coalgebra morphisms. Observe these are the same. One thing you will need to know: the tensor product of two algebras $A \otimes B$ is itself an algebra via the multiplication $m_{A \otimes B} : A \otimes B \otimes A \otimes B \to A \otimes B$ given by $(m_A \otimes m_B) \circ T$ where $T : A \otimes B \otimes A \otimes B \to A \otimes A \otimes B \otimes B$ swaps the middle two factors, and likewise for the coproduct when making the tensor product of two coalgebras into a coalgebra.

Hint: this will be easiest if you write the conditions in terms of commutative diagrams.

(6) Find 8 units of typos in the lecture summaries or assignment solutions. A typo that causes no actual confusion (like a spelling error or a mathematical typo where the intent is entirely clear) is a single unit. A typo of significance is two units.