Preserving and Generalizing χ-boundedness

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The content of this thesis is based on the following:

(a) Two coauthored papers [24, 25], in which I played a major role in producing:

1. Graphs of bounded chordality
   Aristotelis Chaniotis, Babak Miraftab, and Sophie Spirkl
   The Electronic Journal of Combinatorics (2025).
2. Intersections of graphs and χ-boundedness
   Aristotelis Chaniotis, Hidde Koerts, and Sophie Spirkl
   Submitted. Manuscript available at: arxiv:2504.00153

(b) Two ongoing projects:

1. An ongoing project which is joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.
2. An ongoing project which is joint work with Bartosz Walczak.

Several passages are taken verbatim from [24] and [25]. I am the sole writer of these passages in [24] and [25].

Abstract

The notion of χ-boundedness, introduced by Gyárfás in the mid-1980s, captures when, for every induced subgraph of a graph, large chromatic number can occur only due to the presence of a sufficiently large complete subgraph. The study of χ-boundedness is a central topic in graph theory. Understanding which hereditary classes of graphs are χ-bounded is of particular importance for advancing our understanding of how restrictions on the induced subgraphs of a graph affect both its global structure and key parameters such as the clique number and the independence number.

Which classes of graphs are χ-bounded? A method that has been used to prove that a class $\mathcal{C}$ of graphs is χ-bounded proceeds as follows: We prove that $\mathcal{C}$ can be obtained by applying operations that preserve χ-boundedness to already χ-bounded classes. This approach gives rise to the following question: Which operations preserve χ-boundedness?

Given k graphs $G_1, \ldots, G_k$, their *intersection* is the graph $\Big(\cap_{i\in[k]} V(G_i), \cap_{i\in[k]} E(G_i) \Big)$. Given k graph classes $\mathcal{G}_1, \ldots, \mathcal{G}_k$, we call the class $\{G : \forall i \in [k], \exists G_i \in \mathcal{G}_i$ such that, $G = G_1 \cap \ldots \cap G_k\}$ the *graph-intersection* of $\mathcal{G}_1, \ldots, \mathcal{G}_k$. In the mid-1980s, in his seminal paper *"Problems from the world surrounding perfect graphs"*, Gyárfás observed that, due to early results of Asplund and Grünbaum, and Burling, graph-intersection does not preserve χ-boundedness in general, and he raised some questions regarding the interplay between graph-intersection and χ-boundedness. This topic has not received much attention since then. In this thesis, we formalize and explore the connection between the operation of graph-intersection and χ-boundedness.

Let $r \geq 2$ be an integer. We denote by K_r the complete graph on r vertices. The *K_r-free chromatic number* of G, denoted by $\chi_r(G)$, is the minimum size of a partition of $V(G)$ into sets each of which induces a K_r-free graph. Generalizing the notion of χ-boundedness, we say that a class of graphs $\mathcal{C}$ is *χ_r-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$ and every induced subgraph G' of G, we have $\chi_r(G') \leq f(\omega(G'))$, where $\omega(G')$ denotes the clique number of G'. We study the induced subgraphs of graphs with large K_r-free chromatic number.

Finally, we introduce the *fractional K_r-free chromatic number*, and for every $r \geq 2$, we construct K_{r+1}-free intersection graphs of straight line segments in the plane with arbitrarily large fractional K_r-free chromatic number.

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I am deeply grateful to the inter-institutional Graduate Program "Algorithms, Logic and Discrete Mathematics" (ALMA)[1] in Athens. I learned a lot, grew as a mathematician, and had an amazing experience during my studies in ALMA — an experience that was crucial

[1]ALMA is a graduate program leading to a master's degree in the areas of Algorithms, Logic, and Discrete Mathematics. This program is co-organized by the Department of Informatics and Telecommunications and the Department of Mathematics, of the National and Kapodistrian University of Athens, together with the School of Electrical and Computer Engineering, and the School of Applied Mathematical and Physical Sciences, of the National Technical University of Athens.

to my decision to pursue a Ph.D. I would particularly like to thank Panos Rondogiannis, the Director of ALMA during my time in the program, for encouraging me to pursue a Ph.D. and for supporting me to do so. Thank you also to Dimitris Zoros for his contribution to ALMA and in particular for his teaching.

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Finally, I believe that I would go beyond the scope of this brief note if I tried to express my gratitude and love for my parents, Katerina and Nikos. This thesis is dedicated to them.

To my parents,

Katerina and Nikos.

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Part I

Introduction

Chapter 1

Preliminaries

In this chapter, we present some basic definitions, notations, and well-known results that are used throughout this thesis. Additional specialized definitions and notations will be introduced in later chapters, just before they are first used. For terminology and notation not defined in this thesis, we refer the reader to [132].

1. Sets. We denote by $\mathbb{R}$ the set of real numbers, and by $\mathbb{R}_+$ the set of positive real numbers. We denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{N}_+$ the set of positive integers. Let k and k' be positive integers with $k < k'$. We denote the set $\{n \in \mathbb{N}_+ : k \leq n \leq k'\}$ by $[k, k']$. For the set $[1, k']$ we often write $[k']$. Let t be a positive integer. We denote by $[k]^t$ the set $\{(n_1, \ldots, n_t) : \forall i \in [t], n_i \in [k]\}$. Let X be a set. A *k-subset of* X is a subset of X which has size k.

2. Graphs. Unless otherwise stated, graphs in this thesis are finite, undirected, and have no loops or parallel edges. Let G be a graph. We denote the complement of G by G^c. We often denote an edge $\{u, v\} \in E(G)$ by uv.

Let $X \subseteq V(G)$. We denote the subgraph of G which is induced by X, by $G[X]$. Let H be a graph. We say that G is *H-free* (respectively *contains* H) if it contains no (respectively contains an) induced subgraph isomorphic to H. Let $\mathcal{H}$ be a set of graphs. We say that G

is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. We note that throughout this thesis we use the terms *class* and *family* as synonyms for the term "set". We denote by $\{\mathcal{H}\text{-free graphs}\}$ the class of all $\mathcal{H}$-free graphs. We say that a class of graphs is *hereditary* if it is closed under isomorphism and under taking induced subgraphs.

The *neighborhood of X in G*, denoted by $N_G(X)$, is the set $\{u \in V(G) : \exists v \in X, uv \in E(G)\}$. The *closed neighborhood of X in G*, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. When $X = \{u\}$ we write $N_G(u)$ and $N_G[u]$ instead of $N_G(\{u\})$ and $N_G[\{u\}]$. We use $A_G(X)$ to denote the set $V(G) \setminus N_G[X]$. When there is no danger of ambiguity, we omit the subscripts from the notations of neighborhoods and from $A_G(X)$.

For a positive integer t we denote by K_t, P_t, and C_t a complete graph, a path, and a cycle on t vertices, respectively. We note that the *length of a path* is the number of its edges.

The *girth* of G, denoted by $\mathsf{girth}(G)$, is the length of a shortest cycle of G. The *clique number* (respectively the *independence number*) of G, denoted by $\omega(G)$ (respectively by $\alpha(G)$), is the size of a largest clique (respectively independent set) of G. Let r be a positive integer. An *r-clique* of G is a clique of G that have size r. A *triangle* in G is a subgraph induced by a clique of size three.

Let k be a positive integer. A *k-coloring* of G is a function $f : V(G) \to [k]$. A *coloring* of G is a k'-coloring of G for some positive integer k'. Let $r \geq 2$ be an integer. A *K_r-free coloring* of G is a coloring of G such that every color class induces a K_r-free graph. The minimum k such that G has a K_2-free k-coloring is the *K_2-free chromatic number of G*. Equivalently, the K_2-free chromatic number of G is the minimum size of a partition of $V(G)$ into independent sets. When there is no danger of ambiguity, we refer to the K_2-free chromatic number of G simply as the *chromatic number* of G. We denote the K_2-free chromatic number of G by $\chi(G)$. A *k-edge-coloring* of G is a function $f : E(G) \to [k]$. An *edge-coloring* of G is a k-edge-coloring for some k.

Let A and B be disjoint subsets of $V(G)$. We say that A and B are *complete* (respectively *anticomplete*) if for every $a \in A$ and $b \in B$ we have $ab \in E(G)$ (respectively $ab \notin E(G)$). If A and B are not anticomplete, we say that they are *adjacent*. A graph G is *k-partite* if $\chi(G) \leq k$. The graph G is *complete k-partite* if its vertex set can be partitioned into a family of k non-empty independent sets which are pairwise complete to each other. Finally, we say that G is *complete multipartite* if there exists a positive integer k such that G is a complete k-partite graph. A *complete bipartite* graph is a complete 2-partite graph. We use $K_{s,t}$ to denote a complete bipartite graph with parts of sizes s and t. The *biclique number* of G, denoted by $\mathsf{biclique}(G)$, is the largest positive integer t such that G contains $K_{t,t}$ as a (not necessarily induced) subgraph. A *star* is a $K_{1,t}$ for some positive integer t. For a positive integer k and a graph G we denote by kG the graph that consists of k vertex-disjoint copies of G.

3. Results from Ramsey theory. We will need the following classical results from Ramsey theory.

Theorem 1.1 (Ramsey [109])**.** *Let s and t be positive integers. Then there exists an integer $n = n(s,t)$ such that if G is a graph on n vertices, then G contains either a clique of size s or an independent set of size t.*

The *Ramsey number* $R(s,t)$ is the minimum integer such that every graph on $R(s,t)$ vertices contains either a clique of size s or an independent set of size t.

Theorem 1.2 (Ramsey [109])**.** *Let k and t be positive integers. Then there exists an integer $n = n(k,t)$ such that if G is a complete graph on n vertices, then every k-edge-coloring of G results in a monochromatic clique of size t.*

Let k and t be positive integers. We denote by $R_k(t)$ the minimum positive integer such that if G is a complete graph on $R_k(t)$ vertices, then every k-edge-coloring of G results in a monochromatic clique of size t.

Chapter 2

Background and motivation

The study of graph parameters that serve as measures of structural complexity for graphs plays a major role in structural graph theory. A main goal of such a study is to understand those graphs that are highly complex with respect to the graph parameter under study.

The graph parameter which motivates the work of this thesis is the chromatic number. Graphs of small chromatic number have a simple global structure in the sense that such graphs admit a partition into a few trivial pieces. But the notion of chromatic number does not give us an "easy" insight into the structure of graphs of large chromatic number. Questions surrounding the understanding of graphs of large chromatic number are among the most challenging in graph theory. An example of such a question is the famous Hadwiger's conjecture [73] from 1943 which is among the most important open problems in graph theory, and states that if the chromatic number of a graph G is greater than t then G contains a complete graph on $t + 1$ vertices as a minor. Here we are interested in the *local structure* of graphs of large chromatic number. By choosing the notion of induced subgraphs as that of "local structure" of a graph, the main motivation for our work comes from the following question:

Question 1. *What are the graphs whose presence as induced subgraphs is unavoidable in graphs of sufficiently large chromatic number?*

2.1 Chromatic number, complete subgraphs, and χ-boundedness

Since, in a K_2-free coloring, the vertices of a clique need to be pairwise colored with distinct colors, it follows that the chromatic number of a graph is at least its clique number. But, as the example of odd cycles of length at least five witnesses, there are graphs G for which $\chi(G) > \omega(G)$, and thus the clique number of a graph does not determine its chromatic number. Still, in the example of odd cycles the gap between the chromatic number and the clique number is just one, and thus we may expect, as an answer to Question 1, that the presence of a large complete subgraph is unavoidable in graphs of sufficiently large chromatic number. This was disproved, in the late 1940s, in a strong sense: Tutte (alias Blanche Descartes) [42, 43] and Zykov [135], independently of each other, proved that the gap between χ and ω can be arbitrarily large even when $\omega(G) = 2$, and thus there exists no function f such that $\chi(G) \leq f(\omega(G))$ for every graph G.

Nevertheless, by interpreting the chromatic number as a measure of structural complexity of a graph, it is reasonable to suspect/expect that the presence of highly complex induced subgraphs is unavoidable in sufficiently complex graphs. As we will see below, this is not true, again in a strong sense. To get a first glimpse of this phenomenon, we note that Tutte's triangle-free graphs of arbitrarily large chromatic number that we mentioned above [43] (also rediscovered by J. B. Kelly [80, Theorem 5.1]) have girth at least six.

Theorem 2.1 (Tutte [43] and J. B. Kelly [80]). *For every positive integer k there exists a graph G_k such that $\chi(G_k) \geq k$ and $\mathsf{girth}(G_k) \geq 6$.*

So if we are very demanding in our notion of "local" and we are looking at the induced subgraphs with at most five vertices, then it is not true that there exists k such that in graphs of chromatic number at least k these induced subgraphs are "complex". In fact Theorem 2.1 guarantees that for every k there exists a graph G_k such that $\chi(G_k) \geq k$, and all the induced subgraphs of G_k of size at most five are forests and thus have chromatic number at most two.

In 1955, Mycielski [96] presented another construction of triangle-free graphs of large chromatic number, and asked the following question:

Question 2 (Mycielski [96, P131]). *Does there exist, for every pair of natural numbers n and $m \geq 3$, a finite graph that cannot be colored with n colors and that contains, for all $k \in \{3, \ldots, m\}$, no cycle of length k?*

In 1959, Erdős used probabilistic arguments to answer Mycielski's question in the affirmative, yielding what is now a classic result in combinatorics:

Theorem 2.2 (Erdős [47]). *Let $k, l \geq 2$ be integers. Then there exists a graph G with $\chi(G) \geq k$ and $\mathsf{girth}(G) > l$.*

Thus, even if we lower our expectations and we examine larger induced subgraphs of graphs of large chromatic number in order to find "complex local structure," our search (at least in general graphs) is hopeless. To be more precise: By Theorem 2.2, we have that for every choice of l there exist graphs of arbitrarily large chromatic number whose induced subgraphs of size l are forests and thus have chromatic number at most two. In [47] Erdős raised the question for a constructive proof of Theorem 2.2. We should note that very few constructions of graphs of arbitrarily large girth and chromatic number are known. The first one was given by Lovász [91] in 1968 who proved a much more general result for uniform hypergraphs (we discuss his result later on in Section 2.2); for more on these constructions we refer the interested reader to the survey of Nešetřil [97] on the topic.

Recall that in Question 1 we asked *What are the graphs whose presence as induced subgraphs is unavoidable in graphs of sufficiently large chromatic number?* The question remains! But now we have an important new insight: No matter how large the chromatic number of a graph G, we cannot expect to find a single induced subgraph of G of a given bounded size and chromatic number greater than two; large chromatic number is not necessarily a local phenomenon in this sense. The complexity of a graph of large chromatic number may be "spread out" in the graph, and the way that this complexity affects the induced subgraphs of the graph is not necessarily by resulting in a "complex" induced subgraph of bounded size but maybe by imposing a large variety of induced subgraphs that our graph of large chromatic number contains. The above discussion motivates the following question:

Question 3. *What graphs are guaranteed to appear as induced subgraphs in any graph that has sufficiently large chromatic number, while all of its induced subgraphs of bounded size have small chromatic number?*

Here we are interested in the following more specific question:

Question 4. *What graphs are guaranteed to appear as induced subgraphs in any graph that has bounded clique number but a sufficiently large chromatic number?*

We begin with the graphs for which the chromatic number is determined exactly by their clique number. Let G be a graph. We say that G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. What graphs are perfect? As we already mentioned above, the example of odd cycles of length at least five shows that not all graphs are perfect. Hence, the class of perfect graphs is defined by the set of minimal, with respect to induced subgraphs, graphs which are not perfect. This set contains odd cycles of length at least five. The complements of odd cycles of length at least five are also minimal non-perfect graphs, and thus belong in the set of minimal forbidden induced subgraphs of the class of perfect graphs. It was conjectured by Berge [8] in 1961, and proved by Chudnovsky, Robertson, Seymour, and Thomas in 2006, that odd cycles of length at least five and their complements are the only minimal non-perfect graphs:

Theorem 2.3 (Chudnovsky, Robertson, Seymour, and Thomas [30])**.** *A graph is perfect if and only if it contains neither an odd cycle of length at least five nor the complement of such a cycle as an induced subgraph.*

Theorem 2.3 is known as the "Strong Perfect Graph Theorem." So perfect graphs are well-understood, and thus we can add to the list of graphs of Question 4 the odd cycles of length at least five and their complements: At least one such graph appears as an induced subgraph in every graph graph G with $\chi(G) > \omega(G)$.

What about graphs whose chromatic number can only be large due to the presence of a sufficiently large complete subgraph? In 1985, in his seminal paper *"Problems from the world surrounding perfect graphs,"* Gyárfás introduced the notion of χ-boundedness which captures this phenomenon. Following Gyárfás [71] we say that a class of graphs $\mathcal{C}$ is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, we have $\chi(G') \leq f(\omega(G'))$ for every induced subgraph G' of G; in this case f is called a *χ-bounding function* for $\mathcal{C}$. If f can be chosen to be polynomial, then $\mathcal{C}$ is *polynomially χ-bounded*. We note that χ-bounded classes are sometimes called "near-perfect" in the literature.

Question 5. *Which classes of graphs are χ-bounded?*

The existence of triangle-free graphs of arbitrarily large chromatic number shows that the class of all graphs is not χ-bounded. Thus, every hereditary χ-bounded class $\mathcal{C}$ of graphs is described by the, possibly infinite, set of minimal induced subgraphs which do not belong in $\mathcal{C}$. Hence, a possible approach to Question 5 is to try to answer the following:

Question 6. *For which sets of graphs $\mathcal{H}$ is true that the class of all $\mathcal{H}$-free graphs is χ-bounded?*

For example, by the Strong Perfect Graph Theorem, we have that if $\mathcal{H}$ is the set of all odd induced cycles of length at least five and their complements, then the class of all $\mathcal{H}$-free graphs is the class of all perfect graphs, which in turn is χ-bounded by the identity function. Despite the great progress that has been made in the study of χ-boundedness in the last years (see [117, 118] for recent surveys), we are still far from a good understanding of χ-bounded classes of graphs in general, and in particular we are far from answering Question 6.

But something very interesting can be said for the case that the set $\mathcal{H}$, in Question 6, is required to be finite. Theorem 2.2 of Erdős implies that $\mathcal{H}$ should contain a forest. Why is that? Suppose that every graph in $\mathcal{H}$ contains a cycle, and let $g := \max\{g(H) : H \in \mathcal{H}\}$. Let $\mathcal{C}$ be the class of all graphs of girth at least $g+1$. Then for every graph $G \in \mathcal{C}$ we have that $\omega(G) \leq 2$, and by Theorem 2.2 we know that $\mathcal{C}$ contains graphs of arbitrarily large chromatic number. Thus $\mathcal{C}$ is not χ-bounded, but it is $\mathcal{H}$-free. The above observation has the following corollary:

Proposition 2.4. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is χ-bounded, then $\mathcal{H}$ contains a forest.*

Gyárfás [69] and Sumner [125] in the mid 1970s and early 1980s respectively, conjectured that the necessary condition of Proposition 2.4 is also sufficient:

Conjecture 2.5 (Gyárfás [69] and Sumner [125])**.** *Let H be a forest. Then the class of all H-free graphs is χ-bounded.*

This is known as the Gyárfás-Sumner Conjecture, and is one of the most influential and challenging conjectures in the area of χ-boundedness. It has been proved for some simple types of forests but is widely open in general. Before discussing the progress on and around the conjecture, let us make two remarks: First, that in order to prove Conjecture 2.5 it suffices to prove it for trees. This follows from the fact that every forest F is an induced subgraph of a tree T, and thus the class of F-free graphs is contained in the class of T-free graphs. Second, that if we ask for subgraphs instead of induced subgraphs then the question becomes very easy (see [72] for a proof). Strengthening this fact Nguyen, Scott and Seymour [100] recently proved that if in Conjecture 2.5 we ask for path-induced[1] instead of induced trees, then the answer is positive.

Despite considerable efforts, the Gyárfás-Sumner Conjecture has only been proved for some simple trees. It has been proved for paths[2], stars and brooms[3] by Gyárfás [71]; for trees of radius two, where Gyárfás, Szemeredi, and Tuza [72] proved the result in the class of triangle-free graphs, and Kierstead and Penrice [81] proved the general case; for trees that can be obtained from a tree of radius two by subdividing exactly once every edge incident with the root by Kierstead and Zhu [82]; for trees that can be obtained from a tree of radius two by subdividing exactly once some edges incident with the root by Scott and Seymour [119]; for trees that can be obtained by joining two disjoint paths by an edge by Spirkl [123]; and for two other simple families of trees[4] by Chudnovsky, Scott, and Seymour [33].

Let G and G' be graphs. If there exist an edge $uv \in E(G)$ and vertex $w \notin V(G)$ such that $G' = (V(G) \cup \{w\}, (E(G) \setminus \{uv\}) \cup \{uw, wv\})$, then we say that G' can be obtained from G by *subdividing the edge* uv, or simply by an *edge-subdivision*. If there exists a positive integer k and a sequence of graphs $G = G_1, \dots, G_k = G'$ such that for each $i \in [2, k]$ the graph G_i can be obtained by an edge-subdivision from the graph G_{i-1}, then we say that G' is a *subdivision* of G.

Probably the most important weakening of Conjecture 2.5 that is known to be true is the following result by Scott:

[1]If a tree T is a subgraph of a graph G, and there exists a vertex $r \in V(T)$, such that every path of T with r as one of its ends is an induced path of G we say that T is path-induced.

[2]Gerlits (see [69]) first proved that every triangle-free graph which excludes a fixed path as an induced subgraph has bounded chromatic number.

[3]A broom is a graph which is obtained by identifying the center of a star with the end of a path.

[4]Chudnovsky, Scott, and Seymour [33] observed that all the trees for which the conjecture was known before their result have the property that their vertices of degree greater than two have pairwise small distance. In [33] they prove the conjecture for two families of trees that do not have this property.

Theorem 2.6 (Scott [121]). *Let T be a tree. Then the class of all graphs that contain no subdivision of T as an induced subgraph is χ-bounded.*

Rödl proved, but did not publish (see the discussion in [81]), that the Gyárfás-Sumner Conjecture is true in the classes of graphs that exclude a complete bipartite graph as a subgraph. We note that Rödl's original proof of this result was later published in [83, Theorem 2.3].

Recall that we denote by $\mathsf{biclique}(G)$ the largest t, for which G contains a subgraph (not necessarily induced) which is isomorphic to $K_{t,t}$.

Theorem 2.7 (Rödl). *For every forest H, there exists a function f such that $\chi(G) \leq f(\mathsf{biclique}(G))$ for every H-free graph G.*

Kierstead and Penrice [84] strengthened the above result. We need a few definitions. Given an integer $k \geq 0$, we say that a graph G is *k-degenerate* if every subgraph of G has a vertex of degree at most k. The *degeneracy* of G, denoted by $\mathsf{degen}(G)$, is the minimum k for which G is k-degenerate. It is easy to see that $\chi(G) \leq \mathsf{degen}(G) + 1$.

Theorem 2.8 (Kierstead and Penrice [84]). *For every forest H, there exists a function f such that $\mathsf{degen}(G) \leq f(\mathsf{biclique}(G))$ for every H-free graph G.*

Bonamy, Bousquet, Piłipczuk, Rzazewski, Thomassè, and Walczak [11] proved that the above theorem holds with polynomial bounds when H is a path. Scott, Seymour, and Spirkl [120] proved that the Gyárfás-Sumner Conjecture holds with polynomial bounds in the classes of graphs which exclude a complete bipartite graph as a subgraph:

Theorem 2.9 (Scott, Seymour, and Spirkl [120]). *For every forest H, there exists $c > 0$ such that $\mathsf{degen}(G) \leq \mathsf{biclique}(G)^c$ for every H-free graph G.*

In this thesis, we propose a conjecture that extents the Gyárfás-Sumner Conjecture for the K_r-free chromatic number. This conjecture drives much of the work presented in this thesis. In particular, one of our results is a generalization of Theorem 2.8: We prove an analogous result for our conjecture (which extends the Gyárfás-Sumner Conjecture). We will discuss both our conjecture and the relevant results in Section 3.2.

Most of the results on and around the Gyárfás-Sumner Conjecture that we discuss above have been proved using a mixture of Ramsey-theoretic and structural techniques, where by structural techniques here we mean some frameworks for exploring the structure of a graph (see [118]).

Another approach that can be and has been used in order to prove that a class of graphs $\mathcal{C}$ is χ-bounded, is as follows: *We prove that $\mathcal{C}$ can be obtained as the result of the application of an operation which preserves χ-boundedness in χ-bounded classes.* We should say that this approach is an instance of a more general and very successful approach in structural graph theory where graphs belonging to a class that we want to understand are "decomposed" via well-understood decompositions/operations into "simpler" graphs. For example, this "decomposition approach" is the approach that has been used for the proof of the two cornerstones of structural graph theory: The Graph Minor Structure Theorem [112, 113], and the Strong Perfect Graph Theorem that we mentioned above (for a discussion of this approach in the proof of the Strong Perfect Graph Theorem we refer the reader to [122]). This discussion motivates the following question:

Question 7. *Which operations preserve χ-boundedness?*

Among operations that are known to preserve χ-boundedness are the following: gluing along a clique, gluing along a bounded number of vertices [5] (see also [64]), substitution [29], 1-joins [12, 45], and amalgams [105]. In this thesis we consider the operation of the graph-intersection between classes of graphs, which arises from the operation of intersection among a finite number of graphs.

Let k be a positive integer. Given k graphs $G_1, \dots, G_k$, their intersection (respectively union) is the graph $\left(\cap_{i \in [k]} V(G_i), \cap_{i \in [k]} E(G_i)\right)$ (respectively $\left(\cup_{i \in [k]} V(G_i), \cup_{i \in [k]} E(G_i)\right)$). Given k graph classes $\mathcal{G}_1, \dots, \mathcal{G}_k$, we call the class $\{G : \forall i \in [k], \exists G_i \in \mathcal{G}_i \text{ such that } G = G_1 \cap \dots \cap G_k\}$ (respectively, the class $\{G : \forall i \in [k], \exists G_i \in \mathcal{G}_i \text{ such that } G = G_1 \cup \dots \cup G_k\}$) the *graph-intersection* (respectively *graph-union*) of $\mathcal{G}_1, \dots, \mathcal{G}_k$, and we denote this class by $\mathcal{G}_1 \mathbin{\dot{\cap}} \dots \mathbin{\dot{\cap}} \mathcal{G}_k$ (respectively $\mathcal{G}_1 \mathbin{\dot{\cup}} \dots \mathbin{\dot{\cup}} \mathcal{G}_k$). Given the definitions of these operations and the discussion above about operations which preserve χ-boundedness, it is natural to ask whether or not these operations preserve χ-boundedness.

Graph-union preserves χ-boundedness. We begin by discussing a well-known way for obtaining a coloring for the union of a finite number of graphs by making use of given colorings of the individual graphs. Let k be a positive integer, let $G_1, \dots, G_k$ be graphs on the same vertex set V. For each $i \in [k]$, let $f_i : V \to [k_i]$ be a k_i-coloring of G_i. Then the *product coloring obtained from* $f_1, \dots, f_k$ is the function $\prod_{i\in[k]} f_i : V \to \{(j_1, \dots, j_k) : j_i \in [k_i],\ \text{for every } i \in [k]\}$ which is defined as follows: $\prod_{i\in[k]} f_i(v) := (f_1(v), \dots, f_k(v))$, for every $v \in V$. The following well-known result states that if the colorings of the individual graphs are proper, then the corresponding product coloring is proper as well:

Proposition 2.10 (Folklore)**.** *Let k be a positive integer, let $G_1, \dots, G_k$ be graphs on the same vertex set V, and for each $i \in [k]$, let $f_i : V \to [k_i]$ be a proper k_i-coloring of G_i. Then the product coloring obtained from $f_1, \dots, f_k$ is a proper $\left(\prod_{i\in[k]} k_i\right)$-coloring of $G_1 \cup \ldots \cup G_k$. In particular, $\chi(\cup_{i\in[k]} G_i) \leq \prod_{i\in[k]} \chi(G_i)$.*

The following, which states that graph-union preserves χ-boundedness, is an immediate corollary of Proposition 2.10 and of the fact that $\omega(G_1 \cup \ldots \cup G_k) \geq \max_{i\in[k]}\{\omega(G_i)\}$:

Proposition 2.11 (Gyárfás [71, Proposition 5.1 (a)])**.** *If $\mathcal{G}_1, \dots \mathcal{G}_k$ are χ-bounded classes of graphs with χ-bounding functions $f_1, \dots, f_k$ respectively, then $\mathcal{G}_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \mathcal{G}_k$ is a χ-bounded family and $f(x) := \prod_{i\in[k]} f_i(x)$ is a suitable χ-bounding function.*

What about graph-intersection? In this thesis we study the following two questions:

Question 8. *Is it always true that the graph-intersection of two χ-bounded classes of graphs is χ-bounded?*

Question 9. *Is it always true that if a class $\mathcal{A}$ is χ-bounded, then for every positive integer k the class $\dot\bigcap_{i\in[k]}\mathcal{A}$ is χ-bounded?*

To the best of our knowledge, Gyárfás [71, Section 5] first considered the interplay between graph-intersection and χ-boundedness. In recent work, Adenwalla, Braunfeld, Sylvester, and Zamaraev [3] considered this topic in the context of their broader study on Boolean combinations of graphs. Since, as we will see later on in Section 3.1, graph-intersection does not preserve χ-boundedness in general, we focus on understanding under which conditions it does.

Before going to the next section in which we consider the K_r-free chromatic number, we discuss how this notion appears in the study of polynomial χ-boundedness.

Recall that a class $\mathcal{C}$ of graphs is said to be polynomially χ-bounded if $\mathcal{C}$ has a polynomial χ-bounding function. Thus, in polynomially χ-bounded classes, the clique number controls the chromatic number -and so the global structure of the graph- more efficiently. Due to this fact, graphs in polynomially χ-bounded classes often have a nice structure, and a proof of the fact that a class is polynomially χ-bounded often reveals this structure and results in polynomial approximation algorithms for computing the chromatic number in these classes (see the discussion in [71, Section 1.3]). For example, in the class of perfect graphs the problems of determining the independence number, the clique number, and an optimal coloring of a graph can be solved in polynomial time (see [68]). We remark that for every fixed $k \geq 3$, the problem of deciding whether or not $\chi(G) \leq k$ for an input graph, is NP-complete [78].

Polynomial χ-boundedness is of particular interest also because of its connection with the Erdős-Hajnal Conjecture [52, 53]. Let G be a graph and let $S \subseteq V(G)$. In 1947 Erdős [48] proved that there exist graphs G on n vertices such that $\max\{\alpha(G), \omega(G)\} \leq \mathcal{O}(\log n)$. Erdős and Hajnal conjectured the following:

Conjecture 2.12 (Erdős and Hajnal [52, 53])**.** *Let H be a graph. Then there exists a constant $c(H)$ such that* $\max\{\alpha(G), \omega(G)\} \geq n^{c(H)}$ *for every H-free graph G.*

We say that a class $\mathcal{C}$ of graphs has the *Erdős-Hajnal property* if there exists a constant $c > 0$ such that for every $G \in \mathcal{C}$ we have $\max\{\alpha(G), \omega(G)\} \geq n^c$. In this terminology the Erdős-Hajnal Conjecture states that every proper hereditary class of graphs has the Erdős-Hajnal property. It is easy to see that every polynomially χ-bounded class of graphs has the Erdős-Hajnal property. We note that the Erdős-Hajnal Conjecture is one of the deepest open questions in graph theory: For hereditary classes of graphs we do not have (even a conjecture of) an analogue of the Graph Minor Structure Theorem [112, 113]. Understanding how excluding an induced subgraph affects the clique number and independence number of a graph could be a very important step towards understanding better the structure of proper hereditary classes of graphs (for more on this see [117, Section 9]).

Examples of polynomially χ-bounded classes are perfect graphs [30], even-hole-free graphs [35], graphs of bounded clique-width [12], graphs of bounded twin-width [13], rectangle intersection graphs [7, 23], grounded L-graphs [39], and circle graphs [38, 40]. Esperet [55] made the following conjecture:

Conjecture 2.13 (Esperet [55]). *Every χ-bounded class of graphs is polynomially χ-bounded.*

This conjecture was recently disproved by Briański, Davies, and Walczak [16]. Motivated by these results Chudnovsky, Cook, Davies, and Oum [28] considered the following question:

Question 10. *What are the classes of graphs where Esperet's Conjecture is true?*

Following [28], we call a class of graphs $\mathcal{C}$ *Pollyanna* if for every χ-bounded class of graphs $\mathcal{D}$, the class $\mathcal{C} \cap \mathcal{D}$ (we note that here we have the intersection of sets as opposed to the graph-intersection operation) is polynomially χ-bounded; thus Pollyanna graph classes are exactly the proper classes in which Esperet's Conjecture is true.

For the rest of this section, following [28], we call a class $\mathcal{C}$ of graphs r*-good*[for a class of graphs] if there exists a constant k such that for every graph $G \in \mathcal{C}$ and for every induced subgraph H of G we have that if $\omega(H) \leq r$, then $\chi(H) \leq k$. We note that Galvin and Rödl (see [98]) and Esperet (see [118]) independently made the very strong conjecture that a class is χ-bounded if and only if it is 2-good. This conjecture has been disproved by Carbonero, Hompe, Moore and Spirkl [22]. Briański, Davies and Walczak [16], extending the result of Carbonero, Hompe, Moore and Spirkl, proved that r-goodness is a strictly weaker property than χ-boundedness for every $r \geq 2$ (see also the work of Girão, Illingworth, Powierski, Savery, Scott, Tamitegama, and Tan [63]).

Following [28] we say that a class $\mathcal{C}$ of graphs is r*-strongly Pollyanna* if for every r-good class $\mathcal{D}$ the class $\mathcal{C} \cap \mathcal{D}$ is χ-bounded. A class is *strongly Pollyanna* if it is r-strongly Pollyanna for some integer $r \geq 2$.

In [28] Chudnovsky, Cook, Davies, and Oum made the following observation which they used in order to prove that some classes are strongly Pollyanna:

Proposition 2.14 (Chudnovsky, Cook, Davies, and Oum [28]). *Let $\mathcal{C}$ be a class of graphs. If there exists an integer $r \geq 2$ and a polynomial function $f : \mathbb{N}_+ \to \mathbb{N}_+$ such that the vertex set of every graph in $\mathcal{C}$ can be partitioned into $f(\omega(G))$ parts each of which induces a K_r-free graph, then $\mathcal{C}$ is $(r-1)$-strongly Pollyanna.*

In a discussion at the end of the paper [28] Chudnovsky, Cook, Davies, and Oum pointed out the (obvious) connection of Proposition 2.14 with the notions of K_r-free chromatic number and (polynomial) χ_r-boundedness. These notions are central to our work. We define and discuss them in the next section.

2.2 K_r-free chromatic number, complete subgraphs, and χ_r-boundedness

Let G be a graph, and let $r \geq 2$ be an integer. The *K_r-free chromatic number* of G, denoted by $\chi_r(G)$, is the minimum size of a partition of $V(G)$ into sets each of which induces a K_r-free graph. For every graph G we have $\chi(G) = \chi_2(G) \geq \ldots \geq \chi_{\omega(G)-1}(G) \geq \chi_{\omega(G)} \geq \chi_{\omega(G)+1} = 1$.

We remark that K_r-free graphs are well-studied: Let m_r be the number of edges of the complete $(r-1)$-partite graph on n vertices; Turan's theorem [128] states that every K_r-free graph on n vertices has at most m_r edges, and thus minimum degree at most $\frac{(r-2)n}{r-1}$; Andrásfai, Erdős, and Sós [6] proved that, for every $r \geq 3$, every K_r-free graph G of minimum degree greater than $\frac{3r-7}{3r-4}n$ has K_2-free chromatic number at most $(r-1)$; and Brandt [15] proved a structure theorem for edge-maximal K_r-free graphs. Hence, we can interpret the K_r-free chromatic number (for various choice of r) as measures of structural complexity of a graph.

In this thesis we are interested in the local structure of graphs of large K_r-free chromatic number. So, by generalizing Question 1, we ask:

Question 11. *Let $r \geq 2$. What are the graphs whose presence as induced subgraphs is unavoidable in graphs of sufficiently large K_r-free chromatic number?*

As far as we know, the above question has not been studied. Generalizing the fact that the clique number of a graph is a lower bound for its K_2-free chromatic number we have that: $\chi_r(G) \geq \lceil \frac{\omega(G)}{r-1} \rceil$, for every graph G and for every integer $r \geq 2$.

The oldest work on the K_r-free chromatic number that we are aware of generalizes the fact that the K_2-free chromatic number is not upper bounded by a function of the clique number in the class of all graphs. Sachs and Schäuble [116] (see also [115], in which the same result is presented by Sachs in English) in the late 1960s, motivated by the constructions of triangle-free graphs of large chromatic number, showed that the constructions of triangle-free graphs of arbitrarily large chromatic number can be used to inductively construct K_{r+1}-free graphs of clique number r and arbitrarily large K_r-free chromatic number. In fact they proved the following stronger result:

Theorem 2.15 (Sachs and Schäuble [116]). *Let $r \geq 2, \quad k \geq 1$, and $k' \geq k$ be integers. Then there exists a graph $G = G(r,k,k')$ with the following properties:*

1. *$\omega(G) \leq r$;*
2. *$\chi_r(G) = k$; and*
3. *every K_r-free coloring of G with at most k' colors results in at least k monochromatic $(r-1)$-cliques which are pairwise colored with different colors.*

The fact that large complete subgraphs are not unavoidable in graphs of large K_r-free chromatic number is an instance of a much more general result of Folkman who proved the following:

Theorem 2.16 (Folkman [58]). *For each positive integer k and each graph G there is a graph $H(k,G)$ with the following properties:*

1. *$\omega(H(k,G)) = \omega(G)$; and*
2. *For every k-coloring of $H(k,G)$ there exists a color class which is not G-free.*

Thus, the presence of large complete subgraphs is not unavoidable in graphs of large K_r-free chromatic number (obviously these graphs contain K_r). Before going on in refining Question 11 we review research results on and around K_r-free chromatic numbers.

We begin with the work of Broere and Frick [17, 18] which was published in 1990. We note that several results from [17, 18] appear also in the PhD thesis of Frick, from 1986, titled "Generalised colourings of graphs" [61]. In [18] Broere and Frick generalized the aforementioned result of Sachs and Schäuble [116] by proving that for every choice of positive integers k, l and $r \geq 2$ there exists a graph G such that $\chi_r(G) = k$ and $\omega(G) = l$ if and only if $r \leq l \leq k(l-1)$. They also determined the size of the largest complete graph that can be joined[5] to a graph G such that $\chi_r(G) = k$ so that for the resulting graph G' we have $\chi_r(G') = k$. For every graph G Frick calls the sequence $\chi(G) = \chi_2(G) \geq \ldots \geq \chi_{\omega(G)-1}(G) \geq \chi_{\omega(G)} \geq \chi_{\omega(G)+1} = 1$ the sequence of "generalized chromatic numbers" of G. In [61] Frick provided necessary and sufficient conditions for a sequence of positive integers to be the sequence of generalized chromatic numbers of a graph. They also proved the following interesting result:

[5]By "joined" here we mean: Take the disjoint union of G with the complete subgraph and then add edges so that the vertex set of the complete subgraph is complete to $V(G)$.

Theorem 2.17 (Frick [61]). *Let $k' \geq k \geq 1$, $r \geq 3$ be integers. Then there exists a graph G such that $\chi_r(G) = k'$ and $\chi_{r+1}(G) = k$.*

In [17], Broere and Frick studied those graphs in which any two K_r-free colorings induce the same partitions, as well as (k, χ_r)-critical graphs which are graphs G such that $\chi_r(G) = k$ and $\chi_r(G') < k$ for every proper induced subgraph G' of G. For more on the work on the K_r-free chromatic numbers mentioned above we refer the interested reader to the survey of Frick from 1993 [60]. Brown [19] studied the computational complexity of "Generalized graph colorings", that is, colorings in which the color classes satisfy various properties. In this context they proved that for every $r \geq 3$ and for every $k \geq 3$, the problem of deciding whether or not $\chi_r(G) \leq k$, is NP-complete. In a related study Achlioptas [1] studied the complexity of the G-free coloring. In more recent work, in 2017, Karpiński and Piecuch [79] considered the K_3-free chromatic number. Szabó and Zaválnij provided greedy algorithms for triangle-free colorings [126]. Thomassen [127] considered list-coloring without monochromatic triangles.

Much more relevant to our work is the work of Krawczyk and Walczak [89] who, motivated by a well-known conjecture about the number of edges of quasi-planar graphs (we discuss this conjecture in Section 2.4), considered the K_r-free chromatic numbers and the notion of χ_r-boundedness (to our knowledge, their work is the first work in which this notion is studied).

Generalizing the notion of χ-boundedness, for an integer $r \geq 2$, we say that a class of graphs $\mathcal{C}$ is *χ_r-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, we have that $\chi_r(G') \leq f(\omega(G'))$ for every induced subgraph G' of G; in this case f is called a *χ_r-bounding function* for $\mathcal{C}$. If f can be chosen to be polynomial, then $\mathcal{C}$ is *polynomially χ_r-bounded.*

Krawczyk and Walczak [89] showed that a certain class of geometric intersection graphs is not χ_r-bounded for any $r \geq 2$; they also showed that another class of geometric intersection graphs which is not χ-bounded is χ_3-bounded. We state their results in Section 2.3, and point out the connection of their results to the quasi-planar conjecture in Section 2.4.

Recall that in Question 11 we asked: *What are the graphs whose presence as induced subgraphs is unavoidable in graphs of sufficiently large K_r-free chromatic number?* Since the presence of complete subgraphs is not unavoidable in graphs of large K_r-free chromatic number we refine Question 11 and ask:

Question 12. *Which classes of graphs are χ_r-bounded?*

The existence of K_{r+1}-free graphs of arbitrarily large K_r-free chromatic number that we discussed above shows that the class of all graphs is not χ_r-bounded. Thus, every hereditary χ_r-bounded class $\mathcal{C}$ of graphs is described by the, possibly infinite, set of minimal induced subgraphs which do not belong in $\mathcal{C}$. Hence, a possible approach to Question 12 is to try to answer the following:

Question 13. *For which sets of graphs $\mathcal{H}$ is true that the class of all $\mathcal{H}$-free graphs is χ_r-bounded?*

The above question drives much of the work of this thesis. In particular, in this thesis we prove a theorem which provides a set of conditions that a finite set of graphs $\mathcal{H}$ should satisfy when the class of all $\mathcal{H}$-free graphs is χ_r-bounded; we use this result in order to formulate a conjecture for the characterization of the graphs H for which the class of all H-free graphs is χ_r-bounded; and we consider special cases of our conjecture. We discuss these results in Section 3.2.

Constructions of K_{r+1}-free graphs of large K_r-free chromatic number are crucial for our understanding of χ_r-boundedness. In this thesis, we give such a construction for every $r \geq 2$. Our construction is a construction of graphs with a geometric representation and was motivated by a conjecture for the number of edges of graphs with a certain geometric representation. Graphs with geometric representations play an important role in the study of χ-boundedness and our both our result and the results of Krawczyk and Walczak [89] that we mentioned above indicate that they are likely to play an equally interesting role in the study of χ_r-boundedness. In the next section, we discuss some known results on coloring graphs with geometric representations and bounded clique number.

2.3 Coloring graphs with geometric representations

We begin by showing how given a finite family of nonempty sets we can construct two, possibly different, graphs. Let $\mathcal{F}$ be a finite family of nonempty sets, and let F, F' be distinct elements of $\mathcal{F}$. We say that $\{F, F'\}$ is an *intersecting pair of sets* (or simply that F and F' intersect) if $F \cap F' \neq \emptyset$. We say that $\{F, F'\}$ is a *nested pair of sets* (or simply that F and F' are nested) if $F \subseteq F'$ or $F' \subseteq F$. Finally, we say that $\{F, F'\}$ is an *overlapping pair of sets* (or simply that F and F' overlap) if F and F' are intersecting sets, but are not nested. Then the *intersection graph of* $\mathcal{F}$ (respectively the *overlap graph of* $\mathcal{F}$) is the graph on $\mathcal{F}$ in which two distinct vertices $F, F' \in \mathcal{F}$ are adjacent if and only if the pair $\{F, F'\}$ is an intersecting pair of sets (respectively an overlapping pair of sets). A graph G is an *intersection graph* (respectively an *overlap graph*) if it is isomorphic to the intersection graph (respectively to the overlap graph), say G', of a finite family, say $\mathcal{F}$, of non-empty sets. In this case an *intersection model of* G (respectively an *overlap model of* G) is a function which witnesses that G is isomorphic to G'. Classes of intersection and overlap graphs of geometric figures were among the first to be studied from the perspective of χ-boundedness (see also the survey of Kostochka [85] on the topic.)

An *interval graph* is a graph which is isomorphic to the intersection graph of a family of intervals on a linearly ordered set (such as the real line). As Bielecki [10] and Rado [108] proved in the late 1940s, interval graphs are perfect. A *rectangle intersection graph* is a graph which is isomorphic to the intersection graph of a family of axis parallel rectangles in the plane. Bielecki [10] asked whether the chromatic number of every rectangle intersection graph of clique number two is bounded by a constant. In 1960 Asplund and Grünbaum [7], in one of the first results on coloring graphs with bounded clique number, motivated by Bielecki's question, proved that the class of intersection graphs of axis-parallel rectangles in the plane is polynomially χ-bounded (see also [23] for a better bound). Surprisingly, the situation changes in $\mathbb{R}^3$. In 1965, Burling [21] constructed, for every positive integer k, a triangle-free graph G_k which is the intersection graph of a family of axis-aligned boxes in $\mathbb{R}^3$ and has chromatic number greater than k. Thus, Burling [21] proved that the class of intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ is not χ-bounded. We note that until very recently it was an open problem whether or not the graphs constructed by Burling are the only obstruction for the χ-boundedness of intersection graphs with geometric representation on the plane. This was answered in the negative by Pournajafi in [106]. In 2021, Davies [37] constructed intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ with arbitrarily large girth and chromatic number.

Following [110], we define the *boxicity* of G to be the minimum integer k such that G is isomorphic to the intersection graph of a family of axis-aligned boxes in $\mathbb{R}^k$. We denote the boxicity of a graph G by $\mathsf{box}(G)$. Recently, Davies and Yuditsky [41], proved that the Gyárfás-Sumner Conjecture holds with polynomial bounds in every class of graphs of bounded boxicity. In Section 3.1 we discuss a result of this thesis which extends their result.

In the rest of this section, we denote by $\mathcal{I}$ the class of interval graphs. We note that the class of intersection graphs of axis-parallel boxes in $\mathbb{R}^k$ is exactly the class $\dot\bigcap_{i\in[k]}\mathcal{I}$: Observe that given an intersection graph of axis-parallel boxes in $\mathbb{R}^k$, say G, the set of the projections of the boxes in one of the k axes gives rise to an interval graph, and it is easy to see that G is the intersection of these k interval graphs. For the other inclusion the argument is similar. Thus, the class of intersection graphs of axis-parallel rectangles in the plane is the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I}$, and the class of intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ is the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I} \mathbin{\dot\cap} \mathcal{I}$. Hence, Asplund and Grünbaum [7] proved that the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I}$ is χ-bounded, and Burling [21] proved that the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I} \mathbin{\dot\cap} \mathcal{I}$ is not χ-bounded. It follows that the operation of graph-intersection does not preserve χ-boundedness. In particular, both Question 8 and Question 9 have a negative answer. In Section 3.1 we discuss our results in this thesis which focus on understanding the following question: *Under which conditions does graph-intersection preserve χ-boundedness?*

An *interval overlap graph* is a graph which is isomorphic to the overlap graph of a family of intervals on a linearly ordered set (such as the real line). A *circle graph* is an intersection graph of a family of chords on a circle. We note that a graph is an interval overlap graph if and only if it is a circle graph. In 1985 Gyárfás [70] proved that the class of interval overlap graphs is χ-bounded (see also [38] for an improved bound). A *rectangle overlap graph* is a graph which is isomorphic to the overlap graph of a family of axis parallel rectangles on the plane. *Is the class of rectangle overlap graphs χ-bounded?* We revisit this question later on this section. Before answering this question we would like to discuss the research line that led to its answer.

A *segment graph* is a graph isomorphic to the intersection graph of a family of straight line segments on the plane. In the 1970s Erdős (see for example [71, Problem 1.9]) asked whether the class of segment graphs is χ-bounded. In 2014 Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104] answered this question in the negative, by constructing triangle-free segment graphs of arbitrarily large chromatic number. It is worth mentioning that their construction yields the same graphs as Burling's graphs that we mentioned above. In a subsequent paper [103] the same group of authors generalized this result to a wider class of families of sets on the plane. Krawczyk, Pawlik, and Walczak [88] gave a geometric representation of the construction from [103] as rectangle overlap graphs. Thus, they proved that the class of rectangle overlap graphs is not χ-bounded.

In particular Krawczyk, Pawlik, and Walczak [88] gave a geometric representation of the construction from [103] as overlap graphs of a clean directed family of rectangles. Let $\mathcal{F}$ be a family of rectangles. We say that $\mathcal{F}$ is a *directed family of rectangles* if whenever two rectangles $I_1 \times J_1, I_2 \times J_2 \in \mathcal{F}$ intersect we have: Either $J_1 \subseteq J_2$ and $\min(I_2) < \min(I_1)$, or $J_2 \subseteq J_1$ and $\min(I_1) < \min(I_2)$. We say that $\mathcal{F}$ is a *clean family of rectangles* if for every distinct $F_1, F_2 \in \mathcal{F}$ such that F_1 and F_2 overlap, we have $F_3 \notin F_1 \cap F_2$, for every $F_3 \in \mathcal{F}$. Interestingly, in the first result on χ_r-boundedness that we are aware of, Krawczyk and Walczak [89] proved that the class of clean rectangle overlap graphs is χ_3-bounded.

An *L-shape* is a set in the plane which consists of two line segments L_1 and L_2 such that L_1 is parallel to the vertical axis and perpendicular to L_2, and the set $L_1 \cap L_2$ contains exactly one point, which is the bottom endpoint of L_1 and left endpoint of L_2. An *L-graph* is an intersection graph of L-shapes. It is easy to see that overlap graphs of directed families of rectangles are L-graphs. A stretching argument of Middendorf and Pfeiffer [94] shows that L-graphs are segment graphs. Thus, overlap graphs of directed families of rectangles are segment graphs.

In this thesis we generalize the result of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104] by constructing, for every $r \geq 2$, K_{r+1}-free segment graphs of arbitrarily large K_r-free chromatic number. In particular, the graphs that we construct are non-clean directed rectangle overlap graphs.

A *string graph* is a graph isomorphic to an intersection graph of a family of arbitrary curves in the plane. In particular, segment graphs are string graphs. Krawczyk and Walczak [89] constructed, for every $r \geq 2$, K_{r+1}-free string graphs of arbitrarily large K_r-free chromatic number. Their motivation for investigating whether, for some $r \geq 3$, the class of string graphs is χ_r-bounded came from an attempt to prove the quasi-planar graph conjecture which we will discuss in the next section.

2.4 Quasi-planar graphs, segment graphs, and a problem of Erdős, Gallai, and Rogers

Following [59], by *topological graph* we mean a graph drawn in the plane with points as vertices and edges as continuous curves connecting some pairs of vertices, and containing no other vertices than their ends. Let $r \geq 2$ be an integer. We say that a topological graph G is *r-quasi-planar* if no r edges of G are pairwise crossing. The following is a long-standing open question in the area of geometric intersection graphs:

Conjecture 2.18 (Pach, Shahrokhi, and Szegedy [102]). *Let $r \geq 3$ be a positive integer. Then there exists a constant $c_r > 0$ such that every r-quasi-planar graph on n vertices has at most $c_r n$ edges.*

We note that the above conjecture generalizes the fact that planar graphs on $n \geq 3$ vertices have at most $3n - 6$ edges. Conjecture 2.18 has been proved by Ackerman [2] for $r = 3$ and by Agarwal, Aronov, Pach, Pollack, and Sharir [4] for $r = 4$, but it remains open for $r \geq 5$.

Krawczyk and Walczak [89] pointed out a connection of Conjecture 2.18 with the K_r-free chromatic number of string graphs. Fix $r' \geq 2$. Suppose that we can prove that the class of string graphs is $\chi_{r'}$-bounded by a function f, and suppose that the conjecture is known to be true for r'-quasi-planar graphs with a constant $c_{r'} > 0$. Let G be an r-quasi-planar graph, and let $\varepsilon > 0$ be such that for every vertex $v \in V(G)$ if D_v is the disk on the plane with center v and radius ε we have $D_v \cap V(G) = \{v\}$; D_v contains no point other than v in which two or more edges intersect; and after deleting D_v from G we are left with a set of continuous curves (it is easy to see that such an $\varepsilon > 0$ exists). Let $\mathcal{F}$ be the set of continuous curves on the plane that we obtain by deleting the set $\cup_{v \in V(G)} D_v$ from G. Let G' be the intersection graph of $\mathcal{F}$. Then G' is a K_r-free string graph, and thus $\chi_{r'}(G') \leq f(r)$. Thus, the vertex set of G' (equivalently the edge set of G) can be partition into $l \leq f(r)$ sets each of which corresponds to an r'-quasi-planar subgraph of G, and thus (since the conjecture is known to be true for r'-quasi-planar) each of these l parts of the partition of $E(G)$ has size at most $c_{r'}|V(G)|$. Thus, $E(G)$ has size at most $f(r)c_{r'}|V(G)|$, which shows that Conjecture 2.18 holds for r.

In 2017 Krawczyk and Walczak [89] constructed, for every $r \geq 2$, K_{r+1}-free string graphs of arbitrarily K_r-free chromatic number, and thus proved that, for every $r \geq 2$, the class of string graphs is not χ_r-bounded. This result shows that the above approach for Conjecture 2.18 cannot succeed (for any $r' \geq 2$). But what about using this approach for solving Conjecture 2.18 in the special case of segment graphs?

As we discussed in Section 2.3, in 2014 Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104] constructed triangle-free segment graphs of arbitrarily large chromatic number. This construction implies that the class of segment graphs is not χ_2-bounded, and thus the above approach does not work for $r' = 2$, even when we restrict the conjecture to segment graphs (instead of string graphs).

Motivated by the fact that clean rectangle overlap graphs are not χ-bounded but are χ_3-bounded Krawczyk and Walczak [89] suggested that the approach that we discussed above for Conjecture 2.18 may be successful for some $r' \geq 3$.

Question 14. *Is there an integer $r \geq 3$ such that the class of segment graphs is χ_r-bounded? In particular, is there an integer $r \geq 3$ such that the class of non-clean rectangle overlap graphs is χ_r-bounded?*

In this thesis, building on the methods of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104] we generalize their result by constructing, for every $r \geq 2$, classes of K_{r+1}-free segment graphs of arbitrarily large K_r-free chromatic number. Thus, we answer Question 14 in the negative, and hence we show that the approach that we discussed above for Conjecture 2.18 cannot succeed for any $r \geq 2$, even if we restrict Conjecture 2.18 to quasi-planar graphs whose edges are drawn as straight-line segments. We discuss this result in Section 3.3.

Let G and H be graphs, and let $X \subseteq V(G)$. We say that X is an *H-free set* if $G[X]$ is H-free. Motivated by the above, Fox, Pach, and Suk [59] asked the following question:

Question 15 (Fox, Pach, and Suk [59])**.** *Let $r \geq 4$ and n be integers. Is there a constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$?*

In this thesis, we answer Question 15 in the negative. We discuss this result in Section 3.3. We note that Question 15 is a specific instance of a more general question of Erdős and Gallai [49]:

Question 16 (Erdős and Gallai [49]). *Let $r > p \geq 2$ and n be integers. What is the largest integer $k = k(r, p, n)$ such that every K_r-free graph on n vertices contains a K_p-free induced subgraph on k vertices?*

The case $p = r - 1$ was first considered by Erdős and Rogers [54] (see also [65, 95] for recent results). In the case $p = r - 1$ the function $f_r(n) : \mathbb{N}_+ \to \mathbb{N}_+$ that maps n to the largest integer k such that every K_r-free graph on n vertices contains a K_{r-1}-free induced subgraph on k vertices, is called an *Erdős-Rogers function*. Thus, our result on segment graphs that give a negative answer to Question 15, can also be regarded as a result on the study of Erdős-Rogers function on segment graphs.

Chapter 3

Contributions and organization of this thesis

This thesis is organized in four parts. Part I is the introduction. The main body of this thesis is organized into three parts. In Part II we investigate under which conditions the operation of graph-intersection between graph classes preserves χ-boundedness. In Part III we study the induced subgraphs of graphs of large K_r-free chromatic number and bounded clique number. In Part IV we construct, for every $r \geq 2$, K_{r+1}-free segment graphs of arbitrarily large K_r-free chromatic number, and K_{r+1}-free segment graphs of arbitrarily large fractional K_r-free chromatic number. In Section 3.1, Section 3.2, and Section 3.3, of this chapter, we discuss the main results of Parts II, III, and IV respectively.

3.1 Intersections of graphs and χ-boundedness

In Part II of this thesis we investigate under which conditions the operation of graph-intersection between graph classes preserves χ-boundedness. Motivated by results of Asplund and Grünbaum [7] and Burling [21], and by a question of Gyárfás, [71] we begin with studying this question, in Chapter 4, for graphs which can expressed as intersections of chordal graphs.

For the rest of this section we denote by $\mathcal{I}$ the class of interval graphs. As we discussed in Section 2.3 Asplund and Grünbaum [7], proved that the class $\mathcal{I} \Cap \mathcal{I}$ is χ-bounded (see also [23] for a better χ-bounding function), and due to a construction of Burling [21] we know that for every $k \geq 3$ the class $\Cap_{i \in [k]} \mathcal{I}$, is not χ-bounded.

A *hole* in a graph G is an induced cycle of length at least four. A graph is *chordal* if it contains no holes. Given a class of graphs $\mathcal{A}$ and a graph G, we follow Kratochvíl and Tuza [87], and define the *intersection dimension of G with respect to $\mathcal{A}$* to be the minimum integer k such that $G \in \Cap_{i \in [k]} \mathcal{A}$ if such a k exists, and $+\infty$ otherwise. Following McKee and Scheinerman [93] we call the intersection dimension of a graph G with respect to the class of chordal graphs the *chordality* of G, and we denote it by $\mathsf{chor}(G)$. For the rest of this section we denote by $\mathcal{C}$ the class of chordal graphs. Since, for every positive integer n, both the graphs K_n and K_n^- are chordal, it follows that the chordality of every graph is finite (and upper bounded by the number of its non-edges). To the best of our knowledge, chordality was first studied by Cozzens and Roberts [36] under the name rigid circuit dimension.

In the 1970s, Buneman [20], Gavril [62], and Walter [130, 131], proved independently that chordal graphs are exactly the intersection graphs of subtrees in trees. It is easy to see that the intersection graphs of subpaths in paths are exactly the interval graphs, and thus every interval graph is also chordal.

Since the class $\mathcal{I} \Cap \mathcal{I}$ is χ-bounded it is natural to ask whether any proper superclasses of this class are χ-bounded as well. Gyárfás, asked the following question:

Problem 1 (Gyárfás, [71, Problem 5.7])**.** *Is the class $\mathcal{C} \Cap \mathcal{C}$ χ-bounded? In particular, is $\mathcal{C} \Cap \mathcal{I}$ χ-bounded?*

In Section 4.1 we discuss a result of Felsner, Joret, Micek, Trotter and Wiechert [57] which implies that Burling graphs are contained in $\mathcal{C} \Cap \mathcal{I}$, and thus that the answer to Gyárfás' question is negative. In the rest of Chapter 4 we consider two families of subclasses of $\mathcal{C} \Cap \mathcal{C}$, which we prove are χ-bounded. We need to introduce some definitions in order to state our results.

Let G be a graph. A *chordal completion* (respectively *interval completion*) of G is a supergraph of G on the same vertex which is chordal (respectively interval). Since every complete graph is an interval graph, it follows that every graph has an interval and thus a chordal completion.

Let H be a chordal graph. A tree T is a *representation tree* of H if there exists a function $\beta : V(T) \to 2^{V(H)}$ such that for every $v \in V(H)$, the subgraph $T[\{t \in V(T) : v \in \beta(t)\}]$ of T is connected, and H is isomorphic to the intersection graph of the family $\{\{t \in V(T) : v \in \beta(t)\} : v \in V(H)\}$. In this case, we call the pair (T, β) a *representation* of H. By the aforementioned characterization of chordal graphs, it follows that every chordal graph has a representation.

A *tree-decomposition* of G is a representation (T, β) of a chordal completion H of G. Fix a chordal completion H of G and a representation (T, β) of H. For every $t \in V(T)$, we call the set $\beta(t)$ the *bag* of t. It is easy to see that every bag is a clique of H and that every clique of H is contained in a bag of T. We say that (T, β) is a *complete tree-decomposition* of G if for every $t \in V(T)$, the set $\beta(t)$ is a clique of G. If T is a path, then H is an interval completion of G and we call tree-decomposition (T, β) a *path-decomposition* of G. It is easy to see that a graph has a complete tree-decomposition (respectively complete path-decomposition) if and only if it is chordal (respectively interval). The *width* of a tree-decomposition is the clique number of the corresponding chordal completion minus one[1]. The *tree-width* (respectively *path-width*) of G, denoted by $\mathsf{tw}(G)$ (respectively $\mathsf{pw}(G)$) is the minimum width of a tree-decomposition (respectively path-decomposition) of G. That is, $\mathsf{tw}(G) := \min\{\omega(H) - 1 : H \text{ is a chordal completion of } G\}$, and $\mathsf{pw}(G) := \min\{\omega(I) - 1 : I \text{ is an interval completion of } G\}$.

In Section 4.2 we prove the following:

Theorem 3.1. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leq k_i$. If G is a graph such that $G = G_1 \cap G_2$, then G is $\mathcal{O}(\omega(G) \log(\omega(G)))(k_1 + 1)(k_2 + 1)$-colorable.*

We remark that each of the classes which satisfies the assumptions of Theorem 3.1 is a proper superclass of $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$. Towards the proof of Theorem 3.1 we prove a result which, as we point out, can be easily modified to give the following:

[1]The “minus one” in the definition of the width serves so that trees have tree-width one.

Lemma 3.2. *Let $k_1, \ldots, k_l$ be positive integers, and let $G_1, \ldots, G_l$ be chordal graphs such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leq k_i$. If G is a graph such that $G = G_1 \cap \ldots \cap G_l$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leq \prod_{i \in l}(k_i + 1)$ and for every $V \in \mathcal{P}$, the graph $G[V]$ has boxicity at most l.*

As we discussed in Section 2.3, Davies and Yuditsky [41] proved that the Gyárfas-Sumner conjecture holds with polynomial bounds for graphs of bounded boxicity:

Theorem 3.3 (Davies and Yuditsky [41])**.** *For every positive integer d and forest F, the class of all F-free graphs of boxicity at most d is polynomially χ-bounded.*

Now the following strengthening of Theorem 3.3 is an immediate corollary of Lemma 3.2 and Theorem 3.3.

Theorem 3.4. *Let $k_1, \ldots, k_l$ be positive integers, and let $\mathcal{C}$ be the class of all graphs G for which there exist chordal graphs $G_1, \ldots, G_l$ such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leq k_i$, and $G = G_1 \cap \ldots \cap G_l$. Then, the for every forest F the class of F-free graphs in $\mathcal{C}$ is polynomially χ-bounded.*

Let u and v be two vertices of a graph G. Then their *distance*, which we denote by $d_G(u, v)$, is the length of a shortest (u, v)-path in G; we will often omit the subscript G from $d_G(u, v)$ unless there is ambiguity. A *rooted tree* is a tree T with one fixed vertex $r \in V(T)$ which we call the *root* of T. The *height* of a rooted tree T with root r is $h(T, r) := \max\{d(r, t) : t \in V(T)\}$. The *radius* of a tree T, which we denote by $\mathsf{rad}(T)$, is the nonnegative integer $\min\{h(T, r) : r \in V(T)\}$. In Section 4.3 we prove the following:

Theorem 3.5. *Let k be a positive integer, and let G_1 and G_2 be chordal graphs such that the graph G_1 has a representation (T_1, β_1) where $\mathsf{rad}(T_1) \leq k$. If G is a graph such that $G = G_1 \cap G_2$, then $\chi(G) \leq k \cdot \omega(G)$.*

Since graph-intersection does not preserve χ-boundedness in general, it is natural to ask when it does; in the rest of Part II we focus on understanding under which conditions graph-intersection preserves χ-boundedness. To this end we introduce following terminology: Let $\mathcal{A}$ be a class of graphs. We call $\mathcal{A}$ *intersectionwise χ-imposing* if for every class of graphs $\mathcal{B}$

the class $\mathcal{A} \dot{\cap} \mathcal{B}$ is χ-bounded. We call $\mathcal{A}$ *intersectionwise χ-guarding* if for every χ-bounded class of graphs $\mathcal{B}$ the class $\mathcal{A} \dot{\cap} \mathcal{B}$ is χ-bounded. Finally, we call $\mathcal{A}$ *intersectionwise self-χ-guarding* if for every positive integer k the class $\dot{\cap}_{i \in [k]} \mathcal{A}$ is χ-bounded.

In Chapter 5 we prove a characterization of intersectionwise χ-imposing graph classes. Following [31], we say that $\mathcal{A}$ is *colorable* if there exists an integer k such that every graph in $\mathcal{A}$ has chromatic number at most k. The main result of this section is the following:

Theorem 3.6. *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is intersectionwise χ-imposing if and only if $\mathcal{C}$ is colorable.*

In Section 5.2, we prove that classes of graphs which admit a certain decomposition are intersectionwise χ-guarding, and we use this result to prove that the classes of unit interval graphs and of line graphs of bipartite graphs are intersectionwise χ-guarding (we define these classes later on in Section 5.2). We need a few definitions in order to state our result.

Let G be a graph and let $r \geq 2$ be an integer. We say that G is *componentwise r-dependent* if every component of G has independence number at most $r - 1$. Thus, the componentwise r-dependent chromatic number of a graph G is the minimum integer k for which G admits a componentwise r-dependent k-coloring. We say that G is *(t, k, r)-decomposable* if there exist positive integers t, k and r such that G is the union of t graphs of componentwise r-dependent chromatic number at most k. Finally we say that a class of graphs $\mathcal{C}$ is *decomposable* if there exist t, k and r such that every graph in $\mathcal{C}$ is (t, k, r)-decomposable. The main result of Section 5.2 is the following:

Theorem 3.7. *Let $\mathcal{C}$ be a decomposable class of graphs. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

In Chapter 6 we prove that certain χ-bounded classes of graphs are not intersectionwise χ-guarding. We do this by considering the graph-intersections of pairs of χ-bounded graph classes and proving that these graph-intersections contain triangle-free graphs of arbitrarily large chromatic number.

The main results of Section 6.1 are the following two theorems:

Theorem 3.8. *The class of complete multipartite graphs is not intersectionwise χ-guarding.*

Theorem 3.9. *Let $g \geq 3$. Then the class of line graphs of graphs of girth at least g is not intersectionwise χ-guarding. In particular, the class of line graphs is not intersectionwise χ-guarding.*

Following Golumbic [67] we say that a graph G is *trivially perfect graph* if for every induced subgraph H of G the independence number of H equals to the number of maximal cliques of H. We note that the class of trivially perfect graphs is a proper subclass of the class of interval graphs (see for example [14]). We also note that the class of trivially perfect graphs has the following nice characterization in terms of forbidden induced subgraphs:

Theorem 3.10 (Golumbic [67, Theorem 2])**.** *Let G be a graph. Then G is trivially perfect if and only if G is $\{P_4, C_4\}$-free.*

In Section 6.2 we strengthen the fact that, by the result of Felsner, Joret, Micek, Trotter and Wiechert [57], Burling graphs are contained in the graph-intersection of the class of chordal graphs with the class of interval graphs, by proving that Burling graphs are contained in the graph-intersection of the class of trivially perfect graphs with a proper subclass of the class of chordal graphs. In particular, our result implies the following:

Theorem 3.11. *The class of trivially perfect graphs is not intersectionwise χ-guarding.*

In Chapter 7 we focus on the finite sets $\mathcal{H}$ for which the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding. In Section 7.1 we prove the following:

Theorem 3.12. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains:*

- *a forest;*
- *a line graph of a forest;*
- *a complete multipartite graph; and*
- *a trivially perfect graph.*

In Section 7.2 we use Theorem 3.12 in order to characterize the graphs H for which the class of H-free graphs is intersectionwise χ-guarding. The main result of Section 7.2 is the following:

Theorem 3.13. *Let H be a graph. Then the class of H-free graphs is intersectionwise χ-guarding if and only if H is isomorphic to P_2, P_3, or rK_1 for some $r > 0$.*

In Chapter 8, the last chapter of Part II, we focus on intersectionwise self-χ-guarding classes. In Section 8.1 we provide the necessary conditions that a graph H should satisfy when the class of H-free graphs is intersectionwise self-χ-guarding. The main result of Section 8.1 is the following:

Corollary 3.14. *Let H be a graph. If the class of H-free graphs is intersectionwise self-χ-guarding, then H is a linear forest or a star.*

Finally, in Section 8.2, we prove the following theorem which allows us to construct new intersectionwise self-χ-guarding classes from intersectionwise χ-guarding classes which are defined by a finite set of forbidden induced subgraphs:

Theorem 3.15. *Let k and t be a positive integers and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a set of graphs. For every $i \in [t]$ let $r_1^i, \ldots, r_k^i$ be k nonnegative integers and let $\mathcal{H}^i := \{H_j + r_j^i K_2 : j \in [k]\}$. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then the class $\bigcap_{i\in[t]}\{\mathcal{H}^i\text{-free graphs}\}$ is χ-bounded. In particular, for every $i \in [t]$ the class of $\mathcal{H}^i$-free graphs is intersectionwise self-χ-guarding.*

An application of Theorem 3.15 is the following immediate corollary of Theorem 3.13 and Theorem 3.15:

Corollary 3.16. *Let r be a positive integer. Then the class of $(P_3 + rK_2)$-free graphs is intersectionwise self-χ-guarding.*

3.2 Induced subgraphs of graphs of large K_r-free chromatic number

In Part III of this thesis, we study the induced subgraphs of graphs of large K_r-free chromatic number and bounded clique number. We begin by proving, in Chapter 9, necessary conditions that the graphs of a finite set $\mathcal{H}$ should satisfy in order for the class of all $\mathcal{H}$-free graphs to be χ_r-bounded. We first need some definitions in order to state the results of Chapter 9.

A *diamond* is a graph that is isomorphic to the graph that we obtain from K_4 by removing an edge. A *bull* is a graph that can be obtained from a triangle by adding two pendant edges.

Let n be a positive integer. Following Chudnovsky, Cook, Davies, and Oum [28] we say that a graph G is an *n-willow* if there exists an oriented tree T with $V(G) \subseteq V(T)$ such that for all distinct $u, v \in V(G)$, we have that $uv \in E(G)$ if and only if T has a directed path from u to v or from v to u, whose length is not a multiple of n. A graph is a *willow* if it is a n-willow for some positive integer n. For example, the oriented tree $(\{a, b, c, d, e, x\}, \{ab, bc, ad, bx, xe\})$ shows that the bull is a 3-willow.

For an integer $t \geq 2$ we denote by K_t^+ the graph that we obtain from K_t by adding a new vertex that is adjacent to exactly one vertex of K_t. Equivalently, K_t^+ is isomorphic to the graph that we obtain from a path of length two by substituting a complete graph on $t-1$ vertices for one of its leaves.

In Chapter 9 we prove the following:

Theorem 3.17. *Let $\mathcal{H}$ be a finite set of graphs, and let $r \geq 2$ be a positive integer. If the class of $\mathcal{H}$-free graphs is χ_r-bounded, then $\mathcal{H}$ contains: a diamond-free chordal graph which has clique number at most r, a bull-free graph, and a willow. Moreover, if $r \geq 4$, then $\mathcal{H}$ contains a $K^+_{\lceil \frac{r}{2} \rceil + 2}$-free graph.*

Let G and H be graphs with disjoint vertex sets, and let $v \in V(G)$. We denote by $G(v, H)$ the graph that we obtain from the disjoint union of $G \setminus v$ with H by adding edges so that every vertex of H is complete to the set $N_G(v)$. We say that $G(v, H)$ is obtained by *substituting H for v in G*.

Let G be a graph, and let $r \geq 2$ be a positive integer. We say that G is an *r-broadleaved forest* (respectively *r-broadleaved tree*) if it can be obtained from a forest (respectively tree) F by substituting a complete graph on at most $r-1$ vertices for each leaf of F (not necessarily the same size complete graph for every leaf). Thus, a 2-broadleaved tree is a tree.

A *broadleaved forest* (respectively *broadleaved tree*) is an r'-broadleaved forest (respectively an r'-broadleaved tree) for some integer $r' \geq 2$.

A non-empty set $X \subseteq V(G)$ is a *homogeneous set in* G if for every $v \in V(G) \setminus X$ we have that v is either complete or anticomplete to X. A vertex $v \in V(G)$ is *simplicial* if $N_G(v)$ is a clique.

Let H be an r-broadleaved forest (respectively r-broadleaved tree). Let $H' := H[\{v \in V(H) : v \text{ is simplicial}\}]$, and let C be a component of H'. Then we have that $V(C)$ is a clique and a homogeneous set in H and there exists exactly one vertex $v_C \in V(H) \setminus C$ that is complete to $V(C)$. It is easy to see that $d_H(v_C) \in \{0, 1\}$. Let F be the induced subgraph of H that we obtain as follows: for every component C of H', with $d_H(v_C) = 0$ (respectively with $d_H(v_C) = 1$) we delete from H all but two (respectively one) vertices of the set $V(C)$. We call F an *underlying forest* (respectively *underlying tree*) of H. We note that any two underlying forests of H are isomorphic.

We note that a graph is chordal, diamond-free, and bull-free, if and only if it is a broadleaved forest. Thus, the following is an immediate corollary of Theorem 3.17.

Corollary 3.18. *Let H be a graph, and let $r \geq 2$ be a positive integer, such that the class of all H-free graphs is χ_r-bounded. Then H is an r-broadleaved forest, such that every component of H that is not a complete graph is an $(\lceil \frac{r}{2} \rceil + 1)$-broadleaved tree.*

Motivated by the above result, we suggest the following conjecture which extends the Gyárfás-Sumner conjecture to χ_r-bounded classes.

Conjecture 3.19 (Strong Forbidden Broadleaved Forest Conjecture)**.** *Let $r \geq 2$ be an integer, and let H be an r-broadleaved forest, such that every component of H that is not a complete graph is an $(\lceil \frac{r}{2} \rceil + 1)$-broadleaved tree. Then the class of all H-free graphs is χ_r-bounded.*

We also suggest the following weakening of Conjecture 3.19:

Conjecture 3.20 (Forbidden Broadleaved Forest Conjecture)**.** *Let $r \geq 2$ be an integer. Then there exists a function $f : \mathbb{N}_+ \to \mathbb{N}_+$ such that the class of all H-free graphs is $\chi_{f(r)}$-bounded for every r-broadleaved forest H.*

In the rest of Part III, we consider special cases of the above conjectures. Before discussing the results of Part III, we need to introduce some terminology and discuss some useful observations.

Let G be a graph. The *K_r-degree* of a vertex $v \in V(G)$, which we denote by $d^r_G(v)$, is the maximum size of a set of pairwise disjoint cliques, each of size $r-1$, in $N(v)$. When there is no danger of confusion, we may omit the subscript from the notation $d^r_G(v)$. The *maximum K_r-degree* (respectively the *minimum K_r-degree*) of G, denoted by $\Delta_r(G)$ (respectively by $\delta_r(G)$), is the maximum (respectively the minimum) K_r-degree of a vertex of G. A graph G is *(k, K_r)-degenerate* if every non-null subgraph H of G has a vertex v such that $d^r_H(v) \leq k$. The *K_r-degeneracy* of G is the minimum integer k for which G is *(k, K_r)-degenerate*. We denote the K_r-degeneracy of G by $\mathsf{degen}_r(G)$. We note that for every graph G we have $\mathsf{degen}_r(G) \leq \Delta_r(G)$. The following generalizes a well-known result for the K_2-free chromatic number.

Proposition 3.21. *Let G be a graph. Then $\chi_r(G) \leq \mathsf{degen}_r(G) + 1 \leq \Delta_r(G) + 1$.*

Proof of Proposition 3.21. We prove Proposition 3.21 by induction on the number of vertices. The statement holds trivially if $|V(G)| = 1$. Suppose that $|V(G)| > 1$ and that for every graph H with $|V(H)| < |V(G)|$ we have $\chi_r(H) \leq \mathsf{degen}_r(H) + 1$.

Let $k := \mathsf{degen}_r(G)$, and let $v \in V(G)$ be such that $d^r(v) \leq k$. Let $H := G \setminus v$. By the inductive hypothesis, there exists a K_r-free coloring $\phi : V(H) \to [k+1]$ of H. Let $\mathcal{K}$ be a maximal set of monochromatic $(r-1)$-cliques in $N(v)$ that are pairwise colored with different colors. Then $\mathcal{K}$ is a set of pairwise disjoint cliques, and since $d^r(v) \leq k$, we have $|\mathcal{K}| \leq k$.

Let $i \in [k+1]$ be such that $i \notin \{\phi(K) : K \in \mathcal{K}\}$, and let $\phi' : V(G) \to [k+1]$ be the $(k+1)$-coloring of G that we obtain by extending ϕ so that $\phi'(v) := i$. We claim that ϕ' is a K_r-free $(k+1)$-coloring of G. Suppose not. Let K be a monochromatic, with respect to ϕ', clique of size r in G. Since ϕ is a K_r-free coloring of $G \setminus v$, it follows that $v \in K$. Then, the $(r-1)$-clique $K' := K \setminus \{v\}$ is colored by ϕ' (and thus by ϕ) with the color $\phi'(v) = i$ and thus $K' \notin \mathcal{K}$. Then, the set $\mathcal{K} \cup \{K'\}$ contradicts the maximality of $\mathcal{K}$. Hence, ϕ' is a K_r-free $(k+1)$-coloring of G, and thus $\chi_r(G) \leq \mathsf{degen}_r(G) + 1$. This completes the proof of Proposition 3.21. $\square$

The following two propositions generalize well-known results for K_2-degeneracy.

Proposition 3.22. *Let $k, r \geq 2$ be positive integers, and let G be a graph. Then $\mathsf{degen}_r(G) \geq k$ if and only if G contains an induced subgraph H such that $\delta_r(H) \geq k$.*

Proposition 3.23. *Let $r \geq 2$ and k be positive integers, let G be a graph, and let $n := |V(G)|$. Then $\mathsf{degen}_r(G) \leq k$ if and only if there is an ordering $v_1, \dots, v_n$ of the vertices of G such that $d^r(v_1) \leq k$, and for every $i \in [2, n]$ we have $d^r_{G[V(G)\setminus\{v_1,\dots,v_{i-1}\}]}(v_i) \leq k$.*

Let $r, s \geq 2$ be integers. An *r-broadleaved s-star* is an r-broadleaved tree whose underlying tree is isomorphic to $K_{1,s}$. As we mentioned, in Section 2.1, Gyárfás [71] proved that for every $s \geq 2$ the class of $K_{1,s}$-free graphs is χ-bounded; and this is one of the simplest proofs in χ-boundedness. Thus, it was natural for us to try to generalize this result for r-broadleaved s-stars and χ_r-boundedness. Unfortunately, we did not succeed. We include the proof of Gyárfás' result [71] below and discuss why his approach cannot be generalized to our context.

Theorem 3.24 (Gyárfás [71])**.** *Let $s \geq 2$ be a positive integer. Then for every $K_{1,s}$-free graph G we have $\chi(G) \leq R(\omega(G), s)$.*

Proof of Theorem 3.24. Let G be a $K_{1,s}$-free graph. We claim that $\Delta(G) \leq R(\omega(G), s) - 1$. Let $v \in V(G)$. Then $d(v) = |N(v)|$. Since G is $K_{1,s}$-free, it follows that $G[N(v)]$ does not contain an independent set of size s. In addition, $G[N(v)]$ does not contain a clique of size $\omega(G)$, since the union of such a clique with v would result in a clique of size $\omega(G) + 1$ in G. Thus, by Theorem 1.1, we have $|N(v)| \leq R(\omega(G), s) - 1$. Thus, $d(v) \leq R(\omega(G), s) - 1$, and since v is an arbitrary vertex of G we have $\Delta(G) \leq R(\omega(G), s) - 1$. Thus, since $\chi(G) \leq \Delta(G) + 1$, we have $\chi(G) \leq R(\omega(G), s)$. This completes the proof of Theorem 3.24. $\square$

For, $r \geq 3$, for the K_r-free chromatic number, the issue in extending the above proof is that the existence of a large set of pairwise adjacent K_{r-1}'s is not a "forbidden" outcome. In particular, the existence of a large set of pairwise adjacent K_{r-1}'s in a graph does not imply the existence of a large complete subgraph.

Lemma 3.25. *Let $r \geq 2, s \geq 2$ and $t \geq 1$ be integers, let G be a graph, and let $v \in V(G)$ be such that v is not the center of an induced r-broadleaved s-star in G. If $d^r(v) \geq R(t, s)$, then $N(v)$ contains t pairwise disjoint and pairwise adjacent $(r-1)$-cliques.*

Proof of Lemma 3.25. Let $v \in V(G)$ be as in the statement of the lemma. Let $S_1, \dots S_{R(t,s)}$ be a collection of pairwise disjoint $(r-1)$-cliques that are subsets of $N(v)$.

We claim that there exists no s-subset I of $[R(t,s)]$ such that for all distinct $i, j \in I$ the sets S_i and S_j are anticomplete. Suppose not. Then $G[\{v\} \cup \{S_i : i \in I\}]$ is an r-broadleaved s-star in G with v as a center, a contradiction.

It follows, by Theorem 1.1, that $[R(t,s)]$ contains a t-set I' such that for all distinct $i, j \in I'$ the sets S_i and S_j are adjacent. Then $T := \{S_i : i \in I'\}$ is a set of t pairwise disjoint and pairwise adjacent $(r-1)$-subsets of $N(v)$. This completes the proof of Lemma 3.25. □

Since we were not able to prove Conjecture 3.19 or Conjecture 3.20 for general broadleaved stars we considered specific broadleaved stars. Let T be a triangle on $\{v, u, w\}$, and let k be a positive integer. A *k-volcano*, denoted by V_k is a graph isomorphic to the graph $(\{v, u, w\} \cup \{v_1, \dots, v_k\}, E(T) \cup \{vv_i : i \in [k]\})$. A *volcano* is a k-volcano for some k. We remark that volcanoes are the simplest case of 3-broadleaved stars (which are not regular stars). In Section 10.1 we prove that for every positive integer k the class of V_k-free graphs is χ_3-bounded:

Theorem 3.26. *Let k be a positive integer, and let $f_{3.26} : \mathbb{N}_+ \to \mathbb{N}_+$ be the function which is defined as follows:*

$$f_{3.26}(\omega) := 4\omega(10k^2\omega^3)^\omega \cdot [(2^\omega - 2)R(\omega, k) + 1] .$$

Let G be a V_k-free graph. Then $\chi_3(G) \leq f_{3.26}(\omega(G))$.

Thus, Conjecture 3.19, holds for all k-volcanoes.

We note that when $k = 1$, the graph V_1, is called a *paw*. Olariu [101] proved the following structure theorem for paw-free graphs:

Theorem 3.27 (Olariu [101])**.** *A graph G is paw-free if and only if each component of G is triangle-free or complete multipartite.*

We note that if a graph G is paw-free then the neighborhood of every vertex of G is $\{K_2 + K_1\}$-free, and that the $\{K_2 + K_1\}$-free graphs are exactly the complete multipartite graphs. Generalizing this observation we note that if a graph G is V_k-free, then the neighborhood of every vertex of G is $\{K_2 + kK_1\}$-free. Our proof of Theorem 3.26 relies heavily on a structure theorem for V_k-free graphs, which in turn relies on the structure of $\{K_2 + kK_1\}$-free graphs. We need to introduce some definitions in order to state our result.

Let G be a graph, and let $k \geq 0$ and $t \geq 1$ be positive integers. We say that G is a *k-tolerant* t-partite graph if G is t-partite and $V(G)$ admits a partition $\{A_1, \ldots, A_t\}$, which is such that:

- for every $i \in [t]$, we have that A_i is an independent set; and
- for distinct $i, j \in [t]$, and for every $v \in A_i$ we have that the vertex v has at most k non-neighbors in A_j, that is $|A_j \setminus N(v)| \leq k$.

Thus a 0-tolerant t-partite graph is a complete t-partite graph. In Section 10.1 we prove that every $(K_2 + kK_1)$-free graph is "almost" a $(k-1)$-tolerant $\omega(G)$-partite graph. We postpone the exact statement of our theorem for Section 10.1. Using our result for $(K_2 + kK_1)$-free graphs we prove the following theorem for the structure of V_k-free graphs:

Theorem 3.28. *Let k and ω be positive integers, and let G be a V_k-free graph with $\omega(G) \leq \omega$. Then there exists a partition $\mathcal{P}$ of $V(G)$, with the following properties:*

- $|\mathcal{P}| \leq 2(2^{\omega-1} - 1)R(\omega, k) + 1$, *and*
- *for every $P \in \mathcal{P}$ and for every $v \in G[P]$ we have that $G[N_{G[P]}(v)]$ is a $(k-1)$-tolerant t-partite graph, where $t \leq \omega(G[N(v)])$.*

We discussed above that classes of graphs that exclude a star are χ-bounded, and that we were not able to generalize this result for χ_r-boundedness. In Section 10.2 we prove a result which indicates that the case of broadleaved stars may be more difficult to handle in χ_r-boundedness than the case of stars in χ-boundedness.

Let H be an oriented graph. We denote by $\omega(H)$ (respectively $\chi(H)$) the clique number (respectively the chromatic number) of the underlying graph of H. Let $r \geq 2$ be a positive integer. We say that H is $\overrightarrow{\chi_r}$-bounding if there exists a function $f : \mathbb{N}_+ \to \mathbb{N}_+$ such that for every oriented graph G not containing H as an induced subdigraph we have that $\chi(G) \leq f(\omega(G))$. Chudnovsky, Scott, and Seymour [32] proved the following:

Theorem 3.29 (Chudnovsky, Scott, and Seymour [32])**.** *Every oriented star is $\overrightarrow{\chi_2}$-bounding.*

Let v be the unique vertex of degree three of V_1, and let $\overrightarrow{V_1}$ be the orientation of V_1 that we obtain by directing every edge of V_1 which is incident to v outwards from v, and then choosing an orientation for the only edge of V_1 which is not incident to v. The main result of Section 10.2 is the following:

Theorem 3.30. *Let $r \geq 2$ be an integer. Then $\overrightarrow{V_1}$ is not $\overrightarrow{\chi_r}$-bounding.*

We get Theorem 3.30 as an immediate corollary of the following:

Theorem 3.31. *Let $r \geq 2$ be an integer. Then for every integer $k \geq 2$, there exists a graph G_{c_k} such that:*

- *$\chi_r(G_{c_k}) > k$ and $\omega(G_{c_k}) \leq r$; and*
- *G_{c_k} has an orientation which is $\overrightarrow{V_1}$-free.*

In Chapter 11, we consider classes of graphs that exclude a fixed complete bipartite graph as a subgraph. One of the main tools used in our proofs in Chapter 11, is the following lemma, which is an easy corollary of the Kővári–Sós–Turán theorem [86]. The latter theorem states that, for all positive integers $s \geq t$, a graph on n vertices that does not contain a $K_{s,t}$ as a subgraph has $\mathcal{O}(n^{2-\frac{1}{t}})$ edges.

Lemma 3.32. *Let r and t be positive integers. Then there exists a constant $f_{3.32} = f_{3.32}(r,t) \in \mathbb{N}$ with the following property: If G is a graph which contains a collection of $f_{3.32}(r,t)$ pairwise disjoint and pairwise adjacent subsets of vertices, each of size at most r, then G contains a $K_{t,t}$ as a subgraph.*

We note that Lozin and Razgon [92] gave a short constructive proof of Lemma 3.32, based solely on the pigeonhole principle.

Recall Theorem 2.8:

Theorem 2.8 (Kierstead and Penrice [84])**.** *For every forest H, there exists a function f such that $\mathsf{degen}(G) \leq f(\mathsf{biclique}(G))$ for every H-free graph G.*

The main result of Section 11.1 is the following generalization of Theorem 2.8:

Theorem 3.33. *Let $r \geq 2$ be an integer. For every r-broadleaved forest H, there exists a function f such that* $\mathsf{degen}_r(G) \leq f(\mathsf{biclique}(G))$ *for every H-free graph G.*

We note that our proof of Theorem 3.33 follows closely the proof of Scott, Seymour, and Spirkl of the following result which strengthens Theorem 2.8:

Theorem 2.9 (Scott, Seymour, and Spirkl [120])**.** *For every forest H, there exists $c > 0$ such that* $\mathsf{degen}(G) \leq \mathsf{biclique}(G)^c$ *for every H-free graph G.*

Let K_p be a complete graph on p vertices. Let H be the graph that is obtained from K_p as follows: for each $v \in K_p$ we add a copy C_v of qK_{p-1} to K_p so that these copies are pairwise disjoint and pairwise anticomplete. Finally, we add edges to H so that each vertex $v \in K_p$ is complete to $V(C_v)$. We call H a *q-bloomed p-clique.* We call K_p the *base clique* of H. We also call the cliques of C_v the *private cliques* of v. A *bloomed clique* is a q-bloomed p-clique for some positive integers q and p.

The main result of Section 11.2 is the following:

Theorem 3.34. *Let p, q and t be positive integers. Then there exist constants $f_{3.34} = f_{3.34}(p, q, t) \in \mathbb{N}$ and $h_{3.34} = h_{3.34}(p, q, t) \in \mathbb{N}$ with the following property: If G is a graph that does not contain a $K_{t,t}$ as a subgraph or a q-bloomed p-clique as an induced subgraph, then $\chi_{f_{3.34}}(G) \leq h_{3.34}$.*

In Chapter 12, we prove the following result for classes of graphs which exclude a complete multipartite graph as a subgraph and a broadleaved star as an induced subgraph:

Theorem 3.35. *Let $q \geq 2$, $r, s \geq 2$ and $t \geq 1$ be integers. Then there exist constants $f_{3.35} = f_{3.35}(q, r, s, t) \in \mathbb{N}$ and $h_{3.35} = h_{3.35}(q, r, s, t) \in \mathbb{N}$, with the following property: If G is a graph that does not contain a complete q-partite graph with all parts having size t as a subgraph and G does not contain an r-broadleaved s-star as an induced subgraph, then $\Delta_{h_{3.35}}(G) \leq f_{3.35}$. In particular $\chi_{h_{3.35}}(G) \leq f_{3.35} + 1$.*

Let $s, t \geq 2$ be integers. An (s, t)-*bowtie* is a graph isomorphic to the graph that we obtain from the disjoint union of a K_s and a K_t by adding a new vertex complete to everything else. Chudnovsky, Cook, Davies, and Oum [28] proved that the class of $(2, 2)$-bowtie-free graphs is Pollyanna, and asked the following question:

Question 17 (Chudnovsky, Cook, Davies, and Oum [28]). *Is the class of (s,t)-bowtie-free graphs Pollyanna for each $s \geq 3$ and $t \geq 2$?*

In Chapter 13, we give an affirmative answer to Question 17 by proving the following:

Theorem 3.36. *Let $s, t \geq 2$ and $\omega \geq 1$ be integers, with $s \leq t$, and let*

$$f_{3.36} = f_{3.36}(s,t) = (s+1)(t+3s^2-2s)+(t-1) \in \mathbb{N}$$

and

$$h_{3.36} = h_{3.36}(s,t,\omega) = \left\lceil \left\lceil \frac{\omega}{t-1} \right\rceil + e\omega^{s-1} + \omega^{s+1} \right\rceil \in \mathbb{N}.$$

Then, for every (s,t)-bowtie-free graph G with $\omega(G) = \omega$, we have: $\chi_{f_{3.36}}(G) \leq h_{3.36}$. In particular, the class of all (s,t)-bowtie-free graphs is polynomially $\chi_{f_{3.36}(s,t)}$-bounded, and thus strongly Pollyanna.

An important tool for our proof of Theorem 3.36 is the following lemma, which generalizes an observation of Chudnovsky, Cook, Davies, and Oum [28] for $(2,2)$-bowtie-free graphs.

Lemma 3.37. *Let $s \geq 2$ and $t \geq 2$ be integers, let G be an (s,t)-bowtie-free graph, and let G' be the graph obtained by G after deleting every edge that does not belong to a $3s$-clique. Then G' is an $(s, t+3s^2-2s)$-bowtie-free graph.*

3.3 K_{r+1}-free segment graphs of large fractional K_r-free chromatic number

In this section, we discuss the content of Part IV. Recall Question 14:

Question 14. *Is there an integer $r \geq 3$ such that the class of segment graphs is χ_r-bounded? In particular, is there an integer $r \geq 3$ such that the class of non-clean rectangle overlap graphs is χ_r-bounded?*

The first main result of Part IV, which we prove in Chapter 15, is the following which implies a negative answer to Question 14:

Theorem 3.38. *Let k and $r \geq 2$ be positive integers. Then there exists a K_{r+1}-free segment graph G, such that $\chi_r(G) \geq k$.*

We note that the proof of the above is constructive and the graphs that we construct are directed rectangle overlap graphs. In particular, the graphs that we construct for $r = 2$ are the same graphs as the construction of Pawlik, and Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104], which are the same graphs with the triangle-free intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ constructed by Burling [21]; and the graphs that we construct for every $r \geq 3$ are non-clean directed rectangle overlap graphs.

Before proving Theorem 3.38 we introduce, in Chapter 14, the framework of "on-line coloring games" and their "game graphs" and discuss a method which, to the best of our knowledge, was established by Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [103] (see also [88]) and was formalized by Krawczyk and Walczak [89], that can be used in order to construct graphs of bounded clique and large K_r-free chromatic number.

Recall Question 15:

Question 15 (Fox, Pach, and Suk [59])**.** *Let $r \geq 4$ and n be integers. Is there a constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$?*

Our work in Chapter 16 is motivated by the above question, which we answer in the negative.

Let $r \geq 2$ be an integer, and let G be a graph. Let us denote by $\mathcal{F}$ the family of all subsets of $V(G)$ which induce K_r-free graphs. Then we have the following observation for the K_r-free chromatic number of G:

$$\chi_r(G) = \min\{\sum_{S \in \mathcal{F}} x_S : \sum_{v \in S \in \mathcal{F}} x_S \geq 1 \text{ for all } v \in V(G); x_S \in \{0,1\}\}.$$

We define the *fractional K_r-free chromatic number* of G, which we denote by $\chi_r^f(G)$, as the optimal value of the linear programming relaxation of the above integer program. That is:

$$\chi_r^f(G) := \min\{\sum_{S \in \mathcal{F}} x_S : \sum_{v \in S \in \mathcal{F}} x_S \geq 1 \text{ for all } v \in V(G); x_S \geq 0\}.$$

In Section 16.1, Section 16.2 and Section 16.3 we develop a framework which allows us to prove in Section 16.4 the following:

Theorem 3.39. *Let k and $r \geq 2$ be positive integers. Then there exists a K_{r+1}-free segment graph G, such that $\chi_r^f(G) \geq k$.*

We note that since for every integer $r \geq 2$, and for every graph G we have $\chi_r^f(G) \leq \chi_r(G)$, it follows that Theorem 3.39 is a strengthening of Theorem 3.38.

In Section 16.5, we establish a connection between the maximum size of a K_r-free set of vertices and fractional K_r-free chromatic number of segment graphs. Let G be a graph, and $r \geq 2$ be an integer. We denote by $\alpha_r(G)$ the number $\max\{|I| : I \subseteq V(G) \text{ and } G[I] \text{ is } K_r\text{-free}\}$. The main result of Section 16.5 is the following:

Theorem 3.40. *Let $\mathcal{C}$ be the class of segment graphs, and let $r \geq 2$ be a positive integer. Then the following are equivalent:*

1. *There exists a constant $c_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\alpha_r(G) \geq c_r|V(G)|$.*
2. *There exists a constant $C_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\chi_r^f(G) \leq C_r$.*

The following is an immediate corollary of Theorem 3.40 and Theorem 3.39:

Theorem 3.41. *Let $r \geq 2$ and n be integers. Then there exists no constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$.*

Part II

Intersections of graphs and χ-boundedness

Chapter 4

Chordality and χ-boundedness

In this chapter we study classes of graphs of bounded chordality from the perspective of χ-boundedness. In Section 4.1 we discuss a result of Felsner, Joret, Micek, Trotter and Wiechert [57] which implies that Burling graphs are contained in $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$, and thus that the answer to Gyárfás' question is negative. From the characterization of chordal (respectively interval) graphs as intersection graphs of subtrees (respectively subpaths) of trees (respectively paths) that we presented in the Section 3.1, it follows that $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$ is the subclass of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$ in which each of the two chordal graphs in the intersection has a representation tree which is a path. In Section 4.2 we consider the family of subclasses of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$ (and superclasses of $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$) in which each of the two chordal graphs in the intersection has a representation tree of bounded path-width. We prove that these classes are χ-bounded. In Section 4.3, for each positive integer k, we consider the subclass of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$ in which one of the two chordal graphs in the intersection has a representation tree of radius at most k, and we prove that this class is χ-bounded.

4.1 The class $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$ is not χ-bounded

In [44], Dujmovic, Joret, Morin, Norin, and Wood studied graphs which have two tree-decompositions such that "each bag of the first decomposition has a bounded intersec-

This chapter is based on the coauthored paper [25].

tion with each bag of the second decomposition". Following [44], we say that two tree-decompositions (T_1, β_1) and (T_2, β_2) of a graph G are *k-orthogonal* if for every $t_1 \in T_1$ and $t_2 \in T_2$, we have $|\beta_1(t_1) \cap \beta_2(t_2)| \leq k$. Dujmovic, Joret, Morin, Norin, and Wood [44] proved that every graph which belongs to one of the following classes has a tree-decomposition and a path-decomposition that are k-orthogonal:a proper minor-closed class, string graphs with a linear number of crossings in a fixed surface, and for graphs with linear crossing number in a fixed surface). In a more recent work Liu, Norin, and Wood [90] proved that for graphs which exclude a fixed graph as an odd-minor there exists an integer k such that these graphs have a tree-decomposition and a path-decomposition which are k-orthogonal. Here we are interested in connections of this concept with the concept of χ-boundedness.

Observation 4.1 (Dujmović, Joret, Morin, Norin, and Wood [44, Observation 27])**.** *Let G be a graph and k be a positive integer. Then G has two k-orthogonal path-decompositions if and only if G is a subgraph of a graph H such that H has boxicity at most two, and $\omega(H) \leq k$.*

Lemma 4.2. *Let G be a graph and k be a positive integer. Then the following hold:*

1. *The graph G has two k-orthogonal tree-decompositions if and only if G is a subgraph of a graph H such that H has chordality at most two, and $\omega(H) \leq k$.*
2. *The graph G has a tree-decomposition and a path-decomposition which are k-orthogonal if and only if G is a subgraph of a graph H such that $H \in \mathcal{C} \Cap \mathcal{I}$, and $\omega(H) \leq k$.*

Proof of Lemma 4.2. Follows immediately by the corresponding definitions and the facts that every bag of a tree-decomposition is a clique of the corresponding chordal completion, and that every clique of a chordal completion is contained in a bag of the corresponding tree-decomposition. □

The following is an immediate corollary of Observation 4.1 and Lemma 4.2.

Proposition 4.3. *Let $\mathcal{C}$ be the class of chordal graphs and $\mathcal{I}$ be the class of interval graphs. The following hold:*

1. *The class $\mathcal{I} \Cap \mathcal{I}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has two k-orthogonal path-decompositions, we have $\chi(G) \leq f(k)$.*

2. *The class $\mathcal{C} \Cap \mathcal{C}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has two k-orthogonal tree-decompositions, we have $\chi(G) \leq f(k)$.*
3. *The class $\mathcal{C} \Cap \mathcal{I}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has a tree-decomposition and a path-decomposition which are k-orthogonal, we have $\chi(G) \leq f(k)$.*

The authors of [44] posed the following question, for which they conjectured a positive answer.

Problem 2 (Dujmović, Joret, Morin, Norin, and Wood, [44, Open Problem 3])**.** *Is there a function f such that every graph G that has two k-orthogonal tree-decompositions is $f(k)$-colorable?*

By Proposition 4.3, it follows that the above question is equivalent to the first part of the question of Gyárfás that we mentioned in the Section 3.1 (Problem 1), which asks whether the class of all graphs of chordality at most two is χ-bounded.

In [57] Felsner, Joret, Micek, Trotter and Wiechert, answered Problem 2 in the negative, and in particular they answered Gyárfás's question (Problem 1), in the negative.

Felsner, Joret, Micek, Trotter and Wiechert [57] proved the following, which answers in the negative both the questions in Problem 1 and Problem 2.

Theorem 4.4 (Felsner, Joret, Micek, Trotter and Wiechert, [57, Theorem 2])**.** *For every positive integer k, there is a graph with chromatic number at least k which has a tree-decomposition (T, β) and a path-decomposition (P, γ), which are 2-orthogonal. That is, for every $t \in V(T)$ and for every $p \in V(P)$, we have $|\beta(t) \cap \gamma(p)| \leq 2$.*

The following is an immediate corollary of Lemma 4.2 and Theorem 4.4.

Corollary 4.5. *For every positive integer k, there exist a graph $H_k \in \mathcal{C} \Cap \mathcal{I}$ such that H_k is triangle-free and has chromatic number at least k.*

Corollary 4.6. *The class $\mathcal{C} \Cap \mathcal{I}$ is not χ-bounded. In particular, since $\mathcal{I} \subseteq \mathcal{C}$, it follows that the class of all the graphs of chordality at most two is not χ-bounded.*

4.2 Subclasses of $\mathcal{C} \Cap \mathcal{C}$: When each chordal graph has a representation tree of bounded path-width

In Section 4.1, we saw that the class $\mathcal{C} \Cap \mathcal{I}$ is not χ-bounded. From the characterization of chordal (respectively interval) graphs as intersection graphs of subtrees (respectively subpaths) of trees (respectively paths) that we discussed in Section 3.1, it follows that $\mathcal{I} \Cap \mathcal{I}$ is the subclass of $\mathcal{C} \Cap \mathcal{C}$ in which each of the two chordal graphs in the intersection has a representation tree which is a path.

In this subsection we consider the family of subclasses of $\mathcal{C} \Cap \mathcal{C}$ (and superclasses of $\mathcal{I} \Cap \mathcal{I}$) in which each of the two chordal graphs in the intersection has a representation tree of bounded path-width. We prove that these classes are χ-bounded.

Theorem 3.1. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leq k_i$. If G is a graph such that $G = G_1 \cap G_2$, then G is $\mathcal{O}(\omega(G)\log(\omega(G)))(k_1+1)(k_2+1)$-colorable.*

The main step towards our proof of Theorem 3.1 is to prove that the vertex set of a graph G as in the statement of Theorem 3.1 can be partitioned into a constant number of sets so that each of these sets induces a graph of boxicity at most two. Then we use the fact that the class $\mathcal{I} \Cap \mathcal{I}$ is χ-bounded and we color each of these induced subgraphs with a different palette of colors.

Lemma 4.7. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leq k_i$. If G is a graph such that $G = G_1 \cap G_2$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leq (k_1+1)(k_2+1)$ and for every $V \in \mathcal{P}$, the graph $G[V]$ has boxicity at most two.*

In 2021, Chalermsook and Walczak [23] provided an improvement on the upper bound of Asplund and Grünbaum [7] for the chromatic number of graphs of boxicity at most two.

Theorem 4.8 (Chalermsook and Walczak, [23])**.** *Every family of axis-parallel rectangles in the plane with clique number ω is $\mathcal{O}(\omega\log(\omega))$-colorable, and an $\mathcal{O}(\omega\log(\omega))$-coloring of it can be computed in polynomial time.*

Since Theorem 3.1 follows immediately by Lemma 4.7 and Theorem 4.8, in order to prove Theorem 3.1 it remains to prove Lemma 4.7. The main observation that we need is that if a graph has path-width at most k, then it can decomposed into a family of $k+1$ disjoint subgraphs, each of which is a disjoint union of induced paths.

We first need a result about tree-decompositions. Let G be a graph, and let $A, B, X \subseteq V(G)$. We say that X *separates* A *from* B *in* G if for every (A, B)-path P in G we have $V(P) \cap X \neq \emptyset$.

Lemma 4.9 (Robertson and Seymour [111, (2.4)])**.** *Let G be a graph, (T, β) be a tree-decomposition of G, let $\{t_1, t_2\}$ be an edge of T. If T_1 and T_2 are the components of $T \setminus \{t_1, t_2\}$, where $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$, then $\beta(t_1) \cap \beta(t_2)$ separates $V_1 := \bigcup_{t \in V(T_1)} \beta(t)$ from $V_2 := \bigcup_{t \in V(T_2)} \beta(t)$ in G.*

Lemma 4.10. *Let G be a connected graph and let k be a positive integer. If G has path-width at most k, then there exists an induced path Q which is a subgraph of G such that $G \setminus V(Q)$ has path-width at most $k-1$.*

Proof of Lemma 4.10. Consider a path-decomposition (P, β) of G which realizes its path-width. Let $p_1, \ldots, p_l$ be the elements of $V(P)$ enumerated in the order that they appear in P. Let $v_1 \in V(G) \cap \beta(p_1)$ and $v_l \in V(G) \cap \beta(p_l)$, and let Q be an induced (v_1, v_l)-path in G. We define the function $\beta' : V(P) \to 2^{V(G)}$ as follows: for every $p \in V(P)$ we have $\beta'(p) := \beta(p) \setminus V(Q)$. Then (P, β') is a path-decomposition of the graph $G \setminus V(Q)$. Moreover, by Lemma 4.9, it follows that for each $i \in [l]$ we have $V(Q) \cap \beta(p_i) \neq \emptyset$. Thus, the width of (P, β') is at most $k-1$. □

Corollary 4.11. *Let k be a positive integer. If G is a graph of path-width at most k, then there exist (possibly null) induced subgraphs $P_1, \ldots, P_{k+1}$ of G such that the following hold:*

1. *For each $i \in [k+1]$, every component of the graph P_i is a path.*
2. *For each $i \in [2, k+1]$, we have that P_i is an induced subgraph of $G \setminus (V(P_1) \cup \ldots \cup V(P_{i-1}))$, and every component of $G \setminus \cup_{j<i} V(P_j)$ contains exactly one component of P_i.*
3. *$V(G) = \bigcup_{i \in [k+1]} V(P_i)$.*

Proof of Corollary 4.11. We prove the statement by induction on k. If the graph G has path-width equal to one, then G is the disjoint union of paths, and letting $P_1 := G$ we see that the statement of Corollary 4.11 holds.

Let $k > 1$ and suppose that the statement of Corollary 4.11 holds for every positive integer $k' < k$. Let $C_1, \dots, C_l$ be the connected components of G. For each $j \in [l]$, we have that C_j is a connected graph of path-width at most k. Let P_1^j be a subgraph of C_j which is a path as in the statement of Lemma 4.10. Let $P_1 := \cup_{j \in [l]} P_1^j$. Consider the graph $G' := G \setminus V(P_1)$ which, by Lemma 4.10, has path-width at most $k' := k - 1 < k$. Then, by applying the induction hypothesis to the graph G', we obtain subgraphs $P_2, \dots, P_{k+1}$ of G' such that the subgraphs $P_1, \dots, P_{k+1}$ of G satisfy the statement of Corollary 4.11. □

Given a set X, we say that a family $\mathcal{X} := \{X_i\}_{i \in I}$ of subsets of X satisfies the *Helly property* if for every $I' \subseteq I$ the following holds: if $X_i \cap X_j \neq \emptyset$ for all $i, j \in I'$, then we have that $\bigcap_{i \in I'} X_i \neq \emptyset$. The following is a folklore (see, for example, [66, Proposition 4.7]):

Proposition 4.12. *Every family of subtrees of a tree satisfies the Helly property.*

We are now ready to prove Lemma 4.7.

Proof of Lemma 4.7. Let $P_1, \dots, P_{k_1+1}$ and $Q_1, \dots, Q_{k_2+1}$ be subgraphs of T_1 and T_2 respectively, chosen as in Corollary 4.11.

Let X be a subtree of T_1. We define the level of X, denoted by $L_1(X)$, as follows:

$$L_1(X) := \min\{i \in [k_1 + 1] : V(X) \cap V(P_i) \neq \emptyset\}.$$

Similarly, we define the level of a subtree X of T_2 as follows:

$$L_2(X) := \min\{i \in [k_2 + 1] : V(X) \cap V(Q_i) \neq \emptyset\}.$$

Claim 4.12.1. *Let X and Y be subtrees of T_1 such that $L_1(X) = L_1(Y) = l$. Then the following hold:*

1. *Both $X \cap P_l$ and $Y \cap P_l$ are paths; and*
2. *$V(X) \cap V(Y) \neq \emptyset$ if and only if $V(X) \cap V(Y) \cap V(P_l) \neq \emptyset$.*

Similarly for subtrees of T_2.

Proof of Claim 4.12.1. We prove the claim for T_1; the proof for T_2 is identical. Since $L_1(X) = l$, we have that $V(X) \cap (V(P_1) \cup \ldots \cup V(P_{l-1})) = \emptyset$. Thus X is contained in a connected component of the forest $T \setminus (V(P_1) \cup \ldots \cup V(P_{l-1}))$. Let C be this component. By Corollary 4.11, we have that $Z := P_l \cap C$ is a path, and thus $X \cap P_l$ is a path as well. With identical arguments we get that $Y \cap P_l$ is a path.

For the second statement of our claim: The reverse implication is immediate. For the forward implication: Since $V(X) \cap V(Y) \neq \emptyset$, both the subtrees X and Y are contained in the same connected component of the forest $T \setminus (V(P_1) \cup \ldots \cup V(P_{l-1}))$. Let C be this component. By Corollary 4.11, we have that $Z := P_l \cap C$ is a path. Consider the tree C and its family of subtrees $\{X, Y, Z\}$. Since $V(X) \cap V(Z) \neq \emptyset$, $V(Y) \cap V(Z) \neq \emptyset$ and $V(X) \cap V(Y) \neq \emptyset$, by Proposition 4.12, it follows that $V(X) \cap V(Y) \cap V(Z) \neq \emptyset$. In particular $V(X) \cap V(Y) \cap V(P_l) \neq \emptyset$. This concludes the proof of Claim 4.12.1. ■

In what follows in this proof, for every $v \in V(G)$ and $i \in [2]$, we denote by T_i^v the subtree $T_i[\{t \in V(T_i) : v \in \beta_i(t)\}]$ of T_i. For each $i \in [k_1 + 1]$ and for each $j \in [k_2 + 1]$, we define a subset of $V(G)$ as follows:

$$V_{i,j} := \{v \in V(G) : L_1(T_1^v) = i \text{ and } L_2(T_2^v) = j\}.$$

Let $\mathcal{P} := \{V_{i,j}\}_{i\in[k_1+1], j\in[k_2+1]}$ and observe that $\mathcal{P}$ is a partition of $V(G)$.

Claim 4.12.2. *For each $i \in [k_1 + 1]$ the graph $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is an interval graph. Similarly for each $j \in [k_2 + 1]$, and $G_2[\bigcup_{i\in[k_1+1]} V_{i,j}]$.*

Proof of Claim 4.12.2. We prove the claim for G_1; the proof for G_2 is identical. Let $i \in [k_1 + 1]$. For each vertex $v \in G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$, let $P_i^v := P_i \cap T_1^v$. Then, by Claim 4.12.1, we have that P_i^v is a path. Let u and v be distinct vertices of the graph $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$. Then u is adjacent to v in $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ if and only if $V(T_1^v) \cap V(T_1^u) \neq \emptyset$. By Claim 4.12.1, we have that $V(T_1^v) \cap V(T_1^u) \neq \emptyset$ if and only if $V(T_1^v) \cap V(T_1^u) \cap V(P_i) \neq \emptyset$. Since $V(T_1^v) \cap V(T_1^u) \cap V(P_i) = V(P_i^u) \cap V(P_i^v)$, it follows that u is adjacent to v in $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ if and only if $V(P_i^u) \cap V(P_i^v) \neq \emptyset$.

Hence, the graph $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is the intersection graph of the family $\{P_i^v : v \in G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]\}$ of subpaths P_i. Since P_i is the disjoint union of paths it follows that every component of $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is an interval graph, and so $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is an interval graph as well. ■

Let $i \in [k_1+1]$ and $j \in [k_2+1]$. Then, by Claim 4.12.2, we have $G[V_{i,j}] \in \mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$. Hence $\mathcal{P}$ is the desired partition. □

We remark that, using the arguments of the above proof and induction, we can get the following:

Lemma 3.2. *Let $k_1, \dots, k_l$ be positive integers, and let $G_1, \dots, G_l$ be chordal graphs such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \le k_i$. If G is a graph such that $G = G_1 \cap \dots \cap G_l$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \le \prod_{i\in l}(k_i+1)$ and for every $V \in \mathcal{P}$, the graph $G[V]$ has boxicity at most l.*

Davies and Yuditsky [41] proved that the Gyárfas-Sumner conjecture holds with polynomial bounds for graphs of bounded boxicity:

Theorem 3.3 (Davies and Yuditsky [41])**.** *For every positive integer d and forest F, the class of all F-free graphs of boxicity at most d is polynomially χ-bounded.*

Now the following strengthening of Theorem 3.3 is an immediate corollary of Lemma 3.2 and Theorem 3.3.

Theorem 3.4. *Let $k_1, \dots, k_l$ be positive integers, and let $\mathcal{C}$ be the class of all graphs G for which there exist chordal graphs $G_1, \dots, G_l$ such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \le k_i$, and $G = G_1 \cap \dots \cap G_l$. Then, the for every forest F the class of F-free graphs in $\mathcal{C}$ is polynomially χ-bounded.*

4.3 Subclasses of $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$: When at least one chordal graph has a representation tree of bounded radius

For each positive integer k we consider the subclass of $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$ in which one of the two chordal graphs in the intersection has a representation tree of radius at most k, and we prove that this class is χ-bounded.

Theorem 3.5. *Let k be a positive integer, and let G_1 and G_2 be chordal graphs such that the graph G_1 has a representation (T_1, β_1) where $\mathsf{rad}(T_1) \leq k$. If G is a graph such that $G = G_1 \cap G_2$, then $\chi(G) \leq k \cdot \omega(G)$.*

The main observation that we need for the proof of Theorem 3.5 is the following:

Lemma 4.13. *Let G be a chordal graph and k be a positive integer. If G has a representation (T, β) such that $\mathsf{rad}(T) \leq k$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leq k$ and for each $V \in \mathcal{P}$ we have that $G[V]$ is a disjoint union of complete graphs.*

We show how Theorem 3.5 follows from Lemma 4.13.

Proof of Theorem 3.5 assuming Lemma 4.13. Let G be a graph as in the statement of Theorem 3.5, and let $\mathcal{P}$ be a partition of $V(G_1)$ as in the statement of Lemma 4.13.

We claim that for each $V \in \mathcal{P}$, we have $\chi(G[V]) \leq \omega(G)$. Indeed, let $V \in \mathcal{P}$. Then the graph $G_1[V]$ is a disjoint union of complete graphs. Hence, the graph $G[V] = G_1[V] \cap G_2[V]$ is the intersection of a chordal graph with a disjoint union of cliques, and thus a chordal graph. Hence, $\chi(G[V]) \leq \omega(G[V]) \leq \omega(G)$.

For each $V \in \mathcal{P}$, we can color the graph $G[V]$ with a different palette of $\omega(G)$ colors, and obtain a $(k \cdot \omega(G))$-coloring of G. Hence $\chi(G) \leq k \cdot \omega(G)$. $\square$

It remains to prove Lemma 4.13.

Proof of Lemma 4.13. Let r be a vertex of T which, chosen as a root, realizes the radius of T. For each vertex $v \in V(G)$, we denote by T^v the subtree $T[\{t \in V(T) : v \in \beta(t)\}]$ of T. Furthermore, for each subtree X of T, we denote by $L(X)$ the value $\min\{d(r,x) : x \in V(X)\}$, and by $r(X)$ the root of X, which is the unique element of the set $\arg\min_{x \in V(X)} d(r,x)$. We refer to the value $L(X)$ as the level of X.

Let $S := \{T^v : v \in V(G)\}$, and for each $i \in [k]$, let $L_i := \{X \in S : L(X) = i\}$. Observe that, since T has radius at most k, we have that $\{L_i\}_{i \in [k]}$ is a partition of S.

The main observation that we need is that two subtrees X and Y of the same level intersect if and only if they have the same root (and no other common vertex).

Thus, for each level $i \in [k]$, the relation of intersection of subtrees is an equivalence relation in L_i, and the corresponding induced subgraph of G is a disjoint union of complete graphs.

For each $i \in [k]$, let $V_i := \{v \in V(G) : T^v \in L_i\}$. Then $\mathcal{P} := \{V_i : i \in [k] \text{ and } V_i \neq \emptyset\}$ is the desired partition of $V(G)$. $\square$

Chapter 5

First general observations on graph-intersection and χ-boundedness

In this chapter, we prove a characterization of intersectionwise χ-imposing classes of graphs, and we prove that every decomposable class of graphs is intersectionwise χ-guarding. Using the latter result we prove that the classes of unit interval graphs and of line graphs of bipartite graphs are intersectionwise χ-guarding.

5.1 A characterization of intersectionwise χ-imposing graph classes

In this section, we prove Theorem 3.6, which we restate below, which provides a characterization of intersectionwise χ-imposing graph classes.

Theorem 3.6. *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is intersectionwise χ-imposing if and only if $\mathcal{C}$ is colorable.*

We begin with the following observation.

This chapter is based on the coauthored paper [24].

Observation 5.1. *Let $\mathcal{C}$ be an intersectionwise χ-guarding class of graphs. Then $\mathcal{C}$ is χ-bounded.*

Proof of Observation 5.1. Since $\mathcal{C}$ is intersectionwise χ-guarding, it follows that the graph-intersection of $\mathcal{C}$ with the class of complete graphs is χ-bounded. Since this graph-intersection contains the class $\mathcal{C}$, it follows that $\mathcal{C}$ is χ-bounded as well. □

We are now ready to prove Theorem 3.6:

Proof of Theorem 3.6. For the forward direction: Since every intersectionwise χ-imposing class is intersectionwise χ-guarding, it follows by Observation 5.1 that $\mathcal{C}$ is χ-bounded. Let f be a χ-bounding function for $\mathcal{C}$. We claim that $\mathcal{C}$ does not contain arbitrarily large complete graphs. Suppose not. Then the graph-intersection of $\mathcal{C}$ with the class of all graphs contains all graphs. Thus, since $\mathcal{C}$ is intersectionwise χ-imposing, the class of all graphs is χ-bounded which is a contradiction. Let $\omega(\mathcal{C})$ be the maximum size of a complete graph in $\mathcal{C}$, and let $k := f(\omega(\mathcal{C}))$. Then for every graph $G \in \mathcal{C}$, we have that $\chi(G) \leq k$.

For the backward direction: Let k be a positive integer such that $\chi(G) \leq k$ for every graph $G \in \mathcal{C}$. Let $\mathcal{A}$ be a class of graphs, and let $H \in \mathcal{C} \Cap \mathcal{A}$. Then H is a subgraph of a graph in $\mathcal{C}$, and thus $\chi(H) \leq k$. Hence, the class $\mathcal{C} \Cap \mathcal{A}$ is χ-bounded by the function $f(\omega) = k$. This completes the proof of Theorem 3.6. □

5.2 Unions of graphs of bounded componentwise r-dependent chromatic number

In this section, we prove Theorem 3.7 which we restate below. We also use Theorem 3.7 to prove that the classes of unit interval graphs and of line graphs of bipartite graphs are intersectionwise χ-guarding (we define these classes later on in this chapter).

Theorem 3.7. *Let $\mathcal{C}$ be a decomposable class of graphs. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

The main ingredient for the proof of Theorem 3.7 is following result on classes of graphs of bounded independence number:

Lemma 5.2. *Let r be a positive integer. Then the class of rK_1-free graphs is intersectionwise χ-guarding.*

For our proof of Lemma 5.2 we need the following observation:

Proposition 5.3. *Let $\mathcal{C}$ be a class of graphs for which there exists a function g such that for every class $\mathcal{H}$ we have that for every $G \in \mathcal{C}$ and for every $H \in \mathcal{H}$ the following holds: $\omega(H) \leq g(\omega(G \cap H))$. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

Proof of Proposition 5.3. Let $\mathcal{H}$ be a χ-bounded class of graphs, let f be a χ-bounding function for $\mathcal{H}$, let $G \in \mathcal{C}$, and let $H \in \mathcal{H}$. Then $\chi(G \cap H) \leq \chi(H) \leq f(\omega(H)) \leq f(g(\omega(G \cap H)))$. Thus, $f \circ g$ is a χ-bounding function for the class $\mathcal{C} \,\dot{\cap}\, \mathcal{H}$. Hence, $\mathcal{C}$ is intersectionwise χ-guarding. This completes the proof of Proposition 5.3. $\square$

We are now ready to prove Lemma 5.2.

Proof of Lemma 5.2. Let $\mathcal{C}$ be the class of rK_1-free graphs and let $\mathcal{D}$ be a χ-bounded class of graphs with χ-bounding function $f : \mathbb{N} \to \mathbb{N}$. Consider two graphs $G \in \mathcal{C}$ and $H \in \mathcal{D}$. We may assume that $V(G \cap H) = V(G) = V(H)$.

Claim 5.3.1. $\omega(H) < R(\omega(G \cap H) + 1, r)$.

Proof of Claim 5.3.1. Suppose not. That is, there exists a clique $S \subseteq V(H)$ such that $|S| \geq R(\omega(G \cap H) + 1, r)$. Then $(G \cap H)[S] \cong G[S]$. Since G is rK_1-free, it follows that $\omega(G \cap H) \geq \omega((G \cap H)[S]) \geq \omega(G \cap H) + 1$, a contradiction. $\blacksquare$

Now Lemma 5.2 follows by the above claim and Proposition 5.3. This completes the proof of Lemma 5.2. $\square$

Now in order to prove that decomposable classes of graphs are intersectionwise χ-guarding we prove that all the operations needed to create the graphs of a decomposable class starting from rK_1-free graphs preserve the property of being intersectionwise χ-guarding.

Proposition 5.4. *Let $\mathcal{C}_1, \dots, \mathcal{C}_t$ be intersectionwise χ-guarding classes, and let $\mathcal{C}$ be a class of graphs such that for every $G \in \mathcal{C}$ and for every component C of G there exists $i \in [t]$ such that $C \in \mathcal{C}_i$. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

Proof of Proposition 5.4. Let $\mathcal{H}$ be a χ-bounded class of graphs. For each $j \in [t]$, let f_j be a χ-bounding function of the class $\mathcal{C}_j \mathbin{\dot\cap} \mathcal{H}$. Let $H \in \mathcal{H}$, let $G \in \mathcal{C}$, and let $C_1, \dots, C_l$ be the components of G. Let $i \in [l]$ be such that $\chi(G_i \cap H) \geq \chi(G_k \cap H)$ for every $k \in [l]$. Then we have that $\chi(G \cap H) = \chi(G_i \cap H) \leq f_i(\omega(G_i \cap H)) \leq \sum_{j\in[t]} f_j(\omega(G \cap H))$. Thus, $\sum_{j\in[t]} f_j$ is a χ-bounding function of the class $\mathcal{C} \mathbin{\dot\cap} \mathcal{H}$ and hence $\mathcal{C}$ is intersectionwise χ-guarding. This completes the proof of Proposition 5.4. □

The following is an immediate corollary of Lemma 5.2 and Proposition 5.4:

Corollary 5.5. *Let $r \geq 2$ be an integer and $\mathcal{C}$ be a class of componentwise r-dependent graphs. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

Proposition 5.6. *Let k be a positive integer, let $\mathcal{C}_1, \dots, \mathcal{C}_t$ be intersectionwise χ-guarding classes, and let $\mathcal{C}$ be a class of graphs such every $G \in \mathcal{C}$ has a k-coloring f such that for every $j \in [k]$ there exists $i \in [t]$ such that $G[f^{-1}(j)] \in C_i$. Then $\mathcal{C}$ is intersectionwise χ-guarding.*

Proof of Proposition 5.6. Let $\mathcal{H}$ be a χ-bounded class of graphs. For each $i \in [t]$, let f_i be a χ-bounding function of the class $\mathcal{C}_i \mathbin{\dot\cap} \mathcal{H}$. Let $H \in \mathcal{H}$, let $G \in \mathcal{C}$, and let f be a k-coloring f such that for every $j \in [k]$ there exists $i \in [t]$ such that $G[f^{-1}(j)] \in C_i$. Let $j \in [k]$ be such that $\chi(G[f^{-1}(j)] \cap H) \geq \chi(G[f^{-1}(l)] \cap H)$ for every $l \in [k]$. Let $i \in [t]$ be such that $G[f^{-1}(j)] \in \mathcal{C}_i$. Then we have that $\chi(G \cap H) \leq k \cdot \chi(G[f^{-1}(j)] \cap H) \leq k \cdot f_i(\omega\left(G[f^{-1}(j)] \cap H\right)) \leq k \cdot f_i(\omega(G \cap H)) \leq k \cdot \sum_{p\in[t]} f_p(\omega(G \cap H))$. Thus, $k \cdot \sum_{p\in[t]} f_p$ is a χ-bounding function of the class $\mathcal{C} \mathbin{\dot\cap} \mathcal{H}$ and hence $\mathcal{C}$ is intersectionwise χ-guarding. This completes the proof of Proposition 5.6. □

The following is an immediate corollary of Corollary 5.5 and Proposition 5.6:

Corollary 5.7. *Let k and $r \geq 2$ be positive integers and let $\mathcal{C}$ be a class of graphs of componentwise r-dependent chromatic number at most k. Then $\mathcal{C}$ is intersectionwise χ -guarding.*

By the above, in order to prove Theorem 3.7, which states that decomposable classes are intersectionwise χ-guarding, it suffices to prove that graph-union preserves the property of being intersectionwise χ-guarding.

Proposition 5.8. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be intersectionwise χ-guarding classes. Then their graph-union is intersectionwise χ-guarding.*

Proof of Proposition 5.8. Let $\mathcal{C}$ be the graph-union of $\mathcal{C}_1, \ldots, \mathcal{C}_t$. Let $G \in \mathcal{C}$. Let $\mathcal{H}$ be a χ-bounded class and let $H \in \mathcal{H}$. For each $i \in [t]$, let $G_i \in \mathcal{C}_i$ be such that $G = \cup_{i\in[t]} G_i$, and let f_i be a χ-bounding function for the class $\mathcal{C}_i \Cap \mathcal{H}$. Then we have that $\chi(G \cap H) = \chi((\cup_{i\in[t]} G_i) \cap H) = \chi(\cup_{i\in[t]}(G_i \cap H))$. Since, by Proposition 2.10, we have that $\chi(\cup_{i\in[t]}(G_i \cap H)) \leq \prod_{i\in[t]} \chi(G_i \cap H)$, it follows that $\chi(G \cap H) \leq \prod_{i\in[t]} \chi(G_i \cap H) \leq \prod_{i\in[t]} f_i(\omega(G_i \cap H)) \leq \prod_{i\in[t]} f_i(\omega(G \cap H))$ Thus, $\prod_{i\in[t]} f_i$ is a χ-bounding function of the class $\mathcal{C} \Cap \mathcal{H}$ and hence $\mathcal{C}$ is intersectionwise χ-guarding. This completes the proof of Proposition 5.8. □

Now Theorem 3.7 is an immediate corollary of Corollary 5.7 and Proposition 5.8.

A *unit interval* graph is an interval graph which has a representation in which all the intervals have length one.

Lemma 5.9. *The class of unit interval graphs is decomposable.*

Proof of Lemma 5.9. In what follows we prove that every unit interval graph has componentwise 2-dependent chromatic number at most two.

Let $\{I_1, \ldots, I_n\}$ be a family of intervals on the real line such that each has unit length. We may assume that no interval has an integer as an endpoint, and thus every interval contains exactly one integer. Let $\{A, B\}$ be the partition of $\{I_1, \ldots, I_n\}$ which is defined as follows:

A contains an interval I if and only if I contains an even integer, and $B = \{I_1, \ldots, I_n\} \setminus A$. Thus B contains exactly those intervals which contain an odd integer. Let G be the intersection graph of $\{I_1, \ldots, I_n\}$.

Finally we claim that each of $G[A]$ and $G[B]$ is a componentwise 2-dependent graph, that is, a disjoint union of complete graphs. The claim for follows immediately by the observation that any two vertices in $G[A]$ (respectively in $G[B]$) are adjacent if and only if the two corresponding intervals contain the same even (respectively odd) integer. This completes the proof of Lemma 5.9. $\square$

The following is an immediate corollary of Theorem 3.7 and Lemma 5.9.

Corollary 5.10. *The class of unit interval graphs is intersectionwise χ-guarding.*

Let G be a graph. The *line graph* of G, which we denote by $L(G)$, is the graph with vertex set the set $E(G)$ and edge set the set $\{ef : e \cap f \neq \emptyset\}$.

Lemma 5.11. *The class of line graphs of bipartite graphs is decomposable.*

Proof of Lemma 5.11. In what follows we prove that each line graph of a bipartite graph is the union of two componentwise 2-dependent graphs.

Let G be a bipartite graph, and let $\{A_1, A_2\}$ be a bipartition of $V(G)$. Let $\{E_1, E_2\}$ be a partition of $E(L(G))$ which is defined as follows: for each $i \in [2]$, an edge ef of $L(G)$ is in E_i if and only if $e \cap f \subseteq A_i$.

We claim that for each $i \in [2]$ the graph $(E(G), E_i)$ is componentwise 2-dependent, that is, it is the disjoint union of complete graphs. Indeed, let $e, f, g \in E(G)$ and suppose that $ef, fg \in E_i$. Let v be the unique element of $f \cap A_i$. Then $v \in e \cap g$, and thus $eg \in E_i$. Hence the adjacency relation in $(E(G), E_i)$ is an equivalence relation. This completes the proof of Lemma 5.11. $\square$

The following is an immediate corollary of Theorem 3.7 and Lemma 5.11:

Corollary 5.12. *The class of line graphs of bipartite graphs is intersectionwise χ-guarding.*

In Section 6.1 we will show that the class of line graphs is not intersectionwise χ -guarding.

Chapter 6

Classes which are not intersectionwise χ-guarding

In this chapter we prove that certain χ-bounded classes of graphs are not intersectionwise χ-guarding. In particular, we prove that the classes of complete multipartite graphs, line graphs of graphs of large girth, and trivially perfect graphs, are not intersectionwise χ -guarding.

6.1 Line graphs of graphs of large girth and complete multipartite graphs

The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum size of a partition of the edge set of G into matchings. Vizing [129] proved that for every graph G, we have $\chi'(G) \leq \Delta(G) + 1$. Hence, we have the following:

Proposition 6.1 (Vizing [129])**.** *The class of line graphs is χ-bounded by the function $f(\omega) = \omega + 1$.*

This chapter is based on the coauthored paper [24].

The main results of this section are Theorem 3.8 and Theorem 3.9 which we restate:

Theorem 3.8. *The class of complete multipartite graphs is not intersectionwise χ-guarding.*

Theorem 3.9. *Let $g \geq 3$. Then the class of line graphs of graphs of girth at least g is not intersectionwise χ-guarding. In particular, the class of line graphs is not intersectionwise χ-guarding.*

We prove Theorem 3.8 and Theorem 3.9 by showing that the graph-intersection of the class of line graphs of graphs of girth at least g for $g \geq 3$ and the class of complete multipartite graphs contain triangle-free graphs of arbitrarily large chromatic number. We first introduce notions and results we will need to describe the construction of these graphs.

Let D be a digraph. We denote by $\chi(D)$ (respectively by $L(D)$) the chromatic number (respectively the line graph) of the underlying undirected graph of D. Following Harary and Norman [75], the *line digraph* of D, which we denote by $\vec{L}(D)$, is the digraph with $V(\vec{L}(D)) := E(D)$ and $E(\vec{L}(D)) := \{(uv)(vw) : uv, vw \in E(D)\}$. We remark that the underlying undirected graph of $\vec{L}(D)$ is a subgraph of $L(D)$.

We need the following theorem which states that the chromatic number of the underlying undirected graph of the line digraph of a digraph D is lower-bounded by a function of $\chi(D)$:

Theorem 6.2 (Harner and Entringer [76, Theorem 9])**.** *Let D be a digraph. Then $\chi(\vec{L}(D)) \geq \log_2(\chi(D))$.*

Let n and k be integers such that $n > 2k > 2$. Erdős and Hajnal [51] defined the *shift graph* $G(n, k)$ as the graph with vertex set the set of all k-tuples $(t_1, \dots, t_k)$ such that $1 \leq t_1 < \dots < t_k \leq n$, in which the vertices $(t_1, \dots, t_k)$ and $(t'_1, \dots, t'_k)$ are adjacent if and only if $t_{i+1} = t'_i$ for $1 \leq i < k$, or vice versa. Observe that $G(n, k)$ is triangle-free. We additionally consider *directed shift graphs*, orientations of shift graphs where $(t_1, \dots, t_k)(t'_1, \dots, t'_k)$ is an arc exactly if $t_{i+1} = t'_i$ for all $1 \leq i < k$. For integers n and k such that $n > 2k > 2$, we denote the directed shift graph of k-tuples over an alphabet of size n by $\vec{G}(n, k)$. The shift graph $G(n, k)$ is thus the underlying undirected graph of $\vec{G}(n, k)$.

Erdős and Hajnal [51] proved the following result for the chromatic number of shift graphs:

Theorem 6.3 (Erdős and Hajnal [51]). *Let n and k be integers such that $n > 2k > 2$. Then $\chi(G(n,k)) = (1 - o(1)) \log^{(k-1)}(n)$. In particular, $\chi(G(n,2)) = \lceil \log n \rceil$, and for n sufficiently large we have that $\chi(G(n,3)) > k$.*

Hence, for any fixed $k \geq 2$, the shift graphs $(G(n,k))_{n \in \mathbb{N}^+}$ form a class of triangle-free graphs of arbitrarily large chromatic number.

In 2018, Gábor Tardos gave a talk at *Combinatorics: Extremal, Probabilistic and Additive* in São Paulo about work of his and Bartosz Walczak on a conjecture of Erdős and Hajnal. One of their main results is the following:

Theorem 6.4 (Tardos & Walczak). *Let $k \geq 2$, $g \geq 3$ and $c > 0$ be integers. Then there exists an integer $n^* > 0$ such that for each $n \geq n^*$, the shift graph $G(n,k)$ contains a subgraph H_n with girth at least g and $\chi(H_n) \geq c$.*

We will use this result in our construction. A write-up of the proof of Theorem 6.4 may be found in Rodrigo Aparecido Enju's master's dissertation [46] (in Portuguese).

We are now ready to describe the construction showing that the graph-intersection of line graphs of graphs of large girth and complete multipartite graphs contains triangle-free graphs of arbitrarily large chromatic number.

Lemma 6.5. *Let $g \geq 3$ be an integer, let $\mathcal{L}_g$ be the class of line graphs of graphs with girth at least g, and let $\mathcal{C}$ be the class of complete multipartite graphs. Then the class $\mathcal{L}_g \Cap \mathcal{C}$ is not χ-bounded.*

Proof of Lemma 6.5. Let $c \geq 2$ be an integer. Then, by Theorem 6.4, there exists n and a subgraph H of the shift graph $G(n,2)$ such that $\mathsf{girth}(H) \geq g$ and $\chi(H) \geq 2^c$.

Let $\vec{H}$ be the oriented graph that we obtain by orienting the edges of H in the natural "shift" way, as we discussed above. By abusing the notation we denote by $\vec{L}(\vec{H})$ be the graph that we obtain from the directed line graph of H, by relabeling the vertices (and the edges accordingly) so that instead of the form $(x,y)(y,z)$ they have the form (x,y,z). Let $G(\vec{L}(\vec{H}))$ be the underlying undirected graph of $\vec{L}(\vec{H})$. Then $G(\vec{L}(\vec{H}))$ is triangle-free. Also, by Theorem 6.2, we have that:

$$\chi(G(\vec{L}(\vec{H}))) = \chi(\vec{L}(\vec{H})) \geq \log_2(\chi(L(\vec{H}))) = \log_2(\chi(L(H))) \geq \log_2(\chi(2^c)) \geq c.$$

We now show that $G(\vec{L}(\vec{H})) \in \mathcal{L}_g \mathbin{\dot\cap} \mathcal{C}$. Let C be the graph on $V(G(\vec{L}(\vec{H}))) = V(L(\vec{H}))$ in which two vertices are adjacent if and only if their middle entry is different. Then, the partition of $V(C)$ in which two vertices are in the same part if and only if they have the same middle entry, witnesses that C is a complete multipartite graph. Since $L(\vec{H}) \in L_g$, in order to prove $G(\vec{L}(\vec{H})) \in \mathcal{L}_g \mathbin{\dot\cap} \mathcal{C}$, it suffices to prove the following:

Claim 6.5.1. $G(\vec{L}(\vec{H})) = L(\vec{H}) \cap C$.

Proof. Since the graphs $G(\vec{L}(\vec{H})), L(\vec{H})$ and C have the same vertex set, it suffices to prove the following: $E(G(\vec{L}(\vec{H}))) = E(L(\vec{H}) \cap C)$. Observe that $L(\vec{H})$ contains all the edges of $G(\vec{L}(\vec{H}))$, and the only additional edges that it contains are edges between vertices with different middle entries. But these are exactly the non-edges of the complete multipartite graph. ■

Thus, $G(\vec{L}(\vec{H})) \in \mathcal{L}_g \mathbin{\dot\cap} \mathcal{C}$. Since c was arbitrary it follows that the class $\mathcal{L}_g \mathbin{\dot\cap} \mathcal{C}$ contains triangle-free graphs of arbitrarily large chromatic number. This completes the proof of Lemma 6.5. □

Theorem 3.8 and Theorem 3.9 now follow from Lemma 6.5 and the fact that the classes of complete multipartite graphs and line graphs are both χ-bounded.

The following is an immediate corollary of Theorem 3.8:

Corollary 6.6. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains a complete multipartite graph.*

To obtain a similar corollary of Theorem 3.9, we observe the following:

Observation 6.7. *Let $\mathcal{H}$ be finite set of graphs which contains no line graph of a forest. Then there exists g such that the class of all $\mathcal{H}$-free graphs contains all line graphs of graphs of girth at least g.*

The following is then an immediate corollary of Theorem 3.9 and Observation 6.7:

Corollary 6.8. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains the line graph of a forest.*

A *claw* is a $K_{1,3}$.

Corollary 6.9. *The class of claw-free graphs is not intersectionwise χ-guarding.*

Proof of Corollary 6.9. As shown by Van Rooij and Wilf [114], line graphs are claw-free, thus the class of claw-free graphs contains all line graphs. Hence, by Theorem 3.9, the class of claw-free graphs is not intersectionwise χ-guarding. This completes the proof of Corollary 6.9. □

6.2 Trivially perfect graphs

The main result of this section is Theorem 3.11 which we restate:

Theorem 3.11. *The class of trivially perfect graphs is not intersectionwise χ-guarding.*

We prove Theorem 3.11 by showing that the graph-intersection of the class of trivially perfect graphs with a χ-bounded class contains graphs of bounded clique number and arbitrarily large chromatic number. We need some definitions. Let G be a graph, let $l \geq 3$, and let $C = \{v_1, \dots, v_{2l}\}$ be an even cycle of length at least six in G. For $i, j \in [2l]$, we say that the edge v_iv_j is an *odd chord* of C if i and j have different parity; in this case the distance of v_i and v_j in the cycle is odd. A *strongly chordal graph* is a chordal graph in which every even cycle of length at least six has an odd chord. We note that, since chordal graphs are perfect [9], strongly chordal graphs are perfect. In what follows we show that the graph-intersection of the class of trivially perfect graphs with a proper subclass of the class of strongly chordal graphs is not χ-bounded by proving that this graph-intersection contains all Burling graphs (defined below).

As we discussed in Section 2.3, in 1965 Burling [21] introduced a sequence $\{\mathcal{B}_k\}_{k\geq 1}$ of families of axis-aligned boxes in $\mathbb{R}^3$ with the following property:

Theorem 6.10 (Burling [21])**.** *For every positive integer k, the intersection graph G_k of $\mathcal{B}_k$ is triangle-free and has chromatic number at least k.*

Throughout this section, for each k, we denote by G_k the intersection graph of $\mathcal{B}_k$. We call the sequence $\{G_k\}_{k\in\mathbb{N}}$ the *Burling sequence*. Following Pournajafi and Trotignon [107], we say that a graph G is a *Burling graph* if there exists a positive integer k such that G is isomorphic to an induced subgraph of G_k. In what follows in this section we denote by $\mathcal{B}$ the hereditary class of Burling graphs. As we discussed in Section 3.1, Asplund and Grünbaum [7] proved that the 2-fold graph-intersection of the class of interval graphs is χ-bounded, and it follows from Theorem 6.10 that the 3-fold graph-intersection of the class of interval graphs is not χ-bounded. Motivated by these results, Gyárfás [71, Problem 5.7] asked whether the graph-intersection of the class of chordal graphs with the class of interval graphs is χ-bounded. In joint work with Miraftab, and Spirkl pointed out in [25] that a result of Felsner, Joret, Micek, Trotter, and Wiechert [57] implies that Burling graphs are contained in the graph-intersection of the class of chordal graphs with the class of interval graphs, and thus the answer to Gyárfás' question is negative. Here we strengthen this result, by proving the following:

Theorem 6.11. *The class of Burling graphs is a subclass of the graph-intersection of the class of trivially perfect graphs with a proper subclass of the class of strongly chordal graphs.*

We note that Theorem 3.11 follows immediately from Theorem 6.11. Our proof of Theorem 6.11 is based on a characterization of Burling graphs which was proved by Pournajafi and Trotignon in [107], where they defined the hereditary class of derived graphs and proved that this class is the same as the class of Burling graphs.

Let (T, r) be a rooted tree. If there is no danger of ambiguity, we use the notation T for the rooted tree (T, r). The *parent* of a vertex $v \in V(T) \setminus \{r\}$, denoted by $p(v)$, is the neighbor of v which lies in the unique (v, r)-path in T. If $p(v) = u$, then we say that v is a *child* of u. Let $u, v \in V(T)$. We say that u and v are *siblings* if $p(u) = p(v)$. We say that u is an *ancestor* of v if u lies in the in the unique (v, r)-path in T. The *descendants* of a vertex u are all the vertices which have u as an ancestor. Finally, following [107] we say that a *branch* in T is a path $v_1, \ldots, v_k$ such that for each $i \in [k-1]$ the vertex v_i is the parent of the vertex v_{i+1}; in this case we say that *the branch starts* at v_1 and ends at v_k. A *principal branch* is a branch which starts at the root and ends at a leaf of the tree T. Following [107] we say that a *Burling tree* is a 4-tuple (T, r, l, c) in which:

(i) T is a rooted tree and r is its root;

(ii) l is a function associating to each non-leaf vertex v of T one child of v which is called the *last-born* of v;

(iii) c is a function defined on the vertices of T. If v is a non-last-born vertex of T other than the root, then c associates to v the vertex set of a (possibly empty) branch in T starting at the last-born of $p(v)$. If v is a last-born vertex or the root of T, then we define $c(v) = \emptyset$. We call c the *choose function* of T.

Following [107] we say that the *oriented graph fully derived* from the Burling tree (T, r, l, c), which we denote by $A(T)$, is the oriented graph whose vertex set is $V(T)$ and $uv \in E(A(T))$ if and only if v is a vertex in $c(u)$. The *graph fully derived* from the Burling tree (T, r, l, c), which we denote by $G(T)$, is the underlying undirected graph of $A(T)$. A graph (respectively oriented graph) is *derived* from the Burling tree (T, r, l, c) if it is an induced subgraph of $G(T)$ (respectively $A(T)$). A graph G (respectively oriented graph A) is called a *derived graph* (respectively *oriented derived graph*) if there exists a Burling tree (T, r, l, c) such that G (respectively A) is derived from (T, r, l, c).

Theorem 6.12 (Pournajafi and Trotignon [107, Theorem 4.9])**.** *The class of derived graphs is the same as the class of Burling graphs.*

By the above, in order to prove Theorem 6.11, it suffices to prove the following:

Theorem 6.13. *The class of derived graphs is a subclass of the graph-intersection of the class of trivially perfect graphs with a proper subclass of the class of strongly chordal graphs.*

Let (T, r, l, c) be a Burling tree. We denote by $C(T)$ the graph which we obtain from $G(T)$ by adding the necessary edges in order to make the vertex set of every principal branch of T a clique. We also denote by $I(T)$ the graph that we obtain from $G(T)$ by adding edges so that every vertex is adjacent to all of its siblings, and to all the descendants of its last-born sibling.

Observation 6.14. *Let (T, r, l, c) be a Burling tree. Then $G(T) = C(T) \cap I(T)$.*

In what follows in this section, we denote by $\mathcal{C}$ (respectively by $\mathcal{I}$) the closure under induced subgraphs of the class $\{C(T) : (T, r, l, c) \text{ is a Burling tree}\}$ (respectively the closure under induced subgraphs of the class $\{I(T) : (T, r, l, c) \text{ is a Burling tree}\}$). We call the class $\mathcal{C}$ the class of *Burling strongly chordal graphs*.

Observation 6.15. *The class of derived graphs is contained in the class* $\mathcal{C} \Cap \mathcal{I}$*.*

By Observation 6.15, in order to prove Theorem 6.13, it suffices to prove that $\mathcal{C}$ is a proper subclass of the class of strongly chordal graphs, and that $\mathcal{I}$ is the class of trivially perfect graphs.

To this end, we need to introduce some terminology in order to state a characterization of strongly chordal graphs. Let G be a graph and let $u, v \in V(G)$. Following Farber [56], we say u and v are *compatible* if $N[u] \subseteq N[v]$ or $N[v] \subseteq N[u]$, and that a vertex $v \in V(G)$ is *simple* if the vertices in $N[v]$ are pairwise compatible. Farber [56] gave the following characterization of strongly chordal graphs.

Theorem 6.16 (Farber [56, Theorem 3.3])**.** *A graph G is strongly chordal if and only if every induced subgraph of G has a simple vertex.*

We begin with two lemmas that we need in order to prove that $\mathcal{C}$ is a subclass of the class of strongly chordal graphs.

Lemma 6.17. *Let (T, r, l, c) be a Burling tree and let u be an ancestor of v in T. Then u and v are compatible in $C(T)$, in particular $N_{C(T)}[v] \subseteq N_{C(T)}[u]$.*

Proof of Lemma 6.17. The set $N_{C(T)}[v]$ can be partitioned in the following three sets:

- The set N_1 which contains v, the ancestors of v, and the descendants of v;
- the set $N_2 = \{w \in V(T) : wv \in E(A(T))\}$; and
- the set $N_3 = \{w \in V(T) : vw \in E(A(T))\}$.

Since the vertex set of every branch of T is a clique in $C(T)$, it follows that $N_1 \subseteq N_{C(T)}[u]$. We claim that $N_2 \subseteq N_{C(T)}[u]$. Indeed, let $w \in N_2$. Then v lies in a branch of T which starts at $p(w)$. Since u is an ancestor of v, we deduce that either u lies in the $(r, p(w))$-path in T, or u lies in the $(p(w), v)$-path in T. In both cases we have that u is adjacent with w in $C(T)$. Hence, $N_2 \subseteq N_{C(T)}[u]$. We claim that $N_3 \subseteq N_{C(T)}[u]$. Indeed, let $w \in N_3$. Then $w \in c(v)$. Since u is an ancestor of v, we have that u is either equal to $p(v)$ or an ancestor of $p(v)$, and thus w is a descendant of u. Hence, $w \in N_{C(T)}[u]$. By the above it follows that $N_{C(T)}[v] \subseteq N_{C(T)}[u]$. This completes the proof of Lemma 6.17. □

Let (T, r, l, c) be a Burling tree. A *left principal branch* is a principal branch B such that for every $u \in V(T) \setminus V(B)$ we have $c(u) \cap V(B) = \emptyset$.

Lemma 6.18. *Let (T, r, l, c) be a Burling tree. Then T has a left principal branch.*

Proof of Lemma 6.18. We construct a left principal branch of T inductively as follows: Let $v_0 := r$. Let $i \geq 0$ and suppose that we have constructed a path $v_0, \ldots, v_i$. If v_i is a leaf, then we are done. Otherwise, v_i has at least one child. If v_i has at least two children, then we let v_{i+1} be a non-last-born of v_i, otherwise we let $v_{i+1} := l(v_i)$.

Let $B := v_0, \ldots, v_k$ be a principal branch that has been created by the above process. We claim that B is a left principal branch. Let us suppose towards a contradiction that there exists $u \in V(T) \setminus V(B)$ and $v \in V(B)$, such that $v \in c(u)$. Then $l(p(u)) \in V(B)$, but $p(u)$ has a non-last born child, namely u; this contradicts the construction of B. This completes the proof of Lemma 6.18. □

Let (T, r, l, c) be a Burling tree, and let G be a graph which is derived from (T, r, l, c). We may assume that all leaves of T are in G. We call a vertex v of G a *bottom-left* vertex of G if v lies in a left principal branch B of T, and v is a leaf of T.

Observation 6.19. *Let G be a derived graph. Then G has at least one bottom-left vertex.*

Lemma 6.20. *The class $\mathcal{C}$ is a subclass of the class of strongly chordal graphs.*

Proof of Lemma 6.20. We prove that for every Burling tree (T, r, l, c), the graph $C(T)$ is strongly chordal. Let (T, r, l, c) be a Burling tree. In order to prove that $C(T)$ is strongly chordal, by Theorem 6.16, it suffices to prove that every induced subgraph of $C(T)$ has a simple vertex. Let H be an induced subgraph of $C(T)$. By Observation 6.19 we know that $G(T)[V(H)]$ has at least one bottom-left vertex. Let v be such a vertex.

We claim that v is a simple vertex of $C(T)$ and thus a simple vertex of H. By Lemma 6.17, in order to prove that v is a simple vertex in H, it suffices to prove that the neighborhood of v in $C(T)$ is included in a branch of T. Indeed, since v is a bottom-left vertex, we have that v has no in-neighbors in $A(T)[V(H)]$ and no descendant of v is in $A(T)[V(H)]$. Hence, the neighborhood of v in H is included in the principal branch of T which contains the set $c(v)$ in the case that v is not a last-born, and in the principal branch of T which contains v otherwise. This completes the proof of Lemma 6.20. □

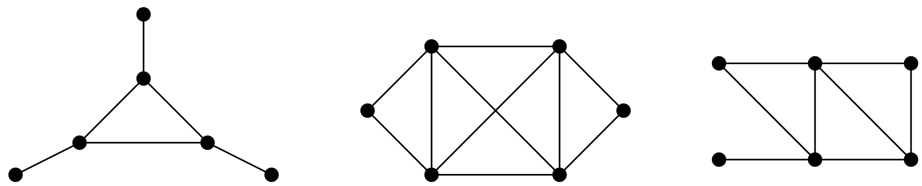

Figure 6.1: From left to right: The net, the complement of the graph H, and the complement of the graph X_{96}. Each of these graphs is a strongly chordal graph which is not a Burling strongly chordal graph.

We say that two vertices of a tree are *collinear* if they both lie in the same principal branch. Since, by Theorem 6.10, Burling graphs (and thus derived graphs) are triangle-free, we have the following useful observation:

Observation 6.21. *Let (T, r, l, c) be a Burling tree and let $a, b, c \in V(T)$ be such that the graph $C(T)[\{a, b, c\}]$ is a triangle. Then at least two of the vertices a, b, c are collinear in T.*

The *net* is the leftmost graph in Figure 6.1.

Lemma 6.22. *Every graph in the class $\mathcal{C}$ is net-free.*

Proof of Lemma 6.22. Suppose not. Then, since $\mathcal{C}$ is hereditary, it follows that $\mathcal{C}$ contains the net. Let $G \in \mathcal{C}$ be isomorphic to the net, and let (T, r, l, c) be a Burling tree such that G is an induced subgraph of $C(T)$. By Observation 6.21, we have that at least two vertices of the triangle of G, say a and b, are collinear. Thus, by Lemma 6.17, we have that a and b are compatible, which is a contradiction. This completes the proof of Lemma 6.22. □

Following the notation of the website `https://www.graphclasses.org` we denote by H^c and X_{96}^c, the second and the third (from left to right) graph which is illustrated in Figure 6.1 respectively. Both H^c and X_{96}^c are strongly chordal graphs. We note that using Lemma 6.17 and Observation 6.21, one can prove that every graph in $\mathcal{C}$ is $\{H^c, X_{96}^c\}$-free.

The following is an immediate corollary of Lemma 6.22 and the fact that the net is a strongly chordal graph:

Corollary 6.23. *The class of Burling strongly chordal graphs is a proper subclass of the class of strongly chordal graphs.*

In Section 7.1 we make use of the following observation:

Proposition 6.24. *The class of trivially perfect graphs is a subclass of the class of Burling strongly chordal graphs.*

Let $G_1, \ldots, G_k$ be graphs. The *disjoint union* of $G_1, \ldots, G_k$, which we denote by $G_1 + \ldots + G_k$, is the graph which has as vertex set (respectively edge set) the disjoint union of the sets $V(G_1), \ldots, V(G_k)$ (respectively of the sets $E(G_1), \ldots, E(G_k)$). In order to prove Proposition 6.24 we use the following characterization of trivially perfect graphs.

Theorem 6.25 (Yan, Chen, and Chang [134, Theorem 3])**.** *The class of trivially perfect graphs is the minimal hereditary class of graphs which contains the graph K_1, and is closed under the following operations:*

(i) disjoint union of two graphs;
(ii) adding a new vertex complete to every other vertex.

Before we proceed to the proof of Proposition 6.24 we need to introduce some notation about functions. For a function f we denote the domain of f by $\mathrm{dom}(f)$. Let f and g be two functions which agree in the set $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, that is for every $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ we have $f(x) = g(x)$. Then we denote by $f \cup g$ the function with $\mathrm{dom}(f \cup g) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$, which is defined as follows:

$$(f \cup g)(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{dom}(f); \\ g(x), & \text{otherwise.} \end{cases}$$

Let k be a positive integer, let f be a function, and let $x_1, \ldots x_k$ be elements such that for every $i \in [k]$ we have that $x_i \notin \mathrm{dom}(f)$; we denote by $f \cup \{(x_1, y_1), \ldots, (x_k, y_k)\}$ the function that we obtain by extending the definition of f to include $f(x_i) = y_i$ for every $i \in [k]$. We are now ready to proceed with the proof of Proposition 6.24.

Proof of Proposition 6.24. Since $\mathcal{C}$ contains K_1, it suffices to show that $\mathcal{C}$ is closed under the two operations which are mentioned in the statement of Theorem 6.25. Let (T_1, r_1, l_1, c_1) and (T_2, r_2, l_2, c_2) be Burling trees on disjoint vertex sets.

We claim that $C(T_1) + C(T_2) \in \mathcal{C}$. Indeed, let T be the tree which we obtain from $T_1 + T_2$ by adding the three new vertices r, r_1' and r_2', and the edges r_1r_1', r_2r_2', rr_1' and rr_2'. Let $l := l_1 \cup l_2 \cup \{(r, r_2'), (r_1', r_1), (r_2', r_2)\}$ and $c := c_1 \cup c_2 \cup \{(r_1', B), (r, \emptyset), (r_2', \emptyset)\}$ where B is a branch in T starting at r_2'. Consider the Burling tree (T, r, l, c). Then $C(T_1) + C(T_2)$ is the induced subgraph of $C(T)$ that we obtain by deleting the vertices r, r_1' and r_2'. This proves that $\mathcal{C}$ is closed under disjoint union.

Let $C'(T_1)$ be the graph that we obtain from $C(T_1)$ by adding a new vertex, say r, which is complete to $V(C(T_1))$. Let T be the tree which we obtain from T_1 by adding the new vertex r and the edge rr_1. Consider the Burling tree $(T, r, l_1 \cup \{(r, r_1)\}, c_1)$. Then $C'(T_1)$ is isomorphic to $C(T)$, and thus $C'(T_1) \in \mathcal{C}$. This proves that $\mathcal{C}$ is closed under operation (ii) in the statement of *Theorem* 6.25. This completes the proof of Proposition 6.24. □

We were not able to fully characterize Burling strongly chordal graphs. We suggest the following:

Problem 3. *Characterize the class of Burling strongly chordal graphs by its forbidden induced subgraphs.*

Let G be an oriented graph which is derived from a Burling tree (T, r, l, c). Following Pournajafi and Trotignon [107] we call a vertex v of G a *top-left* vertex if the following hold:

- the distance of v from r in T is equal to the minimum distance of a vertex of G from r in T; and
- either v is not a last-born of T or v is the only vertex in $\arg\min\{d_T(u, r) : u \in V(G)\}$.

Lemma 6.26 (Pournajafi and Trotignon [107, Lemma 3.1])**.** *Every non-empty oriented graph G derived from a Burling tree (T, r, l, c) contains at least one top-left vertex and every such vertex is a source of G. Moreover, the neighborhood of a top-left vertex is included in a branch of T.*

A *universal vertex* of a graph G is a vertex $v \in V(G)$ which is complete to $V(G) \setminus \{v\}$. We need the following characterization of trivially perfect graphs.

Theorem 6.27 (Wolk [133])**.** *Let G be a graph. Then G is trivially perfect if and only if every connected induced subgraph of G contains a universal vertex.*

We recall the definition of the class $\mathcal{I}$: Let (T, r, l, c) be a Burling tree. We denote by $I(T)$ the graph that we obtain from $G(T)$ by adding edges so that every vertex is adjacent to all of its siblings, and to all the descendants of its last-born sibling. Then $\mathcal{I}$, as we defined it earlier in this section, is the closure under induced subgraphs of the class $\{I(T) : (T, r, l, c) \text{ is a Burling tree}\}$.

Lemma 6.28. *The class $\mathcal{I}$ is a subclass of the class of trivially perfect graphs.*

Proof of Lemma 6.28. We prove that for every Burling tree (T, r, l, c), the graph $I(T)$ is trivially perfect. Let (T, r, l, c) be a Burling tree. By Theorem 6.27 in order to prove that $I(T)$ is trivially perfect it suffices to prove that every connected induced subgraph of $I(T)$ has a universal vertex.

Let H be a connected induced subgraph of $I(T)$. By Lemma 6.26 we know that $A(T)[V(H)]$ has a top-left vertex. Let v be such a vertex. We claim that v is a universal vertex of H. Suppose not. Let $u \in V(H)$ be a vertex which is not adjacent to v. We claim that the vertex set of each connected component of $I(T)$ consists of a set S of siblings in T and the set D of the descendants of the last-born, say l, vertex in S; where S is a clique, and every vertex of S is complete to D. Indeed, our claim follows by the definition of $I(T)$ and the fact that the vertex set of each branch of T is an independent set in $G(T)$. Now since u and v are non-adjacent in H, it follows that $u, v \in D$; that is, both u and v are descendants of l in T. Thus, we have that v is not a top-left vertex, which is a contradiction. This completes the proof of Lemma 6.28. $\square$

Lemma 6.29. *The class of trivially perfect graphs is a subclass of the class $\mathcal{I}$.*

Proof of Lemma 6.29. As in the proof of Proposition 6.24 we use the characterization of trivially perfect graphs from Theorem 6.25. Since $\mathcal{I}$ contains K_1, it suffices to show that $\mathcal{I}$ is closed under the two operations which are mentioned in the statement of Theorem 6.25. Let (T_1, r_1, l_1, c_1) and (T_2, r_2, l_2, c_2) be Burling trees on disjoint vertex sets.

We claim that $I(T_1) + I(T_2) \in \mathcal{C}$. Indeed, let T be the tree which we obtain from $T_1 + T_2$ by adding the three new vertices r, r'_1 and r'_2, and the edges $r_1r'_1, r_2r'_2, rr'_1$ and rr'_2. Let $l := l_1 \cup l_2 \cup \{(r, r'_2), (r'_1, r_1), (r'_2, r_2)\}$ and $c := c_1 \cup c_2 \cup \{(r'_1, B), (r, \emptyset), (r'_2, \emptyset)\}$ where B is a branch in T starting at r'_2. Consider the Burling tree (T, r, l, c). Then $I(T_1) + I(T_2)$ is the induced subgraph of $I(T)$ that we obtain by deleting the vertices r, r_1' and r_2'. This proves that $\mathcal{I}$ is closed under disjoint union.

Let $I'(T_1)$ be the graph that we obtain from $I(T_1)$ by adding a new vertex, say r', which is complete to $V(I(T_1))$. Let T be the tree which we obtain from T_1 by adding the two new vertices r, r' and the edges rr' and rr_1. Let $l := l_1 \cup \{(r, r_1)\}$, and let $c := c_1 \cup \{(r, \emptyset), (r', \emptyset)\}$. Consider the Burling tree (T, r, l, c). Then $I'(T_1)$ is isomorphic to the graph that we obtain from $I(T)$ by deleting r, and thus $I'(T_1) \in \mathcal{C}$. This proves that $\mathcal{I}$ is closed under operation (ii) in the statement of $Theorem$ 6.25. This completes the proof of Lemma 6.29. □

The following is an immediate corollary of Lemma 6.28 and Lemma 6.29:

Corollary 6.30. *The class $\mathcal{I}$ is the class of trivially perfect graphs.*

Now Theorem 6.13 is an immediate corollary of Observation 6.15, Corollary 6.23, and Corollary 6.30. Recall that Theorem 6.13 implies that both the class of trivially perfect graphs and Burling strongly perfect graphs are not intersectionwise χ-guarding. Hence if $\mathcal{H}$ is a finite set of graphs such that the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains a Burling strongly chordal graph and a trivially perfect graph. Since, by Proposition 6.24, every trivially perfect graph is a Burling strongly chordal graph, the condition for $\mathcal{H}$ to contain both a Burling strongly chordal graph and a trivially perfect graph is satisfied when $\mathcal{H}$ contains a trivially perfect graph. We summarize these in the following corollary:

Corollary 6.31. *Let $\mathcal{H}$ be a set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains a trivially perfect graph.*

Chapter 7

On intersectionwise χ-guarding classes which are defined by a finite set of forbidden induced subgraphs

7.1 Necessary conditions a finite set of graphs should satisfy in order to define an intersectionwise χ -guarding class

Recall the following classic result of Erdős that we discussed in Chapter 2:

Theorem 2.2 (Erdős [47]). *Let $k, l \geq 2$ be integers. Then there exists a graph G with $\chi(G) \geq k$ and $\mathsf{girth}(G) > l$.*

The following is an immediate corollary of Theorem 2.2:

Observation 7.1. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is χ-bounded, then $\mathcal{H}$ contains a forest.*

This chapter is based on the coauthored paper [24].

The following is an immediate corollary of Observation 5.1 and Observation 7.1:

Corollary 7.2. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains a forest.*

Corollary 6.6, Corollary 6.8, Corollary 6.31, and Corollary 7.2 give rise to necessary conditions on a finite set $\mathcal{H}$ of graphs for the class of $\mathcal{H}$-free graphs to be intersectionwise χ-guarding. These conditions are summarized in Theorem 3.12 which we restate:

Theorem 3.12. *Let $\mathcal{H}$ be a finite set of graphs. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then $\mathcal{H}$ contains:*

- *a forest;*
- *a line graph of a forest;*
- *a complete multipartite graph; and*
- *a trivially perfect graph.*

7.2 A characterization of H-free intersectionwise χ-guarding classes of graphs

In this section we characterize the graphs H for which the class of H-free graphs is intersectionwise χ-guarding. The main result of this section is Theorem 3.13 which we restate:

Theorem 3.13. *Let H be a graph. Then the class of H-free graphs is intersectionwise χ-guarding if and only if H is isomorphic to P_2, P_3, or rK_1 for some $r > 0$.*

We start by proving the backwards direction of Theorem 3.13. Recall that in Section 5.2 we proved Lemma 5.2 which states that for every positive integer r, the class of rK_1-free graphs is intersectionwise χ-guarding. Thus, since every P_2-free graph is a P_3-free graph, in order to prove the backwards direction it suffices to prove that the class of P_3-free graphs is intersectionwise χ-guarding. To this end we need the following well-known result:

Proposition 7.3 (Folklore)**.** *Let G be a graph. Then G is the disjoint union of complete graphs if and only if G is P_3-free.*

In Section 5.2 we proved Corollary 5.5 which states that every class of componentwise r-dependent graphs is intersectionwise χ-guarding. Since, by Proposition 7.3, every P_3-free graph is componentwise 2-dependent, we get the following:

Corollary 7.4. *The class of P_3-free graphs is intersectionwise χ-guarding.*

Since a graph is complete multipartite if and only if its complement is the disjoint union of complete graphs, Proposition 7.3 implies the following:

Proposition 7.5 (Folklore)**.** *Let G be a graph. Then G is complete multipartite if and only if G is $(K_1 + K_2)$-free.*

We now proceed with proving the characterization of intersectionwise χ-guarding graph classes which are defined by a single forbidden induced subgraph, as stated in Theorem 3.13.

Proof of Theorem 3.13. The backward direction follows directly from Corollary 7.4, the fact that P_2 is an induced subgraph of P_3, and Lemma 5.2.

For the forward direction: Let H be a graph such that the class of H-free graphs is intersectionwise χ-guarding. We first notice that, by Corollary 7.2, H is a forest.

Next, we observe that H has to be claw-free, as otherwise we would obtain a contradiction with Corollary 6.9. Hence, H is a disjoint union of paths. If H consists of multiple components, then each of these is a single vertex, since otherwise H contains $K_1 + K_2$, which contradicts the fact that, by Theorem 3.12, H is complete multipartite, and thus, by Proposition 7.5, H is $(K_1 + K_2)$-free. In this case, H is isomorphic to rK_1 for some $r \geq 2$.

Hence, we may assume that H is a path. H does not contain more than 3 vertices, as any path on at least 4 vertices contains $K_1 + K_2$ as an induced subgraph, leading to a contradiction with Theorem 3.12. Thus, H is P_1 (which equals rK_1 for $r = 1$), P_2, or P_3, each of which has previously been shown to be intersectionwise χ-guarding. This completes the proof of Theorem 3.13. □

Chapter 8

Intersectionwise self-χ-guarding classes

We recall from Section 3.1 that we call a class of graphs $\mathcal{A}$ intersectionwise self-χ-guarding if for every positive integer k the class $\mathop{\bigcap\!\!\!\!\!\cdot}_{i \in [k]} \mathcal{A}$ is χ-bounded. In this section, we study intersectionwise self-χ-guarding classes. In Section 8.1 we focus on intersectionwise self-χ -guarding classes that are defined by one forbidden induced subgraph, and in Section 8.2 we consider ways to construct intersectionwise self-χ-guarding classes from intersectionwise χ-guarding classes.

8.1 H-free intersectionwise self-χ-guarding classes

We begin with an easy observation. Since for every integer k, the k-fold graph-intersection of a class $\mathcal{C}$ contains $\mathcal{C}$, we have the following:

Observation 8.1. *Let $\mathcal{C}$ be a intersectionwise self-χ-guarding class of graphs. Then $\mathcal{C}$ is χ-bounded.*

This chapter is based on the coauthored paper [24].

We note that the converse of Observation 8.1 does not hold. In particular, as we mentioned in Section 3.1, it follows from a result of Burling [21] that the class of interval graphs is not intersectionwise self-χ-guarding, and thus the class of perfect graphs is not intersectionwise self-χ-guarding. The following is an immediate corollary of Observation 8.1 and Observation 7.1.

Corollary 8.2. *Let H be a graph. If the class of H-free graphs is intersectionwise self-χ -guarding, then H is a forest.*

A *chair* is a graph isomorphic to the graph on five vertices which we obtain by identifying one of the vertices of a P_2 with a vertex of degree two of a P_4.

Theorem 8.3. *The following classes of graphs are not intersectionwise self-χ-guarding:*

(i) the class of chair-free graphs.
(ii) the class of $(K_{1,3} + K_1)$-free graphs.

Proof of Theorem 8.3. Let $\mathcal{C}$ be the class of complete multipartite graphs, and let $\mathcal{D}$ the class of line graphs. Then, by Lemma 6.5, the class $\mathcal{C} \mathbin{\cap\!\!\!\!\cdot\,} \mathcal{D}$ is not χ-bounded.

The theorem now follows by the observation that both the class of $(K_{1,3} + K_1)$-free graphs and the class of chair-free graphs contain the class of line graphs (since line graphs are claw-free graphs) and the class of complete multipartite graphs (since complete multipartite graphs are $(K_1 + K_2)$-free graphs). This completes the proof of Theorem 8.3. □

We say that a graph is a *linear forest* if it is a forest in which every component is a path. We are now ready to prove Corollary 3.14 which we restate:

Corollary 3.14. *Let H be a graph. If the class of H-free graphs is intersectionwise self-χ -guarding, then H is a linear forest or a star.*

Proof of Corollary 3.14. Let H be a graph as in the statement. By Corollary 8.2 we have that H is a forest. Suppose that H is not a linear forest. Then a component, say H_1, of H contains a claw. Since, by Theorem 8.3, we have that H is $(K_{1,3} + K_1)$-free, it follows that H is connected. We claim that H_1 is a star. Suppose not. Then H_1 contains a chair, which contradicts Theorem 8.3. This completes the proof of Corollary 3.14. □

Recently, Adenwalla, Braunfeld, Sylvester, and Zamaraev [3], using different terminology, proved that if H is a star, then the class of all H-free graphs is intersectionwise self-χ-guarding.

Theorem 8.4 (Adenwalla, Braunfeld, Sylvester, and Zamaraev [3, Lemma 5.9])**.** *Let t be a positive integer. Then the class of $K_{1,t}$-free graphs is intersectionwise self-χ-guarding.*

In what follows, we discuss a result of Gyárfás [71] which implies that the class of P_4-free graphs is intersectionwise self-χ-guarding. We first need to introduce some terminology. An orientation of an undirected graph G is a *transitive orientation* if the adjacency relation of the resulted directed graph is transitive. A *comparability graph* is a graph which has a transitive orientation.

Theorem 8.5 (Gyárfás, [71, Proposition 5.8])**.** *The class of comparability graphs is intersectionwise self-χ-guarding.*

Jung [77] proved that every P_4-free graph is a comparability graph. Hence, we have the following:

Corollary 8.6. *The class of P_4-free graphs is intersectionwise self-χ-guarding.*

In view of the above we propose the following conjecture:

Conjecture 8.7. *If H is a linear forest, then the class of H-free graphs is intersectionwise self-χ-guarding.*

By Corollary 3.14 and Theorem 8.4 it follows that an affirmative answer to Conjecture 8.7 would imply a complete characterization of the intersectionwise self-χ-guarding graph classes which are defined by a single forbidden induced subgraph. We were not able to decide the following:

Problem 4. *Is the class of P_5-free graphs intersectionwise self-χ-guarding?*

Problem 5. *Is the class of $(P_4 + P_1)$-free graphs intersectionwise self-χ-guarding?*

In Section 8.2 we will show that the class of $(P_3 + rP_2)$-free graphs is intersectionwise self-χ-guarding.

8.2 Intersectionwise self-χ-guarding classes from intersectionwise χ-guarding classes

The main result of this section is Theorem 3.15 which allows us to construct new intersectionwise self-χ-guarding classes from intersectionwise χ-guarding classes which are defined by a finite set of forbidden induced subgraphs:

Theorem 3.15. *Let k and t be a positive integers and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a set of graphs. For every $i \in [t]$ let $r_1^i, \ldots, r_k^i$ be k nonnegative integers and let $\mathcal{H}^i := \{H_j + r_j^i K_2 : j \in [k]\}$. If the class of $\mathcal{H}$-free graphs is intersectionwise χ-guarding, then the class $\dot{\bigcap}_{i \in [t]}\{\mathcal{H}^i\text{-free graphs}\}$ is χ-bounded. In particular, for every $i \in [t]$ the class of $\mathcal{H}^i$-free graphs is intersectionwise self-χ-guarding.*

An application of Theorem 3.15 is the following immediate corollary of Theorem 3.13 and Theorem 3.15 which settles some cases of Conjecture 8.7.

Corollary 3.16. *Let r be a positive integer. Then the class of $(P_3 + rK_2)$-free graphs is intersectionwise self-χ-guarding.*

The following is an immediate corollary of Proposition 7.5 which states that a graph G is complete multipartite if and only if G is $(P_2 + K_1)$-free, and Corollary 3.16:

Corollary 8.8. *The class of complete multipartite graphs is intersectionwise self-χ-guarding.*

We remark that Adenwalla, Braunfeld, Sylvester, and Zamaraev proved the following strengthening of Corollary 8.8:

Proposition 8.9 (Adenwalla, Braunfeld, Sylvester, and Zamaraev [3, Proposition 5.26])**.** *For every positive integer k the k-fold graph-intersection of the class of complete multipartite graphs is χ-bounded by the linear function $f(x) = k^{2^k} x$.*

The rest of Section 8.2 is devoted to the proof of Theorem 3.15, for which we use a technique inspired by previous work of Chudnovsky, Scott, Seymour, and Spirkl [34]: We assume towards a contradiction that there exists a graph $G = G_1 \cap \ldots \cap G_t$ in the class

$\bigcap_{i\in[t]}\{\mathcal{H}^i$-free graphs$\}$ with chromatic number large in terms of its clique number. We then find in G a large collection $\mathcal{B}$ of pairwise disjoint sets of vertices ("boxes") such that for each $B \in \mathcal{B}$ the subgraph which is induced by B has large chromatic number. For a supergraph H of G we consider a pair $\{B, B'\}$ of boxes dense in the graph H if for every $v \in B$ we have that $\chi(G[B' \setminus N_H(v)])$ is small, and for every $v \in B'$ we have that $\chi(G[B \setminus N_H(v)])$ is small. For each pair of boxes $\{B, B'\} \subseteq \mathcal{B}$, and each $i \in [t]$, we ask if the pair $\{B, B'\}$ could be made dense in G_i, possibly by "shrinking" (with respect to their chromatic number in G) the boxes B and B' by a constant factor. This results in a "shrink-resistant" configuration. Now, because we are applying induction on the clique number of G, and neighborhoods of vertices in G have clique number smaller than $\omega(G)$, it follows that G is relatively sparse. By playing this fact against the fact that each G_i is relatively dense (for every edge, its common non-neighbors have small chromatic number by the inductive setup) we get a contradiction.

We are now ready to proceed with the proof of Theorem 3.15.

Proof of Theorem 3.15. We prove the theorem by induction on t. For the base case, where $t = 1$, we prove by induction on $r := \sum_{j\in[k]} r_j^1$ that the class of $\mathcal{H}^1$-free graphs is χ-bounded.

Suppose that $r = 0$. Then $\mathcal{H}^1$ is the class $\mathcal{H}$. Thus, by our assumptions, the class of $\mathcal{H}^1$-free graphs is intersectionwise χ-guarding. Hence, by Observation 5.1, we have that the class of $\mathcal{H}^1$-free graphs is χ-bounded.

Let us suppose that $r > 0$, and let $j \in [k]$ be such that $r_j^1 > 0$. From the induction hypothesis we have that the class of $\{H_1 + r_1^1 K_2, \ldots, H_j + (r_j^1 - 1)K_2, \ldots, H_k + r_k^1 K_2\}$-free graphs is χ-bounded. Let f_{r-1} be a χ-bounding function for this class.

Claim 8.9.1. *The class of $\mathcal{H}^1$-free graphs is χ-bounded by the function $f_r : \mathbb{N} \to \mathbb{N}$ which is defined recursively as follows:*

$$f_r(n) = \begin{cases} 1, & \text{if } n = 1; \\ 2f_r(n-1) + f_{r-1}(n), & \text{otherwise.} \end{cases}$$

Proof of Claim 8.9.1. We prove the claim by induction on $\omega := \omega(G)$. The claim holds trivially when $\omega = 1$. Suppose that $\omega > 1$, and that the claim holds for graphs H such that $\omega(H) < \omega$.

Let G be an $\mathcal{H}^1$-free graph, and let $uv \in E(G)$. Then $\{N(u) \cup N(v), A(u) \cap A(v)\}$ is a partition of $V(G)$, and thus we have that $\chi(G) \leq \chi(G[N(u) \cup N(v)]) + \chi(G[A(u) \cap A(v)])$. Suppose for contradiction that the graph $G[A(u) \cap A(v)]$ contains $H_j + (r_j^1 - 1)K_2$. Then the graph $G[A(u) \cap A(v) \cup \{u, v\}]$ contains $H_j + r_j^1 K_2$, which is a contradiction. Hence, we have that the graph $G[A(u) \cap A(v)]$ is $(H_j + (r_j^1 - 1)K_2)$-free. Thus, by the induction hypothesis, we have that $\chi(G[A(u) \cap A(v)]) \leq f_{r-1}(\omega)$. Also, $\omega(G[N(v)]) < \omega(G)$, and thus, by the induction hypothesis, we have that $\chi(G[N(v)]) \leq f_r(\omega - 1)$. Similarly, $\chi(G[N(u)]) \leq f_r(\omega - 1)$. Finally, putting the above together we have that $\chi(G) \leq \chi(G[N(u) \cup N(v)]) + \chi(G[A(u) \cap A(v)]) \leq 2f_r(\omega - 1) + f_{r-1}(n)$. This completes the proof of Claim 8.9.1. ■

This concludes the proof of the case $t = 1$. Suppose that $t > 1$, and that our statement holds for every positive integer $t' < t$. For each $i \in [t]$, let $\mathcal{A}_i := \dot{\cap}_{j \in [t] \setminus \{i\}} \{\mathcal{H}^j\text{-free graphs}\}$. By the induction hypothesis, it follows that for every $i \in [t]$, the class $\mathcal{A}_i$ is χ-bounded. Thus, since the class $\mathcal{H}$ is intersectionwise χ-guarding, we have that the class $\mathcal{H} \dot{\cap} \mathcal{A}_i$ is χ-bounded. For each $i \in [t]$, let g_i be a χ-bounding function for the class $\mathcal{H} \dot{\cap} \mathcal{A}_i$. Let $f \colon \mathbb{N}_+^{t+1} \to \mathbb{N}_+$ be the function which is defined recursively as follows:

$$f(n, n_1, \ldots, n_t) = \begin{cases} 1, \text{ if } n = 1; \\ \min_{i \in [t]}\{g_i(n) : n_i = 0\}, \text{ if } n \neq 1 \text{ and } \exists i \in [t] \text{ such that } n_i = 0; \\ M(n, n_1, \ldots, n_t) \cdot (t+2)^{t \cdot (R_{2^t}(t+2) - 1)} \cdot R_{2^t}(t+2), \text{ otherwise.} \end{cases}$$

where

$$M(n, n_1, \ldots, n_t) := \max\{t^{t+1} + 1, (t+1)f(n-1, n_1, \ldots, n_t) + 1, C(n, n_1, \ldots, n_t)(t+1)(t+2)\},$$

and

$$C(n, n_1, \ldots, n_t) := \max\{f(n, n_1, \ldots, n_i - 1, \ldots, n_t) : i \in [t]\}.$$

Let $G \in \dot{\cap}_{i \in [t]} \{\mathcal{H}^i\text{-free graphs}\}$ and for every $i \in [t]$ let G_i be an $\mathcal{H}^i$-free graph such that $G = \cap_{i \in [t]} G_i$. For each $i \in [t]$ let $r_i := \sum_{j \in [k]} r_j^i$. In what follows we prove by induction on $\omega(G)$ that:

$$\chi(G) \leq f(\omega(G), r_1, \ldots, r_t). \tag{8.1}$$

If $\omega(G) = 1$, then (8.1) holds trivially. Let $\omega := \omega(G) > 1$, and suppose that for every graph $H \in \dot{\bigcap}_{i\in[t]}\{\mathcal{H}^i\text{-free graphs}\}$ with $\omega(H) < \omega$, we have $\chi(H) \leq f(\omega(H), r_1, \ldots, r_t)$.

We prove, by induction on $\prod_{i\in[t]} r_i$, that G satisfies (8.1). For the basis of the induction we observe that if there exists $i \in [t]$ such that $r_i = 0$, then $G \in \mathcal{H} \mathbin{\dot{\cap}} \mathcal{A}_i$, and thus $\chi(G) \leq \min_{i\in[t]}\{g_i(\omega) : r_i = 0\}$. In particular, G satisfies (8.1) when $\prod_{i\in[t]} r_i = 0$.

Let us suppose that for every $i \in [t]$, we have $r_i \geq 1$, and that for every graph $H \in \dot{\bigcap}_{i\in[t]}\{\mathcal{H}^i\text{-free graphs}\}$, with $\omega(H) \leq \omega$, and for every choice of kt nonnegative integers s^i_j, where $i \in [t]$ and $j \in [k]$, such that $\prod_{i\in[t]}\left(\sum_{j\in[k]} s^i_j\right) < \prod_{i\in[t]} r_i$ we have:

$$\chi(H) \leq f\left(\omega(H), \sum_{j\in[k]} s^1_j, \ldots, \sum_{j\in[k]} s^t_j\right).$$

Let us suppose towards a contradiction that $\chi(G) > f(\omega, r_1, \ldots, r_t)$. For each $i \in [t]$, we denote by c_i the number $f(\omega, r_1, \ldots, r_i - 1, \ldots, r_t)$. Then we have that $C(\omega, r_1, \ldots, r_t) = \max\{c_i : i \in [t]\}$. We denote $C(\omega, r_1, \ldots, r_t)$ by C. We denote $M(\omega, r_1, \ldots, r_t)$ by M. We need to introduce a few new notions: Let X and Y be disjoint subsets of $V(G)$, and let $i \in [t]$. We say that $\{X, Y\}$ is an *i-dense pair* in G if the following hold:

- For all $x \in X$ we have that $\chi(G[Y \setminus N_{G_i}(x)]) \leq c_i$; and
- For all $y \in Y$ we have that $\chi(G[X \setminus N_{G_i}(y)]) \leq c_i$.

Let $\mathcal{X}$ be a family of disjoint subsets of $V(G)$, let $X, Y \in \mathcal{X}$, and let $i \in [t]$ be such that:

- $\{X, Y\}$ is not an i-dense pair;
- There exist $X' \subseteq X$ and $Y' \subseteq Y$, such that:
  - $\chi(G[X']) \geq \frac{1}{t+2}\chi(G[X])$, $\chi(G[Y']) \geq \frac{1}{t+2}\chi(G[Y])$; and
  - $\{X', Y'\}$ is an i-dense pair in G.

Then we say that the pair $\{X, Y\}$ is *i-shrinkable* and we refer to the family $(\mathcal{X} \setminus \{X, Y\}) \cup \{X', Y'\}$ as the family which is obtained from $\mathcal{X}$ by *i-shrinking the pair $\{X, Y\}$ to the pair $\{X', Y'\}$*. If a family $\mathcal{X}$ of disjoint subsets of $V(G)$ contains an i-shrinkable pair for some $i \in [t]$, we say that $\mathcal{X}$ is *shrinkable*; otherwise we say that $\mathcal{X}$ is *unshrinkable*. Finally, in what follows, for disjoint subsets X and Y of $V(G)$ we denote by $I(X, Y)$ the set $\{i \in [t] : \{X, Y\}$ is an i-dense pair in $G\}$.

Claim 8.9.2. *There exists an unshrinkable family $\mathcal{B}$ of pairwise disjoint subsets of $V(G)$ such that the following hold:*

- *$\mathcal{B}$ has size $t+2$;*
- *for every $B \in \mathcal{B}$, we have that $\chi(B) = M$; and*
- *there exists a set $I \subseteq [t]$ such that for each pair $\{B, B'\}$ of distinct subsets in $\mathcal{B}$, we have that $I(B, B') = I$.*

Proof of Claim 8.9.2. Let $\mathcal{B} = \{B_1, \ldots, B_{R_{2^t}(t+2)}\}$ be a partition of $V(G)$ which has size $R_{2^t}(t+2)$, and such that for every $B \in \mathcal{B}$, we have $\chi(B) \geq M \cdot (t+2)^{t \cdot (R_{2^t}(t+2)-1)}$. We observe that, since $\chi(G) > f(\omega, r_1, \ldots, r_t)$, such a partition exists.

Let $\mathcal{B}'$ be the family of disjoint subsets of $V(G)$ which we obtain from $\mathcal{B}$ as follows: We start with $\mathcal{B}' := \mathcal{B}$. For every $i \in [t]$, and for all distinct $l, l' \in [R_{2^t}(t+2)]$, if there exist $A_l \subseteq B_l \in \mathcal{B}'$ and $A_{l'} \subseteq B_{l'} \in \mathcal{B}'$ such that the pair $\{B_l, B_{l'}\}$ is i-shrinkable to the pair $\{A_l, A_{l'}\}$, then we let $\mathcal{B}'$ be the family which is obtained from $\mathcal{B}'$ by i-shrinking the pair $\{B_l, B_{l'}\}$ to the pair $\{A_l, A_{l'}\}$. We repeat this process until $\mathcal{B}'$ is unshrinkable. Since every initial element of $\mathcal{B}$ will be shrunk at most $t \cdot (R_{2^t}(t+2) - 1)$ times, it follows that for every $B' \in \mathcal{B}'$ we have $\chi(B') \geq M$. By restricting the elements of $\mathcal{B}'$ to appropriate subsets, we may assume that for every $B' \in \mathcal{B}'$, we have $\chi(B') = M$.

Let H be the complete graph on $\mathcal{B}'$, and let $\phi : E(H) \to 2^{[t]}$ be a 2^t-edge-coloring of H which is defined as follows: $\phi(\{A, B\}) = I(A, B) \subseteq [t]$. Then, by Theorem 1.2, and the fact that $|V(H)| = R_{2^t}(t+2)$, it follows that H contains a monochromatic, with respect to ϕ, clique of size $t+2$. Let $\mathcal{B}'' \subseteq \mathcal{B}'$ be such a clique, and let I be the color of its edges. Then $\mathcal{B}''$ satisfies our Claim 8.9.2. This completes the proof of Claim 8.9.2. ∎

Let $\mathcal{B} = \{B_1, \ldots, B_{t+2}\}$ be a family of disjoint subsets of $V(G)$ and let I be a subset of $[t]$ as in the statement of Claim 8.9.2.

Claim 8.9.3. *Let B_l and $B_{l'}$ be distinct elements of $\mathcal{B}$. Then for every $x \in B_l$ there exists $i \in [t] \setminus I$ such that $\chi(G[B_{l'} \setminus N_{G_i}(x)]) \geq \frac{1}{t+1} M$.*

Proof of Claim 8.9.3. Let us suppose towards a contradiction that there exists $x \in B_l$ such that for every $i \in [t] \setminus I$ we have that $\chi(G[A_{G_i[B_{l'}]}(x)]) < \frac{1}{t+1}M$. Let $i \in I$. Then, since $\{B_l, B_{l'}\}$ is an i-dense pair, we have that $\chi\left(G[A_{G_i[B_{l'}]}(x)]\right) \leq c_i$. Thus, we have that:

$$\begin{aligned}\chi\left(G[\cup_{i\in I} A_{G_i[B_{l'}]}(x)]\right) &\leq \sum_{i\in I} \chi\left(G[A_{G_i[B_{l'}]}(x)]\right) \leq \sum_{i\in I} c_i \leq |I|C \\ &\leq |I|\frac{t}{t+1}C(t+1)(t+2) \leq |I|\frac{t}{t+1}M.\end{aligned}$$

Hence, we have:

$$\begin{aligned}\chi\left(G[A_{G[B_{l'}]}(x)]\right) &= \chi\left(G\Big[\left(\cup_{i\in I} A_{G_i[B_{l'}]}(x)\right) \cup \left(\cup_{i\in[t]\setminus I} A_{G_i[B_{l'}]}(x)\right)\Big]\right) \\ &\leq \chi\left(G[\cup_{i\in I} A_{G_i[B_{l'}]}(x)]\right) + \chi\left(G[\cup_{i\in[t]\setminus I} A_{G_i[B_{l'}]}(x)]\right) \\ &< |I|\frac{t}{t+1}M + (t-|I|)\frac{M}{t+1} = \frac{t}{t+1}M.\end{aligned}$$

Since, by the induction hypothesis, we have that $\chi(G[N_G(x)]) \leq f(\omega-1, r_1, \ldots, r_t)$, it follows that:

$$\begin{aligned}\chi(G[B_{l'}]) &\leq \chi(G[N_{G[B_{l'}]}(x)]) + \chi(G[A_{G[B_{l'}]}(x)]) \\ &< f(\omega-1, r_1, \ldots, r_t) + \frac{t}{t+1}M \\ &\leq \frac{M}{t+1} + \frac{t\cdot M}{t+1} = M,\end{aligned}$$

which contradicts the fact that $\chi(G[B_{l'}]) = M$. This completes the proof of Claim 8.9.3. ■

By Claim 8.9.3, we have that for every $x \in B_1$ there exists a function $f_x\colon [2,\ldots,t+2] \to [t] \setminus I$, such that for every $l \in [2,\ldots,t+2]$ we have $\chi(G[B_l \setminus N_{G_{f_x(l)}}(x)]) \geq \frac{1}{t+1}M$. Let $g\colon B_1 \to [t]^{t+1}$ with $g(x) = (f_x(2), f_x(3), \ldots, f_x(t+2))$. Then g is a t^{t+1}-coloring of $G[B_1]$. Since $\chi(G[B_1]) = M > t^{t+1}$, it follows that g is not a proper t^{t+1}-coloring of B_1.

Let $\{u, v\} \subseteq B_1$ be an edge of G such that $g(u) = g(v)$. In the $(t+1)$-tuple $g(u)$ at least one element of the set $[t]$ appears more than once. Let $p \in [t]$ be such an element,

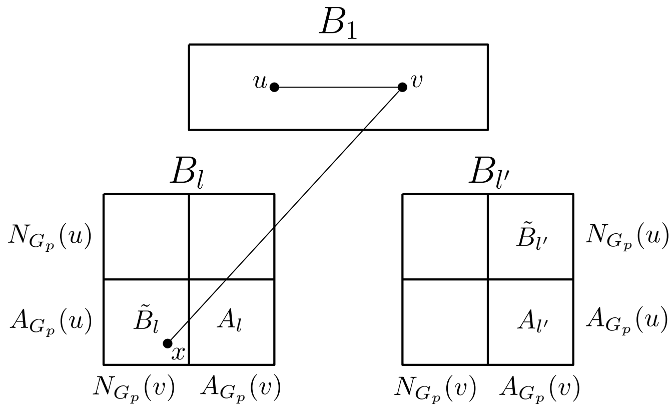

Figure 8.1: An illustration of the situation in the proof of Claim 8.9.5. By the definition of $\tilde{B}_{l'}$ we have that v is anticomplete to $\tilde{B}_{l'}$. Thus both v and x are anticomplete to $\tilde{B}_{l'} \setminus N_{G_p}(x)$. The disjoint union of an induced $H_j + (r_j^p - 1)K_2$ in $\tilde{B}_{l'} \setminus N_{G_p}(x)$ with the edge vx would result in an induced $H_j + r_j^p K_2$ in G_p.

and let $l, l' \in [2, \ldots, t+2]$ be such that $l \neq l'$ and $p = f_u(l) = f_u(l') = f_v(l) = f_v(l')$. Then neither of $\{B_1, B_l\}$ and $\{B_1, B_{l'}\}$ is a p-dense pair. Hence, by Claim 8.9.2, we have that $p \notin I$, and thus, again by Claim 8.9.2, we have that $\{B_l, B_{l'}\}$ is not a p-dense pair.

Let A_l be the set $A_{G_p}(u) \cap A_{G_p}(v) \cap B_l$ and let $A_{l'}$ be the set $A_{G_p}(u) \cap A_{G_p}(v) \cap B_{l'}$.

Claim 8.9.4. $\chi(G[A_l]) \leq c_p$ *and* $\chi(G[A_{l'}]) \leq c_p$.

Proof of Claim 8.9.4. We prove that $\chi(G[A_l]) \leq c_p$; the proof that $\chi(G[A_{l'}]) \leq c_p$ is identical.

Since, by the induction hypothesis, we have that $\prod_{i\in[t]} r_i \geq 1$, it follows that there exists $j \in [k]$ such that $r_j^p \geq 1$. We claim that the graph $G_p[A_l]$ is $(H_j + (r_j^p - 1)K_2)$-free. Indeed, otherwise $G_p[A_l(v) \cup \{u, v\}]$ contains $H_j + r_j^p K_2$, which contradicts the fact that G_p is $\mathcal{H}^p$-free. Hence, by the induction hypothesis, we have that $\chi(G[A_l]) \leq f(\omega, r_1, \ldots, r_p - 1, \ldots, r_t) = c_p$. This completes the proof of Claim 8.9.4. ■

We denote by $\tilde{B}_l$ the set $B_l \setminus (N_{G_p}(u) \cup A_l)$, and by $\tilde{B}_{l'}$ the set $B_{l'} \setminus (N_{G_p}(v) \cup A_{l'})$. Observe that in the graph G_p we have that u is complete to $\tilde{B}_{l'}$ and v is complete to $\tilde{B}_l$.

Claim 8.9.5. *$\{\tilde{B}_l, \tilde{B}_{l'}\}$ is a p-dense pair in G.*

Proof of Claim 8.9.5. We prove that for every $x \in \tilde{B}_l$ we have $\chi(G[\tilde{B}_{l'} \setminus N_{G_p}(x)]) \leq c_p$. The fact that for every $y \in \tilde{B}_{l'}$ we have $\chi(G[\tilde{B}_l \setminus N_{G_p}(y)]) \leq c_p$ follows by symmetry.

Let $x \in \tilde{B}_l$. Since, by the induction hypothesis, we have that $r_p \geq 1$, it follows that there exists $j \in [k]$ such that $r_j^p \geq 1$. We claim that the graph $G_p[\tilde{B}_{l'} \setminus N_{G_p}(x)]$ is $(H_j + (r_j^p - 1)K_2)$-free. Indeed, otherwise $G_p[A_{G_p}(v) \cup \{x, v\}]$ contains $H_j + r_j^p K_2$, which contradicts the fact that G_p is $\mathcal{H}^p$-free. Hence, by the induction hypothesis, we have that $\chi(G[\tilde{B}_{l'} \setminus N_{G_p}(x)]) \leq f(\omega, r_1, \dots, r_p - 1, \dots, r_t) = c_p$. This completes the proof of Claim 8.9.5. ■

Claim 8.9.6. $\chi(G[\tilde{B}_l]) \geq \frac{1}{t+2}\chi(G[B_l])$ *and* $\chi(G[\tilde{B}_{l'}]) \geq \frac{1}{t+2}\chi(G[B_{l'}])$.

Proof of Claim 8.9.6. We prove that $\chi(G[\tilde{B}_l]) \geq \frac{1}{t+2}\chi(G[B_l])$. The proof that $\chi(G[\tilde{B}_{l'}]) \geq \frac{1}{t+2}\chi(G[B_{l'}])$ is identical.

By the definition of g and the choice of l we have that $\chi(G[B_l \cap A_{G_p}(u)]) \geq \frac{1}{t+1}M = \frac{1}{t+1}\chi(G[B_l])$. By Claim 8.9.4, we have that $\chi(G[A_{G_p}(u) \cap A_{G_p}(v) \cap B_l]) \leq c_p$. It follows that:

$$\chi(G[\tilde{B}_l]) \geq \frac{1}{t+1}\chi(G[B_l]) - c_p \geq \frac{1}{t+1}\chi(G[B_l]) - C = \frac{M}{t+1} - C.$$

We also have:

$$\begin{aligned} M &\geq C(t+1)(t+2) \\ (t+2)M - (t+1)M &\geq C(t+1)(t+2) \\ \frac{M}{t+1} - \frac{M}{t+2} &\geq C \\ \frac{M}{t+1} - C &\geq \frac{M}{t+2}. \end{aligned}$$

Since $\chi(G[B_l]) = M$ we have that $\frac{M}{t+2} = \frac{1}{t+2}\chi(G[B_l])$ and thus by the above we have that $\chi(G[\tilde{B}_l]) \geq \frac{1}{t+2}\chi(G[B_l])$, as desired. This completes the proof of Claim 8.9.6. ■

By Claim 8.9.5 and Claim 8.9.6, it follows that the pair $\{B_l, B_{l'}\}$ is p-shrinkable which contradicts the fact that the family $\mathcal{B}$ is unshrinkable. Hence,

$$\chi(G) \leq f(\omega(G), r_1, \ldots, r_t).$$

Let $h : \mathbb{N} \to \mathbb{N}$, be defined as follows:

$$h(n) = f(n, \sum_{j \in [k]} r_j^1, \ldots, \sum_{j \in [k]} r_j^t).$$

Then we proved that $\chi(G) \leq h(\omega(G))$. Hence, h is a χ-bounding function for the class $\bigcap_{i \in [t]} \{\mathcal{H}^i\text{-free graphs}\}$. This completes the proof of Theorem 3.15. $\square$

Part III

Induced subgraphs of graphs with large K_r-free chromatic number

Chapter 9

On χ_r-bounded classes of graphs which are defined by a finite set of forbidden induced subgraphs

In this chapter, we study necessary conditions that the graphs of a finite set $\mathcal{H}$ should satisfy in order for the class of all $\mathcal{H}$-free graphs to be χ_r-bounded. Our results allow us extend the Gyárfás-Sumner conjecture to a conjecture for χ_r-bounded H-free classes.

We begin with introducing some terminology for hypergraphs, and discussing a connection between the chromatic number of hypergraphs and the K_r-free chromatic number of graphs.

Let $r \geq 2$ be an integer. A hypergraph is *r-uniform* if all of its hyperedges have size r. Let $l \geq 2$ be a positive integer. A *cycle in a hypergraph* is a list $(v_1, e_1, \ldots, e_l, v_l)$, where v_i's are distinct vertices and e_i's are distinct hyperedges, and $v_i, v_{i+1} \in e_i$ (indices are modulo l); in this case l is the length of the cycle $(v_1, e_1, \ldots, e_l, v_l)$. The *girth of a hypergraph $\mathcal{H}$* is the minimum length of a cycle in $\mathcal{H}$. Finally, the *chromatic number of a hypergraph $\mathcal{H}$*, denoted by $\chi(\mathcal{H})$, is the minimum size of a partition of the vertex set in which no part contains a hyperedge.

This chapter is based on ongoing joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.

In 1966, generalizing Theorem 2.2, Erdős and Hajnal [50] proved with probabilistic arguments, the existence of hypergraphs of arbitrarily large girth and chromatic number. In 1968, Lovász [91] gave the first constructive proof of this theorem by proving the following stronger result:

Theorem 9.1 (Lovász [91])**.** *For all integers $r, k, l \geq 2$ there exists an r-uniform hypergraph which has chromatic number at least k and girth at least l.*

We note that a shorter proof of the above result was given in 1979 by Nešetřil and Rödl [99], using the operation of amalgamation.

For a graph G we denote by $\mathcal{H}_r(G)$ the r-uniform hypergraph which has as vertex set the set $V(G)$ and as set of hyperedges the set of r-cliques of G.

Observation 9.2. *Let G be a graph and let $r \geq 2$ be an integer. Then $\chi_r(G) = \chi(\mathcal{H}_r(G))$.*

Let $\mathcal{H}$ be a hypergraph. We denote by $G(\mathcal{H})$ the graph on $V(\mathcal{H})$ in which two distinct vertices $u, v \in V(\mathcal{H})$ are adjacent if and only if there exists a hyperedge e of $\mathcal{H}$ such that $u, v \in e$.

Theorem 9.3. *For all integers $r, k, l \geq 2$ there exists a diamond-free graph G such that $\omega(G) \leq r$, $\chi_r(G) \geq k$, and G contains no holes of length smaller than l.*

Proof of Theorem 9.3. Let $r, k, l \geq 2$ be integers. Then, Theorem 9.1, there exists an r-uniform hypergraph which has chromatic number at least k and girth at least l. Let $\mathcal{H}$ be such a hypergraph. Then $G(\mathcal{H})$ satisfies the statement of the theorem. This completes the proof of Theorem 9.3. □

In Proposition 2.4 we observed that the existence of graphs of arbitrarily large girth and chromatic number implies that if the class of H-free graphs is χ-bounded, then H should be a forest. Theorem 9.3 implies the following:

Corollary 9.4. *Let $\mathcal{H}$ be a finite set of graphs, and let $r \geq 2$ be a positive integer. If the class of all $\mathcal{H}$-free graphs is χ_r-bounded, then $\mathcal{H}$ contains a diamond-free chordal graph of clique number at most r.*

In what follows, we strengthen Corollary 9.4. Let $r \geq 2$ be an integer, and let $\mathcal{O}$ be an operation which is applied to one or more graph classes and yields to a new class of graphs. We say that $\mathcal{O}$ *preserves* χ_r*-boundedness* (respectively *preserves polynomial* χ_r*-boundedness*) if whenever the graph classes to which $\mathcal{O}$ is applied are χ_r-bounded (respectively polynomially χ_r-bounded) we have that the resulting class of graphs (after the application of the $\mathcal{O}$) is also χ_r-bounded (respectively polynomially χ_r-bounded). Throughout this section, for a class $\mathcal{C}$, we denote by $\mathcal{C}_{\mathsf{sub}}$ the closure of $\mathcal{C}$ under substitution. Chudnovsky, Penev, Scott, and Trotignon [29] proved that substitution preserves χ-boundedness, and moreover if a class $\mathcal{C}$ is polynomially χ-bounded then $\mathcal{C}_{\mathsf{sub}}$ is polynomially χ-bounded as well. We will show that for every $r \geq 3$ it is not true that for every χ_r-bounded class $\mathcal{C}$ the class $\mathcal{C}_{\mathsf{sub}}$ is χ_r-bounded.

For the remainder of this chapter, given a graph G and a positive integer i, we denote by G^i the graph which is defined inductively as follows: if $i = 1$, then $G^1 = G$; and if $i > 1$, then for each $v \in V(G)$ we create a copy G_v^{i-1} of G^{i-1}, these copies are such that for all distinct $u, v \in V(G)$ we have $V(G_u^{i-1}) \cap V(G_v^{i-1}) = \emptyset$ and $V(G_u^{i-1}) \cap V(G) = \emptyset$; we define G^i to be the graph that we obtain from G by substituting G_v^{i-1} for v in G, for every $v \in V(G)$.

Lemma 9.5. *Let $r \geq 2$ and $k, i \geq 1$ be integers, and let G be a graph such that $\chi_r(G) \geq k$. Then for every $k' < k$, every k'-coloring of G^i yields a monochromatic r^i-clique. In particular, $\chi_{r^i}(G^i) \geq k$.*

Proof of Lemma 9.5. Let G be a graph such that $\chi_r(G) \geq k$. We prove the lemma by induction on i. For $i = 1$ the result follows immediately by the definition of K_r-free chromatic number.

Let $i > 1$, and suppose that the statement of the lemma holds for every $i' < i$. Let $k' < k$ and let ϕ be a k'-coloring of the graph G^i. For every $v \in V(G)$ let G_v^{i-1} be the copy of G^{i-1} in G^i which corresponds to the vertex v. By the inductive hypothesis we have that for every $v \in V(G)$ the restriction of ϕ to $V(G_v^{i-1})$ results in a monochromatic r^{i-1}-clique in G_v^{i-1}. For every $v \in V(G)$, let Q_v be such a monochromatic r^{i-1}-clique in G_v^{i-1}.

Let ϕ' be the k'-coloring of G which assigns to every vertex $v \in V(G)$ the color that ϕ assigns to all the vertices of Q_v. Since $\chi_r(G) \geq k$ and $k' < k$, it follows that ϕ' induces a monochromatic r-clique in G. Let Q be such a clique. Then $G^i[\{\cup_{v \in Q}\}Q_v]$ is a monochromatic, with respect to ϕ, r^i-clique in G. This completes the induction on i. This completes the proof of Lemma 9.5. □

Theorem 9.6. *Let $r \geq 2$ and let $\mathcal{C}$ be a class of triangle-free graphs which contains graphs of arbitrarily large chromatic number. Then the class $\mathcal{C}_{\mathsf{sub}}$, that is, the closure of $\mathcal{C}$ under substitution, is not χ_r-bounded.*

Proof of Theorem 9.6. Let us suppose towards a contradiction that the class $\mathcal{C}_{\mathsf{sub}}$ is χ_r-bounded, and let f be a χ_r-bounding function for $\mathcal{C}_{\mathsf{sub}}$. Let $i = \lceil \log_2 r \rceil$, and let G be a triangle-free graph of chromatic number at least $f(2^i) + 1$. It is easy to see that (as in the proof of Lemma 9.5) $\omega(G^i) = 2^i \geq r$.

By our assumption that $\chi_r(G) \leq f(\omega(G)) = f(2^i)$, it follows that there exists a K_r-free $f(2^i)$-coloring of G^i. Let ϕ be such a coloring. Since $\chi(G) \geq f(2^i) + 1 > f(2^i)$, it follows, by Lemma 9.5, that ϕ induces a monochromatic 2^i-clique in G^i. But, since $2^i \geq r$, this contradicts the fact that ϕ is a K_r-free coloring of G^i. This completes the proof of Theorem 9.6. □

Let $\mathcal{C}$ be the class of all triangle-free graphs. It is easy to see that $\mathcal{C}_{\mathsf{sub}}$ is bull-free. By Theorem 2.1, $\mathcal{C}$ contains graphs of arbitrarily large chromatic number. Thus, by applying Theorem 9.6, we see that for every $r \geq 2$ the class of bull-free graphs is not χ_r-bounded. It follows that:

Corollary 9.7. *Let $\mathcal{H}$ be a finite set of graphs, and let $r \geq 2$ be a positive integer. If the class of $\mathcal{H}$-free graphs is χ_r-bounded, then $\mathcal{H}$ contains a bull-free graph.*

So now we have further insights in the structure of a finite set $\mathcal{H}$ which is such that the class of $\mathcal{H}$-free graphs is χ_r-bounded.

The following result, whose proof we omit, follows by arguments analogous to those used in the proof of Lemma 9.5.

Theorem 9.8. *Let $r \geq 3$ and k be positive integers. Let G be a triangle-free graph such that $\chi(G) > k$, let $t := \lceil \frac{r}{2} \rceil$, and let H be a K_{t+1}-free graph such that $\chi_t(H) > k$. For every $v \in V(G)$ let H_v be a graph isomorphic to H, such that the graphs of the set $\{H_v : v \in V(G)\}$*

are pairwise vertex-disjoint, and each H_v is vertex-disjoint from G. Let G' be the graph that we obtain by substituting, for every $v \in V(G)$, H_v for v in G. Then $\chi_{2t}(G') > k$, G' is K_{2t+1}-free, and G' does not contain an induced subgraph isomorphic to K^+_{t+2}.

The following is an immediate corollary of Theorem 9.8 and the fact (see, for example, Theorem 2.15) that for all integers $k, r \geq 2$ there exists a K_{r+1}-free graph of K_r-free chromatic number at least k.

Corollary 9.9. *Let $\mathcal{H}$ be a finite set of graphs, let $r \geq 3$ be a positive integer, and let $t := \lceil \frac{r}{2} \rceil$. If the class of $\mathcal{H}$-free graphs is χ_{2t}-bounded, then $\mathcal{H}$ contains a K^+_{t+2}-free graph.*

Before proving one more necessary condition for the graphs in the sets $\mathcal{H}$ as above, we would like to mention that Corollary 9.7 has one more interesting consequence. Chudnovsky, Cook, Davies, and Oum [28] proved[1] the following:

Theorem 9.10 (Chudnovsky, Cook, Davies, and Oum [28])**.** *The class of all bull-free graphs is Pollyanna.*

Recall that every polynomially χ_r-bounded class is Pollyanna. Combining Corollary 9.7 and Theorem 9.10 we get the following:

Theorem 9.11. *For every positive integer $r \geq 2$ there are Pollyanna graph classes which are not χ_r-bounded.*

We need the following:

Theorem 9.12 (Chudnovsky, Cook, Davies, and Oum [28])**.** *Let $\mathcal{F}$ be a finite set of graphs none of which is a willow. Then for every positive integer r there is a class $\mathcal{G}$ of $\mathcal{F}$-free graphs that is not χ-bounded, but such that every graph $G \in \mathcal{G}$ with $\omega(G) < r$ has chromatic number at most $\binom{r+1}{3}$.*

It follows that:

[1] We note that Hajebi [74] recently gave a proof of Theorem 9.10 which is surprisingly shorter than the proof of Chudnovsky, Cook, Davies, and Oum [28]. In particular, his proof bypasses the structure theorem for bull-free graphs (see [26, 27]).

Theorem 9.13. *Let $r \geq 2$ be a positive integer, and let $\mathcal{F}$ be a finite set of graphs none of which is a willow. Then the class of all $\mathcal{F}$-free is not χ_r-bounded.*

Proof of Theorem 9.13. Suppose not. Let f be a χ_r-bounding function of the class of all $\mathcal{F}$-free graphs. Let $\mathcal{G}_r$ be a class $\mathcal{F}$-free graphs that is not χ-bounded, but such that every graph $G \in \mathcal{G}$ with $\omega(G) < r$ has chromatic number at most $\binom{r+1}{3}$, as in the statement of Theorem 9.12.

Let $G \in \mathcal{G}_r$, and let $\phi : V(G) \to [f(\omega(G))]$ be a K_r-free $f(\omega(G))$-coloring of G. Since, for every $i \in [f(\omega(G))]$ the graph $G[\phi^{-1}(i)]$ is K_r-free, it follows that $\chi(G[\phi^{-1}(i)]) \leq \binom{r+1}{3}$. Thus,

$$\chi(G) \leq \sum_{i \in [f(\omega(G))]} \chi(G[\phi^{-1}(i)]) \leq f(\omega(G)) \cdot \binom{r+1}{3}.$$

It follows that the function $h(\omega) := f(\omega) \cdot \binom{r+1}{3}$ is a χ-bounding function for the class $\mathcal{G}_r$, contradicting the fact that $\mathcal{G}_r$ is not χ-bounded. This completes the proof of Theorem 9.13. □

Corollary 9.14. *Let $\mathcal{H}$ be a finite set of graphs, and let $r \geq 2$ be a positive integer. If the class of $\mathcal{H}$-free graphs is χ_r-bounded, then $\mathcal{H}$ contains a willow.*

Combining Corollary 9.4, Corollary 9.7, Corollary 9.9, and Corollary 9.14, we get the following:

Theorem 3.17. *Let $\mathcal{H}$ be a finite set of graphs, and let $r \geq 2$ be a positive integer. If the class of $\mathcal{H}$-free graphs is χ_r-bounded, then $\mathcal{H}$ contains: a diamond-free chordal graph which has clique number at most r, a bull-free graph, and a willow. Moreover, if $r \geq 4$, then $\mathcal{H}$ contains a $K^+_{\lceil \frac{r}{2} \rceil + 2}$-free graph.*

Following [28] a *pentagram spider* is a graph G on ten vertices which has a perfect matching M such that $G \setminus M$ has a component isomorphic to K_5. We denote by F_7 a graph which is obtained from a path on seven vertices by adding a new vertex which is complete to the path. Chudnovsky, Cook, Davies, and Oum [28] showed that an F_7 is not a willow, and that pentagram spiders are not willows. The following shows that all the conditions in Theorem 3.17 are indeed necessary:

Observation 9.15. *We note the following:*

- *A bull is a willow, but it is not a diamond-free chordal graph.*
- *A diamond is a willow, but it is bull-free.*
- *An F_7 is bull-free, but not a willow.*
- *Pentagram spiders are diamond-free chordal graphs, but not willows.*

We note that we do not know if the converse of Theorem 3.17 is true. The following is a corollary of Theorem 3.17.

Corollary 3.18. *Let H be a graph, and let $r \geq 2$ be a positive integer, such that the class of all H-free graphs is χ_r-bounded. Then H is an r-broadleaved forest, such that every component of H that is not a complete graph is an $(\lceil \frac{r}{2} \rceil + 1)$-broadleaved tree.*

We suggest the following conjecture which extends the Gyárfás-Sumner conjecture:

Conjecture 3.19 (Strong Forbidden Broadleaved Forest Conjecture)**.** *Let $r \geq 2$ be an integer, and let H be an r-broadleaved forest, such that every component of H that is not a complete graph is an $(\lceil \frac{r}{2} \rceil + 1)$-broadleaved tree. Then the class of all H-free graphs is χ_r-bounded.*

We also suggest the following weakening of Conjecture 3.19:

Conjecture 3.20 (Forbidden Broadleaved Forest Conjecture)**.** *Let $r \geq 2$ be an integer. Then there exists a function $f : \mathbb{N}_+ \to \mathbb{N}_+$ such that the class of all H-free graphs is $\chi_{f(r)}$-bounded for every r-broadleaved forest H.*

We note that similarly with the Gyárfás-Sumner conjecture, in order to prove Conjecture 3.19 and Conjecture 3.20 it suffices to consider r-broadleaved trees.

Chapter 10

Excluding a volcano

In this chapter, we prove that volcanoes satisfy the Strong Forbidden Broadleaved Forest Conjecture, that is Conjecture 3.19. We also prove that an orientation of the 1-volcano is not $\overrightarrow{\chi_r}$-bounding for every $r \geq 2$. As we discussed in Section 3.2, the latter result is in contrast with the fact, due to [32], that orientations of stars are $\overrightarrow{\chi_2}$-bounding.

10.1 Volcano-free classes of graphs are χ_3-bounded

The main result of this section is:

Theorem 3.26. *Let k be a positive integer, and let $f_{3.26} : \mathbb{N}_+ \to \mathbb{N}_+$ be the function which is defined as follows:*

$$f_{3.26}(\omega) := 4\omega(10k^2\omega^3)^\omega \cdot \left[(2^\omega - 2)R(\omega, k) + 1\right].$$

Let G be a V_k-free graph. Then $\chi_3(G) \leq f_{3.26}(\omega(G))$.

As we discussed in Section 3.2, our proof of Theorem 3.26 relies on the following structure theorem for graphs which exclude a fixed volcano as an induced subgraph:

This chapter is based on ongoing joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.

Theorem 3.28. *Let k and ω be positive integers, and let G be a V_k-free graph with $\omega(G) \leq \omega$. Then there exists a partition $\mathcal{P}$ of $V(G)$, with the following properties:*

- $|\mathcal{P}| \leq 2(2^{\omega-1} - 1)R(\omega, k) + 1$, *and*
- *for every $P \in \mathcal{P}$ and for every $v \in G[P]$ we have that $G[N_{G[P]}(v)]$ is a $(k-1)$-tolerant t-partite graph, where $t \leq \omega(G[N(v)])$.*

After proving Theorem 3.28, the main step towards completion of the proof of Theorem 3.26:

Lemma 10.1. *Let k be a positive integer, and let $f_{10.1} : \mathbb{N}_+ \to \mathbb{N}_+$ be the function which is defined as follows:*

$$f_{10.1}(\omega) := 4\omega(10k^2\omega^3)^\omega.$$

Let G be a V_k-free graph such that for every $v \in V(G)$, there exists an integer t_v such that $G[N(v)]$ is a $(k-1)$-tolerant t_v-partite graph, where $t_v \leq \omega(G) - 1$. Then $\chi_3(G) \leq f_{10.1}(\omega(G))$.

In what follows, we often make use of the following simple observation for k-tolerant multipartite graphs:

Observation 10.2. *Let $k \geq 0$ and $t \geq 2$ be integers, let G be a k-tolerant t-partite graph, and let $A_1, \ldots, A_t$ be a partition which witnesses this. If there exists an independent set S of G such that $|S| \geq tk + 1$, then there exists $i \in [t]$ such that $S \subseteq A_i$.*

We need to introduce some terminology for the rest of this section. Let A and B be disjoint subsets of $V(G)$. We say that $\{A, B\}$ is a *k-nice* pair if every vertex of A has at most k non-neighbors in B, and every vertex of B has at most k non-neighbors in A. The following lemma is the main ingredient for the proof of Theorem 3.28:

Lemma 10.3. *Let k be a positive integer, and let G be a $(K_2 + kK_1)$-free graph. Then, there exists a partition $\{A, B\}$ of $V(G)$ such that $G[A]$ is a $(k-1)$-tolerant $\omega(G)$-partite graph, and $|B| \leq (2^\omega - 1 - \omega)R(\omega + 1, k)$.*

Proof of Lemma 10.3. Suppose that $\omega(G) = 1$. Then by letting $A := V(G)$ and $B = \emptyset$, we have that $\{A, B\}$ satisfies the lemma. Thus, we may assume that $\omega(G) \geq 2$. Let

$\omega := \omega(G) \geq 2$, and let K be an ω-clique in G. Let $\{K_1, \ldots, K_{2^\omega - 1}\}$ be the set of all proper subsets of K. Let $R := V(G) \setminus K$. For each $i \in [2^\omega - 1]$, let $A_i := \{u \in R : N_K(u) = K_i\}$. Then, by relabeling the sets $\{A_i\}_{i \in [2^\omega - 1]}$ and $\{K_i\}_{i \in [2^\omega - 1]}$ if needed, there exists an integer $s \leq 2^\omega - 1$ such that $\{A_1, \ldots, A_s\}$ is a partition of R.

Let $i \in [s]$. We claim that if $|K_i| \leq \omega - 2$, then $\alpha(G[A_i]) \leq k - 1$. Suppose not. Let $\{v_1, \ldots, v_k\}$ be an independent set in $G[A_i]$, and let $\{u, v\} \subseteq K \setminus K_i$ with $u \neq v$. Then $G[\{u, v\} \cup \{v_1, \ldots, v_k\}]$ is a $K_2 + kK_1$ in G contradicting the fact that G is a $(K_2 + kK_1)$-free graph. Hence, we have $\alpha(G[A_i]) \leq k - 1$, and thus $|A_i| \leq R(\omega + 1, k)$. Let $B := \cup_{i:|K_i| \leq \omega - 2} A_i$. Then, $|B| = \sum_{i:|K_i| \leq \omega - 2} |A_i| \leq (2^\omega - 1 - \omega) R(\omega + 1, k)$.

By relabeling the sets $\{A_i\}_{i \in [s]}$ and $\{K_i\}_{i \in [s]}$ if needed, we can assume that $\{K_1, \ldots, K_\omega\}$ is the set of all $(\omega - 1)$-subsets of K.

Let $i \in [\omega]$. We claim that A_i is an independent set. Suppose not. Let $\{u, v\}$ be an edge in A_i. Then $G[K_i \cup \{u, v\}]$ is an $(\omega + 1)$-clique in G, contradicting the fact that $\omega(G) = \omega$.

Let $i, j \in [\omega]$ be distinct. We claim that $\{A_i, A_j\}$ is a $(k - 1)$-nice pair. Suppose not. Without loss of generality we may assume that there exists a vertex in $u \in A_i$ which has at least k non-neighbors in A_j. Let $v_1, \ldots, v_k \in A_j$ be such that for each $l \in [k]$ we have that $uv_l \notin E(G)$. Let $v_{A_j} \in K$ be the unique vertex of K which is anticomplete to A_j. Then $G[\{v_{A_j}, u, v_1, \ldots, v_k\}]$ is a $K_2 + kK_1$ in G contradicting the fact that G is a $(K_2 + kK_1)$-free graph.

For every $i \in [\omega]$, let v_{A_i} be the unique vertex of K which is anticomplete to A_i, let $A'_i = A_i \cup \{v_{A_i}\}$, and let $A := \cup_{i \in [\omega]} A'_i$. Then it follows from the above that $G[A]$ is a $(k - 1)$-tolerant ω-partite graph. Now $\{A, B\}$ is the desired partition of $V(G)$. This completes the proof of Lemma 10.3. □

We will also need the following well-known result (see for example [83]):

Lemma 10.4 (Folklore)**.** *Let k be a positive integer, let D be a digraph such that every vertex has out-degree at most k, and let G be the underlying undirected graph of D. Then $\chi(G) \leq 2k + 1$.*

We are now ready to prove Theorem 3.28, which we restate:

Theorem 3.28. *Let k and ω be positive integers, and let G be a V_k-free graph with $\omega(G) \leq \omega$. Then there exists a partition $\mathcal{P}$ of $V(G)$, with the following properties:*

- $|\mathcal{P}| \leq 2(2^{\omega-1} - 1)R(\omega, k) + 1$, *and*
- *for every $P \in \mathcal{P}$ and for every $v \in G[P]$ we have that $G[N_{G[P]}(v)]$ is a $(k-1)$-tolerant t-partite graph, where $t \leq \omega(G[N(v)])$.*

Proof of Theorem 3.28. Since G is V_k-free, we have that for every $v \in V(G)$, the graph $G[N(v)]$ is $(K_2 + kK_1)$-free. For every $v \in V(G)$, let $\{A_v, B_v\}$ be the partition of $N(v)$ that we get from Lemma 10.3.

Then, for every $v \in V(G)$ we have that $G[A_v]$ is a $(k-1)$-tolerant $\omega(G[N(v)])$-partite graph, and that

$$\begin{aligned} |B_v| &\leq \left(2^{\omega(G[N(v)])} - 1 - \omega(G[N(v)])\right) R(\omega(G[N(v)]) + 1, k) \\ &\leq (2^{\omega-1} - 1)R(\omega, k). \end{aligned}$$

Let D be the digraph on $V(G)$ with set of arcs the set $\cup_{v \in V_G}\{vu : u \in B_v\}$. Then D is a digraph with out-degree at most:

$$\max_{v \in V(G)} |B_v| \leq (2^{\omega-1} - 1)R(\omega, k).$$

Hence, by Lemma 10.4, there exists a K_2-free coloring of the underlying undirected graph of D which uses at most $2(2^{\omega-1} - 1)R(\omega, k) + 1$ colors. Let ϕ be such a coloring, and let $\mathcal{P}$ be the corresponding to ϕ partition of $V(G)$. Let $P \in \mathcal{P}$, and let $v \in G[P]$. We claim that $G[N_{G[P]}(v)]$ is a $(k-1)$-tolerant t-partite graph, where $t \leq \omega(G[N(v)])$. Indeed, for every $v \in V(G)$, we have that $G[N_{G[P]}(v)]$ is an induced subgraph of $G[A_v]$, and hence $G[N_{G[P]}(v)]$ is a $(k-1)$-tolerant t-partite graph, where $t \leq \omega(G[N(v)])$. Thus, $\mathcal{P}$ is the desired partition of $V(G)$. This completes the proof of Theorem 3.28. □

For our proof of Lemma 10.1 we need the following:

Lemma 10.5. *Let k and $t \geq 2$ be positive integers, and let G be a V_k-free graph. Let A be an independent set, let $v \in A$ be such that $G[N(v)]$ is isomorphic to a $(k-1)$-tolerant t-partite graph H, and let $\{H_1, \dots, H_t\}$ be a partition of $V(H)$ which witnesses that H is a $(k-1)$-tolerant t-partite graph. If for every $i \in [t]$ there exists $H'_i \subseteq H_i$, such that $|H'_i| \geq (k+2)(k-1) + 1$ and $\{A, H'_i\}$ is a $(k-1)$-nice pair, then $G[V(H) \cup A]$ is $(k-1)$-tolerant $(t+1)$-partite graph.*

Proof of Lemma 10.5. In order to prove that $G[V(H)\cup A]$ is a $(k-1)$-tolerant $(t+1)$-partite graph, it suffices to prove that for every $i \in [t]$ we have that $\{A, H_i\}$ is a $(k-1)$-nice pair. Let $i \in [t]$.

We claim that every $u \in A$ has at most $k-1$ non-neighbors in H_i. Let $u \in A$, we may assume that $u \neq v$, since otherwise our claim holds. Let us suppose towards a contradiction that $|A(u) \cap H_i| \geq k$. Let $v_1, \ldots, v_k \in A(u) \cap H_i$. Since $\{A, H_i'\}$ is a $(k-1)$-nice pair, and $|H_i'| \geq (k+2)(k-1)+1$ we have that $N(u) \cap H_i' \neq \emptyset$. Let $w \in N(u) \cap H_i'$. Let $j \neq i$. Note that $|A(u) \cap H_j'| \leq k-1$, that $|A(w) \cap H_j'| \leq k-1$, and that for every $l \in [k]$ we have that $|A(v_l) \cap H_j'| \leq k-1$. Hence, since $|H_j'| \geq (k+2)(k-1)+1$, we have that $H_j' \setminus \Big(A(u) \cup A(w) \cup \big(\cup_{l\in[k]} A(v_l)\big)\Big) \neq \emptyset$. Let $w' \in H_j' \setminus \Big(A(u) \cup A(w) \cup \big(\cup_{l\in[k]} A(v_l)\big)\Big)$. Then $G[\{u, w, w'\} \cup \{v_1, \ldots, v_k\}]$ is a k-volcano in G contradicting the fact that G is V_k-free.

Let $u \in H_i$. We claim that $|A(u) \cap A| \leq k-1$. If $u \in H_i'$, then our claim follows from the fact that $\{A, H_i'\}$ is a $(k-1)$-nice pair. Suppose that $u \in H_i \setminus H_i'$. Let us suppose towards a contradiction that $|A(u) \cap A| \geq k$. Let $v_1, \ldots, v_k \in A(u) \cap A$. Let $j \neq i$. Since $|H_j'| \geq (k+2)(k-1)+1$, and both $\{A, H_j'\}$ and $\{H_i, H_j'\}$ are $(k-1)$-nice pairs, there exists $u' \in H_j'$ such that $u' \in N(u)$, and for every $l \in [k]$ we have $u' \in N(v_l)$. Then, $G[\{v, u, u'\} \cup \{v_1, \ldots, v_k\}]$ is k-volcano in G contradicting the fact that G is V_k-free. This completes the proof of Lemma 10.5. □

We are now ready to prove Lemma 10.1 which we restate:

Lemma 10.1. *Let k be a positive integer, and let $f_{10.1} : \mathbb{N}_+ \to \mathbb{N}_+$ be the function which is defined as follows:*

$$f_{10.1}(\omega) := 4\omega(10k^2\omega^3)^{\omega}.$$

Let G be a V_k-free graph such that for every $v \in V(G)$, there exists an integer t_v such that $G[N(v)]$ is a $(k-1)$-tolerant t_v-partite graph, where $t_v \leq \omega(G)-1$. Then $\chi_3(G) \leq f_{10.1}(\omega(G))$.

Proof of Lemma 10.1. For each vertex $v \in V(G)$ let H_v be the $(k-1)$-tolerant t_v-partite graph $G[N(v)]$, where $t_v \leq \omega(G)$. For each graph H_v we fix a partition P_v of $V(H_v)$ which witnesses that H_v is a $(k-1)$-tolerant t_v-partite graph, where $t_v \leq \omega(G)-1$. Let

$g_k : \mathbb{N}_+ \to \mathbb{N}_+$ be the function that is defined as follows: $g_k(\omega) = \max\{2\omega, (k+2)\} \cdot (k-1)+1$. For each induced subgraph G' of G, let

$$l(G') := \max_{v \in V(G')} \left\{ \left| \{A \in P_v : |A \cap V(G')| \geq g_k(\omega(G'))\} \right| \right\}.$$

By Theorem 3.28, we have that $l(G') \leq \omega(G') - 1$. Consider the function $f : \mathbb{N}_+ \to \mathbb{N}_+$ which is defined as follows:

$$f_k(l, \omega) := \begin{cases} 1, & \text{if } \omega < 3 \\ \Big(2(\omega-1) \cdot g_k(\omega) + 1\Big) \cdot f_k(l, \omega - 1), & \text{if } \omega \geq 3 \text{ and } l < \omega - 1 \\ 2 \cdot (\omega+1) f_k(\omega - 2, \omega), & \text{if } \omega \geq 3 \text{ and } l = \omega - 1 \end{cases}$$

Let G' be an induced subgraph of G. In what follows we prove by induction[1] on $(l, \omega(G'))$ that $\chi_3(G') \leq f_k(l(G'), \omega(G'))$.

For the base case: let $\omega(G') < 3$. Then, by assigning the same color to every vertex of the graph G', we get a K_3-free coloring which uses one color. Thus, in this case we have

$$\chi_3(G') \leq f_k(l(G'), \omega(G')),$$

as desired.

Let us suppose that $\omega(G') \geq 3$, and that for every induced subgraph H of G with $(l(H), \omega(H)) < (l(G'), \omega(G'))$ we have $\chi_3(H) \leq f_k(l(H), \omega(H))$.

Claim 10.5.1. *If $l(G') < \omega(G') - 1$, then there exists a positive integer $j \leq 2(\omega(G') - 1) \cdot g_k(\omega(G')) + 1$ and a partition $\{A_1, \ldots, A_j\}$ of $V(G')$, such that for every $i \in [j]$ we have $\omega(G'[A_i]) \leq l(G'[A_i]) + 1 < \omega(G')$.*

Proof of Claim 10.5.1. For every $v \in V(G')$, and for every $u \in N(v)$ let us denote by $H_v(u)$ the unique set in $\{A \cap V(G') : A \in P_v\}$ which contains the vertex u. Let D be the digraph on $V(G')$ with the following set of arcs:

$$\{vu : u \in N_{G'}(v) \text{ and } |H_v(u)| < g_k(\omega(G'))\}.$$

[1] We note that $(a, b) < (a', b')$ if and only if ($a < a'$ and $b \leq b'$) or ($a \leq a'$ and $b < b'$).

Then every vertex of D has out-degree at most $(\omega(G')-1)\cdot g_k(\omega(G'))$. Thus, by Lemma 10.4, there exists a K_2-free coloring of the underlying undirected graph of D with $j \leq 2(\omega(G')-1)\cdot g_k(\omega(G'))+1$ colors. Let $\{A_1,\dots,A_j\}$ be a partition of $V(G')$ that corresponds to such a coloring. Then, for every $i\in[j]$, for every $v\in A_i$, and for every $A\in P_v$ such that $|A\cap V(G')|\neq\emptyset$, we have $|A\cap V(G')|\geq g_k(\omega(G'))$. It follows that for all $i\in[j]$ and $v\in A_i$ we have:

$$|\{A\in P_v : A\cap A_i\neq\emptyset\}| \leq |\{A\in P_v : |A\cap V(G')|\geq g_k(\omega(G'))\}| \leq l(G')$$

and so $\omega(N_{A_i}(v))\leq l(G')$ which implies $\omega(G[A_i])\leq l(G')+1<\omega(G')$. This completes the proof of Claim 10.5.1. ■

Suppose first that $l(G')<\omega(G')-1$. Then, let $j\leq 2(\omega(G')-1)\cdot g_k(\omega(G'))+1$ be a positive integer, and let a partition $\{A_1,\dots,A_j\}$ of $V(G')$, as in the statement of Claim 10.5.1. Note that for every $i\in[j]$ we have $(l(G'[A_i]),\omega(G'[A_i]))<(l(G'),\omega(G'))$. Thus, by the induction hypothesis, for every $i\in[j]$ we have that:

$$\begin{aligned}\chi_3(G'[A_i]) &\leq f_k(l(G'[A_i]),\omega(G'[A_i]))\\ &\leq f_k(l(G'),\omega(G')-1).\end{aligned}$$

Hence, we have that

$$\chi_3(G')\leq [2(\omega(G')-1)\cdot g_k(\omega(G'))+1]\, f_k(l(G'),\omega(G')-1),$$

as desired. Now, we may assume that $l(G')=\omega(G')-1$.

Claim 10.5.2. *Let $S\subseteq V(G')$ be such that $G'[S]$ does not contain a $(k-1)$-tolerant $(l(G')+1)$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$. Then, $\chi_3(G'[S])\leq 2\cdot f_k(l(G')-1,\omega(G'))$.*

Proof of Claim 10.5.2. Let $S_1\subseteq S$ be the set of all vertices $v\in S$ which are such that $G'[N_{G'}(v)\cap S]$ is a $(k-1)$-tolerant $l(G')$-partite graph with all parts of size at least $g_k(\omega(G'))$. Let $S_2:=S\setminus S_1$.

We note that $l(G'[S_2])<l(G')$ and thus, by the induction hypothesis, we have that:

$$\chi_3(G'[S_2])\leq f_k(l(G'[S_2]),\omega(G'[S_2]))\leq f_k(l(G')-1,\omega(G')).$$

We claim that $l(G'[S_1]) < l(G')$. Suppose not. Let $v \in S_1$ be such that $G'[N_{G'}(v) \cap S_1]$ is a $(k-1)$-tolerant $l(G')$-partite graph with all parts of size at least $g_k(\omega(G'))$, and let $\{A_1, \dots, A_{l(G')}\}$ be a partition of $N_{G'}(v) \cap S_1$ which witnesses this.

Let $u \in A_{l(G')}$. Since $u \in S_1$. We have that $G'[N_{G'}(u) \cap S]$ is a $(k-1)$-tolerant $l(G')$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$. Let $\{B_1, \dots, B_{l(G')}\}$ be a partition of $N_{G'}(u) \cap S$ which witnesses this. For each $i \in [l(G')]$, let $A'_i := N_{G'}(u) \cap A_i$. Then, for each $i \in [l(G') - 1]$ we have that:

$$\begin{aligned} |A'_i| &\geq |A_i| - (k-1) \\ &\geq g_k(\omega(G')) - (k-1) \\ &\geq 2\omega(k-1) + 1 - (k-1) \\ &\geq (\omega - 1)(k-1) + 1 \\ &\geq l(G')(k-1) + 1. \end{aligned}$$

Hence, by Observation 10.2, we may assume that for each $i \in [l(G') - 1]$, we have $A'_i \subseteq B_i$.

We claim that $G'[(S \cap N_{G'}(u)) \cup A_{l(G')}]$ is a $(k-1)$-tolerant $(l(G')+1)$-partite graph with all parts of size at least $g_k(\omega(G'))$. It suffices to prove that $\{A_{l(G')}, B_{l(G')}\}$ is a $(k-1)$-nice pair, then our claim follows immediately by Lemma 10.5.

Let $w \in A_{l(G')}$. We claim that w has at most $k-1$ non-neighbors in $B_{l(G')}$. Suppose not. Let $v_1, \dots v_k \in A(w) \cap B_{l(G')}$. Let $w' \in N_{G'}(w) \cap (\cap_{i \in [k]} N_{G'}(v_i)) \cap A'_{l(G')-1} \subseteq B_{l(G')-1}$. Then $G'[\{v, w, w'\} \cup \{v_1, \dots v_k\}]$ is k-volcano in G', contradicting the fact that G is V_k-free.

Let $w \in B_{l(G')}$. We claim that w has at most $k-1$ non-neighbors in $A_{l(G')}$. Suppose not. Let $v_1, \dots v_k \in A(w) \cap A_{l(G')}$. Let $w' \in N_{G'}(w) \cap (\cap_{i \in [k]} N_{G'}(v_i)) \cap A'_{l(G')-1} \subseteq B_{l(G')-1}$. Then $G'[\{u, w, w'\} \cup \{v_1, \dots v_k\}]$ is k-volcano in G', contradicting the fact that G is V_k-free.

Thus, $G'[(S \cap N_{G'}(u)) \cup A_{l(G')}]$ is a $(k-1)$-tolerant $(l(G)+1)$-partite induced subgraph of $G'[S]$ with all parts of size at least $g_k(\omega(G'))$, contradicting the fact that $G'[S]$ does not contain such an induced subgraph. Thus, $l(G'[S_1]) < l(G')$. Hence, by the induction hypothesis, we have that:

$$\chi_3(G'[S_1]) \leq f_k(l(G'[S_1]), \omega(G'[S_1])) \leq f_k(l(G') - 1, \omega(G')).$$

Finally, we have that:

$$\chi_3(G'[S]) \leq \chi_3(G'[S_1]) + \chi_3(G'[S_2]) \leq 2 \cdot f_k(l(G') - 1, \omega(G')).$$

This completes the proof of Claim 10.5.2. ■

We need to introduce some terminology that we use in the rest of this proof. Given a graph G and an induced subgraph H of G we say that H is (k, l, C)-*good* if it has the following properties:

(i) H is a k-tolerant l-partite graph;
(ii) every part of H has size at least C; and
(iii) $|V(H)|$ is maximum subject to (i) and (ii).

We may assume that G' contains at least one $(k-1)$-tolerant $(l(G')+1)$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$, since otherwise, by Claim 10.5.2, we have that $\chi_3(G') \leq 2 \cdot f_k(l(G') - 1, \omega(G'))$, and we are done. Let $s \geq 1$ be a positive integer, and let $L_1, \ldots, L_s$ be a sequence of induced subgraphs of G' which is defined inductively as follows:

- L_1 is a $(k-1, l(G')+1, g_k(\omega(G')))$-good induced subgraph of G'.
- Let $i \geq 1$, and let us suppose that the induced subgraphs $L_1, \ldots, L_i$ have been defined.
- If $G' \setminus \left(\cup_{j \in [i]} L_j\right)$ contains a $(k-1)$-tolerant $(l(G')+1)$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$, then we let L_{i+1} be a $(k-1, l(G')+1, g_k(\omega(G')))$-good induced subgraph of $G' \setminus \left(\cup_{j \in [i]} L_j\right)$, otherwise we let $s := i$.

Let $L := \cup_{i \in [s]} V(L_i)$ Let $R := V(G') \setminus L$. It follows, from Claim 10.5.2 and the fact that $G'[R]$ does not contain a $(k-1)$-tolerant $(l(G')+1)$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$, that

$$\chi_3(G'[R]) \leq 2 \cdot f_k(l(G') - 1, \omega(G')).$$

We now focus on coloring $G'[L]$. For each $j \in [s]$, let $L_j^1, \ldots, L_j^{l(G')+1}$ be the parts of the graph L_j. For each $i \in [l(G')+1]$, we denote by M_i be the graph $G'[\cup_{j \in [s]} V(L_j^i)]$.

Claim 10.5.3. *For each $i \in [l(G')+1]$, the graph M_i does not contain a $(k-1)$-tolerant $(l(G')+1)$-partite induced subgraph with all parts of size at least $g_k(\omega(G'))$.*

Proof of Claim 10.5.3. Suppose not. Without loss of generality, we may assume that $i = 1$. Let A be such an induced subgraph of M_1, and let $A_1, \dots, A_{l(G')+1}$ be the parts of A which witness that it is a $(k-1)$-tolerant $(l(G')+1)$-partite graph. Let $j \in [s]$ be minimum such that $L_j^1 \cap V(A) \neq \emptyset$. Without loss of generality, we may assume that $L_j^1 \cap A_1 \neq \emptyset$. Let $v \in L_j^1 \cap A_1$. Consider the $(k-1)$-tolerant graph $F_v := H_v \cap G'$. Let $F_v^1, \dots, F_v^t$ be the parts of F_v.

For each $i \in [2, \dots, l(G')+1]$, let $K_j^i := N_{G'}(v) \cap L_j^i \subseteq V(F_v)$. Since $v \in L_j^1$, and $G'[L_j]$ is a $(k-1)$-tolerant $(l(G')+1)$-partite graph with all parts of size at least $g_k(\omega(G'))$, we have $|K_j^i| \geq g_k(\omega(G')) - (k-1)$. Thus, since each K_j^i is an independent subset of $V(F_v)$ of size at least $g_k(\omega(G')) - (k-1)$, by Observation 10.2, we have that for all distinct $i, i' \in [2, \dots, l(G')+1]$ there exist distinct $p, p' \in [t]$ such that $K_j^i \subseteq F_v^p$ and $K_j^{i'} \subseteq F_v^{p'}$.

For each $i \in [2, \dots, l(G')+1]$, let $A_i' := N_{G'}(v) \cap A_i \subseteq V(F_v)$. Since $v \in A$, and $G'[A]$ is a $(k-1)$-tolerant $(l(G')+1)$-partite graph with all parts of size at least $g_k(\omega(G'))$, we have $|A_i'| \geq g_k(\omega(G')) - (k-1)$. Thus, since each A_i' is an independent subset of $V(F_v)$ of size at least $g_k(\omega(G')) - (k-1)$, by Observation 10.2, it follows that for all distinct $i, i' \in [2, \dots, l(G')+1]$ there exist distinct $p, p' \in [t]$ such that $A_i' \subseteq F_v^p$ and $A_{i'}' \subseteq F_v^{p'}$.

It follows by the above that $t = l(G')$. Without loss of generality, by relabeling, we may assume that for each $i \in [2, \dots, l(G')+1]$ we have $K_j^i \subseteq H_v^{i-1}$ and $A_i' \subseteq H_v^{i-1}$. Note that:

- L_j^1 is an independent set which contains v;
- $F_v \subseteq N(v)$; and
- since $G'[L_j]$ is a $(k-1)$-tolerant graph, for each $i \in [2, \dots, l(G')+1]$ we have that $\{L_j^1, K_j^i\}$ is a $(k-1)$-nice pair.

It follows, by Lemma 10.5, that $G'[L_j^1 \cup \left(\cup_{i \in [2, l(G')+1]} F_v^i\right)]$ is a $(k-1)$-tolerant $(l(G')+1)$-partite graph. In particular, $J' := G'[L_j^1 \cup \left(\cup_{i \in [2, l(G')+1]} A_i' \cup K_j^i\right)]$ is a $(k-1)$-tolerant $(l(G')+1)$-partite subgraph of $G' \setminus \left(\cup_{j' \in [j]} L_{j'}\right)$.

We claim that for each $i \in [2, l(G')+1]$ we have that $|A_i' \cup K_j^i| > |L_j^i|$. Indeed, let $i \in [2, l(G')+1]$. Then the sets $A_i' \subseteq M_1$ and $L_j^i \subseteq M_i$ are disjoint. Thus,

$$|F_v^i| \geq |A_i'| + |K_j^i| \geq |L_j^i| - (k-1) + g_k(\omega(G')) - (k-1) > |L_j^i|.$$

Hence, we have that $|V(J')| > |V(L_j)|$ contradicting the choice of the graph L_j. This completes the proof of Claim 10.5.3. ■

It follows, by Claim 10.5.2 and Claim 10.5.3, that for every $i \in [l(G')+1]$,we have that

$$\chi_3(M_i) \leq 2 \cdot f_k(l(G')-1, \omega(G')).$$

Thus, we have that:

$$\chi_3(G'[L]) \leq 2 \cdot (l(G')+1) f_k(l(G')-1, \omega(G')).$$

Hence, we conclude that:

$$\begin{aligned}\chi_3(G') &\leq \chi_3(G'[L]) + \chi_3(G'[R]) \\ &\leq 2 \cdot (l(G')+1) f_k(l(G')-1, \omega(G')) + 2 \cdot f_k(l(G')-1, \omega(G')) \\ &\leq 2 \cdot (l(G')+2) f_k(l(G')-1, \omega(G')) = f_k(l(G'), \omega(G')),\end{aligned}$$

as desired. This concludes the proof by induction of the fact that for every induced subgraph G' of G we have:

$$\chi_3(G') \leq f_k(l(G'), \omega(G')). \tag{10.1}$$

Claim 10.5.4. *Let k be a positive integer. For every positive integer ω, and every integer $l \in [1, \omega-1]$, we have that*

$$f_k(l, \omega) \leq 4\omega(10k^2\omega^3)^{\omega} = f_{10.1}(\omega).$$

Proof of Claim 10.5.4. First we note that for every positive integer ω we have:

$$\begin{aligned}g_k(\omega) &= \max\{2\omega, (k+2)\} \cdot (k-1) + 1 \\ &\leq \max\{2k\omega, k^2+k\} \\ &\leq \max\{2k\omega, (k^2+k)\omega\} \\ &\leq (k^2+3k)\omega \\ &\leq 4k^2\omega.\end{aligned}$$

We now proceed on proving the statement of the claim by induction on ω.

For the base case: if $\omega < 3$, then $f_k(l, \omega) = 1 \leq (10\, k^2\, \omega^3)^\omega$, as desired. Let $\omega \geq 3$ and assume that for all $\omega' < \omega$ and all $0 \leq l' \leq \omega' - 1$, we have:

$$f_k(l', \omega') \leq 4\omega'(10\, k^2\, \omega'^3)^{\omega'}.$$

We distinguish two cases.

Case 1: $l < \omega - 1$. By the definition of the function f_k we have:

$$f_k(l, \omega) = \Big[2(\omega - 1)g_k(\omega) + 1\Big]\ f_k\Big(l,\ \omega - 1\Big).$$

Applying the inductive hypothesis to $f_k(l, \omega - 1)$ gives

$$f_k(l, \omega - 1)\ \leq\ 4\,(\omega - 1)\,\Big(10\,k^2\,(\omega - 1)^3\Big)^{\omega - 1}.$$

Hence

$$\begin{aligned} f_k(l, \omega)\ &\leq\ (9\,k^2\,\omega^2)\cdot 4\,(\omega - 1)\,\Big(10\,k^2\,(\omega - 1)^3\Big)^{\omega - 1} \\ &=\ 36\,k^2\,\omega^2\,(\omega - 1)\,\Big(10\,k^2\,(\omega - 1)^3\Big)^{\omega - 1} \\ &\leq (10\,k^2\,\omega^3)^\omega. \end{aligned}$$

Thus,

$$f_k(l, \omega)\ \leq\ \Big(10\,k^2\,\omega^3\Big)^\omega\ \leq\ 4\,\omega\,\Big(10\,k^2\,\omega^3\Big)^\omega,$$

as required.

Case 2: $l = \omega - 1$. Then

$$f_k(\omega - 1, \omega) = 2(\omega + 1)\, f_k(\omega - 2, \omega).$$

Applying the Case 1 bound to $f_k(\omega - 2, \omega)$ yields:

$$f_k(\omega - 2, \omega) \leq (10k^2\omega^3)^\omega.$$

Also $2(\omega + 1) \leq 4\omega$ for $\omega \geq 3$ and $k \geq 1$. Thus,

$$f_k(\omega - 1, \omega) \leq 4\omega \cdot (10k^2\omega^3)^\omega.$$

This completes the proof of Claim 10.5.4. ∎

It follows by Equation 10.1 and Claim 10.5.4 that

$$\chi_3(G) \leq f_{10.1}(\omega).$$

This completes the proof of Lemma 10.1. □

We are now ready to prove Theorem 3.26 which we restate:

Theorem 3.26. *Let k be a positive integer, and let $f_{3.26} : \mathbb{N}_+ \to \mathbb{N}_+$ be the function which is defined as follows:*

$$f_{3.26}(\omega) := 4\omega(10k^2\omega^3)^\omega \cdot \left[(2^\omega - 2)R(\omega, k) + 1\right].$$

Let G be a V_k-free graph. Then $\chi_3(G) \leq f_{3.26}(\omega(G))$.

Proof of Theorem 3.26. Let $\omega := \omega(G)$. We apply Theorem 3.28: Let $\mathcal{P}$ be a partition of $V(G)$, as in the statement of Theorem 3.28.

For every $P \in \mathcal{P}$, by Lemma 10.1, we have that:

$$\chi_3(G[P]) \leq f_{10.1}(\omega(G[P])) = 4\omega(G[P])(10k^2\omega(G[P])^3)^\omega(G[P]) \leq f_{10.1}(\omega) = 4\omega(10k^2\omega^3)^\omega.$$

Recall that, by Theorem 3.28, we have that: $|\mathcal{P}| \leq (2^\omega - 2)R(\omega, k) + 1$.

Thus, we have that:

$$\begin{aligned}\chi_3(G) &\leq \sum_{P \in \mathcal{P}} \chi_3(G[P]) \\ &\leq 4\omega(10k^2\omega^3)^\omega \cdot |\mathcal{P}| \leq 4\omega(10k^2\omega^3)^\omega \cdot [(2^\omega - 2)R(\omega, k) + 1] = f_{3.26}(\omega).\end{aligned}$$

This completes the proof of Theorem 3.26. □

10.2 An orientation of V_1 which is not $\overrightarrow{\chi_r}$-bounding for any $r \geq 2$

The main result of this section is Theorem 3.30 which we restate:

Theorem 3.30. *Let $r \geq 2$ be an integer. Then $\overrightarrow{V_1}$ is not $\overrightarrow{\chi_r}$-bounding.*

As we discussed in Chapter 3, we get Theorem 3.30 as an immediate corollary of Theorem 3.31, which we restate:

Theorem 3.31. *Let $r \geq 2$ be an integer. Then for every integer $k \geq 2$, there exists a graph G_{c_k} such that:*

- *$\chi_r(G_{c_k}) > k$ and $\omega(G_{c_k}) \leq r$; and*
- *G_{c_k} has an orientation which is $\overrightarrow{V_1}$-free.*

Proof of Theorem 3.31. Let $k \geq 2$ be a positive integer, let $p := k2^{k(r-1)}$, and let $c_k := k(r-1)+1$. We construct a sequence $\{G_0, G_1, \ldots, G_{c_k}\}$ of graphs, as follows:

- We define G_0 to be a graph on p isolated vertices; and
- For each $i \in [c_k]$, we form the graph G_i from G_{i-1} as follows: For every complete multipartite subgraph $H \subseteq G_{i-1}$ with at most $r-1$ parts, we add p new vertices $v_H^1, \ldots, v_H^p$, each complete to $V(H)$; for each new vertex v_H^s, we orient every edge $v_H^s x$ so that x is an out-neighbor of v_H^s. We leave all other edges and their orientations from G_{i-1} unchanged.

Claim 10.5.1. *For every $i \in \{0, 1, \ldots, c_k\}$ we have $\omega(G_i) \leq r$. In particular $\omega(G_{c_k}) \leq r$.*

Proof of Claim 10.5.1. We prove the claim by induction on i. For the basis of induction, let $i = 0$. Then, since $V(G_0)$ is an independent set, we have $\omega(G_0) = 1 \leq r$. Let $i \geq 1$, and suppose that for every nonnegative integer $i' < i$ we have $\omega(G_{i'}) \leq r$. Let us suppose towards a contradiction that G_i contains an $(r+1)$-clique K. Since $\omega(G_{i-1}) \leq r$, it follows that $K \cap (V(G_i) \setminus V(G_{i-1})) \neq \emptyset$. Since no two vertices in $V(G_i) \setminus V(G_{i-1})$ are adjacent in G_i, it follows that $|K \cap (V(G_i) \setminus V(G_{i-1}))| = 1$. In particular, there exists a complete multipartite subgraph H with at most $r-1$ parts of G_{i-1}, and a vertex $v_H \in G_i$ such that $K \cap (V(G_i) \setminus V(G_{i-1})) = v_H$. Thus, $V(H)$ contains an r-clique, contradicting the fact that H can be partitioned in $r-1$ independent sets. ■

Claim 10.5.2. *For every $i \in \{0, 1, \dots, c_k\}$ the oriented graph G_i is $\overrightarrow{V_1}$-free.*

Proof of Claim 10.5.2. Since G_0 is edgeless it is obviously $\overrightarrow{V_1}$-free. Let us suppose towards a contradiction that there exists $i \in [c_k]$ such that G_i contains $\overrightarrow{V_1}$. Let v_H be the unique vertex of degree three in the copy of $\overrightarrow{V_1}$ in G_i. We note that the out-neighborhood of v_H is $V(H)$. Thus, the underlying undirected graph of the complete multipartite subgraph H of G_{i-1} contains $K_2 + K_1$ contradicting Proposition 7.5. ■

Let us suppose towards a contradiction that $\chi_r(G_{c_k}) \leq k$. Let f be a K_r-free k-coloring of G_{c_k}. Note that by restricting the domain of f we may view it as a coloring of the graphs $G_0, \dots, G_{c_k-1}$. We introduce some notation and terminology that we need for the rest of the proof. For every complete l-partite graph H with $l \leq r-1$, we use $P(H)$ to denoted the unique partition which witnesses that H is a complete l-partite graph, and we define the *value of* H, denoted by $\mathrm{val}(H)$, as follows:

$$\mathrm{val}(H) = \sum_{P \in P(H)} |f(P)|.$$

For a color $c \in [k]$, and a complete l-partite graph H with $l \leq r-1$, we denote by $n_c(P(H))$, the number of parts in $P(H)$ thst contain a vertex colored with c by f. We note that:

$$\mathrm{val}(H) = \sum_{c \in [k]} n_c(P(H)). \tag{10.2}$$

For the rest of the proof our goal is to show that the graph G_{c_k-1} contains a complete l-partite subgraph H with $l \leq r-1$, such that $\mathrm{val}(H) = k(r-1)$. Then, by Equation 10.2 and pigeonhole principle, every color c appears in at least (and thus in exactly) $r-1$ of the parts of the partition $P(H)$ of H. Hence, the graph H contains monochromatic $(r-1)$-cliques in all k colors. Thus, if v is one of the p vertices $v_H^1, \dots, v_H^p$ of G_{c_k} which are complete to $V(H)$, then v together with a monochromatic $(r-1)$-clique of color $f(v)$ of the graph H will induce a monochromatic r-clique, contradicting the fact that f is a K_r-free coloring of G_{c_k}. To this end we prove the following:

Claim 10.5.3. *For every $i \in \{0, 1, \dots, c_k - 1\}$ the graph G_i contains a complete l-partite subgraph H_i with $l \leq r - 1$, such that:*

- $\mathrm{val}(H_i) \geq i$*; and*
- *for every $P \in P(H_i)$ and for every $c \in [k]$, if $P \cap f^{-1}(c) \neq \emptyset$, then $|P \cap f^{-1}(c)| \geq 2^{k(r-1)-i}$.*

Proof of Claim 10.5.3. We prove the claim by induction on i. For the basis of the induction: let $i = 0$. By pigeonhole principle, there exists at least one color $c \in [k]$ and a set S of $2^{k(r-1)}$ vertices of G_0 all colored with c. Consider the subgraph $H_0 := G[S]$ of G_0. Then since $V(G_0)$ is an independent set, we have that H_0 is trivially a complete multipartite graph with one part, we also have $\mathrm{val}(H_0) = 1 \geq 0$.

Let us suppose that $i \geq 1$, and that for every nonnegative integer $i' < i$, the statement of the claim holds. Let H_{i-1} be a complete l-partite subgraph H_i with $l \leq r - 1$ as in the statement of the claim. We show how H_i is obtained from H_{i-1}. We distinguish the following two cases:

Case 1: $l = r - 1$. Let $P(H) = \{P_1, \dots, P_{r-1}\}$. For every $c \in [k]$ and for every $j \in [r-1]$, let P_j^c be defined as follows: if $P_j \cap f^{-1}(c) \neq \emptyset$, then let P_j^c is an arbitrary but fixed subset of size $2^{k(r-1)-i}$ of the set $P_j \cap f^{-1}(c)$; otherwise $P_j^c = \emptyset$. Let

$$H' := G_i[\cup\{P_j^c : j \in [r-1] \text{ and } c \in [k]\}].$$

Then, as the partition $P(H') := \{P_j \cap V(H') : j \in [r-1]\}$ witnesses, H' is a complete $(r-1)$-partite subgraph of G_{i-1}. We note that for every $j \in [r-1]$ we have that $\sum_{c \in [k]} |f(P_j^c)| = |f(P_j)|$. Thus,

$$\mathrm{val}(H') = \sum_{j \in [r-1]} |f(P_j \cap V(H'))| = \sum_{j \in [r-1]} \sum_{c \in [k]} |f(P_j^c)| = \sum_{j \in [r-1]} |f(P_j)| = \mathrm{val}(H_{i-1}).$$

Let $v_{H'}^1, \dots, v_{H'}^p$ be the p vertices of G_i whose neighborhood in G_i is exactly the set $V(H')$. Since $p = k2^{k(r-1)}$, there exists $c' \in [k]$ such that f assigns c' to at least $2^{k(r-1)}$ of the vertices $v_{H'}^1, \dots, v_{H'}^p$. Let $c' \in [k]$ be such a color, and let $M_{c'}$ be a subset of size $2^{k(r-1)}$ of $\{v_{H'}^1, \dots, v_{H'}^p\} \cap f^{-1}(c')$.

We claim that there exists $j \in [r-1]$ such that $P_j \cap f^{-1}(c') = \emptyset$. Suppose not. For each $j \in [r-1]$ let $v^{c'}_{P_j} \in P_j \cap f^{-1}(c')$. Then $\{v^{c'}_{P_j} : j \in [r-1]\}$ is a monochromatic $(r-1)$-clique of color c' in G_i. Let $v \in M_{c'}$. Then $\{v\} \cup \{v^{c'}_{P_j} : j \in [r-1]\}$ is a monochromatic r-clique of color c' in G_i contradicting the fact that f is a K_r-free coloring. Thus, there exists $j \in [r-1]$ such that $P_j \cap f^{-1}(c') = \emptyset$. Let $j \in [r-1]$ be such that $P_j \cap f^{-1}(c') = \emptyset$. Let

$$H_i := (H' \setminus P_j) \cup G_i[(P_j \setminus V(H')) \cup M_{c'}\}].$$

We claim that H_i is a complete $(r-1)$-partite graph. Indeed, this follows by the observations that $H' \setminus P_j$ is a complete $(r-2)$-partite graph; $(P_j \setminus V(H')) \cup M_{c'}$ is an independent set; and the set $(P_j \setminus V(H')) \cup M_{c'}$ is complete to $V(H') \setminus P_j$ in the graph G_i. Consider the partition $P(H_i) := \{P_s \cap V(H') : s \in [r-1] \setminus \{j\}\} \cup \{(P_j \setminus V(H')) \cup M_{c'}\}$ of $V(H_i)$ which witnesses that H_i is a complete $(r-1)$-partite graph. Then, for the value of H_i we have:

$$\begin{aligned}
\text{val}(H_i) &= |f((P_j \setminus V(H')) \cup M_{c'})| + \sum_{s \in [r-1] \setminus \{j\}} |f(P_s \cap V(H'))| \\
&= |f(P_j)| + |f(M_{c'})| + \sum_{s \in [r-1] \setminus \{j\}} \sum_{c \in [k]} |f(P^c_s)| \\
&= |f(P_j)| + 1 + \sum_{s \in [r-1] \setminus \{j\}} |f(P_s)| \\
&= 1 + \sum_{s \in [r-1]} |f(P_s)| \\
&= 1 + \text{val}(H_{i-1}) \\
&= i.
\end{aligned}$$

Finally let $Q \in P(H_i)$ and $c \in [k]$. We claim that if $Q \cap f^{-1}(c) \neq \emptyset$, then $|Q \cap f^{-1}(c)| \geq 2^{k(r-1)-i}$. Indeed, suppose first that $Q = P_s \cap V(H')$ for some $s \in [r-1] \setminus \{j\}$. Then $Q \cap f^{-1}(c) = P^c_s$, and since $|P^c_s| = 2^{k(r-1)-i}$, our claim follows. Suppose that $Q \cap f^{-1}(c) = (P_j \cap f^{-1}(c)) \setminus V(H')$. Then

$$|Q \cap f^{-1}(c)| \geq |P_j \cap f^{-1}(c)| - |P^c_j| = 2^{k(r-1)-(i-1)} - 2^{k(r-1)-i} = 2^{k(r-1)-i}$$

Otherwise we have that $Q = (P_j \setminus V(H')) \cup M_{c'}$ and thus $Q \cap f^{-1}(c) = M_{c'}$, and since $|M_{c'}| = 2^{k(r-1)} \geq 2^{k(r-1)-i}$ the claim holds.

It follows by the above that H_i, as defined in this case, satisfies all the desired properties.

Case 2: $l < r-1$. Let $v^1_{H_{i-1}}, \ldots, v^p_{H_{i-1}}$ be the p vertices of G_i whose neighborhood in G_i is exactly the set $V(H_{i-1})$. Then, since $p = k2^{k(r-1)}$, there exists $c' \in [k]$ such that $|\{v^1_{H_{i-1}}, \ldots, v^p_{H_{i-1}}\} \cap f^{-1}(c')| \geq 2^{k(r-1)}$. Let $c' \in [k]$ be such a color, and let $M_{c'}$ be a subset of $\{v^1_{H_{i-1}}, \ldots, v^p_{H_{i-1}}\} \cap f^{-1}(c')$ such that $|M_{c'}| = 2^{k(r-1)}$. Let

$$H_i := H_{i-1} \cup G_i[M_{c'}].$$

Then, the partition $P(H_{i-1}) \cup \{M_{c'}\}$ witnesses that H_i is a complete $(l+1)$-partite graph.

For the value of H_i we have that:

$$\text{val}(H_i) = |f(M_{c'})| + \sum_{P \in P(H_{i-1})} |f(P)| = 1 + \text{val}(H_{i-1}) = 1 + (i-1) = i.$$

Finally, it follows by the induction hypothesis and the fact that $|M_{c'}| = 2^{k(r-1)}$, that for every $Q \in P(H_{i-1}) \cup \{M_{c'}\}$ and for every $c \in [k]$ such that $Q \cap f^{-1}(c) \neq \emptyset$, we have that $|Q \cap f^{-1}(c)| \geq 2^{k(r-1)-i}$. It follows by the above that H_i, as defined in this case, satisfies all the desired properties. This completes the proof of Claim 10.5.3. ■

It follows, by Claim 10.5.3, that the graph G_{c_k-1} contains a complete l-partite subgraph H with $l \leq r-1$, such that $\text{val}(H) = k(r-1)$, and thus G_{c_k} contains a monochromatic r-clique, contradicting the fact that f is a K_r-free coloring of G_{c_k}. Hence, $\chi_r(G_{c_k}) > k$. This completes the proof of Theorem 3.31. □

Chapter 11

Excluding a complete bipartite subgraph

In this chapter, we study the K_r-free chromatic number of graphs that exclude a fixed complete bipartite graph as a subgraph. Our main result, which we prove in Section 11.1, is that for any fixed broadleaved tree H, the K_r-degeneracy of every H-free graph G is upper bounded by a function of the biclique number of G. In Section 11.2 we prove that if H is a diamond-free chordal graph of a certain type namely a bloomed clique, that is not a broadleaved forest, then the K_r-free chromatic number of every H-free graph G is upper bounded by a function of the biclique number of G.

11.1 Excluding a complete bipartite subgraph and an induced broadleaved forest

The main result of this section is Theorem 3.33, which we restate:

Theorem 3.33. *Let $r \geq 2$ be an integer. For every r-broadleaved forest H, there exists a function f such that $\mathsf{degen}_r(G) \leq f(\mathsf{biclique}(G))$ for every H-free graph G.*

This chapter is based on ongoing joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.

Our proof of Theorem 3.33, follows closely the method of Scott, Seymour, and Spirkl from [120], but additional care is required in two places: First with finding and adding broadleaved stars, and second with our notion of degeneracy which, for $r \geq 3$, does not provide a linear bound on the number of edges. We note that much of the terminology and notions that we introduce in this section are analogous to/follows the terminology and notions of [120].

We recall that a rooted tree is a pair (T, s) where T is a tree and s is a vertex of T which is called the root of T. Let (T, s) be a rooted tree. Recall that the height of (T, s) is the length of a longest (s, t)-path in T, and that if t and t' are two adjacent vertices of T, such that t lies in the unique (s, t')-path in T, then we say that t' is a child of t. Following [120], the *spread of a rooted tree* (T, s) is the maximum number of number of children of a vertex $t \in V(T)$.

Let $r \geq 2$ be an integer, let H be an r-broadleaved tree, and let T be an underlying tree of H. If T has a vertex that is not a leaf, let s be such a vertex, otherwise let s be any of the two leaves of T. We call (H, s) a *rooted r-broadleaved tree*, and we refer to s as the *root of the broadleaved tree H*. In this case, we say that (T, s) is an *underlying rooted tree of the rooted broadleaved tree* (H, s). Let (H, s) be a rooted r-broadleaved tree. Let $u, v \in V(T)$. We say that v is a (T, s)*-child* of u in H, if v is a child of u in T. When there is no danger of confusion, we say child instead of (T, s)-child. The *height of the rooted r-broadleaved tree* (respectively *spread of the rooted r-broadleaved tree*) (H, s) is the height (respectively the spread) of (T, s).

In this section, we prove the following, which implies Theorem 3.33:

Theorem 11.1. *Let $r \geq 2, t, \zeta$ and η be positive integers. Then there exists a constant $f_{11.1} = f_{11.1}(r, t, \zeta, \eta) \in \mathbb{N}$ with the following property: Let (H, s) be a rooted r-broadleaved tree of spread at most ζ, and height at most η. If G is an H-free graph which does not contain $K_{t,t}$ as a subgraph, then $\mathsf{degen}_r(G) \leq f_{11.1}$.*

We start with rooted r-broadleaved trees of height one and spread at least two, that is with broadleaved stars:

Lemma 11.2. *Let $r \geq 2, \zeta \geq 2$ and $t \geq 1$ be integers. Then there exists a constant $f_{11.2} = f_{11.2}(r, \zeta, t) \in \mathbb{N}$ with the following property: Let G be a graph which does not contain* *a $K_{t,t}$ as a subgraph. For every $v \in V(G)$, if v is not the center of an r-broadleaved ζ-star in G, then $d_r(G) \leq f_{11.2}$. In particular, if G contains no r-broadleaved ζ-star as an induced subgraph, then $\Delta_r(G) \leq f_{11.2}$.*

Proof of Lemma 11.2. We claim that the constant

$$f_{11.2}(r, \zeta, t) = R(f_{3.32}(r-1, t), \zeta) - 1$$

satisfies the statement of the lemma. Suppose not. Then, by Lemma 3.25, $V(G)$ contains $f_{3.32}(r-1,t)$ pairwise disjoint and pairwise adjacent $(r-1)$-subsets. Then, by Lemma 3.32, G contains $K_{t,t}$ as a subgraph, contradicting our assumptions for G. This completes the proof of Lemma 11.2. □

We note that, in contrast to the result of Scott, Seymour, and Spirkl [120], in our proof of Theorem 3.33 we were not able to derive polynomial bounds; this is because we were not able to derive polynomial bounds for Lemma 11.2.

Problem 6. *Is it true that for every pair of integers $r \geq 3$ and $\zeta \geq 2$, and for every r-broadleaved ζ-star H, there exists $k > 0$ such that $\mathsf{degen}_r(G) \leq \mathsf{biclique}(G)^k$ for every H-free graph G.*

We need to introduce some additional notions. Let (H, s) be a rooted r-broadleaved tree and let (T, s) be an underlying rooted tree of (H, s). Let ζ and η be positive integers. We say that (H, s) is a (ζ, η)-*uniform rooted r-broadleaved tree* if:

- for every $v \in V(H)$ if v has a (T, s)-child, then v has exactly ζ many (T, s)-children;
- for every $v \in V(H)$ if v has no (T, s)-child, then v is joined to s by a path of length exactly η; and
- every leaf of T belongs to an r-clique of H.

We note that our definition of uniform rooted broadleaved tree above is analogous to the definition of uniform rooted trees in [120].

Let ζ and $\eta \geq 2$ be positive integers, and let (H, s) be a (ζ, η)-uniform rooted r-broadleaved tree. Let H' be a graph that is obtained from H as follows:

- for every vertex $v \in V(H)$ with $d(s, v) \leq \eta - 2$ we add to H a copy G_v of $\zeta \cdot K_{r-1}$ on a new vertex set; and
- We add edges so that every $v \in V(H)$ with $d(s, v) \leq \eta - 2$ is complete to $V(G_v)$.

Then we call (H', s) an *enriched (ζ, η)-uniform rooted r-broadleaved tree.* If (T, s) is the underlying rooted tree of (H, s), then we also refer to (T, s) as the underlying rooted tree of (H', s). An *enriched $(\zeta, 1)$-uniform rooted r-broadleaved tree* is just a $(\zeta, 1)$-uniform rooted r-broadleaved tree. An *enriched uniform rooted r-broadleaved tree* is an enriched (ζ, η)-uniform rooted r-broadleaved tree for some positive integers ζ and η. Let (H', s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree, let $v \in V(H')$, and let S be an $(r-1)$-clique of $H' \setminus v$. If S is complete to v and anticomplete to $V(H') \setminus (S \cup \{v\})$, then we say that S is a *clique-child* of v in H'.

Observation 11.3. *Let ζ and η be positive integers, let (H, s) be a rooted r-broadleaved tree of spread at most ζ, and height at most η, and let (H', s') be an enriched (ζ, η)-uniform rooted r-broadleaved tree. Then H' contains an induced subgraph isomorphic to H. Thus for every graph G if G is H-free, then G is H'-free.*

Observation 11.4. *Let $\zeta \geq 1$ and $\eta \geq 1$ be integers, let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree, and let $f_{11.4} = f_{11.4}(\zeta, \eta) \in \mathbb{N}$, be the constant which is defined as follows:*

$$f_{11.4} = f_{11.4}(\zeta, \eta) := 1 + \sum_{i=1}^{\eta} \zeta^i + (r-1) \sum_{j=1}^{\eta-1} \zeta^j.$$

Then we have that $f_{11.4} = |V(H)|$ and that $f_{11.4} \leq (r+1)\eta\zeta^\eta$.

Let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree. An *enriched uniform rooted r-broadleaved subtree of* (H, s) is an enriched uniform rooted r-broadleaved tree (H', s) such that H' is an induced subgraph of H. When there is no danger of confusion, we may refer to H' simply as a subtree.

The following lemma is the analogue for enriched uniform rooted broadleaved trees of the result of Scott, Seymour, and Spirkl [120, (2.1)] for uniform rooted trees.

Lemma 11.5. *Let $k, r \geq 2$ $\zeta, \eta \geq 1$ be integers. Then there exists a constant $f_{11.5} = f_{11.5}(k, r, \zeta, \eta) \in \mathbb{N}$ with the following property: Let G be a graph, and for each $i \in [k]$ let (H_i, s_i) be an enriched $(f_{11.5}, \eta)$-uniform rooted r-broadleaved tree which is a subgraph* *of G. If for all distinct $i, j \in [k]$ we have $s_i \notin V(H_j)$, then for each $i \in [k]$, there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H'_i, s_i) of (H_i, s_i), such that the subgraphs $H'_1 \dots H'_k$ of G are pairwise vertex-disjoint.*

Proof of Lemma 11.5. We claim that the following constant satisfies the statement of the lemma:

$$f_{11.5}(k, r, \eta, \zeta) = (k-1) \cdot f_{11.4}(\zeta, \eta) + \zeta$$

Let $j \leq k$ be maximum such that for each $i \in [j]$, there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H'_i, s_i) of (H_i, s_i), such that the graphs $H'_1 \dots H'_j$ are pairwise vertex-disjoint. Let $X = V(H'_1) \cup \dots \cup V(H'_j)$. Then, by Observation 11.4, we have that $|X| \leq j \cdot f_{11.4}(\zeta, \eta)$.

Let us suppose towards a contradiction that $j < k$. Then, each vertex of (H_{j+1}, s_{j+1}) with a child (respectively with a clique-child) has at least

$$(k-1) f_{11.4}(\zeta, \eta) - j f_{11.4}(\zeta, \eta) + \zeta \geq \zeta$$

children (respectively clique-children) not in X. Since $s_{j+1} \notin X$, it follows that there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H'_{j+1}, s_{j+1}) of (H_{j+1}, s_{j+1}) such that $V(H'_{j+1}) \cap V(X) = \emptyset$. This contradicts the choice of j. Thus $j = k$. This completes the proof of Lemma 11.5. □

Lemma 11.6. *Let $r \geq 2, t \geq 1, k \geq 2, \zeta \geq 1$ and $\eta \geq 1$ be integers. Then there exist constants $f_{11.6} = f_{11.6}(k, r, t, \zeta, \eta) \in \mathbb{N}$ and $h_{11.6} = h_{11.6}(k, r, t, \zeta, \eta) \in \mathbb{N}$, with the following property: Let G be a graph that does not contain a $K_{t,t}$ as a subgraph, and for each $i \in [f_{11.6}]$ let (H_i, s_i) be an enriched $(h_{11.6}, \eta)$-uniform rooted r-broadleaved tree which is a subgraph of G. If for all distinct $i, j \in [f_{11.6}]$ we have $s_i \notin V(H_j)$, then there exists $I \subseteq [f_{11.6}]$ such that $|I| = k$ and for each $i \in I$, there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H'_i, s_i) of (H_i, s_i), such that the subgraphs $\{H'_i : i \in I\}$ of G are pairwise vertex-disjoint and pairwise anticomplete in G.*

Proof of Lemma 11.6. We claim that the following constants satisfy the statement of the theorem:

$$f_{11.6} := R\left(k, f_{3.32}(f_{11.4}(\zeta, \eta), t)\right),$$

and

$$h_{11.6} := f_{11.5}(f_{11.6}, r, \zeta, \eta).$$

Indeed, by Lemma 11.5, we have that for each $i \in [f_{11.6}]$ there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H'_i, s_i) of (H_i, s_i), such that the graphs $H'_1 \dots H'_{f_{11.6}}$ are pairwise vertex-disjoint.

Let K be a graph with vertex set $[f_{11.6}]$ where for each pair of distinct $i, j \in [f_{11.6}]$ we have that $ij \in E(K)$ if and only if $V(H'_i)$ is adjacent to $V(H'_j)$ in G.

We claim that K does not contain a clique of size $f_{3.32}(f_{11.4}(\zeta, \eta), t)$. Suppose not. Let K' be such a clique. Then $\{V(H'_i) : i \in K'\}$ is a family of $f_{3.32}(f_{11.4}(\zeta, \eta), t)$ vertex disjoint and pairwise adjacent $f_{11.4}(\zeta, \eta)$-subsets of $V(G)$. Thus, by Lemma 3.32, it follows that G contains a $K_{t,t}$ as a subgraph, contradicting our assumptions for G. Hence, K does not contain a clique of size $f_{3.32}(f_{11.4}(\zeta, \eta), t)$.

Since $|V(K)| = f_{11.6} = R\left(k, f_{3.32}(f_{11.4}(\zeta, \eta), t)\right)$ it follows that K contains an independent set of size k. Let I be such a set. Then I is the desired k-subset of $[f_{11.6}]$. This completes the proof of Lemma 11.6. □

Let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree, and let (T, s) be an underlying rooted tree of (H, s). For every $v \in V(H)$ we denote by $\mathfrak{C}_{(H,s)}(v)$ the following set of all vertices $u \in V(H)$, such that: u is a child of v in (H, s), or there exists a clique-child S of v in (H, s) such that $u \in S$. We note that the definition of sets $\mathfrak{C}_{(H,s)}(v)$ is interdependent of the choice of the tree T, that is, for every tree T' such that (T', s) is an underlying rooted tree of (H, s) we get the same sets $\mathfrak{C}_{(H,s)}(v)$. When there is no danger of confusion we write $\mathfrak{C}(v)$ instead of $\mathfrak{C}_{(H,s)}(v)$.

Observation 11.7. *Let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree. Then, for every $v \in V(H)$, we have that $|\mathfrak{C}(v)| \in \{0, (r-1)\zeta, r\zeta\}$.*

Let $t \geq 1$ be an integer. Let G be a graph, and let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved which is a subgraph of G. Let $v \in V(H)$ be such that $\mathfrak{C}(v) \neq \emptyset$. Similarly with [120], we say that a vertex $u \in V(G) \setminus V(H)$ is *t-bad for v in* (H, s) if

$$|N(u) \cap \mathfrak{C}(v)| > \frac{t-1}{t} |\mathfrak{C}(v)|.$$

If there exists $v \in V(H)$ such that u is t-bad for v in (H, s), then we say that u is *t-bad for* (H, s).

Lemma 11.8. *Let x, t, ζ, η and $r \geq 2$ be positive integers, and let $f_{11.8} = f_{11.8}(x, t, r, \zeta, \eta) := xt\zeta \in \mathbb{N}$. Then $f_{11.8}$ has the following property: Let G be a graph, let (H, s) be an enriched $(f_{11.8}, \eta)$-uniform rooted r-broadleaved tree which is a subgraph of G, and let S be an x-subset of $V(G) \setminus V(H)$. If no vertex in S is t-bad for (H, s), then there is an enriched (ζ, η)-uniform rooted r-broadleaved subtree (H', s) of (H, s) such that $N_G(S) \cap V(H) \subseteq \{s\}$.*

Sketch of the proof of Lemma 11.8. For every vertex of (H, s) that has a child $v \in N_G(S)$ (respectively a clique-child K such that $K \cap N_G(S) \neq \emptyset$) we delete v from H (respectively we delete all the vertices of K from H). Let F be the resulting subgraph of H. Then any enriched (ζ, η)-uniform rooted r-broadleaved subtree (H', s) of (H, s) that is a subgraph of F satisfies the lemma, and it is easy to see that such an (H', s) exists. □

We note that Lemma 11.9 and Corollary 11.10 are analogous to [120, (2.3)].

Lemma 11.9. *Let ζ, η and $r, t \geq 2$ be positive integers, where t divides ζ. Let G be a graph which does not contain $K_{t,t}$ as a subgraph, let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree which is a subgraph of G, and let $w \in V(H)$. Then at most $t - 1$ vertices in $V(G) \setminus V(H)$ are t-bad for w in (H, s).*

Proof of Lemma 11.9. Let $w \in V(H)$, and let us suppose towards a contradiction that there exist distinct $u_1, \ldots, u_t \in V(G) \setminus V(H)$, such that for each $i \in [t]$, we have that u_i is t-bad for w. Then, for each $i \in [t]$, we have that u_i has at least $\frac{t-1}{t}|\mathfrak{C}(w)| + 1$ neighbors in $\mathfrak{C}(w)$, and thus

$$|A(u_i) \cap \mathfrak{C}(w)| \leq |\mathfrak{C}(w)| - (\frac{t-1}{t}|\mathfrak{C}(w)| + 1) = \frac{|\mathfrak{C}(w)|}{t} - 1.$$

Hence, $|\bigcup_{i \in [t]} A(u_i) \cap \mathfrak{C}(w)| \leq t(\frac{|\mathfrak{C}(w)|}{t} - 1) = |\mathfrak{C}(w)| - t$. Thus, there is a t-subset of $\mathfrak{C}(w)$ which is complete to the set $\{u_1, \ldots, u_t\}$, contradicting the fact that G does not contain $K_{t,t}$ as a subgraph. This completes the proof of Lemma 11.9. □

The following is an immediate corollary of Lemma 11.9 and Observation 11.4.

Corollary 11.10. *Let ζ, η and $r, t \geq 2$ be positive integers, where t divides ζ, and let G be a graph which does not contain $K_{t,t}$ as a subgraph. If (H, s) is an enriched (ζ, η)-uniform rooted r-broadleaved which is a subgraph of G, then at most $(t-1) \cdot f_{11.4}(\zeta, \eta)$ vertices in $V(G) \setminus V(H)$ are t-bad for (H, s).*

Let G be a graph, and let (H, s) be an enriched (ζ, η)-uniform rooted r-broadleaved tree such that H is an induced subgraph of G. We call (H, s) a (ζ, η)-*limb* in G.

Lemma 11.11. *Let $r, t \geq 2, \zeta,$ and $\eta \geq 2$ be positive integers. Then there exist constants $f_{11.11} = f_{11.11}(r, t, \zeta, \eta) \in \mathbb{N}$, $g_{11.11} = g_{11.11}(r, t, \zeta, \eta) \in \mathbb{N}$, and $h_{11.11} = h_{11.11}(r, t, \zeta, \eta) \in \mathbb{N}$ with the following property. Let G be a graph such that every vertex of G is a root of an $(f_{11.11}, 1)$-limb. If G contains neither a $K_{t,t}$ as a subgraph nor a (ζ, η)-limb, then for every $u \in V(G)$ we have that u is adjacent to at most $h_{11.11} - 1$ vertices v with the following property: v is a root of an $(g_{11.11}, \eta - 1)$-limb (J_v, v) of G, $u \notin V(J_v)$, and u is not t-bad for (J_v, v).*

Proof of Lemma 11.11. Let $f := f_{11.8}((r-1)\zeta, t, r, \zeta, \eta)$. We claim that the constants

$$f_{11.11} := \zeta + f + (t-1) \cdot f_{11.4}(\zeta, \eta) + f \cdot f_{11.4}(f, \eta - 1),$$

$$g_{11.11} := f_{11.8}(1, t, r, h_{11.6}(f, r, t, f, \eta - 1), \eta - 1),$$

and

$$h_{11.11} := f_{11.8}(1, t, r, \zeta, \eta) \cdot \Big(f_{11.6}(f, r, t, f, \eta - 1) - 1\Big) + f_{11.6}(f, r, t, f, \eta - 1),$$

satisfy the lemma.

Let us suppose towards a contradiction that there exist a vertex $u \in V(G)$, an $h_{11.11}$-subset V_0 of $N(u)$, and a collection $S_0 = \{(J_v, v) : v \in V_0\}$, such that for every $v \in V_0$ we have that (J_v, v) is an $(g_{11.11}, \eta - 1)$-limb in G, that $u \notin V(J_v)$, and that u is not t-bad for (J_v, v). We will derive a contradiction by showing that u is the root of a (ζ, η)-limb of G.

We note that, by the choice of $g_{11.11}$, we have that for every $v \in V_0$ there exists a $(h_{11.6}(f, r, t, f, \eta - 1), \eta - 1)$ subtree (J'_v, v) of (J_v, v) such that $N(u) \cap V(J'_v) = \{v\}$. Let $S'_0 := \{(J'_v, v) : v \in V_0\}$. By the choice of the size of $h_{11.11}$, it follows that there exist $f_{11.6}(f, r, t, f, \eta - 1)$-subsets S_1 of S'_0, and V_1 of V_0 such that for any two distinct $(J'_v, v), (J'_w, w) \in S_1$ we have $v \notin V(J'_w)$.

By Lemma 11.6, there exist an f-subset V_2 of V_1, and a collection $S_2 = \{(J''_v, v) : v \in V_2\}$ of f pairwise vertex disjoint and pairwise anticomplete $(f, \eta - 1)$-limbs, such that for each $v \in V_2$, we have that (J''_v, v) is a subtree of (J'_v, v).

Let $X := \cup_{v \in V_2} V(J''_v)$. Since, by our assumptions, u is the root of an $(f_{11.11}, 1)$-limb, there exists a set of $f_{11.11}$ pairwise disjoint $(r-1)$-cliques which are subsets of $N(u)$. Let S_u be such a set. We claim that there exists a $(\zeta + f + (t-1) \cdot f_{11.4}(\zeta, \eta))$-subset S'_u of S_u such that for every $v \in V_2$ and for every $(r-1)$-clique $K \in S'_u$ we have that $V(J''_v) \cap K = \emptyset$. Let $A := \{K \in S_u : \exists v \in V_2, V(J''_v) \cap K \neq \emptyset\}$. By Observation 11.4, we have that $|X| = f \cdot f_{11.4}(f, \eta - 1)$. Thus, $|A| \leq f \cdot f_{11.4}(f, \eta - 1)$. Hence,

$$\begin{aligned} |S_u \setminus A| &\geq f_{11.11} - f \cdot f_{11.4}(f, \eta - 1) \\ &= [\zeta + f + (t-1) \cdot f_{11.4}(\zeta, \eta) + f \cdot f_{11.4}(f, \eta - 1)] - f \cdot f_{11.4}(f, \eta - 1) \\ &= \zeta + f + (t-1) \cdot f_{11.4}(\zeta, \eta). \end{aligned}$$

Let S'_u be such a $(\zeta + f + (t-1) \cdot f_{11.4}(\zeta, \eta))$-subset of S_u, and let S''_u be an f-subset of S'_u.

Let F be the graph that we obtain from $G[\{u\} \cup \{\cup_{K \in S_{u''}} K\} \cup X]$ by deleting all the edges of the set

$$\{e \in E(G) : e \cap \{\cup_{K \in S''_u} K\} \neq \emptyset \text{ and } e \cap X \neq \emptyset\}.$$

Then (F, u) is an enriched (f, η)-uniform rooted r-broadleaved tree, and F is a subgraph of G. By Corollary 11.10 we know that at most $(t-1) \cdot f_{11.4}(\zeta, \eta)$ vertices of $V(G) \setminus V(F)$ are t-bad for F. Note that $|S'_u \setminus S''_u| \geq \zeta + (t-1) \cdot f_{11.4}(\zeta, \eta)$. Thus, there exists a ζ-subset C of $S'_u \setminus S''_u$, such that for every $K \in C$ and for every $v \in K$ we have that v is not t-bad for F.

By Lemma 11.8, there exists an enriched (ζ, η)-uniform rooted r-broadleaved subtree (F', u) of (F, u) such that for every $K \in C$ we have that $N(K) \cap V(F') = \{u\}$. Then,

$$\Big(G[\{u\} \cup \{\cup_{K \in C} K\} \cup X], u\Big)$$

is a (ζ, η)-limb in G, contradicting the fact that G contains no (ζ, η)-limb. This completes the proof of Lemma 11.11. $\square$

The following lemma uses the proof technique of [120, (2.4)].

Lemma 11.12. *Let $r, t \geq 2, \zeta$, and $\eta \geq 2$ be positive integers. Then there exist constants $f_{11.12} = f_{11.12}(r, t, \zeta, \eta) \in \mathbb{N}$ and $g_{11.12} = g_{11.12}(r, t, \zeta, \eta) \in \mathbb{N}$ with the following property. Let H be an r-broadleaved tree of spread at most ζ, and height at most η, and let G be an H-free graph which does not contain $K_{t,t}$ as a subgraph. If every vertex of G is a root of an $(g_{11.12}, 1)$-limb in G, then* $\mathsf{degen}(G) \leq f_{11.12}$.

Proof of Lemma 11.12. We start with the following claim, which is the main observation that we need for the proof of the lemma.

Claim 11.12.1. *There exists a constant $f_{11.12.1} = f_{11.12.1}(r, t, \zeta, \eta) \in \mathbb{N}$ with the following property: Let G' be an induced subgraph of G such that every vertex of G' is a root of an $(g_{11.11}(r, t, \zeta, \eta), \eta - 1)$-limb of G, and let $F := \{e \in E(G) : e \cap V(G') \neq \emptyset\}$. Then $|F| \leq f_{11.12.1}|V(G)|$.*

Proof of Claim 11.12.1. For each $v \in V(G')$, there exists at least one $(g_{11.11}(r, t, \zeta, \eta), \eta - 1)$-limb in G with root v; let (J_v, v) be such a $(g_{11.11}(r, t, \zeta, \eta), \eta - 1)$-limb in G. For each edge $e \in F$, select an end of e from the set $e \cap V(G')$, and call it the *head* of e.

- Let B be the set of all edges $uv \in F$ with head v, such that $u \notin V(J_v)$, and u is not t-bad for (J_v, v);
- Let C be the set of all edges $uv \in F$ with head v, such that $u \notin V(J_v)$, and u is t-bad for (J_v, v);
- Let D be the set of all edges $uv \in F$ with head v, such that $u \in V(J_v)$.

Thus, every edge of F belongs to exactly one of the sets B, C, and D. By Observation 11.3, we have that G' does not contain a (ζ, η)-limb. Thus, by Lemma 11.11, for each vertex $u \in V(G)$, there are at most $h_{11.11}(r, t, \zeta, \eta) - 1$ edges $uv \in B$ with head v; and so

$$|B| \leq (h_{11.11}(r, t, \zeta, \eta) - 1)|V(G)|.$$

For each $v \in V(G')$, by Corollary 11.10, there are at most $(t-1) \cdot f_{11.4}\Big(g_{11.11}(r, t, \zeta, \eta), \eta - 1\Big)$ edges $uv \in C$ with head v, and so

$$|C| \leq (t - 1) \cdot f_{11.4}\Big(g_{11.11}(r, t, \zeta, \eta), \eta - 1\Big)|V(G')|.$$

For each $v \in V(G')$, since (J_v, v) is induced, we have that the set of neighbors of v in (J_v, v) is exactly the set $|\mathfrak{C}_{(J_v,v)}(v)| \leq r \cdot g_{11.11}(r, t, \zeta, \eta)$. Thus, we have that:

$$|D| \leq r \cdot g_{11.11}(r, t, \zeta, \eta)|V(G')|.$$

Thus, since $|V(G')| \leq |V(G)|$, we have that:

$$|F| \leq \Big(h_{11.11}(r, t, \zeta, \eta) + t \cdot f_{11.4}\Big(g_{11.11}(r, t, \zeta, \eta), \eta - 1\Big) + r \cdot g_{11.11}(r, t, \zeta, \eta)\Big)|V(G)|.$$

Then the constant $f_{11.12.1}(r, t, \zeta, \eta) := h_{11.11}(r, t, \zeta, \eta) + (t-1) \cdot f_{11.4}\Big(g_{11.11}(r, t, \zeta, \eta), \eta - 1\Big) + r \cdot g_{11.11}(r, t, \zeta, \eta)$ satisfies the claim. This completes the proof of Claim 11.12.1. ■

Let $g_{11.12}(r, t, \zeta, \eta) := \max\{f_{11.11}(r, t, \zeta, \eta), g_{11.11}(r, t, \zeta, \eta)\}$, and let $f_{11.12}(r, t, \zeta, \eta)$ be defined recursively as follows:

$$f_{11.12}(r, t, \zeta, \eta) := \begin{cases} 2 \cdot f_{11.12.1}(r, t, \zeta, \eta), & \text{if } \eta = 2 \\ 2 \cdot \Big(f_{11.12}(r, t, g_{11.11}(r, t, \zeta, \eta), \eta - 1) + f_{11.12.1}(r, t, \zeta, \eta)\Big), & \text{else} \end{cases}$$

In what follows, we prove by induction on η that $\mathsf{degen}(G) \leq f_{11.12}(r, t, \zeta, \eta)$. For the basis of the induction let $\eta = 2$. By the choice of $g_{11.12}$ we have that every vertex of G is a root of an $(g_{11.11}(r, t, \zeta, \eta), 1)$-limb of G. Hence, by Claim 11.12.1, we have that $E(G) \leq f_{11.12.1}(r, t, \zeta, 2)|V(G)|$. Thus, there exists a vertex $v \in V(G)$, such that $d(v) \leq 2 \cdot f_{11.12.1}(r, t, \zeta, 2)$. This also holds for every non-null induced subgraph of G. Hence, $\mathsf{degen}(G) \leq 2 \cdot f_{11.12.1}(r, t, \zeta, 2) = f_{11.12}(r, t, \zeta, 2)$, as desired.

Let $\eta \geq 3$, and suppose that the statement of the lemma holds for every r-broadleaved tree of height at most $\eta - 1$.

Let P be the set of all vertices of G which are roots of an $(g_{11.11}(r, t, \zeta, \eta), \eta - 1)$-limb of G, and let $Q := V(G) \setminus P$. Then $\{P, Q\}$ is a partition of $V(G)$. Let A be the set of all edges with both ends in Q, and let $B := E(G) \setminus A$. Then $\{A, B\}$ is a partition of the edges of G. By the induction hypothesis we have that $\mathsf{degen}(G[Q]) \leq f_{11.12}(r, t, g_{11.11}(r, t, \zeta, \eta), \eta - 1)$. Thus,

$$|A| \leq f_{11.12}(r, t, g_{11.11}(r, t, \zeta, \eta), \eta - 1)|Q| \leq f_{11.12}(r, t, g_{11.11}(r, t, \zeta, \eta), \eta - 1)|V(G)|.$$

By Claim 11.12.1, we have that $|B| \leq f_{11.12.1}(r,t,\zeta,\eta)|V(G)|$. Hence, we have that:

$$|E(G)| \leq \Big(f_{11.12}(r,t,g_{11.11}(r,t,\zeta,\eta),\eta-1) + f_{11.12.1}(r,t,\zeta,\eta)\Big)|V(G)|.$$

Thus, there exists a vertex $v \in V(G)$ such that:

$$d(v) \leq 2 \cdot \Big(f_{11.12}(r,t,g_{11.11}(r,t,\zeta,\eta),\eta-1) + f_{11.12.1}(r,t,\zeta,\eta)\Big) = f_{11.12}(r,t,\zeta,\eta).$$

Since this is also true for every non-null induced subgraph of G, it follows that $\mathsf{degen}(G) \leq f_{11.12}(r,t,\zeta,\eta)$. This concludes the induction on η. This completes the proof of Lemma 11.12. □

We are now ready to prove Theorem 11.1, which we restate:

Theorem 11.1. *Let $r \geq 2, t, \zeta$ and η be positive integers. Then there exists a constant $f_{11.1} = f_{11.1}(r,t,\zeta,\eta) \in \mathbb{N}$ with the following property: Let (H,s) be a rooted r-broadleaved tree of spread at most ζ, and height at most η. If G is an H-free graph which does not contain $K_{t,t}$ as a subgraph, then $\mathsf{degen}_r(G) \leq f_{11.1}$.*

Proof of Theorem 11.1. We may assume that $t \geq 2$ since otherwise the theorem holds trivially. Let $f_{11.1}$ be defined as follows:

$$f_{11.1}(r,t,\zeta,\eta) := \begin{cases} f_{11.2}(r,\zeta,t), & \text{if } \eta = 1 \\ \max\{f_{11.2}(r,g_{11.12}(r,t,\zeta,\eta),t)-1, f_{11.12}(r,t,\zeta,\eta)\} & \text{else} \end{cases}$$

Suppose that $\eta = 1$. Then, by Lemma 11.2, we have that $\Delta_r(G) \leq f_{11.2}(r,\zeta,t)$, and thus

$$\mathsf{degen}_r(G) \leq \Delta_r(G) \leq f_{11.2}(r,\zeta,t) = f_{11.1}(r,t,\zeta,1),$$

as desired. Thus, we may assume that $\eta \geq 2$.

In what follows, we construct an ordering of the vertices of G which witnesses that $\mathsf{degen}_r(G) \leq f_{11.1}(r,t,\zeta,\eta)$. Let $v_1,\ldots,v_i$ be a maximal, with respect to i, sequence of vertices of G such that: $d^r_G(v_1) \leq f_{11.2}(r,g_{11.12}(r,t,\zeta,\eta),t)-1$, and for every $j \in \{2,\ldots,i\}$ we have that:

$$d^r_{G\setminus\{v_1,\ldots,v_{i-1}\}}(v) \leq f_{11.2}(r,g_{11.12}(r,t,\zeta,\eta),t)-1.$$

We note that the set $\{v_1, \dots, v_i\}$ may be empty. Let $G' := G \setminus \{v_1, \dots, v_i\}$. Then, by the maximality of $v_1, \dots, v_i$, we have that $\delta_r(G') \geq f_{11.2}(r, g_{11.12}(r, t, \zeta, \eta), t)$. Thus, by Lemma 11.2, we have that every vertex of G' is a root of a $(g_{11.12}(r, t, \zeta, \eta), 1)$-limb in G'. Thus, by Lemma 11.12, we have that $\mathsf{degen}(G') \leq f_{11.12}(r, t, \zeta, \eta) \leq f_{11.1}(r, t, \zeta, \eta)$. Let $n := |V(G')|$, and let $u_1, \dots, u_n$ be an ordering of the vertices of $V(G')$ which witnesses that $\mathsf{degen}(G') \leq f_{11.12}(r, t, \zeta, \eta) \leq f_{11.1}(r, t, \zeta, \eta)$. Then, $v_1, \dots, v_i, u_1, \dots, u_n$ is an ordering of the vertices of $V(G)$, which witnesses that $\mathsf{degen}_r(G) \leq \max\{f_{11.2}(r, g_{11.12}(r, t, \zeta, \eta), t) - 1, f_{11.12}(r, t, \zeta, \eta)\} = f_{11.1}(r, t, \zeta, \eta)$. This completes the proof of Theorem 11.1. □

11.2 Excluding a complete bipartite subgraph and an induced bloomed clique

We recall the definition of bloomed cliques. Let K_p be a complete graph on p vertices. Let H be the graph that is obtained from K_p as follows: for each $v \in K_p$ we add a copy C_v of qK_{p-1} to K_p so that these copies are pairwise disjoint and pairwise anticomplete. Finally, we add edges to H so that each vertex $v \in K_p$ is complete to $V(C_v)$. We call H a q-bloomed p-clique. We call K_p the base clique of H, and we call the cliques of C_v the private cliques of v. A bloomed clique is a q-bloomed p-clique for some positive integers q and p. We note that bloomed cliques are diamond-free chordal graphs but not bull-free graphs. In particular, bloomed cliques are not broadleaved trees. The main result of this section is Theorem 3.34 which we restate:

Theorem 3.34. *Let p, q and t be positive integers. Then there exist constants $f_{3.34} = f_{3.34}(p, q, t) \in \mathbb{N}$ and $h_{3.34} = h_{3.34}(p, q, t) \in \mathbb{N}$ with the following property: If G is a graph that does not contain a $K_{t,t}$ as a subgraph or a q-bloomed p-clique as an induced subgraph, then $\chi_{f_{3.34}}(G) \leq h_{3.34}$.*

Let G be a graph which contains a q-bloomed p-clique H as a subgraph (not necessarily induced), and let K_p be the base clique of H. We say that H is a *semi-induced q-bloomed p-clique subgraph* in G if the following hold: for every distinct $v_1, v_2 \in V(H)$, if $v_1v_2 \in E(G) \setminus E(H)$, then there exist distinct vertices $v_1', v_2' \in K_p$ such that for each $i \in [2]$ there exists a private clique C_i of v_i' such that $v_i \in C_i$.

We first need the following lemma:

Lemma 11.13. *Let p, q and t be positive integers. Then, there exists a constant $f_{11.13} = f_{11.13}(p, q, t) \in \mathbb{N}$ with the following property: If G is a graph such that G does not contain $K_{t,t}$ as a subgraph, and G contains a semi-induced q-bloomed $f_{11.13}$-clique subgraph, then G contains q-bloomed p-clique as an induced subgraph.*

Proof of Lemma 11.13. We claim that $f_{11.13} = R(f_{3.32}(q(p-1), t), p)$ satisfies the statement of the lemma. Let H be a semi-induced q-bloomed $f_{11.13}$-clique subgraph in G. Let K be the base clique of H. For each vertex $v \in K$ let $C_v := N_H(v) \setminus K$. Let F be the graph on $\{C_v : v \in K\}$ where two vertices C_v and C_w are adjacent in F if and only if the sets C_v and C_w are adjacent in G. Since $|V(F)| = |K| = R(f_{3.32}(q(p-1), t), p)$ it follows that F contains either an independent set of size p or a clique of size $f_{3.32}(q(p-1), t)$.

Let us suppose towards a contradiction that F contains a clique of size $f_{3.32}(q(p-1), t)$. Let C be such a clique. Then, C is a collection of $f_{3.32}(q(p-1), t)$ pairwise disjoint and pairwise adjacent $q(p-1)$-subsets of $V(G)$. Thus, by Lemma 3.32, it follows that G contains a $K_{t,t}$ as a subgraph, which is a contradiction.

Thus, F contains an independent set of size p. Let A be such a set. Let $K' := \{v \in K : C_v \in A\}$. Then $G[K' \cup \{C_v : v \in K'\}]$ is a q-bloomed p-clique in G. This completes the proof of Lemma 11.13. □

We are now ready to prove Theorem 3.34.

Proof of Theorem 3.34. We claim that the constants

$$f_{3.34} := f_{11.13}(p, q, t),$$

and

$$h_{3.34} := 2\Big((f_{11.13} - 1) \cdot (R(f_{3.32}(f_{11.13} - 1, t)) - 1, q + 1)\Big) + 1$$

satisfy the statement of the theorem. Suppose that $\omega(G) < f_{3.34}$. Then, by coloring all the vertices of G with the same color we have a $K_{f_{3.34}}$-free coloring of G which witnesses that $\chi_{f_{3.34}}(G) = 1 \leq h_{3.34}$. Thus, we may assume that $\omega(G) \geq f_{3.34}$.

We need some terminology for the rest of the proof. Let K be an $f_{3.34}$-clique, and let $v \in K$. We say that v is K-*happy* if $N(v) \setminus K$ contains q cliques of size $(f_{3.34} - 1)$ which together with $K \setminus \{v\}$ form a set of $q+1$ pairwise disjoint and pairwise anticomplete $(f_{3.34} - 1)$-cliques. If for an $f_{3.34}$-clique K and a vertex $v \in K$ we have that v is not K-happy, then we say that v is K-*unhappy*. We note that if for a $f_{3.34}$-clique K we have that every $v \in K$ is K-happy, then $G[N[K]]$ contains a semi-induced q-bloomed $f_{3.34}$-clique subgraph in G. Indeed, for every $v \in K$ let $C_v \subseteq N(v) \setminus K$ be a set of q pairwise disjoint and pairwise anticomplete $(f_{3.34} - 1)$-cliques, so that C_v is anticomplete to $K \setminus \{v\}$. In particular, the sets $\{C_v : v \in K\}$ are pairwise disjoint. Then, $G[K \cup \{C_v : v \in K\}]$ is a semi-induced q-bloomed $f_{3.34}$-clique subgraph in G. By the above observation, Lemma 11.13, and the fact that G contains no q-bloomed p-clique as an induced subgraph, it follows that there is no $f_{3.34}$-clique K in G such that for every $v \in K$ we have that v is K-happy. Thus, we may assume that every $f_{3.34}$-clique K contains at least one K-unhappy vertex. For every $f_{3.34}$-clique K let v_K be an arbitrary but fixed K-unhappy vertex.

Claim 11.13.1. *For every $v \in V(G)$, there exists a set $X_v \subseteq N(v)$, such that:*

- $|X_v| \leq (f_{3.34} - 1) \cdot (R(f_{3.32}(f_{3.34} - 1, t), q + 1) - 1)$*; and*
- *For every $f_{3.34}$-clique K such that $v_K = v$ we have that $X_v \cap (K \setminus \{v\}) \neq \emptyset$.*

Proof of Claim 11.13.1. Let $v \in V(G)$, let $C := \{K : v_K = v\}$, and let S be a maximal set of pairwise disjoint $(f_{3.34} - 1)$-cliques in $C' := \{K \setminus \{v\} : K \in C\}$. We claim that S contains no $q+1$ pairwise disjoint and pairwise anticomplete $(f_{3.34} - 1)$-cliques. Suppose not. Let S' be a $(q+1)$-subset of S such that the cliques in S' are pairwise disjoint and pairwise anticomplete. Let $K' \in S'$. Then, v is $K' \cup \{v\}$-happy, contradicting that $K' \cup \{v\} \in C$. We claim that $|S| < R(f_{3.32}(f_{3.34} - 1, t), q + 1)$. Suppose not. Then, S contains $f_{3.32}(f_{3.34} - 1, t)$ pairwise adjacent $(f_{3.34} - 1)$-cliques. Thus, by Lemma 3.32, it follows that G contains a $K_{t,t}$ as a subgraph, contradicting our assumptions for G. Hence, $|S| \leq R(f_{3.32}(f_{3.34} - 1, t), q + 1) - 1$. Let $X_v := \cup_{K \in S} K$. We note that:

$$|X_v| = (f_{3.34} - 1) \cdot |S| \leq (f_{3.34} - 1) \cdot (R(f_{3.32}(f_{3.34} - 1, t), q + 1) - 1).$$

We also note that, by the maximality of S, it follows that $X_v \cap K \neq \emptyset$ for every $K \in C'$. This completes the proof of Claim 11.13.1. ■

For every $v \in V(G)$ let $X_v \subseteq N(v)$ be as in the statement of Claim 11.13.1. Let $b := (f_{3.34} - 1) \cdot (R(f_{3.32}(f_{3.34} - 1, t), q + 1) - 1)$. Let D be the following digraph: $(V(G), \{vv' : v' \in X_v\})$. Then, D has out-degree at most b. It follows, by Lemma 10.4, the that the underlying graph of D has a K_2-free coloring which uses at most $2b + 1 = h_{3.34}$ colors. Let $\phi : V(G) \to [h_{3.34}]$ be such a coloring. We claim that ϕ is a $K_{f_{3.34}}$-free coloring of the graph G. Suppose not. Let $i \in [h_{3.34}]$ be such that $\phi^{-1}(i)$ contains an $f_{3.34}$-clique. Let $K \subseteq \phi^{-1}(i)$ be an $f_{3.34}$-clique. Let $v \in X_{v_k} \cap K$. Then, $\{v_K, v\} \subseteq \phi^{-1}(i)$ contradicting the fact that ϕ is a K_2-free coloring of the underlying graph of D. The $K_{f_{3.34}}$-free coloring ϕ of G witnesses that $\chi_{f_{3.34}}(G) \leq h_{3.34}$. This completes the proof of Theorem 3.34. $\square$

Chapter 12

Excluding a complete multipartite subgraph and an induced broadleaved star

The main result of this chapter is Theorem 3.35, which we restate:

Theorem 3.35. *Let $q \geq 2$, $r, s \geq 2$ and $t \geq 1$ be integers. Then there exist constants $f_{3.35} = f_{3.35}(q, r, s, t) \in \mathbb{N}$ and $h_{3.35} = h_{3.35}(q, r, s, t) \in \mathbb{N}$, with the following property: If G is a graph that does not contain a complete q-partite graph with all parts having size t as a subgraph and G does not contain an r-broadleaved s-star as an induced subgraph, then $\Delta_{h_{3.35}}(G) \leq f_{3.35}$. In particular $\chi_{h_{3.35}}(G) \leq f_{3.35} + 1$.*

We begin with the following lemma, in which we describe a structure, with parameters two integers $t \geq 1$ and $q \geq 2$, and show that its presence in a graph implies the existence of a complete q-partite subgraph with all of its parts of size t.

Lemma 12.1. *Let $t \geq 1$ and $q \geq 2$ be integers, and let G be a graph. Let $S_1, \dots, S_q$ be pairwise disjoint tq-subsets of $V(G)$. For every $i \in [q]$, let $S_i = \{v_1^i, \dots, v_{tq}^i\}$. For every*

This chapter is based on ongoing joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.

$p \in [tq]$, let $Q_p = \{v_p^i : v_p^i \in S_i\}$. If there exist a bipartite graph G_1 on parts $A = \{a_p : p \in [tq]\}$ and $B = \{b_p : p \in [tq]\}$, and a bipartite graph G_2 on parts $C = \{c_i : i \in [q]\}$ and $D = \{d_i : i \in [q]\}$, such that the following hold:

- *for every $i, j \in [q]$ with $i < j$ there exists an isomorphism $\phi_{ij} : V(G_1) \to S_i \cup S_j$ which witnesses that the bipartite graph*
$$\Big(S_i \cup S_j, \{v_x^i v_z^j : x, z \in [tq] \text{ and } v_x^i v_z^j \in E(G)\}\Big)$$
is isomorphic to G_1, and ϕ_{ij} is such that for every $p \in [tq]$ we have that $\phi_{ij}(a_p) = v_p^i$ and that $\phi_{ij}(b_p) = v_p^j$; and
- *for every $p, p' \in [tq]$ with $p < p'$ there exists an isomorphism $\psi_{pp'} : V(G_2) \to Q_p \cup Q_{p'}$ which witnesses that the bipartite graph*
$$\Big(Q_p \cup Q_{p'}, \{v_p^i v_{p'}^j : i, j \in [q] \text{ and } v_p^i v_{p'}^j \in E(G)\}\Big)$$
is isomorphic to G_2, and $\psi_{pp'}$ is such that for every $i \in [q]$ we have that $\psi_{pp'}(c_i) = v_p^i$ and that $\psi_{pp'}(d_i) = v_{p'}^i$; and
- *there exist distinct $p, p' \in [tq]$ such that $a_p b_{p'} \in E(G_1)$.*

Then G contains a complete q-partite graph with parts of size t as a subgraph.

Proof of Lemma 12.1. Let $p, p' \in [tq]$ be distinct such that $a_p b_{p'} \in E(G_1)$. By symmetry we may assume that $p < p'$.

Claim 12.1.1. *For every $s, s' \in [tq]$ with $s < s'$ we have that $a_s b_{s'} \in E(G_1)$.*

Proof of Claim 12.1.1. Let $s, s' \in [tq]$ with $s < s'$. To prove that $a_s b_{s'} \in E(G_1)$ we fix a bipartite graph between "rows" (the S_i's), here we choose the bipartite graph between S_1 and S_2. Then using ϕ_{12} we identify the "copy" of $a_s b_{s'} \in E(G_1)$ in this bipartite graph; using $\psi_{ss'}^{-1}$ and $\psi_{pp'}$ we deduce the equivalence of $a_s b_{s'} \in E(G_1)$ with $v_p^1 v_{p'}^2 \in E(G)$, and finally using ϕ_{12}^{-1} we prove the equivalence of $a_s b_{s'} \in E(G_1)$ with $a_p b_{p'} \in E(G_1)$.

$$\begin{aligned}
a_s b_{s'} \in E(G_1) &\iff \phi_{12}(a_s)\phi_{12}(b_{s'}) \in E(G) \\
&\iff v_s^1 v_{s'}^2 \in E(G) \\
&\iff \psi_{ss'}^{-1}(v_s^1)\psi_{ss'}^{-1}(v_{s'}^2) \in E(G_2) \\
&\iff c_1 d_2 \in E(G_2) \\
&\iff \psi_{pp'}(c_1)\psi_{pp'}(d_2) \in E(G) \\
&\iff v_p^1 v_{p'}^2 \in E(G) \\
&\iff \phi_{12}^{-1}(v_p^1)\phi_{12}^{-1}(v_{p'}^2) \\
&\iff a_p b_{p'} \in E(G_1).
\end{aligned}$$

Since $a_p b_{p'} \in E(G_1)$, it follows by the above that $a_s b_{s'} \in E(G_1)$. This completes the proof of Claim 12.1.1. ■

For each $i \in [q]$ let $P_i := \{v^i_{(i-1)t+s} : s \in [t]\}$. Then for every pair of distinct $i, j \in [q]$, the isomorphism ϕ_{ij} and Claim 12.1.1 witness that $G[P_i \cup P_j]$ contains as a subgraph a complete bipartite graph with bipartition $\{P_i, P_j\}$. Thus, $G[\{P_i : i \in [q]\}]$ contains a complete q-partite graph with all parts of size t as a subgraph. This completes the proof of Lemma 12.1. □

The following lemma is the main step towards the proof of Theorem 3.35. In the proof of Theorem 3.35 we apply Lemma 12.2 in the neighborhood of a vertex of a graph which excludes an induced broadleaved star and a complete multipartite subgraph, in order to upper bound its K_r-degree by a function of the clique number of the graph.

Lemma 12.2. *Let q, r, t, x and ω be positive integers. Then there exist constants $f_{12.2} = f_{12.2}(q, r, t, x, \omega) \in \mathbb{N}$ and $h_{12.2} = h_{12.2}(q, r, t, x, \omega) \in \mathbb{N}$ with the following property: If G is a graph and $S_1, \dots, S_{f_{12.2}}$ are pairwise disjoint $h_{12.2}$-subsets of $V(G)$, then at least one of the following holds:*

(i) G contains as a subgraph a complete q-partite graph with all parts of size t; or
(ii) G contains an ω-clique; or
(iii) there exists an x-subset I of $\{1, \dots, f_{12.2}\}$, such that for each $i \in I$ there exists a set $S'_i \subseteq S_i$, with $|S'_i| \geq r$, and the sets $\{S'_i\}_{i \in I}$ are pairwise anticomplete.

Proof of Lemma 12.2. Let $m = \max\{\omega, x, q\}$, $m' = \max\{\omega, r, tq\}$. We claim that $h_{12.2} := R_{2^{m^2}}(m')$ and $f_{12.2} := R_{2^{h^2_{12.2}}}(m)$ satisfy the lemma.

The proof proceeds by two successive Ramsey-coloring arguments that aim to obtain a collection of tq-subsets of the elements of a q-subset of $\{S_i : i \in [f_{12.2}]\}$, which satisfy the assumptions of Lemma 12.1. This will yield outcome (i) from the statement of our lemma. Outcomes of the Ramsey-coloring arguments that are not favorable for this goal will result in outcomes (ii) and (iii). For each $i \in [f_{12.2}]$, let $S_i = \{v^i_y : y \in [h_{12.2}]\}$. Let $A = \{a_y : y \in [h_{12.2}]\}$, and let $B = \{b_y : y \in [h_{12.2}]\}$.

Claim 12.2.1. *There exists an m-subset I_1 of $[f_{12.2}]$ and a bipartite graph G_1 with bipartition $\{A, B\}$ such that: for every $i, j \in I_1$ with $i < j$ there exists an isomorphism $\phi_{ij} : V(G_1) \to S_i \cup S_j$ which witnesses that the bipartite graph*

$$\Big(S_i \cup S_j, \{v^i_y v^j_z : y, z \in [h_{12.2}] \text{ and } v^i_y v^j_z \in E(G)\}\Big)$$

is isomorphic to G_1, and ϕ_{ij} is such that for every $p \in [h_{12.2}]$ we have that $\phi_{ij}(a_p) = v^i_p$ and that $\phi_{ij}(b_p) = v^j_p$.

Proof of Claim 12.2.1. Let K be the complete graph on $[f_{12.2}]$. Note that the number of bipartite graphs with bipartition $\{A, B\}$ is $2^{h^2_{12.2}}$. Let

$$\phi : E(K) \to \{\text{all the bipartite graphs with bipartition } \{A, B\}\},$$

be a $2^{h^2_{12.2}}$-edge-coloring of K, which is defined as follows: for each $1 \leq i < j \leq f_{12.2}$ we set

$$\phi(ij) = (A \cup B, \{a_y b_z : v^i_y v^j_z \in E(G)\}).$$

Since $|V(K)| = f_{12.2} = R_{2^{h^2_{12.2}}}(m)$, Ramsey's theorem (Theorem 1.2) gives a monochromatic m-clique $I_1 \subseteq [f_{12.2}]$. Let I_1 be such an m-clique, let $e \in E(K[I_1])$, and let $G_1 = \phi(e)$.

For every $i, j \in I_1$ with $i < j$ let $\phi_{ij} : V(G_1) \to S_i \cup S_j$ be the function which is defined as follows: for every $p \in [h_{12.2}]$ we have $\phi_{ij}(a_p) := v^i_p$ and $\phi_{ij}(b_p) := v^j_p$. It is immediate that ϕ_{ij} is an isomorphism which witnesses that the bipartite graph $\Big(S_i \cup S_j, \{v^i_y v^j_z : y, z \in [h_{12.2}] \text{ and } v^i_y v^j_z \in E(G)\}\Big)$ is isomorphic to G_1.

Then the m-subset I_1 of $[f_{12.2}]$, the bipartite graph G_1, and the isomorphisms $\{\phi_{ij} : i, j \in I_1 \text{ and } i < j\}$ satisfy the statement of our claim. This completes the proof of Claim 12.2.1. ■

For each $y \in [h_{12.2}]$, let $Q_y := \{v_y^i : i \in I_1\}$. Let $C := \{c_i : i \in I_1\}$, and let $D := \{d_i : i \in I_1\}$.

Claim 12.2.2. *There exists an m'-subset I_2 of $[h_{12.2}]$ and a bipartite graph G_2 with bipartition $\{C, D\}$ such that: for every $p, p' \in I_2$ with $p < p'$ there exists an isomorphism $\psi_{pp'} : V(G_2) \to Q_p \cup Q_{p'}$ which witnesses that the bipartite graph*

$$\left(Q_p \cup Q_{p'}, \{v_p^i v_{p'}^j : i, j \in I_1 \text{ and } v_p^i v_{p'}^j \in E(G)\}\right)$$

is isomorphic to G_2, and $\psi_{pp'}$ is such that for every $i \in I_1$ we have that $\psi_{pp'}(c_i) = v_p^i$ and that $\psi_{pp'}(d_i) = v_{p'}^i$.

Proof of Claim 12.2.2. Let K be the complete graph on $[h_{12.2}]$. Note that the number of bipartite graphs with bipartition $\{C, D\}$ is 2^{m^2}. Let

$$\psi : E(K) \to \{\text{all the bipartite graphs with bipartition } \{C, D\}\},$$

be a 2^{m^2}-edge-coloring of K, which is defined as follows: for each $1 \leq p < p' \leq h_{12.2}$ we set

$$\psi(pp') = \left(C \cup D, \{c_y d_z : v_p^y v_{p'}^z \in E(G)\}\right).$$

Since $|V(K)| = h_{12.2} = R_{2^{m^2}}(m')$, Ramsey's theorem (Theorem 1.2) gives a monochromatic m'-clique $I_2 \subseteq [h_{12.2}]$. Let I_2 be such an m'-clique, let $e \in E(K[I_2])$, and let $G_2 = \psi(e)$.

For every $p, p \in I_2$ with $p < p'$ let $\psi_{pp'} : V(G_2) \to Q_p \cup Q_{p'}$ be the function which is defined as follows: for every $i \in I_1$ we have that $\psi_{pp'}(c_i) = v_p^i$ and that $\psi_{pp'}(d_i) = v_{p'}^i$. It is immediate that $\psi_{pp'}$ is an isomorphism which witnesses that the bipartite graph $\left(Q_p \cup Q_{p'}, \{v_p^i v_{p'}^j : i, j \in I_1 \text{ and } v_p^i v_{p'}^j \in E(G)\}\right)$ is isomorphic to G_2.

Then, I_2, the bipartite graph G_2, and the isomorphisms $\{\psi_{pp'} : p, p' \in I_2 \text{ and } p < p'\}$ satisfy the statement of our claim. This completes the proof of Claim 12.2.2. ■

Suppose that $E(G_1[\{a_y : y \in I_2\} \cup \{b_y : y \in I_2\}]) = \emptyset$. Then, since $m \geq x$ and $|I_2| \geq r$, letting I be an arbitrary but fixed x-subset of I_1, and letting $S_i' := \{v_p^i \in S_i : p \in I_2\}$ for each $i \in I$, we get the outcome (iii) from the statement of the lemma. Hence, we may assume that $E(G_1[\{a_y : y \in I_2\} \cup \{b_y : y \in I_2\}]) \neq \emptyset$.

Let $a_y b_z \in E(G_1[\{a_y : y \in I_2\} \cup \{b_y : y \in I_2\}])$. Suppose that $y = z$. Then the set $\{v^i_y : i \in I_1\}$ is an m-clique in G. Since $m \geq \omega$, in this case we get outcome (ii) from the statement of the lemma. Thus, we may assume that $y \neq z$. It follows, by the exact same arguments as in the proof of Claim 12.1.1, that for all $s, s' \in I_2$ with $s < s'$ we have $a_s b_{s'} \in E(G_1)$.

Let I'_1 (respectively I'_2) be an arbitrary but fixed q-subset (respectively tq-subset) of I_1 (respectively I_2). For each $i \in I'_1$, let $S'_i := \{v^i_p \in S_i : p \in I'_2\}$. Let $G'_1 := G_1[\{a_y : y \in I'_2\} \cup \{b_y : y \in I'_2\}]$ and $G'_2 := G_2[\{c_i : i \in I'_1\} \cup \{d_i : i \in I'_1\}]$. Since, for all $s, s' \in I_2$ with $s < s'$ we have $a_s b_{s'} \in E(G_1)$, it follows that for all $s, s' \in I'_2$ with $s < s'$ we have $a_s b_{s'} \in E(G'_1)$.

Then, the graphs G'_1 and G'_2, and the collections of isomorphisms $\{\phi_{ij}|_{V(G'_1)} : i, j \in I'_1 \text{ and } i < j\}$, and $\{\psi_{pp'}|_{V(G'_2)} : p, p' \in I'_2 \text{ and } p < p'\}$, witness that the q sets S'_i (each of size tq) satisfy the assumptions of Lemma 12.1. Thus, G contains as a subgraph a complete q-partite graph with parts of size t. This completes the proof of Lemma 12.2. □

We are now ready to prove Theorem 3.35, which we restate:

Theorem 3.35. *Let $q \geq 2$, $r, s \geq 2$ and $t \geq 1$ be integers. Then there exist constants $f_{3.35} = f_{3.35}(q, r, s, t) \in \mathbb{N}$ and $h_{3.35} = h_{3.35}(q, r, s, t) \in \mathbb{N}$, with the following property: If G is a graph that does not contain a complete q-partite graph with all parts having size t as a subgraph and G does not contain an r-broadleaved s-star as an induced subgraph, then $\Delta_{h_{3.35}}(G) \leq f_{3.35}$. In particular $\chi_{h_{3.35}}(G) \leq f_{3.35} + 1$.*

Proof of Theorem 3.35. Let $\omega := \omega(G)$. We claim that the constants

$$f_{3.35} := f_{12.2}(q, r-1, t, s, \omega) - 1 \text{ and } h_{3.35} := h_{12.2}(q, r-1, t, s, \omega) + 1$$

satisfy the statement of the theorem.

Suppose not. Then $\Delta_{h_{3.35}}(G) > f_{3.35}$. Let $v \in V(G)$ be such that $d_{h_{3.35}}(v) \geq f_{12.2}(q, r-1, t, s, \omega)$. Then $N(v)$ contains $f_{12.2}(q, r-1, t, s, \omega)$ pairwise disjoint cliques, each of size $h_{3.35} - 1 = h_{12.2}(q, r-1, t, s, \omega)$.

Let $S_1, \ldots, S_{f_{12.2}(q,r-1,t,s,\omega)}$ be pairwise disjoint $h_{12.2}(q, r-1, t, s, \omega)$-cliques of $G[N(v)]$. Since $G[N(v)]$ does not contain a complete q-partite graph with all parts having size t as a subgraph, and $\omega(G[N(v)]) < \omega$, it follows, by Lemma 12.2, that there exist an s-subset I of $\{1, \ldots, h_{12.2}(q, r-1, t, s, \omega)\}$, such that for each $i \in I$ there exists a set $S'_i \subseteq S_i$, with $|S'_i| \geq r-1$, and the sets $\{S'_i\}_{i \in I}$ are pairwise anticomplete. Let $S := \cup_{i \in I} S'_i$. Then $G[S \cup \{v\}]$ is an r-broadleaved s-star contradicting that G does not contain an r-broadleaved s-star as an induced subgraph. Hence, $\Delta_{h_{3.35}}(G) \leq f_{3.35}$. Thus, by Proposition 3.21, we have that $\chi_{h_{3.35}}(G) \leq f_{3.35} + 1$. This completes the proof of Theorem 3.35. □

Chapter 13

Excluding a bowtie

We recall that given two integers $s, t \geq 2$, an (s, t)-bowtie is a graph isomorphic to the graph we obtain from the disjoint union of a K_s and a K_t by adding a new vertex complete to everything else. Thus, bowties are exactly the broadleaved trees that can be obtained from a path of length two by substituting a complete graph on at least 2 vertices for each of the two leaves of the path. The main result of this chapter is that the Forbidden Broadleaved Forest Conjecture (Conjecture 3.20) holds with polynomial bounds for bowties. In particular, we prove Theorem 3.36 which we restate:

Theorem 3.36. *Let $s, t \geq 2$ and $\omega \geq 1$ be integers, with $s \leq t$, and let*

$$f_{3.36} = f_{3.36}(s,t) = (s+1)(t+3s^2-2s) + (t-1) \in \mathbb{N}$$

and

$$h_{3.36} = h_{3.36}(s,t,\omega) = \left\lceil \left\lceil \frac{\omega}{t-1} \right\rceil + e\omega^{s-1} + \omega^{s+1} \right\rceil \in \mathbb{N}.$$

Then, for every (s,t)-bowtie-free graph G with $\omega(G) = \omega$, we have: $\chi_{f_{3.36}}(G) \leq h_{3.36}$. In particular, the class of all (s,t)-bowtie-free graphs is polynomially $\chi_{f_{3.36}(s,t)}$-bounded, and thus strongly Pollyanna.

This chapter is based on ongoing joint work with Taite LaGrange, Mathieu Rundström, and Sophie Spirkl.

We begin the preparation for the proof of Theorem 3.36 with the following lemma, in which we show how to color a maximum clique and its neighborhood in a bowtie-free graph.

Lemma 13.1. *Let $s \geq 2$ and $t \geq 2$ be integers, with $s \leq t$. Let K be a maximum clique in an (s,t)-bowtie-free graph G. Then $\chi_t(G[N[K]]) \leq \lceil \frac{\omega(G)}{t} \rceil + e \cdot \omega(G)^{s-1} + \omega(G)^{s+1}$.*

Proof of Lemma 13.1. We may assume that $\omega(G) \geq t$, and thus $|K| \geq t$, since otherwise we have $\chi_t(G[N[K]]) = 1$.

For each $K' \subseteq K$ with $|K'| \leq s-1$, we denote by $A_{K'}$ the, possibly empty, subset of $N(K)$ that is complete to $K \setminus K'$ and anticomplete to K' in G.

For every s-subset $K' \subseteq K$ and for every $v \in K \setminus K'$ we denote by $B_{K',v}$ the set of all vertices in $N(K)$ that are adjacent to v and anticomplete to K'. We note that for every vertex $u \in N(K)$, either exists a set $K' \subseteq K$ with $|K'| \leq s-1$ such that $u \in A_{K'}$, or u has at least s non-neighbors in K, and thus there exist a vertex $v \in K$ and an s-subset K' of K such that $u \in B_{K',v}$. Thus, we have:

$$V(G[N[K]]) \subseteq K \cup \Big(\bigcup_{\substack{K' \subseteq K \\ |K'| \leq s-1}} A_{K'} \Big) \cup \Big(\bigcup_{\substack{K' \subseteq K \\ |K'| = s \\ v \in K}} B_{K',v} \Big).$$

Hence,

$$\chi_t(G[N[K]]) \subseteq \chi_t(G[K]) + \chi_t\Big(G\Big[\Big(\bigcup_{\substack{K' \subseteq K \\ |K'| \leq s-1}} A_{K'} \Big)\Big]\Big) + \chi_t\Big(G\Big[\Big(\bigcup_{\substack{K' \subseteq K \\ |K'| = s \\ v \in K}} B_{K',v} \Big)\Big]\Big). \tag{13.1}$$

We claim that for every $K' \subseteq K$ with $|K'| \leq s-1$ we have $\omega(G[A_{K'}]) < s$. Suppose not. Let $S \subseteq A_{K'}$ be an s-clique. Then, since S is complete to $K \setminus K'$, we have that the set $(K \setminus K') \cup S$ is a clique in G. Thus, $\omega(G) \geq |(K \setminus K') \cup S| \geq \omega(G) + 1$, which is a contradiction. Hence, for every $K' \in N_1$ we have $\omega(G[A_{K'}]) < s \leq t$. Thus, $\chi_t(G[A_{K'}]) \leq 1$, and hence:

$$\chi_t\Big(G\Big[\bigcup_{\substack{K' \subseteq K \\ |K'| \leq s-1}} A_{K'} \Big]\Big) \leq \sum_{\substack{K' \subseteq K \\ |K'| \leq s-1}} \chi_t(G[A_{K'}]) \leq \sum_{\substack{K' \subseteq K \\ |K'| \leq s-1}} 1 = \sum_{k=1}^{s-1} \binom{\omega(G)}{k} \leq e\omega(G)^{s-1}. \tag{13.2}$$

Let $K' \subseteq K$ be such that $|K'| = s$, and let $v \in K$. We claim that $G[B_{K',v}]$ is K_t-free. Suppose not. Let $S \subseteq B_{K',v}$ be a t-clique. Then $G[K' \cup \{v\} \cup S]$ is an (s,t)-bowtie in G, contradicting that G is (s,t)-bowtie-free. Thus, $\chi_t(G[B_{K',v}]) \leq 1$. Observe that $|\{B_{K,v} : K' \subseteq K; |K'| = s; v \in K\}| = \omega(G) \cdot \binom{\omega(G)-1}{s} \leq \omega(G) \cdot (\omega(G)-1)^s \leq \omega^{s+1}$. Thus,

$$\chi_t\Big(G\Big[\Big(\bigcup_{\substack{K' \subseteq K \\ |K'|=s \\ v \in K}} B_{K',v}\Big)\Big]\Big) \leq \sum_{\substack{K' \subseteq K \\ |K'|=s \\ v \in K}} \chi_t(G[B_{K',v}]) \leq \omega(G)^{s+1} \cdot 1 = \omega(G)^{s+1}. \tag{13.3}$$

Since, $\chi_t(G[K]) = \lceil \frac{\omega(G)}{t-1} \rceil$, by Equation 13.1, Equation 13.2 and Equation 13.3, we have:

$$\chi_t(G[N[K]]) \leq \lceil \frac{\omega(G)}{t-1} \rceil + e\omega(G)^{s-1} + \omega(G)^{s+1}.$$

This completes the proof of Lemma 13.1. □

As we discussed in Section 3.2, an important tool for the proof of Theorem 3.36 is Lemma 3.37 which we restate and prove below:

Lemma 3.37. *Let $s \geq 2$ and $t \geq 2$ be integers, let G be an (s,t)-bowtie-free graph, and let G' be the graph obtained by G after deleting every edge that does not belong to a $3s$-clique. Then G' is an $(s, t+3s^2-2s)$-bowtie-free graph.*

Proof of Lemma 3.37. Let us suppose towards a contradiction that G' contains an $(s, t + 3s^2 - 2s)$-bowtie H as an induced subgraph. Let C and C' be the cliques of size s and $t + 3s^2 - 2s$ of H respectively, and let x be the unique vertex of H which is complete to $C \cup C'$.

We claim that for every $v \in C$ we have $N_G(v) \cap C' \leq 3s - 2$. Indeed, we may assume that $N_G(v) \cap C' \neq \emptyset$. Let $u \in N_G(v) \cap C'$. We observe that the edge vu of G belongs to the clique $\{v\} \cup (N_G(v) \cap C')$. Suppose for contradiction that $|N_G(v) \cap C'| \geq 3s - 1$. Then $|\{v\} \cup (N_G(v) \cap C')| \geq 3s$. Thus, $vu \in E(G')$ which contradicts the fact that $vu \notin E(H)$ and H is an induced subgraph of G'. Hence, for every $v \in C$ we have $|N_G(v) \cap C'| \leq 3s - 2$.

Let $C'' := C' \setminus N(C)$. It follows by the above that $|C''| \geq (t + 3s^2 - 2s) - s(3s - 2) = t$. Thus, $G[C \cup \{x\} \cup C'']$ contains an (s,t)-bowtie, which contradicts the fact that G is an (s,t)-bowtie-free graph. □

We are now ready to prove Theorem 3.36 which we restate:

Theorem 3.36. *Let $s, t \geq 2$ and $\omega \geq 1$ be integers, with $s \leq t$, and let*

$$f_{3.36} = f_{3.36}(s,t) = (s+1)(t+3s^2-2s) + (t-1) \in \mathbb{N}$$

and

$$h_{3.36} = h_{3.36}(s,t,\omega) = \left\lceil \left\lceil \frac{\omega}{t-1} \right\rceil + e\omega^{s-1} + \omega^{s+1} \right\rceil \in \mathbb{N}.$$

Then, for every (s,t)-bowtie-free graph G with $\omega(G) = \omega$, we have: $\chi_{f_{3.36}}(G) \leq h_{3.36}$. In particular, the class of all (s,t)-bowtie-free graphs is polynomially $\chi_{f_{3.36}(s,t)}$-bounded, and thus strongly Pollyanna.

Proof of Theorem 3.36. Let G be a graph as in the statement of Theorem 3.36. We may assume that $\omega(G) \geq f_{3.36}$ since otherwise we have $\chi_{f_{3.36}}(G) = 1$, and we are done.

Let G' be the graph that we obtain from G after deleting every edge of G which is not contained in a $3s$-clique. Then, by Lemma 3.37, G' is $(s, t+3s^2-2s)$-bowtie-free. We remark that $\omega(G') = \omega(G) = \omega$.

We claim that every $K_{f_{3.36}}$-free coloring of G' is a $K_{f_{3.36}}$-free coloring of G. Indeed, let $\phi : V(G') \to \mathbb{N}$ be a $K_{f_{3.36}}$-free coloring of G', and let $\phi^{-1}(i)$ be a color class of ϕ. Suppose for contradiction that $G[\phi^{-1}(i)]$ contains an $f_{3.36}$-clique, say S. Since $G'[S]$ is not a clique, it follows that there exists an edge $\{u,v\} \subseteq S$ such that $uv \in E(G)$ and $uv \notin E(G')$. Since $uv \notin E(G')$, it follows that uv is not in a $3s$-clique of G contradicting the fact that $\{u,v\} \subseteq S$.

By the above, it follows that in order to prove the statement of the theorem it suffices to prove that $\chi_{f_{3.36}}(G') \leq h_{3.36}$. In what follows, we prove this by induction on $|G'|$.

For the basis of the induction, suppose that $|V(G')| < f_{3.36}$. Then, $\chi_{f_{3.36}}(G') = 1$. Let us suppose that $|V(G')| \geq f_{3.36}$, and that if H is an $(s, t+3s^2-2s)$-bowtie-free graph with $|V(H)| < |V(G')|$, such that every edge of H is contained in a $3s$-clique, then $\chi_{f_{3.36}}(H) \leq h_{3.36}(s,t,\omega(H))$.

Let K be an ω-clique in G', and let $R := V(G') \setminus N_{G'}[K]$. By Lemma 13.1, we have that $\chi_t(G'[N_{G'}[K]]) \leq h_{3.36}(s,t,\omega)$. Let $\phi_1 : N_{G'}[K] \to [h_{3.36}(s,t,\omega)]$ be a K_t-free coloring that witness this. By the induction hypothesis we have that $\chi_{f_{3.36}}(G'[R]) \leq h_{3.36}(s,t,\omega(G[R])) \leq h_{3.36}(s,t,\omega)$. Let $\phi_2 : R \to [h_{3.36}(s,t,\omega)]$ be a $K_{f_{3.36}}$-free coloring that witnesses $\chi_{f_{3.36}}(G'[R]) \leq h_{3.36}(s,t,\omega)$.

Let $\phi := \phi_1 \cup \phi_2$. We claim that ϕ is a $K_{f_{3.36}}$-free coloring of G'. Suppose not. Let C be a monochromatic clique of size $f_{3.36}$ in G'.

We claim that $C \cap K = \emptyset$. Suppose not. Then $C \subseteq N_{G'}[K]$ contradicting the definition of $\phi|_{N[K]} = \phi_1$. Thus, we have that $C \subseteq N_{G'}(K) \cup R$.

Since, by the definition of $\phi|_{N[K]} = \phi_1$, there is no monochromatic t-clique in $N_{G'}(K)$, we have that $|C \cap N_{G'}(K)| \leq t-1$ and hence we have:

$$|C \cap R| \geq f_{3.36}(s,t) - (t-1) = (s+1)(t+3s^2-2s) + (t-1) - (t-1) = (s+1)(t+3s^2-2s).$$

Since, by the definition of $\phi|_R = \phi_2$, R contains no monochromatic $f_{3.36}$-clique, it follows that $C \nsubseteq R$. Hence, $|C \cap N_{G'}(K)| \geq 1$.

Claim 13.1.1. *Let $x \in N_{G'}(K)$. If $N_{G'}(x) \cap R$ contains a $(t+3s^2-2s)$-clique, then $|N_{G'[K]}(x)| < s$.*

Proof of Claim 13.1.1. Let $x \in N_{G'}(K)$ and let S_1 be an $(t+3s^2-2s)$-clique in $N_{G'}(x) \cap R$. Let us suppose towards a contradiction that $|N_{G'[K]}(x)| \geq s$. Let S_2 be a subset of size s of $N_{G'[K]}(x)$. Then $G'[S_1 \cup \{x\} \cup S_2]$ is an $(s, t+3s^2-2s)$-bowtie in G' which contradicts the fact that G' is $(s, t+3s^2-2s)$-bowtie-free. This completes the proof of Claim 13.1.1. ■

Let $x \in C \cap N_{G'}(K)$, and let $y \in N_{G'[K]}(x)$. Recall that every edge of G' is contained in a $3s$-clique. Let Q be a $3s$-clique in G' $\{x,y\} \subseteq Q$. Let $Q' := Q \setminus K$. Since $|N_{G'[K]}(x)| < s$, we have $|Q \cap K| < s$. Thus, $|Q'| \geq 2s$. Let $Q_1 := \{v \in Q' : |N_{G'[C \cap R]}(v)| \geq t+3s^2-2s\}$ and let $Q_2 := Q' \setminus Q_1$.

We claim that $|Q_1| < s$. Suppose not. Then, by Claim 13.1.1, we have that each vertex in $Q_1 \subseteq C \cap N_{G'}(K)$ has at most $s-1$ neighbors in K. Let A_1 be an s-clique that is a subset of Q_1. Then $|N_{G'(K)}(A_1)| \leq s^2$. Thus, $|K \setminus N_{G'}(A_1)| \geq f_{3.36} - s^2 \geq t+3s^2-2s$. Let A_2 be a $(t+3s^2-2s)$-subset of $K \setminus N_{G'}(A_1)$. Then $G'[A_1 \cup \{y\} \cup A_2]$ is an $(s, t+3s^2-2s)$-bowtie in G' which contradicts the fact that G' is $(s, t+3s^2-2s)$-bowtie-free. Hence, $|Q_1| < s$. Thus, since $|Q'| \geq 2s$, we have $|Q_2| \geq s$.

Let X be an s-subset of Q_2. Now, by the definition of Q_2, we have $|N_{G'[C\cap R]}(X)| \leq s(t+3s^2-2s)$. Since $|C \cap R| \geq (s+1)(t+3s^2-2s)$ we have that:

$$\begin{aligned} |C \cap R| - |N_{G'[C\cap R]}(X)| &\geq (s+1)(t+3s^2-2s) - s(t+3s^2-2s) \\ &= (t+3s^2-2s). \end{aligned}$$

Let Y be a $(t+3s^2-2s)$-subset of $|C \cap R| - |N_{G'[C\cap R]}(X)|$. Then $G'[X \cup \{x\} \cup Y]$ is an $(s, t+3s^2-2s)$-bowtie in G' contradicting that G' is $(s, t+3s^2-2s)$-bowtie-free. Hence, there is no monochromatic $f_{3.36}$-clique in G'. Thus, ϕ is an $K_{f_{3.36}}$-free coloring of G. Since $\phi : V(G') \to [h_{3.36}(s,t,\omega)]$, it follows that $\chi_{f_{3.36}}(G') \leq h_{3.36}(s,t,\omega)$. This completes the induction on $|V(G')|$. This completes the proof of Theorem 3.36. □

Part IV

K_{r+1}-free segment graphs with large fractional K_r-free chromatic number

Chapter 14

On-line coloring games and their game graphs

In this chapter, we discuss a method for constructing graphs of bounded clique number, arbitrarily large K_r-free chromatic number, and additional desirable properties such as membership in a specific class and having a certain geometric representation. To our knowledge, this method was first observed by Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [103] (see also [88]) and was later formalized by Krawczyk and Walczak [89]. The method relies on a connection between "on-line graph coloring games" and the properties of colorings of their so-called "game graphs".

An *on-line coloring game* is a game which is played by two deterministic players the *Presenter* and the *Algorithm* in rounds. In each round, Presenter introduces a new vertex, and, for each of the vertices presented in the previous rounds (if any), declares whether or not this vertex is adjacent to the new vertex. Then, in the same round, Algorithm assigns a color to the new vertex.

Let G be a graph and let $\mathcal{C}$ be a family of nonempty sets. An injective function $\mu : V(G) \to \mathcal{C}$ is called a *representation of G in $\mathcal{C}$* if the edges of G are defined in terms of μ. For example, such a representation may be an intersection (or an overlap) model of G.

All the results of this chapter are included in or follow immediately from the paper [89] by Krawczyk and Walczak.

Variants of the on-line coloring game which we defined above may arise by imposing restrictions on Presenter and/or Algorithm. Bellow we mention some possible restrictions for Presenter and Algorithm that will be useful for our purposes:

(i) For each round $i \in [n]$ the graph that Presenter has built so far belongs to a specific class of graphs.
(ii) Presenter is required to build (round-by-round together with the graph G) a representation of G in a specific family of non-empty sets.
(iii) For a fixed finite set of positive integers X, we have that Algorithm is restricted to use "colors" only from the set X.

Let $\mathcal{C}$ be a family of non-empty sets, and let G be an on-line coloring game with a representation in $\mathcal{C}$, as in (ii) above. Let n be a positive integer, G be a graph, $\mu : V(G) \to \mathcal{C}$ be a function, and $\prec$ be a linear order on $V(G)$. If the graph G and the representation μ can occur in n rounds of the game G, with the vertices of G introduced by Presenter in the game according to $\prec$, then we say that $(G, \mu, \prec)$ is an *n-round presentation scenario* of G.

A *rooted forest* is a forest in which every connected component is a rooted tree. A graph G is a *game graph* of G if there exist: a rooted forest F on the vertex set $V(G)$, a representation $\mu : V(G) \to \mathcal{C}$, and a linear order $\prec$ on $V(G)$, such that the following hold:

(i) For each $v \in V(G)$, let P_v be the unique path in F from a root to v. Then

$$\left(G[V(P_v)],\, \mu\big|_{V(P_v)},\, \prec\big|_{V(P_v)}\right)$$

is a $|V(P_v)|$-round presentation scenario of G.
(ii) For every edge $uv \in E(G)$, either u is an ancestor of v or v is an ancestor of u in F.

Here is the first connection between an on-line coloring game and its game graphs that we would like to point out:

Observation 14.1 (Krawczyk and Walczak [89]). *Let G be an on-line coloring game, and let G be a game graph of G. It follows from condition (i) in the definition of game graphs that $\omega(G) = \max\{\omega(G[V(P_v)])\colon v \in V(G)\}$. In particular if one of the restrictions for Presenter in G is that the graph they build at each round should be K_r-free for some integer $r \geq 2$, then $\omega(G) \leq r$.*

Let G be an on-line coloring game and let Σ be a strategy of Presenter for G. Then, following [89], we say that Σ is a *finite strategy* if the total number of presentation scenarios that can occur in G when Presenter plays according to Σ is finite.

The following lemma, on which the proofs of the main results of this part rely heavily, gives the method that we mentioned in the introduction of this chapter for constructing graphs with bounded clique number and large K_r-free chromatic number. Krawczyk and Walczak [89] proved this lemma for on-line K_2-free coloring games but their proof works as is for K_r-free coloring games.

Lemma 14.2 (Krawczyk and Walczak [89, Lemma 2.2])**.** *Let $r \geq 2$ be an integer, let X be a finite set of positive integers, let G be an on-line coloring game in which Algorithm is restricted to use colors only from the set X, and suppose that Presenter has a finite strategy Σ such that for every presentation scenario S that can occur while Presenter plays according to Σ, Presenter can force Algorithm to create a monochromatic r-clique. Let $\mathcal{S}(\Sigma)$ be the set of all presentation scenarios of Σ. Then there exists a game graph G of G such that $\chi_r(G) > |X|$ and $|V(G)| = |\mathcal{S}(\Sigma)|$.*

Proof of Lemma 14.2. Since Σ is a finite strategy, we have that $\mathcal{S}(\Sigma)$ is a finite set. For each $S \in \mathcal{S}(\Sigma)$ let $v(S)$ be the last vertex which is presented by Presenter in the scenario S. Let $V := \{v(S) : S \in \mathcal{S}(\Sigma)\}$.

Let F be the graph on V such that for distinct presentation scenarios $S, S' \in \mathcal{S}(\Sigma)$ we have that $v(S)v(S') \in E(F)$ if and only if S is the presentation scenario that describes the situation (constructed graph, representation, and order in which the vertices are presented) in the scenario S' of G in the round exactly before the presentation of the vertex $v(S')$, or vice versa. It is easy to see that F is a forest, whose roots are exactly the presentation scenarios whose graph has only one vertex.

Let G be the graph on V such that for distinct presentation scenarios $S, S' \in \mathcal{S}(\Sigma)$ we have that $v(S)v(S') \in E(G)$ if and only if S in an ancestor of S' in F and $v(S)v(S')$ is an edge of the graph corresponding to the presentation scenario S', or vice versa.

Finally, we define $\mu : V(G) \to \mathcal{C}$ as follows: for every $S \in \mathcal{S}(\Sigma)$, $\mu(v(S)) := \mu_S(v(S))$, where $\mu_{S'}$ is the representation that Presenter defines in the scenario S'. From the above and the definition of game graphs, it follows that G is a game graph of G.

We claim that $\chi_r(G) > |X|$. Suppose not. Let $\phi : V(G) \to [|X|]$ be a K_r-free coloring of G. Consider the following strategy, call it A, of Algorithm against Presenter's strategy Σ for the game G. When a new vertex, say v, is presented, this is the last vertex of a presentation scenario S. Since Presenter plays according to Σ we have that $S \in \mathcal{S}(\Sigma)$. Thus, $v(S) \in V(G)$. Algorithm assigns to v the color $\phi(v_S)$. When Algorithm plays according to A in S, Algorithm produces a K_r-free coloring of the graph presented in S which uses colors only from the set $|X|$. This contradicts our assumptions for Σ. Hence, we have $\chi_r(G) > |X|$. This completes the proof of Lemma 14.2. $\square$

We will also need the following observation:

Observation 14.3 (Krawczyk and Walczak [89])**.** *Let* G *be an on-line coloring game. The game graphs of* G *only depend on the restrictions of Presenter.*

Chapter 15

K_{r+1}-free segment graphs of arbitrarily large K_r-free chromatic number

In this chapter, using the method that we discussed in Chapter 14, and building on the methods of Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [104], we prove Theorem 3.38 which we restate:

Theorem 3.38. *Let k and $r \geq 2$ be positive integers. Then there exists a K_{r+1}-free segment graph G, such that $\chi_r(G) \geq k$.*

15.1 An on-line K_r-free coloring game on K_{r+1}-free interval overlap graphs

Throughout this chapter we denote by $\mathcal{I}$ the set of all closed intervals of $\mathbb{R}$. Let $I \in \mathcal{I}$. We denote by $l(I)$ (respectively, by $r(I)$) the left end (respectively, the right end) of I, that is, $l(I) := \min\{x : x \in I\}$ and $r(I) := \max\{x : x \in I\}$. Let $\mathcal{J} \subseteq \mathcal{I}$ and let $x \in \mathbb{R}$. Then,

This chapter is based on ongoing joint work with Bartosz Walczak.

throughout this chapter, we denote by $\mathcal{J}(x)$ the set of intervals in $\mathcal{J}$ which contain the point x, that is $\mathcal{J}(x) := \{I \in \mathcal{J} : x \in I\}$.

Let r be a positive integer, and let X be a finite set of positive integers. Then we denote by $\mathrm{IOV}_r(X)$ the on-line coloring game, with the following properties:

(i) In every round of the game the graph G which has been built by the Presenter so far is a K_{r+1}-free interval overlap graph;
(ii) Presenter, in addition to the graph, builds an overlap model $\mu : V(G) \to \mathcal{I}$ of G;
(iii) If an interval I' is presented, by Presenter, in a later round than an interval I, then $l(I') > l(I)$; and
(iv) Algorithm is restricted to use "colors" only from the set X.

We denote by $\mathcal{G}(\mathrm{IOV}_r(X))$ the class of game graphs of $\mathrm{IOV}_r(X)$. It is clear, by the definition of game graphs and the definition of $\mathrm{IOV}_r(X)$, that the game graphs of $\mathrm{IOV}_r(X)$ only depend on the restriction (i) to Presenter.

In [103], Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak, considered the game $\mathrm{IOV}_2(X)$ without the restriction (iv) that we impose to Algorithm (to use colors only from a specific palette of colors). They imposed to Algorithm the restriction that the coloring that they create should remain K_2-free throughout the game. We denote this game by IOV_2. They characterized the game graphs of IOV_2, as follows:

Theorem 15.1 (Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [103])**.** *Let G be a game graph of* IOV_2*. Then G is a K_3-free rectangle overlap graph.*

Krawczyk, Pawlik, and Walczak [88] strengthened the above by showing:

Theorem 15.2 (Krawczyk, Pawlik, and Walczak [88])**.** *Let G be a game graph of* IOV_2*. Then G is a K_3-free directed rectangle overlap graph.*

As we discussed in Section 2.3, every directed rectangle overlap graph is a segment graph. We note that the only difference on the restrictions that Presenter has in the game IOV_2 compared to the game $\mathrm{IOV}_r(X)$ is the clique number of the graph that Presenter builds. The following theorem is an immediate corollary of Theorem 15.2, Observation 14.3 and Observation 14.1.

Theorem 15.3. *Let $r \geq 2$ be a positive integer, let X be a finite subset of $\mathbb{N}_+$, and let G be a game graph of $\mathrm{IOV}_r(X)$. Then G is a K_{r+1}-free directed rectangle overlap graph, and thus a K_{r+1}-free segment graph.*

It follows by Lemma 14.2 that to prove Theorem 3.38 it suffices to prove the following:

Theorem 15.4. *Let k and r be a positive integers, and let X be a k-subset of $\mathbb{N}_+$. In the game $\mathrm{IOV}_r(X)$, Presenter has a finite strategy $\Sigma(r, X)$ to force Algorithm to create a monochromatic r-clique.*

15.2 A Presenter's strategy for $\mathrm{IOV}_r(X)$

We deduce Theorem 15.4 as an immediate corollary of the following stronger statement, which facilitates an inductive proof.

Lemma 15.5. *Let $k, r \in \mathbb{N}_+$, let X be a k-subset of $\mathbb{N}_+$, and let R be a closed interval of $\mathbb{R}$ which has positive length. In the game $\mathrm{IOV}_r(X)$, Presenter has a finite strategy $\Sigma(r, X, R)$ to construct a family of intervals $\mathcal{J}$ together with a point $x \in R$, that satisfy the following conditions:*

(i) every interval of $\mathcal{J}$ is contained in the interior of R; and
(ii) Algorithm is forced to create a monochromatic r-clique on the graph $G[\mu^{-1}(\mathcal{J}(x))]$.

Proof of Lemma 15.5. We prove the statement of the lemma by induction on r. For the base case, let $r = 1$. Presenter presents an arbitrary but fixed interval I in the interior of R, and an arbitrary point $x \in I \subseteq R$. Then $\mathcal{J} := \{I\}$ and x satisfy the statements of the lemma.

Now let $r \geq 2$ and suppose that for every positive integer r' with $r' < r$ we have: for every positive integer k', for every k'-subset X' of $\mathbb{N}_+$, and for every closed interval R' of positive length, Presenter has a finite strategy $\Sigma(r', X', R')$ as in the statement of the lemma.

Claim 15.5.1. *Let $i \in \{1, \dots, k\}$ be an integer, and let R' be a closed interval of $\mathbb{R}$ which has positive length. Then Presenter has a finite strategy $\Sigma'(r, X, R', i)$ for $\mathrm{IOV}_r(X)$, to construct a family of intervals $\mathcal{J}_i$ with the property that every interval of $\mathcal{J}_i$ is contained in the interior of R', together with a point x_i in the interior of R', such that one of the following holds:*

(a) $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ contains at least i monochromatic $(r-1)$-cliques which are pairwise colored with different colors, and $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ contains no r-clique.
(b) $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ contains a monochromatic r-clique.

Proof of Claim 15.5.1. We prove the claim by induction on i. For $i = 1$, we define $\Sigma'(r, X, R', 1)$ to be the strategy $\Sigma(r-1, X, R')$. Strategy $\Sigma(r-1, X, R')$ gives outcome (a) of $\Sigma'(r, X, R', 1)$. Let $i \geq 2$ and suppose that the claim holds for smaller values of i.

We describe the strategy $\Sigma'(r, X, R', i)$. Presenter starts by playing according to $\Sigma'(r, X, R', i-1)$. If outcome (b) of $\Sigma'(r, X, R', i-1)$ occurs, then Presenter stops, and $\mathcal{J}_{i-1}$ and x_{i-1} are the desired set of intervals and point respectively. Thus, we may assume that outcome (a) of $\Sigma'(r, X, R', i-1)$ occurs. Let $Q_1, \dots, Q_{i-1}$ be $(i-1)$ monochromatic $(r-1)$-cliques of the graph $G[\mu^{-1}(\mathcal{J}_{i-1}(x_{i-1}))]$ which are pairwise colored with different colors. Let $Y \subseteq X$ be the set of colors of these cliques. Thus, $|Y| = i - 1$.

Let $r_{min} := \min\{r(I) : I \in J_{i-1}(x_{i-1})\}$, and let $R'' := [x_{i-1}, r_{min}]$. Observe that any interval, introduced by Presenter, that lies entirely in the interior of R'', is pairwise nested with every interval of the set $J_{i-1}(x_{i-1})$.

Sub-Claim 15.5.1.1. *Presenter can now create a family of intervals $\mathcal{J}^*$, all contained in the interior of R'', and a point $x^* \in R''$, such that one of the following hold:*

(1) Algorithm is forced to create a monochromatic $(r-1)$-clique in $G[\mu^{-1}(\mathcal{J}^*(x^*))]$ with a color from the set $X \setminus Y$, and the graph $G[\mu^{-1}(\mathcal{J}^*(x^*))]$ contains no r-clique.
(2) Algorithm is forced to create a monochromatic r-clique in $G[\mu^{-1}(\mathcal{J}^*(x^*))]$ with a color from the set X.

Proof of Sub-Claim 15.5.1.1. Note that, by the induction hypothesis, we have the strategy $\Sigma(r-1, X \setminus Y, R'')$. If Presenter could "simply" play this strategy, then we would get outcome (1). Unfortunately Presenter cannot "simply" play this strategy since in the game

$\text{IOV}_r(X)$ Algorithm can use any of the colors of the set X, and is not restricted, by any means, to use only colors from the set $X \setminus Y$. In what follows, we show that we can overcome this difficulty.

Algorithm 1 "Simulation of $\Sigma(r-1, X \setminus Y, R'')$" as part of the strategy $\Sigma'(r, X, R', i)$ when outcome (a) of $\Sigma'(r, X, R', i-1)$ occurs.

1: $\mathcal{L} \leftarrow \emptyset$.
2: $x_{\mathcal{L}} \leftarrow$ novalue.
3: $I, x_{\mathcal{L}} \leftarrow$ next interval and next point according to the strategy $\Sigma(r-1, X \setminus Y, R'')$ "applied" to the intervals of the set $\mathcal{L}$ (with the coloring, with colors from $X \setminus Y$, created by the Algorithm) and the point $x_{\mathcal{L}}$.
4: $l'' \leftarrow \max\{\{x_{i-1}\} \cup \{l(J) : J \in \mathcal{L} \text{ and } l(J) < l(I)\} \cup \{r(J) : J \in \mathcal{L} \text{ and } r(J) < l(I)\}\}$.
5: Presenter plays according to the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$. Let $\mathcal{A}$ and z be the resulting set of intervals and point, respectively.
6: **if** $G[\mu^{-1}(\mathcal{A}(z))]$ contains a monochromatic r-clique **then**
7:     **return** $\mathcal{A}$ and z ▷ *This is the second outcome of* Sub-Claim 15.5.1.1.
8: **else**
9:     Let $Q_1, \ldots, Q_{i-1}$ be distinctly monochromatic $(r-1)$-cliques in $G[\mu^{-1}(\mathcal{A}(z))]$.
10:     **if** $\exists i' \in [i-1]$, every vertex of $Q_{i'}$ is colored with a color $c \in X \setminus Y$ **then**
11:         **return** $\mathcal{A}$ and z ▷ *This is the first outcome of* Sub-Claim 15.5.1.1.
12:     **else**
13:         Presenter now introduces the interval $[z, r(I)]$.
14:         **if** Algorithm colors the interval $[z, r(I)]$ with a color $c \in Y$ **then**
15:             **return** $\mathcal{A}$ and z. ▷ *This is the second outcome of* Sub-Claim 15.5.1.1.
16:         **else**
17:             $\mathcal{L} \leftarrow \mathcal{L} \cup \{[z, r(I)]\}$
18:             **if** $\mathcal{L}(x_{\mathcal{L}})$ contains a monochromatic $(r-1)$-clique in a color from $X \setminus Y$ **then**
19:                 **return** $\mathcal{L}$ and $x_{\mathcal{L}}$. ▷ *This is the first outcome of* Sub-Claim 15.5.1.1.
20:             **else**
21:                 Go to line 3.

Presenter continues according to item 15.2. We observe that:

- By the choice of l'' in line 4 of item 15.2, we have that if an interval of the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$ (from line 5) does not contain z, then it does not overlap with any of the intervals presented outside of $\Sigma'(r, X, [l'', l(I)], i-1)$.

- if an interval, say I^*, of the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$ (from line 5) contains z, then the only interval of those presented outside of $\Sigma'(r, X, [l'', l(I)], i-1)$ in the game which may overlap with I^* is the interval which is introduced in line 13, and the interval in line 13 is introduced only if the graph $G[\mu^{-1}(\mathcal{A}(z))]$ is K_r-free. Thus, the graph $G[\mu^{-1}(\mathcal{A}(z))]$ remains K_{r+1}-free.
- Finally, we note that the interval overlap graph of $\mathcal{L}$ is isomorphic to the interval overlap graph that we get from the intervals that we compute for the strategy $\Sigma(r-1, X \setminus Y, R'')$ without "extending" them to the left as we do in line 17. This is because, by the choice of l'' and the fact that $z \geq l''$, the interval I (as in line 17) overlaps with the same subset of intervals of $\mathcal{L}$ as the interval $[z, r(I)]$.

The above observations show that item 15.2 is a valid (according to the rules of the game) way for Presenter to continue in $\Sigma'(r, X, R', i)$. This completes the proof of Sub-Claim 15.5.1.1.

■

The set of intervals $\mathcal{J}_i := \mathcal{J}_{i-1} \cup \mathcal{J}^*$ and the point $x_i := x^*$ (as in the statement of Sub-Claim 15.5.1.1) satisfy the requirements of $\Sigma'(r, X, R', i)$. This completes the induction on i. This completes the proof of Claim 15.5.1. ■

We describe the strategy $\Sigma(r, X, R)$. Let $\Sigma'(r, X, R, k)$, $\mathcal{J}_k$ and x_k, be as in the statement of Claim 15.5.1. Presenter plays according to $\Sigma'(r, X, R, k)$. If outcome (b) of $\Sigma'(r, X, R, k)$ occurs, then Presenter stops. In this case $\mathcal{J}_k$ and x_k is the desired set of intervals and point respectively. Suppose that outcome (a) of $\Sigma'(r, X, R, k)$ occurs. Presenter plays an interval I, such that $l(I) = x_k$, I is contained in R, and I overlaps with all the intervals of the set J_k. Let $\mathcal{J} := \mathcal{J}_k \cup \{I\}$. We note that since $G[\mu^{-1}(\mathcal{J}(x_k))]$ contains no r-clique, Presenter is allowed to present the interval I, that is, the graph remains K_{r+1}-free. Now, let $i \in X$ be the color that Algorithm assigns to I. Recall that $G[\mu^{-1}(\mathcal{J}_k(x_k))]$ contains monochromatic $(r-1)$-cliques in all colors of X. Let Q_i be the $(r-1)$-subset of $\mathcal{J}_k$ such that $G[\mu^{-1}(Q_i)]$ is an $(r-1)$-clique with all of its vertices colored with i by Algorithm. Then $G[\mu^{-1}(Q_i \cup \{I\})]$ is a monochromatic r-clique. Thus, the family of intervals $\mathcal{J}$ and the point x_k satisfy the requirements of the strategy $\Sigma(r, X, R)$. This completes the induction on r. This completes the proof of Lemma 15.5. □

Chapter 16

K_{r+1}-free segment graphs of bounded clique number and arbitrarily large fractional K_r-free chromatic number

The main result of this chapter is Theorem 3.39 which we restate:

Theorem 3.39. *Let k and $r \geq 2$ be positive integers. Then there exists a K_{r+1}-free segment graph G, such that $\chi_r^f(G) \geq k$.*

Similarly to our proof of Theorem 3.38 which states the existence of K_{r+1}-free segment graphs of arbitrarily large K_r-free chromatic number, we will deduce Theorem 3.39 as a corollary of the existence of an appropriate Presenter's strategy for an appropriate on-line game. We find it more convenient, for our purposes, to work with a measure-theoretic definition of the fractional K_r-free chromatic number. In Section 16.1, we introduce the measure assignments of graphs and the K_r-free measure assignment number, and we prove that the K_r-free measure assignment number of a graph equals to its fractional K_r-free chromatic number. In Section 16.2, we introduce the on-line measure assignment games. In Section 16.3, we define an on-line measure assignment game whose game graphs are directed rectangle overlap graphs, and discuss how the proof of Theorem 3.39 can be reduced to

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the proof of the existence of an appropriate Presenter's strategy for this on-line measure assignment game. In Section 16.4, we prove the existence of the desired Presenter's strategy, and thus we prove Theorem 3.39. Finally, in Section 16.5, we show how Theorem 3.39 implies a negative answer to the following question that we discussed in Section 2.4:

Question 15 (Fox, Pach, and Suk [59])**.** *Let $r \geq 4$ and n be integers. Is there a constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$?*

16.1 A measure-theoretic definition of fractional K_r-free chromatic number

In the rest of this chapter, we use Lebesgue measure and some related measure-theoretic notions (such as *measurable sets*). We refer the reader to [124] for the definitions of these measure-theoretic notions. Throughout this chapter, we denote by $\mathcal{M}$ the set of all Lebesgue measurable subsets of $\mathbb{R}$, and for each $A \in \mathcal{M}$ we denote by $\lambda(A)$ the Lebesgue measure of A.

Let G be a graph, and let ℓ be a positive integer. An *ℓ-wide measure assignment of G* is a function $f : V \to \mathcal{M}$ such that for every $v \in V(G)$ we have $\lambda(f(v)) \geq \ell$. A 1-wide measure assignment of G is simply called a *measure assignment of G*. Let $r \geq 2$ be an integer and let f be an ℓ-wide measure assignment of G. We say that f is a *K_r-free ℓ-wide measure assignment of G* if the following holds: for every r-clique Q of G we have $\bigcap_{v \in Q} f(v) = \emptyset$. A 1-wide K_r-free measure assignment of G is simply called a *K_r-free measure assignment of G*.

Observation 16.1. *Let G be a graph, let $r \geq 2, \ell$ be positive integers, and let $M \in \mathbb{R}_+$. Then there exists a K_r-free measure assignment $f : V(G) \to [0, M] \cap \mathcal{M}$ of G if and only if there exists an ℓ-wide K_r-free measure assignment $f' : V(G) \to [0, \ell \cdot M] \cap \mathcal{M}$ of G.*

Let $r \geq 2$ be an integer. The *K_r-free measure assignment number* of a graph G, which we denote by $\chi_r^\lambda(G)$, is defined as follows:

$$\chi_r^\lambda(G) := \min\{M \in \mathbb{R}_+ : \exists \text{ is a } K_r\text{-free measure assignment } f : V(G) \to [0, M] \cap \mathcal{M}\}.$$

Proposition 16.2. *Let $r \geq 2$ be an integer and let G be a graph. Then $\chi_r^f(G) = \chi_r^\lambda(G)$.*

Proof of Proposition 16.2. Let $\mathcal{F}$ be the set of all subsets of $V(G)$ which induce K_r-free subgraphs. Let $(\bar{x}_S)_{S\in\mathcal{F}}$ be an optimal solution for the following:

$$\min\{\sum_{S\in\mathcal{F}} x_S : \sum_{v\in S\in\mathcal{F}} x_S \geq 1 \text{ for all } v \in V(G); x_S \geq 0\}. \tag{16.1}$$

Then, $\chi_r^f(G) = \sum_{S\in\mathcal{F}} \bar{x}_S$. Let $M := \sum_{S\in\mathcal{F}} \bar{x}_S$. Let $\{S_1, \ldots, S_k\}$ be an enumeration of the elements of $\mathcal{F}$. For an open interval I in $\mathbb{R}$ we denote by $r(I)$ the right endpoint of its closure. Let $I_1 \subseteq \mathbb{R}$ be the open interval $(0, \bar{x}_{S_1})$. For each $i \in \{2, \ldots, k\}$, let I_i be the open interval $(r(I_{i-1}), r(I_{i-1}) + \bar{x}_{S_i})$. Observe that $r(I_k) = M$. Let $f : V(G) \to [0, M] \cap \mathcal{M}$ be defined as follows: $f(v) = \cup\{I_{S_i} : v \in S_i\}$.

Claim 16.2.1. $\chi_r^\lambda(G) \leq \chi_r^f(G)$.

Proof of Claim 16.2.1. It suffices to prove that f is a K_r-free measure assignment of G, then we will have $\chi_r^\lambda(G) \leq M = \chi_r^f(G)$.

First, we note that each I_i is a Lebesgue measurable set, with measure: $\lambda(I_i) = |I_i| = \bar{x}_{S_i}$, and the sets $I_1, \ldots, I_k$ are pairwise disjoint. Hence for each $v \in V(G)$ we have that the set $f(v) = \bigcup_{\{i:v\in S_i\}} I_i$ which is assigned to v by f is a finite union of disjoint measurable intervals, and so measurable, In particular, for every $v \in V(G)$, we have:

$$\lambda\Big(f(v)\Big) = \sum_{\{i:v\in S_i\}} \lambda(I_i) = \sum_{\{i:v\in S_i\}} |I_i| = \sum_{\{i:v\in S_i\}} \bar{x}_{S_i} \geq 1,$$

where the last inequality follows by the constrains of the linear-programming problem 16.1.

Let us suppose towards a contradiction that there exists an r-clique, say Q, such that $\bigcap_{v\in Q} f(v) \neq \emptyset$. Let $t \in \bigcap_{v\in Q} f(v)$. Since the sets $I_1, \ldots, I_k$ are pairwise disjoint, we have that $t \in I_j$ for exactly one index j. But then $v \in S_j$ for every $v \in Q$, so $Q \subseteq S_j$, contradicting the fact that $G[S_j]$ is K_r-free. Hence, we have $\bigcap_{v\in Q} f(v) = \emptyset$. This completes the proof of Claim 16.2.1. ■

Let $M > 0$ be such that $\chi_r^\lambda(G) = M$ and let $f : V(G) \to [0, M] \cap \mathcal{M}$ be a K_r-free measure assignment of G which witnesses this. For every $S \in \mathcal{F}$ let

$$A_S := \Big(\cap_{v \in S} f(v) \Big) \cap \Big(\cap_{v \notin S} ([0, M] \setminus f(v) \Big)$$

and let

$$x_S^* = \lambda(A_S).$$

Claim 16.2.2. $(x_S^*)_{S \in \mathcal{F}}$ *is a feasible solution for the linear-programming problem 16.1, and that* $\sum_{S \in \mathcal{F}} x_S^* = M$*. Thus,* $\chi_r^f(G) \le \chi_r^\lambda(G)$*.*

Proof of Claim 16.2.2. For every $S \in \mathcal{F}$ the set A_S is a finite intersection of measurable sets and thus a measurable set. Hence, for every $S \in \mathcal{F}$ we have $x_S^* = \lambda(A_S) \ge 0$.

For every $t \in M$, let $S(t) := \{v \in V(G) : t \in f(v)\}$. We claim that for every $t \in M$ the set S_t is K_r-free. Suppose not. Let Q be an r-clique in $G[S(t)]$. Then, $t \in \cap_{v \in Q} f(v)$, contradicting the fact that f is K_r-free measure assignment. Thus, for every $t \in M$ the set S_t is K_r-free.

We claim that $\{A_S : S \in \mathcal{F}\}$ is a partition of $[0, M]$. Indeed, this follows by the fact that for every $t \in M$ we have that $t \in A_S$ if and only if $S = S(t)$. Thus, the $\{A_S : S \in \mathcal{F}\}$ is a family of pairwise disjoint non-empty subsets of $[0, M]$ and since, by the above, we have that for every t the set $S(t)$ is K_r-free, it follow that for every t there exists we have $A_{S(t)} \in \mathcal{F}$. Thus, $\{A_S : S \in \mathcal{F}\}$ is a partition of $[0, M]$.

Since, $\{A_S : S \in \mathcal{F}\}$ is a partition of $[0, M]$, it follows that $\lambda(\cup_{S \in \mathcal{F}} A_S) = \lambda([0, M]) = M$.

Let $v \in V(G)$. We have:

$$\sum_{S \ni v} x_S^* = \sum_{S \ni v} \lambda(A_S) = \lambda\Big(\bigcup_{S \ni v} \lambda(A_S) \Big) = \lambda\Big(\{ t \in [0, M] : v \in S(t) \} \Big) = \lambda\Big(f(v) \Big) \ge 1,$$

where the last inequality follows from the definition of K_r-free measure assignment. This completes the proof of Claim 16.2.2. ■

This completes the proof of Proposition 16.2 □

16.2 On-line measure assignment games

An *on-line measure assignment game* is a game which is played by two deterministic players the Presenter and the Algorithm in rounds. In each round, Presenter introduces a new vertex, and, for each of the vertices presented in the previous rounds (if any), declares whether or not this vertex is adjacent to the new vertex. Then, in the same round, Algorithm assigns a Lebesgue measurable subset to the new vertex.

We note that, similarly with on-line coloring games, variants of the on-line measure assignment game which we defined above may arise by imposing restrictions on Presenter and/or Algorithm. In the following we mention some possible restrictions for Presenter and Algorithm that will be useful for our purposes:

(i) For each round $i \in [n]$ the graph that Presenter has built so far belongs to a specific class of graphs.
(ii) Presenter is required to build (round-by-round together with the graph G) a representation of G in a specific family of non-empty sets.
(iii) For some positive integer ℓ the measure assignment that Algorithm creates is required to be ℓ-wide.
(iv) For a fixed set $X \in \mathcal{M}$, Algorithm is restricted to use, for the measure assignment that they construct, only measurable subsets of X.

The notions of presentation scenario, game graphs, and finite strategy of Presenter are defined as in Chapter 14. The following lemma is the analogue of Lemma 14.2 for on-line measure assignment games. We omit its proof as it is essentially identical to the proof of Lemma 16.3.

Lemma 16.3. *Let $r \geq 2$ and $\ell \geq 1$ be integers, let M be a positive real number, let G be an on-line measure assignment game in which Algorithm is restricted to create an ℓ-wide measure assignment using only measurable subsets of $[0, M]$, and suppose that Presenter has a finite strategy Σ such that for every presentation scenario S that can occur while Presenter plays according to Σ, Presenter can force Algorithm to create an r-clique Q such that $\lambda(\bigcap_{v \in Q} f(v)) > 0$, where f is the measure assignment created by Algorithm. Let $\mathcal{S}(\Sigma)$ be the set of all presentation scenarios of Σ. Then there exists a game graph G of G such that $\chi_r^\lambda(G) > \frac{M}{\ell}$ and $|V(G)| = |\mathcal{S}(\Sigma)|$.*

16.3 An on-line measure assignment game on K_{r+1}-free interval overlap graphs

Throughout this chapter, we denote by $\mathcal{I}$ the set of all closed intervals of $\mathbb{R}$. Let $I \in \mathcal{I}$. We denote by $l(I)$ (respectively, by $r(I)$) the left end (respectively, the right end) of I, that is, $l(I) := \min\{x : x \in I\}$ and $r(I) := \max\{x : x \in I\}$. Let $\mathcal{J} \subseteq \mathcal{I}$ and let $x \in \mathbb{R}$. Then, throughout this chapter, we denote by $\mathcal{J}(x)$ the set of intervals in $\mathcal{J}$ that contain the point x, that is, $\mathcal{J}(x) := \{I \in \mathcal{J} : x \in I\}$.

Let $r \in \mathbb{N}_+$, let $M \in \mathbb{R}_+$, and let $X \in [0, M] \cap \mathcal{M}$. Then, we denote by $\mathrm{FIOV}_r(X)$, the on-line measure assignment game, with the following properties:

(i) In every round of the game the graph G which has been built by the Presenter so far is a K_{r+1}-free interval overlap graph;
(ii) Presenter, in addition to the graph, builds an overlap model $\mu : V(G) \to \mathcal{I}$ of G;
(iii) If an interval I' is presented, by Presenter, in a subsequent round than an interval I, then $l(I') > l(I)$; and
(iv) Algorithm creates a $(2r-1)$-wide measure assignment which assigns to every vertex of G a measurable set from X.

The following is an immediate corollary of Observation 14.3 and Theorem 15.3:

Theorem 16.4. *Let $r \in \mathbb{N}_+$, let $M \in \mathbb{R}_+$, let $X \in [0, M] \cap \mathcal{M}$, and let G be a game graph of* $\mathrm{FIOV}_r(X)$. *Then, G is a K_{r+1}-free directed rectangle overlap graph, and thus a K_{r+1}-free segment graph.*

Recall that our goal in this chapter is to prove the following:

Theorem 3.39. *Let k and $r \geq 2$ be positive integers. Then there exists a K_{r+1}-free segment graph G, such that $\chi_r^f(G) \geq k$.*

It follows by Lemma 16.3 and Proposition 16.2, and Observation 16.1, that to prove Theorem 3.39 it suffices to prove the following:

Theorem 16.5. *Let $r \in \mathbb{N}_+$, let $M \in \mathbb{R}_+$, and let $X \in [0, M] \cap \mathcal{M}$. In the game* $\mathrm{FIOV}_r(X)$, *Presenter has a finite strategy $\Sigma(r, X)$ to force Algorithm to create an r-clique Q such that $\lambda(\bigcap_{v \in Q} f(v)) > 0$.*

16.4 A Presenter's strategy for $\mathrm{FIOV}_r(X)$

We deduce Theorem 16.5 as an immediate corollary of the following stronger statement, which facilitates an inductive proof.

Lemma 16.6. *Let $r \in \mathbb{N}_+$, let $M \in \mathbb{R}_+$, let $X \in [0, M] \cap \mathcal{M}$, and let R be a closed interval of $\mathbb{R}$ of positive length. In the game $\mathrm{FIOV}_r(X)$, Presenter has a finite strategy $\Sigma(r, X, R)$ to construct a family of intervals $\mathcal{J}$ together with a point $x \in R$, that satisfy the following conditions:*

(i) every interval of $\mathcal{J}$ is contained in the interior of R; and
(ii) There exists a set $\mathcal{Q}$ of r-cliques in $G[\mu^{-1}(\mathcal{J}(x))]$ such that $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq 1$.

Proof of Lemma 16.6. We prove the statement of the lemma by induction on r. For the base case $r = 1$, Presenter presents an arbitrary but fixed interval I in the interior of R, and an arbitrary point $x \in I \subseteq R$. Then the set of intervals $\mathcal{J} := \{I\}$, the point x, and the set $\mathcal{Q} := \{\{\mu^{-1}(I)\}\}$ satisfy the statement of the lemma. To see this, observe that by the rules of the game $\mathrm{FIOV}_1(X)$ the measure assignment f which is created by Algorithm is required to be a $(2 \cdot 1 - 1)$-wide measure assignment. Thus we have $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) = \lambda(f(\mu^{-1}(I))) \geq 2r - 1 = 1$.

Now let $r \geq 2$ and suppose that for every positive integer r' with $r' < r$, we have: for every $X' \in \mathcal{M}$ such that $\lambda(X') > 0$, and for every closed interval R' of positive length Presenter has a finite strategy $\Sigma(r', X', R')$ as in the statement of the lemma.

Claim 16.6.1. *Let $i \in \{1, \ldots, \lceil\lambda(X)\rceil + 1\}$ be an integer, and let R' be a closed interval of $\mathbb{R}$ which has positive length. Then Presenter has a finite strategy $\Sigma'(r, X, R', i)$ for $\mathrm{FIOV}_r(X)$, to construct a family of intervals $\mathcal{J}_i$ with the property that every interval of $\mathcal{J}_i$ is contained in the interior of R', together with a point x_i in the interior of R', such one of the following holds:*

(a) $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ contains a set $\mathcal{Q}$ of $(r-1)$-cliques such that $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq i$, and $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ contains no r-clique.
(b) There exists a set $\mathcal{Q}$ of r-cliques in $G[\mu^{-1}(\mathcal{J}_i(x_i))]$ such that $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq 1$.

Proof of Claim 16.6.1. We prove the claim by induction on i. For $i = 1$, we define $\Sigma'(r, X, R', 1)$ to be the strategy $\Sigma(r-1, X, R')$. Let $i \geq 2$ and suppose that the claim holds for smaller values of i.

We describe the strategy $\Sigma'(r, X, R', i)$. Presenter starts by playing according to $\Sigma'(r, X, R', i-1)$. If outcome (b) of $\Sigma'(r, X, R', i-1)$ occurs, then Presenter stops, and $\mathcal{J}_{i-1}$ and x_{i-1} are the desired set of intervals and point respectively. Thus, we may assume that outcome (a) of $\Sigma'(r, X, R', i-1)$ occurs. Let $\mathcal{Q}_{i-1}$ be a set of $(r-1)$-cliques in the graph $G[\mu^{-1}(\mathcal{J}_{i-1}(x_{i-1}))]$ such that $\lambda(\bigcup_{Q \in \mathcal{Q}_{i-1}} \bigcap_{v \in Q} f(v)) \geq i-1$. Let $Y := \bigcup_{Q \in \mathcal{Q}_{i-1}} \bigcap_{v \in Q} f(v) \subseteq X$. Thus, $\lambda(Y) \geq i-1$.

Let $r_{min} := \min\{r(I) : I \in J_{i-1}(x_{i-1})\}$, and let $R'' := [x_{i-1}, r_{min}]$. Observe that any interval, introduced by Presenter, that lies entirely in the interior of R'', is pairwise nested with every interval of the set $J_{i-1}(x_{i-1})$.

Sub-Claim 16.6.1.1. *Presenter can now create a family of intervals $\mathcal{J}^*$, all contained in the interior of R'', and a point $x^* \in R''$, such that one of the following hold:*

(1) There exists a set $\mathcal{Q}$ of $(r-1)$-cliques in $G[\mu^{-1}(\mathcal{J}(x))]$ such that for every $Q \in \mathcal{Q}$ and for every $v \in Q$ we have that $f(v) \subseteq X \setminus Y$; $\lambda(\bigcup_{Q \in \mathcal{Q}} \bigcap_{v \in Q} f(v)) \geq 1$; and the graph $G[\mu^{-1}(\mathcal{J}^*(x^*))]$ contains no r-clique.
(2) There exists a set $\mathcal{Q}$ of $(r-1)$-cliques in $G[\mu^{-1}(\mathcal{J}(x))]$ such that $\lambda(\bigcup_{Q \in \mathcal{Q}} \bigcap_{v \in Q} f(v)) \geq i$, and $G[\mu^{-1}(\mathcal{J}(x))]$ contains no r-clique.
(3) There exists a set $\mathcal{Q}$ of r-cliques in $G[\mu^{-1}(\mathcal{J}(x))]$ such that $\lambda(\bigcup_{Q \in \mathcal{Q}} \bigcap_{v \in Q} f(v)) \geq 1$.

Proof of Sub-Claim 16.6.1.1. Note that, by the induction hypothesis, we have the strategy $\Sigma(r-1, X \setminus Y, R'')$. If Presenter could "simply" play this strategy, then we would get outcome (1). Unfortunately, Presenter cannot "simply" play this strategy since in the game $\text{FIOV}_r(X)$ Algorithm can use any measurable subset of X, and is not restricted, by any means, to use only subsets of the set $X \setminus Y$ for the measure assignment that they construct. Below, we show that we can overcome this difficulty.

Presenter continues according to Algorithm 2. We observe that:

- By the choice of l'' in line 4 of Algorithm 2, we have that if an interval of the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$ (from line 5) does not contain z, then it does not overlap with any of the intervals of presented outside of $\Sigma'(r, X, [l'', l(I)], i-1)$.

Algorithm 2 "Simulation of $\Sigma(r-1, X \setminus Y, R'')$" as part of the strategy $\Sigma'(r, X, R', i)$ when outcome (a) of $\Sigma'(r, X, R', i-1)$ occurs.

1: $\mathcal{L} \leftarrow \emptyset$.
2: $x_{\mathcal{L}} \leftarrow \mathsf{novalue}$.
3: $I, x_{\mathcal{L}} \leftarrow$ next interval and next point according to the strategy $\Sigma(r-1, X \setminus Y, R'')$ "applied" to the intervals of the set $\mathcal{L}$ (with the $(2(r-1)-1)$-wide measure assignment $f' := f - Y$), and the point $x_{\mathcal{L}}$.
4: $l'' \leftarrow \max\{\{x_{i-1}\} \cup \{l(J) : J \in \mathcal{L} \text{ and } l(J) < l(I)\} \cup \{r(J) : J \in \mathcal{L} \text{ and } r(J) < l(I)\}\}$.
5: Presenter plays according to the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$. Let $\mathcal{A}$ and z be the resulting set of intervals and point, respectively.
6: **if** $G[\mu^{-1}(\mathcal{A}(z))]$ contains a set $\mathcal{Q}$ of r-cliques such that $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq 1$ **then**
7:     **return** $\mathcal{A}$ and z ▷ *This is the third outcome of* Sub-Claim 16.6.1.1*.*
8: **else**
9:     Let $\mathcal{Q}$ be a set of $(r-1)$-cliques in $G[\mu^{-1}(\mathcal{A}(z))]$ such that $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq i-1$.
10:     $Z \leftarrow \bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)$.
11:     **if** $\lambda(Y \cup Z) \geq i$ **then**
12:         **return** $\mathcal{A} \cup \mathcal{J}_{i-1}$ and z ▷ *This is the second outcome of* Sub-Claim 16.6.1.1*.*
13:     **else**
14:         Presenter now introduces the interval $[z, r(I)]$.
15:         **if** $\lambda(f(\mu^{-1}([z, r(I)])) \cap Z) \geq 1$ **then**
16:             **return** $\mathcal{A}$ and z. ▷ *This is the third outcome of* Sub-Claim 16.6.1.1*.*
17:         **else**
18:             $\mathcal{L} \leftarrow \mathcal{L} \cup \{[z, r(I)]\}$
19:             **if** there exists a set $\mathcal{Q}$ of $(r-1)$-cliques in $G[\mu^{-1}(\mathcal{L}(x_{\mathcal{L}}))]$ such that for every $Q \in \mathcal{Q}$ and for every $v \in Q$ we have that $f(v) \subseteq X \setminus Y$ and $\lambda(\bigcup_{Q\in\mathcal{Q}} \bigcap_{v\in Q} f(v)) \geq 1$ **then**
20:                 **return** $\mathcal{L}$ and $x_{\mathcal{L}}$. ▷ *This is the first outcome of* Sub-Claim 16.6.1.1*.*
21:             **else**
22:                 Go to line 3.

- If an interval, say I^*, of the strategy $\Sigma'(r, X, [l'', l(I)], i-1)$ (from line 5) contains z, then the only interval of those presented outside of $\Sigma'(r, X, [l'', l(I)], i-1)$ in the game which may overlap with I^* is the interval which is introduced in line 14; and the interval in line 14 is introduced only if the graph $G[\mu^{-1}(\mathcal{A}(z))]$ is K_r-free. Thus, the graph that Presenter builds remain K_{r+1}-free throughout the game.
- Finally, we note that, throughout the game, the interval overlap graph of $\mathcal{L}$ is isomorphic to the interval overlap graph that we get from the intervals that we compute for the strategy $\Sigma(r-1, X \setminus Y, R'')$ without "extending" them to the left as we do in line 18. This is because, by the choice of l'' and the fact that $z \geq l''$, the interval I (as in line 3) overlaps with the same subset of intervals of $\mathcal{L}$ as the interval $[z, r(I)]$.
- We justify line 16: Let $v := \mu^{-1}([z, r(I)])$. Let $\mathcal{Q}' := \{Q \cup \{v\} : Q \in \mathcal{Q}\}$. Then, $\mathcal{Q}'$ is a set of r-cliques, all contained in the graph $G[\mu^{-1}(\mathcal{A}(z))]$. Furthermore,

$$\lambda\Big(\bigcup_{Q \in \mathcal{Q}'} \bigcap_{u \in Q} f(u)\Big) = \lambda\Big(\bigcup_{Q \in \mathcal{Q}} \Big(\bigcap_{u \in Q} f(u) \cap f(v)\Big)\Big) =$$

$$\lambda\Big(\Big(\bigcup_{Q \in \mathcal{Q}} \bigcap_{u \in Q} f(u)\Big) \cap f(v)\Big) = \lambda(Z \cap f(v)) \geq 1.$$

- We prove that the function $f'(v) := f(v) \setminus Y$ defined on the vertices which correspond to the intervals of the set $\mathcal{L}$ is a $2(r-1)-1$-wide measure assignment. Whenever we add an interval to the set $\mathcal{L}$ in line 18 we have:

$$\lambda(f(\mu^{-1}([z, r(I)]) \cap Y) \leq \lambda(f(\mu^{-1}([z, r(I)]) \cap Z) + \lambda(Y \setminus Z) < 1 + 1 = 2.$$

  Thus, $\lambda(f(\mu^{-1}([z, r(I)]) \setminus Y) \geq 2(r-1) - 1$.

The above observations show that Algorithm 2 is a valid (according to the rules of the game) way for Presenter to continue in $\Sigma'(r, X, R', i)$. This completes the proof of Sub-Claim 16.6.1.1. ■

The set of intervals $\mathcal{J}_i := \mathcal{J}_{i-1} \cup \mathcal{J}^*$ and the point $x_i := x^*$ (as in the statement of Sub-Claim 16.6.1.1) satisfy the requirements of $\Sigma'(r, X, R', i)$. This completes the induction on i. This completes the proof of Claim 16.6.1. ■

Observe that the strategy $\Sigma'(r, X, R, \lceil\lambda(X)\rceil + 1)$ for the game $\text{FIOV}_r(X)$ is impossible to result to the first outcome of Claim 16.6.1 since no subset of X has Lebesgue measure greater than or equal to $\lambda(X)\rceil + 1$. Thus, the strategy $\Sigma'(r, X, R, \lceil\lambda(X)\rceil + 1)$ for the game $\text{FIOV}_r(X)$, results to the second outcome of Claim 16.6.1. That is, $\Sigma'(r, X, R, \lceil\lambda(X)\rceil + 1)$ for the game $\text{FIOV}_r(X)$ results to a family of intervals $\mathcal{J}$ together with a point $x \in R$, that satisfy the following conditions:

(i) every interval of $\mathcal{J}$ is contained in the interior of R; and
(ii) There exists a set $\mathcal{Q}$ of r-cliques in $G[\mu^{-1}(\mathcal{J}(x))]$ such that $\lambda(\bigcup_{Q\in\mathcal{Q}}\bigcap_{v\in Q} f(v)) \geq 1$.

We define $\Sigma(r, X, R) := \Sigma'(r, X, R', \lceil\lambda(X)\rceil + 1)$. This completes the induction on r. This completes the proof of Lemma 16.6. □

16.5 On linear K_r-free sets in K_{r+1}-free segment graphs

As we discussed in Section 3.3, our motivation to consider the fractional K_r-free chromatic number of segment graphs comes from Question 15, which we restate:

Question 15 (Fox, Pach, and Suk [59])**.** *Let $r \geq 4$ and n be integers. Is there a constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$?*

The main result of this section is the following:

Theorem 3.41. *Let $r \geq 2$ and n be integers. Then there exists no constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$.*

We will deduce Theorem 3.41 as an immediate corollary of Theorem 3.40 and the following:

Theorem 3.40. *Let $\mathcal{C}$ be the class of segment graphs, and let $r \geq 2$ be a positive integer. Then the following are equivalent:*

1. *There exists a constant $c_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\alpha_r(G) \geq c_r|V(G)|$.*
2. *There exists a constant $C_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\chi^f_r(G) \leq C_r$.*

The rest of this section is devoted to the proof Theorem 3.39.

To this end, we need to introduce some terminology. Let G be a graph, and let $v \in V(G)$. Let G' be the graph which is obtained from G by adding a new vertex v' which is complete to $N_G(v)$ and adjacent to no other vertex, that is, $G' := (V(G) \cup \{v'\}, (G) \cup \{v'u : u \in N_G(v)\})$. Then we call the vertex v' of G' a *non-adjacent clone* of v in G', and we say that G' is obtained from G be taking a non-adjacent clone of v. We say that a class $\mathcal{C}$ of graphs is *closed under taking non-adjacent clones* if for every for $G \in \mathcal{C}$ if G' is a graph which is obtained from G by taking non-adjacent clones of a vertex of G, then $G' \in \mathcal{C}$.

Proposition 16.7. *The class of segment graphs is closed under taking non-adjacent clones.*

Sketch of the proof of Proposition 16.7. Let G be a segment graph, let $v \in V(G)$, and let G' be a graph which is obtained from G by adding k non-adjacent clones of v. Consider the representation $\mathcal{F}$ of G with segments, and let z_v be the segment which corresponds to v. Let $\mathcal{F}'$ be the family of segments on the plane that we obtain by adding k new segments to $\mathcal{F}$, such that: these segments are pairwise disjoint; each of these has equal length with z_v; each of these is parallel to z_v; and each of these intersects a segment $z \in \mathcal{F}$ if and only if z_z intersects z. Then $\mathcal{F}'$ witnesses that G' is segment graph. □

It follows, by Proposition 16.7 and the fact that the class of segment graphs is closed under taking induced subgraphs, that in order to prove Theorem 3.40 it suffices to prove the following:

Theorem 16.8. *Let $\mathcal{C}$ be a class of graphs which is closed under induced subgraphs and under taking non-adjacent clones, and let $r \geq 2$ be an integer. Then the following are equivalent:*

1. *There exists a constant $c_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\alpha_r(G) \geq c_r|V(G)|$.*
2. *There exists a constant $C_r > 0$ such that for every graph $G \in \mathcal{C}$ we have $\chi_r^f(G) \leq C_r$.*

Proof of Theorem 16.8. We first prove that the second statement implies the first. Let $G \in \mathcal{C}$, and let $\mathcal{F}$ be the family of all subsets of $V(G)$ which induce K_r-free subgraphs. By linear programming duality, we have:

$$\chi_r^f(G) = \max\Big\{\sum_{v\in V(G)} y_v :\ y_v \geq 0,\ \text{and}\ \sum_{v\in S} y_v \leq 1,\ \forall\ S \in \mathcal{F}\Big\}.$$

Let

$$y_v \ := \ \frac{1}{\alpha_r(G)}, \quad \text{for all } v \in V(G).$$

Then clearly $y_v \geq 0$. Moreover, for every $S \in \mathcal{F}$ we have:

$$\sum_{v\in S} y_v = |S| \cdot \frac{1}{\alpha_r(G)} \ \leq \frac{\alpha_r(G)}{\alpha_r(G)} = 1,$$

since $|S| \leq \alpha_r(G)$. Thus, the dual constraints are satisfied. Let us call y the feasible solution that we defined above. The objective value of y is:

$$\sum_{v\in V(G)} y_v = |V(G)| \cdot \frac{1}{\alpha_r(G)} = \frac{|V(G)|}{\alpha_r(G)}.$$

Thus,

$$\chi_r^f(G) \geq \frac{|V(G)|}{\alpha_r(G)}.$$

Rearranging yields

$$\alpha_r(G) \geq \frac{|V(G)|}{\chi_r^f(G)} \geq \frac{|V(G)|}{C_r},$$

Let $c_r := 1/C_r$. Then $\alpha_r(G) \geq c_r|V(G)|$. Since, G was an arbitrary graph of the class $\mathcal{C}$, it follows that: for every $G \in \mathcal{C}$ we have $\alpha_r(G) \geq c_r|V(G)|$.

For the other direction. Let K be a positive integer such that $\frac{1}{K} < c_r$, and let us suppose towards a contradiction that there exists $G \in \mathcal{C}$ such that $\chi_r^f(G) > K$. Let $y = (y_v)_{v\in V(G)}$ be an optimal solution for the following:

$$\max\Big\{\sum_{v\in V(G)} y_v :\ y_v \geq 0,\ \text{and}\ \sum_{v\in S} y_v \leq 1,\ \forall\ S \in \mathcal{F}\Big\}.$$

We may assume that each y_v is rational. Then there exists an integer Z, and, for every $v \in V(G)$, a positive integer z_v, such that $y_v = \frac{z_v}{Z}$ for every $v \in V(G)$. Then, we have:

$$\sum_{v\in V(G)} z_v = Z \sum_{v\in V(G)} y_v = Z\chi_r^f(G) > ZK.$$

Let G' be the graph that we obtain from G as follows: for every vertex $v \in V(G)$ if $z_v > 0$ we add to G $z_v - 1$ non-adjacent clones of v, otherwise we delete v. Then, $G' \in \mathcal{C}$.

We claim that $\alpha_r(G') \leq Z$. Let $S' \subseteq G'$ be such that $G'[S']$ is K_r-free, and let S be the set of all vertices $v \in V(G)$ such that S' contains v or at least one clone of v. Then, $|S'| = \sum_{v \in S} z_v$. Observe that $G[S]$ is K_r-free. Then, by the feasibility of $(y_v)_{v \in V(G)}$, we have that: $\sum_{v \in S} y_v \leq 1$. Thus,

$$|S'| = \sum_{v \in S} z_v = Z \sum_{v \in S} y_v \leq Z \cdot 1 = Z.$$

This completes the proof that $\alpha_r(G') \leq Z$. For the number of vertices of the graph G', we have:

$$|V(G')| \geq \sum_{v \in V(G)} z_v = Z \sum_{v \in V(G)} y_v = Z\chi_r^f(G) > ZK.$$

Thus,

$$\frac{\alpha_r(G')}{|V(G')|} \leq \frac{Z}{ZK} = \frac{1}{K} < c_r,$$

Hence, we have: $\alpha_r(G') < c_r|V(G)|$, which is a contradiction. This completes the proof of Theorem 16.8. □

Theorem 3.41, which we restate below, is an immediate corollary of Theorem 3.40 and Theorem 3.39.

Theorem 3.41. *Let $r \geq 2$ and n be integers. Then there exists no constant $c_r > 0$ such that every K_{r+1}-free segment graph on n vertices contains a K_r-free set of size $c_r n$.*

References

[1] Demetrios Achlioptas. "The complexity of G-free colourability". In: *Discrete Mathematics* 165 (1997), pp. 21–30.

[2] Eyal Ackerman. "On the maximum number of edges in topological graphs with no four pairwise crossing edges". In: *Proceedings of the twenty-second annual symposium on Computational geometry*. 2006, pp. 259–263.

[3] Sarosh Adenwalla, Samuel Braunfeld, John Sylvester, and Viktor Zamaraev. "Boolean combinations of graphs". In: *arXiv preprint arXiv:2412.19551* (2024).

[4] Pankaj K Agarwal, Boris Aronov, János Pach, Richard Pollack, and Micha Sharir. "Quasi-planar graphs have a linear number of edges". In: *Combinatorica* 17.1 (1997), pp. 1–9.

[5] Noga Alon, Daniel Kleitman, Carsten Thomassen, Michael Saks, and Paul Seymour. "Subgraphs of large connectivity and chromatic number in graphs of large chromatic number". In: *Journal of graph theory* 11.3 (1987), pp. 367–371.

[6] Béla Andrásfai, Paul Erdős, and Vera T Sós. "On the connection between chromatic number, maximal clique and minimal degree of a graph". In: *Discrete Mathematics* 8.3 (1974), pp. 205–218.

[7] Edgar Asplund and Branko Grünbaum. "On a Coloring Problem". In: *Mathematica Scandinavica* 8 (1960), pp. 181–188.

[8] Claude Berge. "Farbung von Graphen, deren samtliche bzw. deren ungerade Kreise starr sind". In: *Wissenschaftliche Zeitschrift* (1961).

[9] Claude Berge. "Les problemes de coloration en théorie des graphes". In: *Annales de l'ISUP*. Vol. 9. 2. 1960, pp. 123–160.

[10] Adam Bielecki. “Problèmes”. In: *Colloquium Mathematicae* 1.4 (1948), pp. 331–336.

[11] Marthe Bonamy, Nicolas Bousquet, Michal Piłipczuk, Pawel Rzazewski, Stéphan Thomassé, and Bartosz Walczak. “Degeneracy of P_t-free and $C_{\geq t}$-free graphs with no large complete bipartite subgraphs”. In: *Journal of Combinatorial Theory, Series B* 152 (2022), pp. 353–378.

[12] Marthe Bonamy and Michał Pilipczuk. “Graphs of bounded cliquewidth are polynomially χ-bounded”. In: *Advances in Combinatorics* (2020).

[13] Romain Bourneuf and Stéphan Thomassé. “Bounded twin-width graphs are polynomially χ-bounded”. In: *Advances in Combinatorics* (2025).

[14] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. “Graph classes: a survey”. SIAM, 1999.

[15] Stephan Brandt. “On the structure of graphs with bounded clique number”. In: *Combinatorica* 23.4 (2003), pp. 693–696.

[16] Marcin Briański, James Davies, and Bartosz Walczak. “Separating polynomial χ-boundedness from χ-bounde-dness”. In: *Combinatorica* (2023), pp. 1–8.

[17] Izak Broere and Marietjie Frick. “On the order of uniquely (k, m)-colourable graphs”. In: *Discrete mathematics* 82.3 (1990), pp. 225–232.

[18] Izak Broere and Marietjie Frick. “Two results on generalized chromatic numbers”. In: *Quaestiones Mathematicae* 13.2 (1990), pp. 183–190.

[19] Jason I Brown. “The complexity of generalized graph colorings”. In: *Discrete Applied Mathematics* 69.3 (1996), pp. 257–270.

[20] Peter Buneman. “A characterisation of rigid circuit graphs”. In: *Discrete Mathematics* 9.3 (1974), pp. 205–212.

[21] James Perkins Burling. “On coloring problems of families of prototypes”. PhD thesis. University of Colorado, Boulder, 1965, p. 66.

[22] Alvaro Carbonero, Patrick Hompe, Benjamin Moore, and Sophie Spirkl. “A counterexample to a conjecture about triangle-free induced subgraphs of graphs with large chromatic number”. In: *Journal of Combinatorial Theory, Series B* 158 (2023), pp. 63–69.

[23] Parinya Chalermsook and Bartosz Walczak. "Coloring and maximum weight independent set of rectangles". In: *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM. 2021, pp. 860–868.

[24] Aristotelis Chaniotis, Hidde Koerts, and Sophie Spirkl. "Intersections of graphs and χ-boundedness". In: *arXiv preprint arXiv:2504.00153* (2025).

[25] Aristotelis Chaniotis, Babak Miraftab, and Sophie Spirkl. "Graphs of bounded chordality". In: *The Electronic Journal of Combinatorics* 32.4 (2025), P4.7.

[26] Maria Chudnovsky. "The structure of bull-free graphs I—Three-edge-paths with centers and anticenters". In: *Journal of Combinatorial Theory, Series B* 102.1 (2012), pp. 233–251.

[27] Maria Chudnovsky. "The structure of bull-free graphs II and III—A summary". In: *Journal of Combinatorial Theory, Series B* 102.1 (2012), pp. 252–282.

[28] Maria Chudnovsky, Linda Cook, James Davies, and Sang-il Oum. "Reuniting χ-boundedness with polynomial χ-boundedness". In: *Journal of Combinatorial Theory, Series B* 176 (2026), pp. 30–73.

[29] Maria Chudnovsky, Irena Penev, Alex Scott, and Nicolas Trotignon. "Substitution and χ-boundedness". In: *Journal of Combinatorial Theory, Series B* 103.5 (2013), pp. 567–586.

[30] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. "The strong perfect graph theorem". In: *Annals of mathematics* (2006), pp. 51–229.

[31] Maria Chudnovsky, Alex Scott, and Paul Seymour. "Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings". In: *Journal of Combinatorial Theory, Series B* 150 (2021), pp. 195–243.

[32] Maria Chudnovsky, Alex Scott, and Paul Seymour. "Induced subgraphs of graphs with large chromatic number. XI. Orientations". In: *European Journal of Combinatorics* 76 (2019), pp. 53–61.

[33] Maria Chudnovsky, Alex Scott, and Paul Seymour. "Induced subgraphs of graphs with large chromatic number. XII. Distant stars". In: *Journal of Graph Theory* 92.3 (2019), pp. 237–254.

[34] Maria Chudnovsky, Alex Scott, Paul Seymour, and Sophie Spirkl. “Pure pairs. I. Trees and linear anticomplete pairs”. In: *Advances in Mathematics* 375 (2020), p. 107396.

[35] Maria Chudnovsky and Paul Seymour. “Even-hole-free graphs still have bisimplicial vertices”. In: *Journal of Combinatorial Theory, Series B* 161 (2023), pp. 331–381.

[36] Margaret Barry Cozzens and Fred S. Roberts. “On dimensional properties of graphs”. In: *Graphs and Combinatorics* 5.1 (1989), pp. 29–46.

[37] James Davies. “Box and segment intersection graphs with large girth and chromatic number”. In: *Advances in Combinatorics* (2021).

[38] James Davies. “Improved bounds for colouring circle graphs”. In: *Proceedings of the American Mathematical Society* 150.12 (2022), pp. 5121–5135.

[39] James Davies, Tomasz Krawczyk, Rose McCarty, and Bartosz Walczak. “Grounded L-graphs are polynomially χ-bounded”. In: *Discrete & Computational Geometry* 70.4 (2023), pp. 1523–1550.

[40] James Davies and Rose McCarty. “Circle graphs are quadratically χ-bounded”. In: *Bulletin of the London Mathematical Society* 53.3 (2021), pp. 673–679.

[41] James Davies and Yelena Yuditsky. “Polynomial Gyárfás-Sumner conjecture for graphs of bounded boxicity”. In: *arXiv preprint arXiv:2407.16882* (2024).

[42] Blanche Descartes. “A three colour problem”. In: *Eureka* 9 (1947), p. 21.

[43] Blanche Descartes. “Solution to advanced problem no. 4526”. In: *Amer. Math. Monthly* 61.352 (1954), p. 216.

[44] Vida Dujmovic, Gwenaël Joret, Pat Morin, Sergey Norin, and David R. Wood. “Orthogonal tree decompositions of graphs”. In: *SIAM Journal on Discrete Mathematics* 32.2 (2018), pp. 839–863.

[45] Zdeněk Dvořák and Daniel Král’. “Classes of graphs with small rank decompositions are χ-bounded”. In: *European Journal of Combinatorics* 33.4 (2012), pp. 679–683.

[46] Rodrigo Aparecido Enju. “Uma conjectura de Erdős e Hajnal”. Master’s dissertation. Universidade de São Paulo, 2023.

[47] Paul Erdős. “Graph theory and probability”. In: *Canadian Journal of Mathematics* 11 (1959), pp. 34–38.

[48] Paul Erdős. “Some remarks on the theory of graphs”. In: *Bull. Amer. Math. Soc.* 53.12 (1947), pp. 292–294.

[49] Paul Erdős and Tibor Gallai. “On the minimal number of vertices representing the edges of a graph”. In: *Publ. Math. Inst. Hungar. Acad. Sci* 6.18 (1961), pp. 1–203.

[50] Paul Erdős and András Hajnal. “On chromatic number of graphs and set-systems”. In: *Acta Math. Acad. Sci. Hungar* 17.61-99 (1966), p. 1.

[51] Paul Erdős and András Hajnal. “On chromatic number of infinite graphs”. In: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*. 1968, pp. 83–98.

[52] Paul Erdős and András Hajnal. “On spanned subgraphs of graphs”. In: *Graphentheorie und Ihre Anwendungen (Oberhof, 1977), www. renyi. hu/p erdos/1977-19. pdf* (1977).

[53] Paul Erdős and András Hajnal. “Ramsey-type theorems”. In: *Discrete Applied Mathematics* 25.1-2 (1989), pp. 37–52.

[54] Paul Erdős and C. Ambrose Rogers. “The construction of certain graphs”. In: *Canadian Journal of Mathematics* 14 (1962), pp. 702–707.

[55] Louis Esperet. “Graph colorings, flows and perfect matchings”. Habilitation Thesis. Université Grenoble Alpes, 2017.

[56] Martin Farber. “Characterizations of strongly chordal graphs”. In: *Discrete Mathematics* 43.2-3 (1983), pp. 173–189.

[57] Stefan Felsner, Gwenaël Joret, Piotr Micek, William T. Trotter, and Veit Wiechert. “Burling graphs, chromatic number, and orthogonal tree-decompositions”. In: *Electronic Notes in Discrete Mathematics* 61 (2017), pp. 415–420.

[58] Jon Folkman. “Graphs with monochromatic complete subgraphs in every edge coloring”. In: *SIAM Journal on Applied Mathematics* 18.1 (1970), pp. 19–24.

[59] Jacob Fox, János Pach, and Andrew Suk. “Quasiplanar graphs, string graphs, and the Erdős–Gallai problem”. In: *European Journal of Combinatorics* 119 (2024), p. 103811.

[60] Marietjie Frick. “A survey of (m, k)-colorings”. In: *Annals of Discrete Mathematics*. Vol. 55. Elsevier, 1993, pp. 45–57.

[61] Marietjie Frick. “Generalised colourings of graphs”. PhD thesis. University of Johannesburg (South Africa), 1986.

[62] Fănică Gavril. “The intersection graphs of subtrees in trees are exactly the chordal graphs”. In: *Journal of Combinatorial Theory, Series B* 16.1 (1974), pp. 47–56.

[63] António Girão, Freddie Illingworth, Emil Powierski, Michael Savery, Alex Scott, Youri Tamitegama, and Jane Tan. “Induced subgraphs of induced subgraphs of large chromatic number”. In: *Combinatorica* 44.1 (2024), pp. 37–62.

[64] António Girão and Bhargav Narayanan. “Subgraphs of large connectivity and chromatic number”. In: *Bulletin of the London Mathematical Society* 54.3 (2022), pp. 868–875.

[65] Lior Gishboliner, Oliver Janzer, and Benny Sudakov. “Induced subgraphs of K_r-Free graphs and the Erdős-Rogers problem”. In: *Combinatorica* 45.2 (2025), pp. 1–20.

[66] Martin Charles Golumbic. “Algorithmic graph theory and perfect graphs”. Elsevier, 2004.

[67] Martin Charles Golumbic. “Trivially perfect graphs”. In: *Discrete Mathematics* 24.1 (1978), pp. 105–107.

[68] Martin Grötschel, László Lovász, and Alexander Schrijver. “The ellipsoid method and its consequences in combinatorial optimization”. In: *Combinatorica* 1.2 (1981), pp. 169–197.

[69] András Gyárfás. “On Ramsey covering-numbers”. In: *Infinite and Finite Sets* 2 (1975), pp. 801–816.

[70] András Gyárfás. “On the chromatic number of multiple interval graphs and overlap graphs”. In: *Discrete mathematics* 55.2 (1985), pp. 161–166.

[71] András Gyárfás. “Problems from the world surrounding perfect graphs”. In: *Zastosowania Matematyki* 19 (1987). Proceedings of the International Conference on Combinatorial Analysis and its Applications, (Pokrzywna, 1985), pp. 413–441.

[72] András Gyárfás, Endre Szemeredi, and Zs Tuza. “Induced subtrees in graphs of large chromatic number”. In: *Discrete Mathematics* 30.3 (1980), pp. 235–244.

[73] Hugo Hadwiger. “Über eine klassifikation der streckenkomplexe”. In: *Vierteljschr. Naturforsch. Ges. Zürich* 88.2 (1943), pp. 133–142.

[74] Sepehr Hajebi. “Bull-free graphs and χ-boundedness”. In: *arXiv preprint arXiv:2504.21093* (2025).

[75] Frank Harary and Robert Z. Norman. “Some properties of line digraphs”. In: *Rendiconti del circolo matematico di palermo* 9 (1960), pp. 161–168.

[76] Charles C. Harner and Roger C. Entringer. “Arc colorings of digraphs”. In: *Journal of Combinatorial Theory, Series B* 13.3 (1972), pp. 219–225.

[77] Heinz Adolf Jung. “On a class of posets and the corresponding comparability graphs”. In: *Journal of Combinatorial Theory, Series B* 24.2 (1978), pp. 125–133.

[78] Richard M. Karp. “Reducibility among combinatorial problems”. In: *Complexity of Computer Computations* (1972), pp. 85–103.

[79] Michał Karpiński and Krzysztof Piecuch. “On vertex coloring without monochromatic triangles”. In: *International Computer Science Symposium in Russia*. Springer. 2018, pp. 220–231.

[80] John B Kelly and LM Kelly. “Paths and circuits in critical graphs”. In: *American Journal of Mathematics* 76.4 (1954), pp. 786–792.

[81] Hal A Kierstead and Stephen G Penrice. “Radius two trees specify χ-bounded classes”. In: *Journal of Graph Theory* 18.2 (1994), pp. 119–129.

[82] Hal A Kierstead and Yingxian Zhu. “Radius three trees in graphs with large chromatic number”. In: *SIAM Journal on Discrete Mathematics* 17.4 (2004), pp. 571–581.

[83] Hal A. Kierstead and V. Rodl. “Applications of hypergraph coloring to coloring graphs not inducing certain trees”. In: *Discrete Mathematics* 150.1-3 (1996), pp. 187–193.

[84] Henry A Kierstead and Stephen G Penrice. “Radius two trees specify χ-bounded classes”. In: *Journal of Graph Theory* 18.2 (1994), pp. 119–129.

[85] Alexandr Kostochka. “Coloring intersection graphs of geometric figures with a given clique number”. In: *Contemporary mathematics* 342 (2004), pp. 127–138.

[86] Tamás Kovári, Vera Sós, and Pál Turán. “On a problem of K. Zarankiewicz”. In: *Colloquium Mathematicum*. Vol. 3. 1. Polska Akademia Nauk. Instytut Matematyczny PAN. 1954, pp. 50–57.

[87] Jan Kratochvil and Zsolt Tuza. “Intersection dimensions of graph classes”. In: *Graphs and Combinatorics* 10.2-4 (1994), pp. 159–168.

[88] Tomasz Krawczyk, Arkadiusz Pawlik, and Bartosz Walczak. “Coloring triangle-free rectangle overlap graphs with $\mathcal{O}(\log \log n)$ colors”. In: *Discrete & Computational Geometry* 53 (2015), pp. 199–220.

[89] Tomasz Krawczyk and Bartosz Walczak. “On-line approach to off-line coloring problems on graphs with geometric representations”. In: *Combinatorica* 37.6 (2017), pp. 1139–1179.

[90] Chun-Hung Liu, Sergey Norin, and David R. Wood. “Product Structure and Tree-Decompositions”. In: *arXiv preprint arXiv:2410.20333* (2024).

[91] László Lovász. “On chromatic number of finite set-systems”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 19 (1968), pp. 59–67.

[92] Vadim Lozin and Igor Razgon. “Tree-width dichotomy”. In: *European Journal of Combinatorics* 103 (2022), p. 103517.

[93] Terry A. McKee and Edward R. Scheinerman. “On the chordality of a graph”. In: *Journal of graph theory* 17.2 (1993), pp. 221–232.

[94] Matthias Middendorf and Frank Pfeiffer. “The max clique problem in classes of string-graphs”. In: *Discrete mathematics* 108.1-3 (1992), pp. 365–372.

[95] Dhruv Mubayi and Jacques Verstraete. “On the order of Erdős-Rogers functions”. In: *arXiv preprint arXiv:2401.02548* (2024).

[96] Jan Mycielski. “Sur le coloriage des graphs”. In: *Colloquium Mathematicae*. Vol. 3. 2. 1955, pp. 161–162.

[97] Jaroslav Nešetřil. “A combinatorial classic—sparse graphs with high chromatic number”. In: *Erdős Centennial, Springer* (2013), pp. 383–407.

[98] Jaroslav Nešetřil. “Teorie grafů”. 1st. Praha: Státní Nakladatelství Technické Literatury, 1979.

[99] Jaroslav Nešetřil and Vojtěch Rödl. “A short proof of the existence of highly chromatic hypergraphs without short cycles”. In: *Journal of Combinatorial Theory, Series B* 27.2 (1979), pp. 225–227.

[100] Tung Nguyen, Alex Scott, and Paul Seymour. “A note on the Gyárfás–Sumner conjecture”. In: *Graphs and Combinatorics* 40.2 (2024), p. 33.

[101] Stephan Olariu. “Paw-free graphs”. In: *Information Processing Letters* 28.1 (1988), pp. 53–54.

[102] J Pach, F Shahrokhi, and M Szegedy. “Applications of the crossing number”. In: *Algorithmica* 16 (1996), pp. 111–117.

[103] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T Trotter, and Bartosz Walczak. “Triangle-free geometric intersection graphs with large chromatic number”. In: *Discrete & Computational Geometry* 50.3 (2013), pp. 714–726.

[104] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T Trotter, and Bartosz Walczak. “Triangle-free intersection graphs of line segments with large chromatic number”. In: *Journal of Combinatorial Theory, Series B* 105 (2014), pp. 6–10.

[105] Irena Penev. “Amalgams and χ-boundedness”. In: *Journal of Graph Theory* 84.1 (2017), pp. 57–92.

[106] Pegah Pournajafi. “A note on χ-unbounded classes of geometric graphs”. In: *arXiv preprint arXiv:2507.14884* (2025).

[107] Pegah Pournajafi and Nicolas Trotignon. “Burling graphs revisited, part I: New characterizations”. In: *European Journal of Combinatorics* 110 (2023), p. 103686.

[108] Richard Rado. “Covering theorems for ordered sets”. In: *Proceedings of the London Mathematical Society* 2.1 (1948), pp. 509–535.

[109] Frank P. Ramsey. “On a Problem of Formal Logic”. In: *Proceedings of the London Mathematical Society* 2.1 (1930), pp. 264–286.

[110] Fred S. Roberts. “On the boxicity and cubicity of a graph”. In: *Recent progress in combinatorics* 1.1 (1969), pp. 301–310.

[111] Neil Robertson and Paul D. Seymour. “Graph minors. II. Algorithmic aspects of tree-width”. In: *Journal of algorithms* 7.3 (1986), pp. 309–322.

[112] Neil Robertson and Paul D. Seymour. “Graph minors. V. Excluding a planar graph”. In: *Journal of Combinatorial Theory, Series B* 41.1 (1986), pp. 92–114.

[113] Neil Robertson and Paul D. Seymour. “Graph minors. XVI. Excluding a non-planar graph”. In: *Journal of Combinatorial Theory, Series B* 89.1 (2003), pp. 43–76.

[114] Arnoud Caspar Maria van Rooij and Herbert S. Wilf. “The interchange graph of a finite graph”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 16 (1965), pp. 263–269.

[115] Horst Sachs. “Finite Graphs (Investigations and Generalizations concerning the Construction of Finite Graphs Having Given Chromatic Number and No Triangles)”. In: *Recent Progress in Combinatorics, Proceedings of the Third Waterloo Conference on Combinatorics, May 1968*. 1968, pp. 175–184.

[116] Horst Sachs and Manfred Schäuble. “Konstruktion von Graphen mit gewissen Färbungseigenschaften”. In: *Beitr. Graphentheorie, Internat. Kolloq., Manebach*. 1967, pp. 131–136.

[117] Alex Scott. “Graphs of large chromatic number”. In: *Proceedings of the ICM*. 2022.

[118] Alex Scott and Paul Seymour. “A survey of χ-boundedness”. In: *Journal of Graph Theory* 95.3 (2020), pp. 473–504.

[119] Alex Scott and Paul Seymour. “Induced subgraphs of graphs with large chromatic number. XIII. New brooms”. In: *European Journal of Combinatorics* 84 (2020), p. 103024.

[120] Alex Scott, Paul Seymour, and Sophie Spirkl. “Polynomial bounds for chromatic number. I. Excluding a biclique and an induced tree”. In: *Journal of Graph Theory* 102.3 (2023), pp. 458–471.

[121] Alex D Scott. “Induced trees in graphs of large chromatic number”. In: *Journal of Graph Theory* 24.4 (1997), pp. 297–311.

[122] Paul Seymour. “How the proof of the strong perfect graph conjecture was found”. In: *Gazette des Mathematiciens* 109 (2006), pp. 69–83.

[123] Sophie Theresa Spirkl. “Cliques, stable sets, and coloring in graphs with forbidden induced subgraphs”. PhD thesis. Princeton University, 2018.

[124] Elias M Stein and Rami Shakarchi. “Real analysis: measure theory, integration, and Hilbert spaces”. Princeton University Press, 2009.

[125] David P. Sumner. “Subtrees of a graph and chromatic number”. In: *The Theory and Applications of Graphs, (G. Chartrand, ed.), John Wiley & Sons, New York* 557 (1981), p. 576.

[126] S Szabó, B Zaválnij, and S Arumugam. “Greedy algorithms for triangle free coloring”. In: *AKCE International Journal of Graphs and Combinatorics* 9.2 (2012), pp. 169–186.

[127] Carsten Thomassen. “2-list-coloring planar graphs without monochromatic triangles”. In: *Journal of Combinatorial Theory, Series B* 98.6 (2008), pp. 1337–1348.

[128] Paul Turán. “On an extremal problem in graph theory”. In: *Mat. Fiz. Lapok* 48 (1941), pp. 436–452.

[129] Vadim G. Vizing. “On an estimate of the chromatic class of a p-graph”. In: *Diskretnyi Analiz* 3 (1964), pp. 25–30.

[130] James R. Walter. “Representations of chordal graphs as subtrees of a tree”. In: *Journal of graph theory* 2.3 (1978), pp. 265–267.

[131] James R. Walter. “Representations of rigid cycle graphs”. English. PhD thesis. Wayne State University, 1972, p. 107.

[132] Douglas B. West. “Combinatorial mathematics”. Cambridge University Press, 2020.

[133] Elliot S. Wolk. “The comparability graph of a tree”. In: *Proceedings of the American Mathematical Society* 13.5 (1962), pp. 789–795.

[134] Jing-Ho Yan, Jer-Jeong Chen, and Gerard J. Chang. “Quasi-threshold graphs”. In: *Discrete applied mathematics* 69.3 (1996), pp. 247–255.

[135] Alexander Aleksandrovich Zykov. “On some properties of linear complexes”. In: *Matematicheskii sbornik* 66.2 (1949), pp. 163–188.

Glossary of special terms and notations

(T, s)-child, 117
(ζ, η)-uniform rooted r-broadleaved tree, 118
(ζ, η)-limb, 123
(k, K_r)-degenerate, 35
(k, K_r)-degenerate — (for a graph), 35
(k, l, C)-good, 106
(s, t)-bowtie, 40
(t, k, r)-decomposable — (for a graph), 30
G^i, 93
H-free — (for a graph), 2
H-free set, 24
K-happy, 130
K-unhappy, 130
K_r-free measure assignment of G, 157
K_r-free ℓ-wide measure assignment of G, 157
K_2-free chromatic number of G, 3
K_r-degeneracy — (of a graph), 35
K_r-degree — (of a vertex), 35
K_r-free chromatic number, 16
K_r-free coloring, 3
K_r-free measure assignment number, 157
K_t^+, 33
L-graph, 22