CO 330 LECTURE 7 SUMMARY

FALL 2026

Summary

We set up and proved the sum, product, and string lemmas that you saw in math 239 using the language we've developed in this course. Here are the statements.

We define
$$\mathcal{Z} = \{\bullet\}$$
to be the combinatorial class contaning one object of weight 1 and no other objects. We define
$$\mathcal{E}$$
to be the combinatorial class containing one object of weight 0 and no other objects.

Let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes. If $\mathcal{B} \cup \mathcal{C} = \emptyset$ then
$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$$
is a combinatorial class with weight function $w_{\mathcal{A}}(a) = \begin{cases} w_{\mathcal{B}}(a) & a \in \mathcal{B} \\ w_{\mathcal{C}}(a) & a \in \mathcal{C} \end{cases}$ and $A(x) = B(x) + C(x)$. Without any disjointness condition
$$\mathcal{A} = \mathcal{B} \times \mathcal{C}$$
is a combinatorial class with weight function $w_{\mathcal{A}}((b, c)) = w_{\mathcal{B}}(b) + w_{\mathcal{C}}(c)$ and $A(x) = B(x)C(x)$.

We use the notation $\mathcal{A}^k = \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{k \text{ times}}$ and $\mathcal{A}^0 = \mathcal{E}$.

If $\mathcal{A}$ is a combinatorial class with no element of weight 0 then
$$\mathcal{C} = \mathsf{Seq}(\mathcal{A}) = \mathcal{E} \cup \mathcal{A} \cup \mathcal{A}^2 \cup \cdots$$
is a combinatorial class with weight function $w_{\mathcal{A}}((a_1, \ldots, a_r)) = w_{\mathcal{A}}(a_1) + \cdots + w_{\mathcal{A}}(a_r)$ and $C(x) = \frac{1}{1-A(x)}$. This is our version of the string lemma.

We'll write
$$\mathsf{Seq}_{(\text{condition})}(\mathcal{A}) = \bigcup_{i \text{ satisfying condition}} \mathcal{A}^i.$$

We did plane rooted trees $\mathcal{T} \leftrightharpoons \mathcal{Z} \times \mathsf{Seq}(\mathcal{T})$ as an example.

Next time

Next time you will play with some examples.

References

- `https://enumeration.ca/toolbox/combinatorial-constructions/`
- Chapter 4 of `https://melczer.ca/330/WagnerNotes.pdf`.