# Visualizing solutions to Dyson–Schwinger equations


Karen Yeats
Boston University



October 4, 2007
BU Mathematical Physics Seminar


## An equation

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

Important special cases

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$

$$2\gamma_1(x) = \left(\frac{x}{3} + \frac{x^2}{4}\right) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$$2\gamma_1(x) = \left(\frac{x}{3} + \frac{x^2}{4} + (-0.0312 + 0.06037)x^3 + (-0.6755 + 0.05074)x^4\right) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$$\gamma_1^+(x) = P^+(x) - \gamma_1^+(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^+(x)$$
$$\gamma_1^-(x) = P^-(x) - \gamma_1^-(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^-(x)$$



## Getting there

In the context of renormalization Hopf algebras consider

$$X(x) = \mathbb{I} - \sum_{k\geq 1} x^k p(k) B_+^k(X(x)Q(x)^k)$$

where $Q(x) = X(x)^r$ with $r < 0$ an integer. Associate with each $B_+$ a

$$F^k(\rho).$$

Write the combination ($X \mapsto G$, $B_+^k \mapsto F^k$, $\rho$ marks the insertion place) as $G(x, L) = \sum \gamma_k(x)L^k$ with $\gamma_k(x) = \sum_{j\geq k} \gamma_{k,j}x^j$.

Systems of equation are similar but messier.



## Extracting $\gamma_k(x)$

Writing

$$G(x, L) = \sum \gamma_k(x)L^k,$$

we know from Connes and Kreimer [2] that if

$$\sigma_1 = \partial_L \phi(S \star Y)|_{L=0}$$

and

$$\sigma_n = \frac{1}{n!} m^{n-1} \underbrace{(\sigma_1 \otimes \cdots \otimes \sigma_1)}_{n} \Delta^{n-1}$$

then

$$\gamma_k(x) = \sigma_k(X(x))$$

where $\phi$ is the renormalized Feynman rules, $m$ is multiplication, $S$ is the antipode, and $Y$ is the grading operator.

## The $\gamma_k(x)$ recursion

But $\sigma_1$ only sees the linear part of the Hopf algebra so we can use $\Delta_{\mathsf{lin}}$ in place of $\Delta$.

$$\Delta_{\mathsf{lin}}^{n-1} = \underbrace{(P_{\mathsf{lin}} \otimes \cdots \otimes P_{\mathsf{lin}})}_{n} \Delta^{n-1}$$

where $P_{\mathsf{lin}}$ projects onto the linear part of the Hopf algebra, that is, kills disjoint unions of graphs.

Calculate

$$\Delta_{\mathsf{lin}} X = P_{\mathsf{lin}} X \otimes P_{\mathsf{lin}} X + P_{\mathsf{lin}} Q \otimes x\partial_x X$$

where $P_{\mathsf{lin}} Q = -sP_{\mathsf{lin}} X$

So

$$\gamma_k(x) = \frac{1}{k}\gamma_1(x)(1 - sx\partial_x)\gamma_{k-1}(x),$$



## The messy $\gamma_1$ recursion

Rewrite the (analytic) Dyson-Schwinger equation using the usual tricks

- plug in $\sum \gamma_k L^k$
- use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
- switch the order of $\int$ and $\partial$

$$\gamma \cdot L = \sum p(k) x^k (1 + \gamma \cdot \partial_{-\rho})^{sk+1} (1 - e^{-L\rho}) F^k(\rho)\Big|_{\rho=0}$$

where $\gamma \cdot U = \sum \gamma_k U^k$.

Take an $L$ derivative and set $L = 0$ to get

$$\gamma_1 = \sum p(k) x^k (1 + \gamma \cdot \partial_{-\rho})^{sk+1} \rho F^k(\rho)\Big|_{\rho=0}$$

This determines $\gamma_1$ recursively, but messily.



## The nice $\gamma_1$ recursion

Assume $\rho F^k(\rho) = r_k/(1-\rho)$

Take two $L$ derivatives of the DSE and set $L = 0$ to get

$$\begin{aligned} 2\gamma_2 &= -\sum_k p(k) x^k (1 + \gamma \cdot \partial_{-\rho})^{sk+1} r_k \frac{\rho}{1-\rho}\Big|_{\rho=0} \\ &= -\gamma_1 + \sum x^k p(k) r_k. \end{aligned}$$

Write $P(x) = \sum x^k p(k) r_k$ and use the other recursion:

$$\gamma_1 = P(x) - \gamma_1(1 - sx\partial_x)\gamma_1.$$



## Where it all began

Broadhurst and Kreimer [1]; a bit of massless Yukawa theory.

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right),$$

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k \cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}.$$

Combine to get

$$\begin{aligned} G(x, L) = 1 - \frac{x}{q^2}\int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2)(k+q)^2} \\ - \cdots |_{q^2=\mu^2} \end{aligned}$$

where $L = \log(q^2/\mu^2)$.

So

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x).$$

## Solved

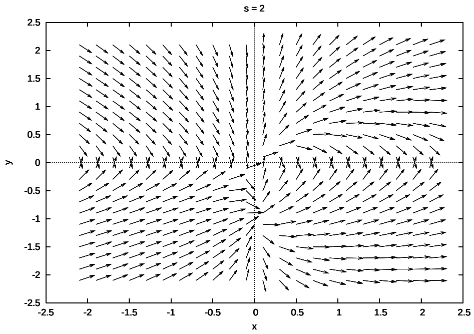
Broadhurst and Kreimer [1] solved this Dyson-Schwinger equation by clever rearranging and recognizing the resulting asymptotic expansion. Today Maple can solve it.

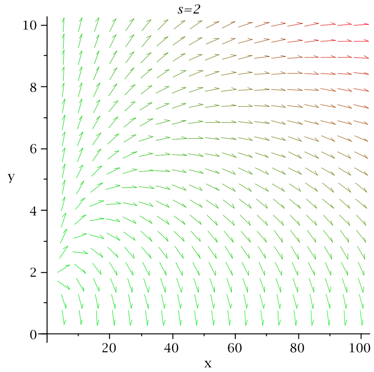
$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$

gives

$$\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C$$








## The $P(x) = x$ family

The last bastion of exact solutions,

$$\gamma_1(x) = x - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x).$$

$s = 1$: $\gamma_1(x) = x + xW\left(C\exp\left(-\frac{1+x}{x}\right)\right)$,

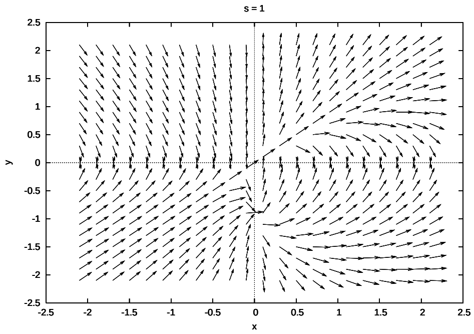
$s = 2$: $\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C$

$s = 3/2$: $A\left(X\right) - x^{1/3}2^{1/3}A'\left(X\right) = C\left(B\left(X\right) - x^{1/3}2^{1/3}B'\left(X\right)\right)$
where $X = \frac{1+\gamma_1(x)}{2^{2/3}x^{2/3}}$

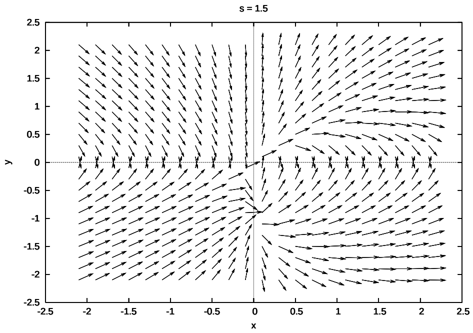
$s = 3$: $(\gamma_1(x)+1)A\left(X\right) - 2^{2/3}A'\left(X\right) =$
$C\left((\gamma_1(x)+1)B\left(X\right) - 2^{2/3}B'\left(X\right)\right)$
where $X = \frac{(1+\gamma_1(x))^2+2x}{2^{4/3}x^{2/3}}$

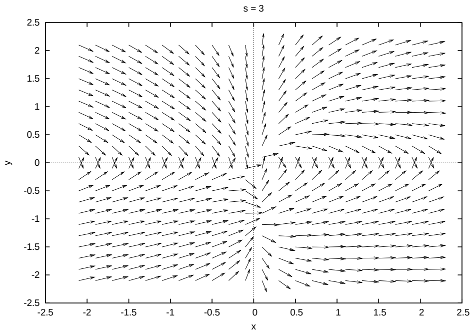
where $A$ is the Airy Ai function, $B$ the Airy Bi function and $W$ the Lambert W function.






Note the solution $\gamma_1(x) = x$. The overall look is similar to the $s = 2$ case.



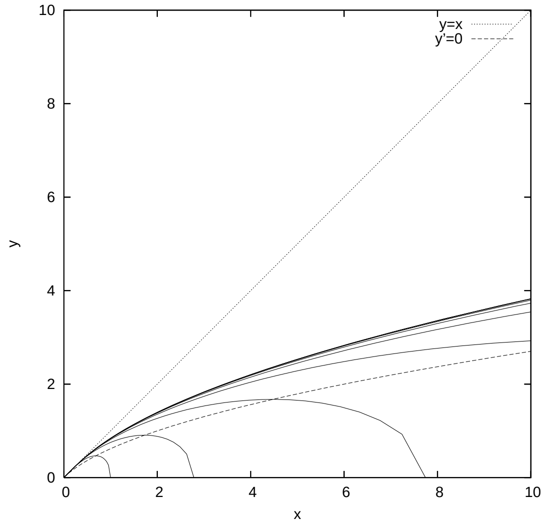



*Show $s$ animation here.*



# $s = 1$ revisited

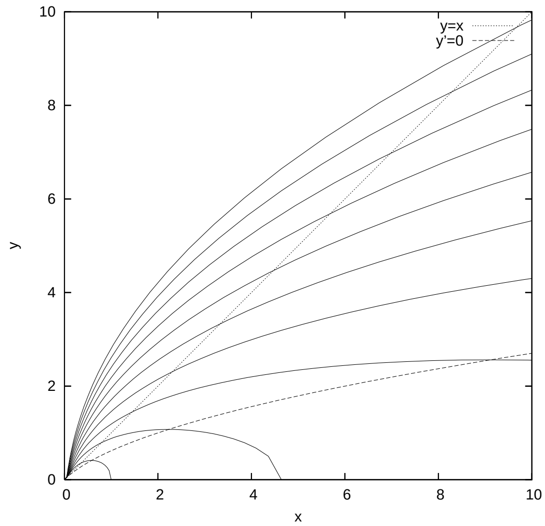
The solution $\gamma_1(x) = x$ appears to be a separatrix. But then again, maybe not






# $s = 2$ revisited

For $s = 2$ it's not so clear that there is a separating curve.



## QED as a single equation

By the Baker, Johnson, Willey analysis we can reduce to a single equation for the photon propagator.

$$2\gamma_1(x) = P(x) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$s = 1$ gives a term $B_+(\mathbb{I})$ independent of $X$ to take into account the fact that the photon propagator can not be inserted into the one loop graph.

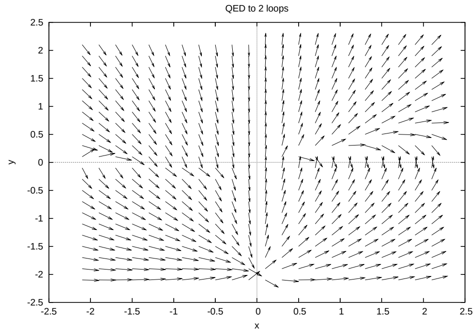
To 2 loops

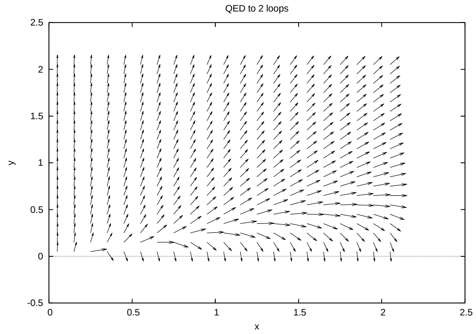
$$P(x) = \frac{x}{3} + \frac{x^2}{4}$$

To 4 loops we need to correct the primitives for our setup

$$\frac{x}{3}+\frac{x^2}{4}+(-0.0312+0.06037)x^3+(-0.6755+0.05074)x^4$$

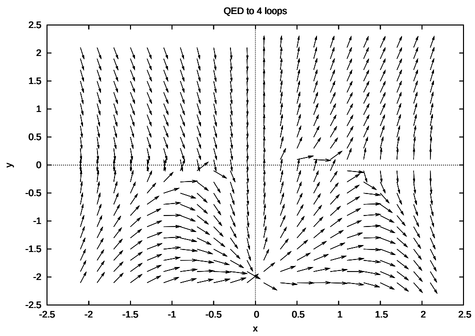


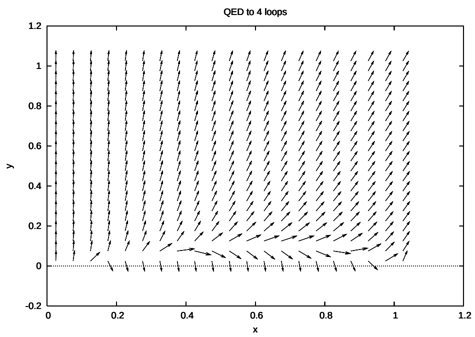





At 4 loops $P(0.992\ldots) = 0$ changing everything.






*Show 5 loops animation here.*

Thanks to David Uminsky and Cameron Morland for help with plots, to Dirk Kreimer for being wiser and more clever than I am, and to the NSF.

# References

[1] D.J. Broadhurst and D. Kreimer. Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality. *Nucl.Phys. B*, 600:403–422, 2001. arXiv:hep-th/0012146.

[2] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216:215–241, 2001. arXiv:hep-th/0003188.