

CO 430/630 W26 ASSIGNMENT 2 SOLUTIONS

- (1) So this comes out of a paper of mine, see <https://arxiv.org/pdf/2412.14036> Lemma 4, but with a few simplifications. I won't repeat all the details, but here are the key points.

In the paper there are red and blue chords, but for this question there are only red ones, so any time the paper talks about red or blue, just think red.

- (a) This part of the question is explained in the three paragraphs before Lemma 4. The idea is if there's a bare point that tells you which side is the outside of all the red chords. If there is a black chord, that tells you which side is the outside of all the red chords. If there are at least two red chords then that also tells you which side is the outside (take a dual tree!). In the paper there's still one case that isn't covered, but in this question I got rid of that case for you by asking you to take $n \geq 3$.
- (b) The only difference here is you get rid of the explicit 2s in the two displays on p6 since those came from the two colours red and blue, where in this question there is only red. Otherwise the argument is the same. The key idea is either the root is bare in which case you have a sequence of bare points and non-crossing complete chord diagram pieces (which you know are Catalan), or you have a Catalan piece containing the root (which you can get the generating series for by pointing), and the rest is the same kind of sequence as the other case.
- (2) We'll use a transfer matrix. We need to keep track of the last two letters of these words.

The matrix is 7x7. We'll use the order $aa, ba, ca, bb, cb, ac, cc$ obtaining

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

so $\det(I - xA) = 1 - 2x - x^4 - x^5$. Now we just need enough initial terms to solve for a generic numerator which will be of degree at most 4 in this case. So by hand compute there is one string of length 0 in this class, three of length 1, $3 \cdot 3 - 2 = 7$ of length 2, $3 \cdot 3 \cdot 3 - 2 - 12 + 1 = 14$ of length 3 (a little inclusion exclusion, just pad out the forbidden ones of size 2 and be careful of the one overlap), and now the more annoying but fundamentally the same inclusion exclusion counting of length 4 giving $3 \cdot 3 \cdot 3 \cdot 3 - 6 + 2 - 6 - 3 \cdot 3 \cdot 3 - 3 \cdot 3 \cdot 3 + 4 + 6 + 2 = 29$.

Thus we have $(1 + 3x + 7x^2 + 14x^3 + 29x^4 + O(x^5))(1 - 2x - x^4 - x^5)$ equals the numerator, and so expanding we get

$$\frac{1 + x + x^2}{1 - 2x - x^4 - x^5}$$

for the generating series we want.

- (3) (a) Let's forget about the mod n part of the set up to start and we'll bring it back shortly. For many classes of permutations the transfer matrix method is not applicable because each a_i depends on all the others. However in this case there are only three choices for a_i namely $i - 1$, i or $i + 1$, and none of those could have been used prior to a_{i-2} , so a_i depends only on a_{i-1} and a_{i-2} making the problem suitable to the transfer matrix method.

So let's set it up. The digraph will have vertices pairs $(\alpha, \beta) \in \{-1, 0, 1\}^2$ for which it is possible to have $a_i - i = \alpha$ and $a_{i+1} - i - 1 = \beta$. An edge goes from (α, β) to (β, γ) if it is possible to have $a_i - i = \alpha, a_{i+1} - i - 1 = \beta, a_{i+2} - i - 2 = \gamma$. From a walk in this digraph we can reconstruct a unique permutation corresponding to it by taking $1 + \alpha_1, 2 + \alpha_2, \dots$ as the permutation in one line notation, provided that $\alpha_1 \neq -1$ and $\alpha_{n+2} \neq 1$.

Now let's bring back the cyclic condition - that is just saying the walks need to be closed walks, in the usual way.

Finally, then we compute the matrix just by checking the possibilities and we get

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the order is $(-1, -1), (-1, 0), (-1, 1), (0, 0), (0, 1), (1, -1), (1, 1)$.

$Q(\lambda) = \det(I - \lambda A) = (1 - \lambda)^2(1 - \lambda - \lambda^2)$ and so the generating series we want is

$$-\frac{\lambda Q'(\lambda)}{Q(\lambda)} = \frac{2\lambda}{1 - \lambda} + \frac{\lambda(1 + 2\lambda)}{1 - \lambda - \lambda^2}$$

Note there's a "standard" startup issue where the cyclic condition maybe doesn't quite do what you want on very small examples, so if you like you can adjust the generating series by hand to account for that, or not and just take the definition that this formalism gives you rather than that naive definition as to what the "right" answer is on those small examples.

- (b) This sequence is 2 more than the sequence of Lucas numbers (A000032) and those are interesting because they count many things including matchings in cycle graphs.
- (4) (a) These objects are sometimes called *vertebrates*, with the idea that the unique path joining the two marked points is the vertebral column and there is a rooted tree hanging off each vertex of the vertebral column. Note also the vertebral column is directed from the first marked point to the second marked point. The

two marked points of the vertebral column could coincide, but the vertebral column can't be empty. This gives the decomposition

$$\mathcal{V} = \mathcal{L}_{>0}[\mathcal{T}^\bullet]$$

- (b) We have $\mathcal{N} = \mathcal{S}[\mathcal{T}^\bullet]$ by thinking of an endofunction by drawing the graph where each element of the underlying set has a directed edge to its image under the endofunction. The resulting directed graphs are composed of directed cycles with trees hanging off them, which is what the decomposition above says. The empty endofunction comes when the empty permutation is composed (and then there's nothing to do in the composition). So

$$\mathcal{N}_{>0} = \mathcal{S}_{>0}[\mathcal{T}^\bullet]$$

- (c) We know $\mathcal{S}_{>0}$ and $\mathcal{L}_{>0}$ are numerically equivalent, so the previous two parts tell us that \mathcal{V} and $\mathcal{N}_{>0}$ are numerically equivalent. We can directly compute the number of endofunctions, on a set of size n there are n^n of them since each element can map to any other element. So for $n > 0$ the number of vertebrates on n vertices is also n^n . Now we defined $\mathcal{V} = (\mathcal{T}^\bullet)^\bullet$ so the number of vertebrates on n vertices is n^2 times the number of free trees on n vertices. Therefore the number of free trees on n vertices is n^{n-2} , which is Cayley's formula.
- (5) From the questions I have gotten I think there are many ways to do this, and that's fine. I didn't ask for a particular way, so any correct way is fine.

Using notation from the previous question we have $\mathcal{N} = \mathcal{S}[\mathcal{T}^\bullet]$, and rewriting permutations by cycle decomposition (set of cycles) we have $\mathcal{N} = \mathcal{E}[\mathcal{C}[\mathcal{T}^\bullet]]$.

In this decomposition, we can incorporate the fixed-point free condition by forcing the cycles to be of length at least 2.

$$\mathcal{N}^f = \mathcal{E}[\mathcal{C}_{\geq 2}[\mathcal{T}^\bullet]]$$

So $N^f(x) = \exp(\log(1/(1-T^\bullet(x))) - T^\bullet(x))$. Also from standard tree stuff $T^\bullet(x) = x \exp(T^\bullet(x))$ and now we can use LIFT to get

$$[x^n]N^f(x) = [x^n] \frac{1}{1-x} e^{-x} (1-x) e^{nx} = \frac{(n-1)^n}{n!}$$

- (6) First we want to find the exponential generating series of $\mathcal{E}_k[\mathcal{B}]$. Suppose X is a finite set of size n . Without loss of generality take $X = \{1, 2, \dots, n\}$. Then we have that $\mathcal{E}_k[\mathcal{B}]_X$ consists of elements of the form $\{B_1, \dots, B_k\}$ where B_1, \dots, B_k are \mathcal{B} -structures, and since labelling means that there can be no duplicates among the entries of the set, there are always exactly $k!$ lists (B_1, B_2, \dots, B_k) , (B_2, B_1, \dots, B_k) , etc giving $\{B_1, \dots, B_k\}$. The species of the lists of length k is \mathcal{B}^k , so we see that $|\mathcal{E}_k[\mathcal{B}]_X| = |\mathcal{B}_X^k|/k!$ and hence $\mathcal{E}_k[\mathcal{B}]$ has exponential generating series $B(x)^k/k!$.

From this we get that $\mathcal{E}[\mathcal{B}]$ has exponential generating series $\sum_{k \geq 0} B(x)^k/k! = \exp(B(x))$ as we expected.

Now $|\mathcal{A}[\mathcal{B}_X]| = \sum_{k \geq 0} |\mathcal{E}_k[\mathcal{B}]_X| |\mathcal{A}_k|$ since it consists of the set partitions of X with any number of parts $k \geq 0$ and for each such set partition we have an \mathcal{A} -structure on the set of the parts and a \mathcal{B} -structure on each part. So we have that the exponential

generating series of $\mathcal{A}[\mathcal{B}_X]$ is

$$\sum_{n \geq 0} |\mathcal{A}[\mathcal{B}_n]| \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{k \geq 0} |\mathcal{E}_k[\mathcal{B}]_n| |\mathcal{A}_k| \frac{x^n}{n!} = \sum_{k \geq 0} |\mathcal{A}_k| \frac{B(x)^k}{k!} = A(B(x))$$