CO 430/630 W26 ASSIGNMENT 2

DUE **FRIDAY** MARCH 6 AT 10AM IN CROWDMARK

Do any 4 of the 6 problems!

The point of the assignment is for you to learn by doing the problems. I'm sure you can look up answers to many of these in various ways, but you largely defeat the purpose and undercut your own learning if you look things up instead of thinking. Having said that, you may use any resources, animate or inanimate, **provided you credit them**. In particular credit any use of generative AI, even mundane uses like in generating tikz for figures.

(1) A *partial rooted chord diagram* on n points is a rooted chord diagram where not all the points are necessarily in a chord, so a not-necessarily-perfect matching of $1, 2, \dots, n$. A *cyclic interval* of $1, 2, \dots, n$ is either an interval of $1, 2, \dots, n$ or a set of the form $\{j, j+1, \dots, n, 1, \dots, i-1, i\}$ (an interval going across n, 1).

If we mark one end of a chord then that distinguishes the *inside* of the chord, namely the cyclic interval from the marked end to the other end, from the *outside* of the chord, namely the cyclic interval from the other end back to the marked end.

Consider those partial rooted chord diagrams which are noncrossing, where the chords are coloured either red or black, the red chords each have one end marked, all points strictly inside a red chord are part of a black chord, and all black chords are inside of some red chord.

Call the class of these diagrams for $n \geq 3$, $\mathcal{C}$.

   (a) Prove that given $C \in \mathcal{C}$, we can reconstruct which end of each red chord of C was marked just from the information of the chords and their colours.
   (b) Calculate the bivariate ordinary generating series of $\mathcal{C}$ counting by number of points and number of chords. You may use the expression for the generating series of noncrossing rooted chord diagrams without justification.

(2) Consider words on the alphabet $\{a, b, c\}$ with no ab, bc, bbb, or acb consecutive substrings. What is the ordinary generating series for such words counted by length? You may use your favorite computer algebra system for the determinants and polynomial manipulations.

(3) (a) Use the transfer matrix method to find the generating series for permutations where when written in one line notation $a_1 a_2 \cdots a_n$ we have that $a_i \equiv 0, \pm 1 \mod n$. *Hint, the choice of a_i really only depends on a_{i-1} and a_{i-2}, cyclically.*
   (b) Look up the counting sequence from the previous part on the OEIS and say one other interesting thing about it.

(4) Let's prove Cayley's formula that the number of labelled trees on $\{1, 2, \dots, n\}$ is n^{n-2} by species/symbolic method.
   (a) Let $\mathcal{T}$ be the species of trees (without root or order information, sometimes called free trees.) Let $\mathcal{V} = (\mathcal{T}^\bullet)^\bullet$. Give a decomposition of $\mathcal{V}$ using the fact that any element of $\mathcal{V}$ will have a unique path between the two pointed vertices.
   (b) Give a decomposition of nonempty endofunctions in terms of $\mathcal{S}$ and $\mathcal{T}$.

(c) Find a numerical equivalence between species involving the two previous parts and use this to prove Cayley's formula.

(5) Let $\mathcal{N}^f$ be the species of fixed point free endofunctions. Determine the exponential generating series of $\mathcal{N}^f$ and the number $|\mathcal{N}^f_X|$ for $|X| = n$. *Hint, part b of the previous question may be a good inspiration.*

(6) Let $\mathcal{A}$ and $\mathcal{B}$ be species with $\mathcal{B}$ connected. Prove that the exponential generating series of $\mathcal{A}[\mathcal{B}]$ is $A(B(x))$.