CO 330 LECTURE 27 SUMMARY

FALL 2026

Summary

We began our discussion of Boltzmann sampling.

For this section we will need to think of the generating function not just as a formal power series but also as series in the sense of calculus. Since the operations agree, so long as the series does converge, we can do this.

We noted some examples that many of the generating series we like have nonzero radius of convergence.

Now for a little probability set up.

Definition 1. *A* discrete probability space *with sample space $\mathcal{S}$ and probability function P is a countable set $\mathcal{S}$ and a function P from the set of subsets of $\mathcal{S}$ to $[0,1]$ satisfying*

- $P(\mathcal{S}) = 1$
- *If* $A = \{a_1, a_2, \ldots\}$ *then* $P(A) = \sum_{i=1}^{\infty} P(a_i)$.

This is strictly simpler than the general probability space situation, but is what we need.

Note that in order to define P it suffices to specify $P(s)$ for all $s \in \mathcal{S}$.

We want a particular discrete probability space where each element of a combinatorial class is weighted by its contribution to the generating function.

Definition 2. *Let $\mathcal{C} \neq \emptyset$ be a combinatorial class and let $\rho > 0$ be the radius of convergence of $C(x)$. The* Boltzmann model *at x for $0 < x < \rho$ is a discrete probability space with sample space $\mathcal{C}$ and probability function given by*

$$P_x(c) = \frac{x^{|c|}}{C(x)} \quad \text{for } c \in \mathcal{C}$$

We checked that this is a discrete probability space. We also checked that probabilities are uniform by size in the Boltzmann model in the following sense:

Proposition 3. *Let $\mathcal{C} \neq \emptyset$ be a combinatorial class and $0 < x < \rho$ where ρ is the radius of convergence of $C(x)$. Let P_x be the probability function of the Boltzmann model. Suppose $c, c' \in \mathcal{C}$ with $|c| = |c'|$ then $P_x(c) = P_x(c')$.*

The proof is direct from the definitions, and also by a direct calculation we can calculate the expected size

Proposition 4. *Let $\mathcal{C}$, ρ, x, P_x be as above. The expected size is*

$$E_x = x\frac{C'(x)}{C(x)}$$

If our combinatorial class has

$$\lim_{x \to \rho^-} x \frac{C'(x)}{C(x)} = \infty$$

then we're in good shape because for n sufficiently large we'll be able to solve for a value of x such that $E_x = n$ which means that the Boltzmann model at x will have expected size n. Now in class, I said this slightly wrong, becuse I said that $\lim_{x \to 0+} x \frac{C'(x)}{C(x)} = 0$, but that's not true. If $c_0 \neq 0$ then it is true that $\lim_{x \to 0+} x \frac{C'(x)}{C(x)} = \frac{0 \cdot C'(0)}{c_0} = 0$, but if $c_0 = 0$ then let j be the smallest index with $c_j \neq 0$, then we have $\lim_{x \to 0+} x \frac{C'(x)}{C(x)} = \frac{c_j}{c_j} = 1$, so we're fine to solve $E_x = n$ for any integer $n > 0$, but I didn't say that last part about $c_0 = 0$ in class and I should have.

Next time

Next time we'll write code to actually build a Boltzmann sampler.

References

- `https://enumeration.ca/asymptotics/random-generation/`