CO 330 LECTURE 5 SUMMARY

FALL 2026

Summary

We continued discussing the generalized binomial theorem (which we'd given the preliminary defs for and statement of last time). The proof is by the formal analogue of Taylor's theorem.

There are a lot of little details to check and some of them I left for you as exercises or will put on the next assignment. Here are the pieces.

Definition 1. *Let R be an integral domain where all positive integers have inverses. Then the* formal exponential *in $R[[x]]$ is*

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

and the formal logarithm *in $R[[x]]$ is*

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n$$

The various exp and log rules are true but all need to be proved in this context. I left those as exercises for you. We also need to know when we can compose formal power series. This is important and we'll return to it in lecture very soon.

Next we defined the formal derivative

Definition 2. *Let R be an integral domain and let $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$. The* formal derivative *of A is*

$$A'(x) = \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n$$

Your usual calculus rules hold in this context (but you need to prove them all!) We proved the sum and product rule, both of which are simple calculations

$$(A(x)+B(x))' = \sum_{n\geq 0} n(a_n + b_n)x^{n-1} = \sum_{n\geq 0} (na_n + nb_n)x^n n - 1 = A'(x) + B'(x)$$

$$\begin{aligned}
(A(x)B(x))' &= \sum_{n\geq 0} n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^{n-1} \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n} (ka_k) b_{n-k} \right) x^{n-1} + \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k ((n-k) b_{n-k}) \right) x^{n-1} \\
&= A'(x)B(x) + A(x)B'(x)
\end{aligned}$$

Others are similar but more intricate. Some will be on your next assignment. Often for proving exp and log properties it can be useful to use the differential equation for exp, which is quite easy in the formal context:

$$\exp(x)' = \sum_{n\geq 0} \frac{(n+1)}{(n+1)!}x^n = \sum_{n\geq 0} \frac{x^n}{n!} = \exp(x)$$

Formal Taylor's theorem we only sketched but iterating the formal derivative (formally use induction) we get

$$A^{(k)}(x) = \sum_{n\geq 0} (n+1)(n+2)\cdots(n+k)a_{n+k}x^n$$

then evaluating at 0 to get the constant term we have $A^{(k)}(0) = k!a_k$ and solving for a_k gives the result.

The next topic we began was discussing when operations on formal power series are valid. This will let us finally talk about when composition makes sense and also explain why we can evaluate at $x = 0$ even though we can't generally evaluate formal power series.

This approach is a slightly quirky approach to the issue which I think avoids certain misunderstandings, but we'll also discuss the more typical approach next time.

We say an operation $\mathcal{F} : R[[x]]^k \to R[[x]]$ is *valid* if for all $n \in \mathbb{Z}_{\geq 0}$ there exists an $m \in \mathbb{Z}_{\geq 0}$ such that for any input series $f_i(x) = \sum_{n\geq 0} a_n^{(i)} x^n$ the coefficient

$$[x^n]\mathcal{F}(f_1(x), f_2(x), \ldots, f_k(x))$$

depends only on $a_0^{(1)}, a_1^{(1)}, \ldots, a_m^{(1)}, a_0^{(2)}, a_1^{(2)}, \ldots, a_m^{(2)}, \ldots, a_0^{(k)}, a_1^{(k)}, \ldots, a_m^{(k)}$.

The point is that to calculate a particular output coefficient you only need a finite amount of the input. We discussed how $+$ and $\cdot$ are valid (both with $m = n$). We discussed how formal derivative is also valid (in this case with $m = n + 1$). We also discussed how evaluation at $x = 1$ is not valid since to calculate the value we need an infinite number of input coefficients.

Next time

Next time you'll think together in groups about validity. We will finally talk about composition and then we'll talk about convergence in the ring of formal power series (which is the more conventional way to deal with these issues around validity.)

References

- Here's a different set of notes which covers the part with the generalized binomial theorem quite well: `https://arxiv.org/abs/2205.00879`.
- Other regerences as last time: `https://enumeration.ca/toolbox/generating-functions/`
- The beginning of chapter 7 of `https://melczer.ca/330/WagnerNotes.pdf`.