K-Theory for C*-Algebras and for Topological Spaces

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

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1 Introduction

The K-theory of C*-algebras is the study of a collection of abelian groups $K_n(A)$ that are invariants of a C*-algebra A for $n \in \mathbb{N}$. In this paper we will focus on the group $K_0(A)$. The map K_0 taking a C*-algebra to an abelian group can be viewed as a covariant functor from the category of C*-algebras to the category of abelian groups with some additional properties. We will follow [4] for this part of the theory.

The K-theory is useful in distinguishing C*-algebras. The class of AF-algebras is completely classified by their K_0 groups. In general, the K_0 group is not a complete invariant for all C*-algebras, but it is an important part of the classification program of C*-algebras.

Topological K-theory is the "original version" of K-theory, introduced by Sir Michael Atiyah. We will follow his classical text [1]. Topological K-theory is the study of a collection of abelian groups $K^n(X)$ that are invariants of a locally compact Hausdorff space X. Unlike the case of C*-algebras, the map K^0 is a contravariant functor from the category of locally compact Hausdorff spaces to the category of abelian groups.

It is well-known that there is a contravariant functor mapping the category of unital C*-algebras bijectively onto the category of compact Hausdorff spaces that reverses the direction of morphisms. We will see that $K^0(X) \cong K_0(C(X))$ for every compact Hausdorff space X. Furthermore, the functors K_0 and K^0 preserve morphisms by reversing their directions. This result can be extended to non-unital C*-algebras and locally compact Hausdorff spaces, where $K^0(X) \cong K_0(C_0(X))$ for every locally compact Hausdorff space X. This correspondence is explained in [6].

The reader is assumed to be familiar with the basics of C*-algebras and topological bundles. If one needs a review on these subjects, we recommend [2] for C*-algebras and the introductory chapter of [6] for vector bundles.

2 K-theory of C*-algebras

Definition 2.1. Let A be a C*-algebra. For $n, m \in \mathbb{N}$, let $M_{m,n}(A)$ be the set of all $m \times n$ matrices with entries in A. If $m = n$, write $M_{n,n}(A) = M_n(A)$, then $M_n(A)$ is a C*-algebra with the involution $(a^*)_{ij} = (a_{ji})^*$.

Definition 2.2. Let A be a C*-algebra. For $n \in \mathbb{N}$ we define $\mathcal{P}_n(A)$ to be the set of all projections in $M_n(A)$. For $n \leq m$, there is a natural embedding of $\mathcal{P}_n(A)$ into $\mathcal{P}_m(A)$ given by

$$p \mapsto \mathrm{Diag}(p, 0_{m-n}) = p \oplus 0_{m-n}.$$

Define $\mathcal{P}_\infty(A) = \varinjlim_n \mathcal{P}_n(A)$ as the direct limit of this inclusion. We can also think of it as $\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$.

Note 2.3. It might be more notationally clear to write p as an element in $\mathcal{P}_n(A)$ for $n \in \mathbb{N}$, and let $[p]$ denote its equivalence class in the direct limit $\mathcal{P}_\infty(A)$. But there are two more equivalence relations to be quotiented by later, and to save ourselves from the nested square brackets, p will denote a finite matrix as well as its equivalence class in $\mathcal{P}_\infty(A)$, or, an $\aleph_0 \times \aleph_0$ matrix with finitely many non-zero entries.

Definition 2.4. Let $\sim_0$ be the relation on $\mathcal{P}_\infty(A)$ given by the following: for $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say $p \sim_0 q$ if there exists $v \in M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$. The relation $\sim_0$ is called the **Murray - von Neumann equivalence**.

Remark 2.5. A matrix $v \in M_{m,n}(A)$ for some $m, n \in \mathbb{N}$ such that v^*v and vv^* are both projections is called a partial isometry. In the special case that $A = B(H)$ for some Hilbert space H, then v is a partial isometry if and only if it maps $(\ker v)^\perp$ isometrically onto im v. If T is a partial isometry in $B(H)$, then TT^* is the projection onto im T and T^*T is the projection onto $(\ker T)^\perp$.

Example 2.6. Let H be an infinite dimensional Hilbert space. Since $H \cong H \oplus H$, there exists some $T \in B(H \oplus H)$ such that $T|_{H\oplus 0}$ is an isometry from $H \oplus 0$ onto $H \oplus H$, and $T|_{0\oplus H} = 0$. Then $TT^* = I_{H\oplus H}$ and $T^*T = P_{H\oplus 0}$. Note that T can be considered as an element in $B(H \oplus H)$ as well as an element in $M_2(B(H))$. In the latter case

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix}$$

for some $T_1, T_2 \in B(H)$. If we let $S = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, then $SS^* = I_1 \in M_1(B(H))$ and $S^*S = I_2 \in M_2(B(H))$. So $I_2 \sim_0 I_1$.

Lemma 2.7. *Let A be a C^*-algebra, let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ for some $n, m \in \mathbb{N}$, and suppose there exists $v \in M_{m,n}(A)$ for which $v^*v = p$ and $vv^* = q$. Then $v = qv = vp = qvp$.*

Proof. Let $w = (1-q)v$, then

$$w^*w = v^*(1-q)(1-q)v = v^*(1-q)v = v^*v - v^*vv^*v = p - pp = 0.$$

However $\|w\|^2 = \|w^*w\| = 0$, which implies that $w = 0$. So $0 = w = v - qv$. This implies that $v = qv$. The case $v = pv$ is proved similarly. Lastly,

$$qvp = (qv)p = vp = v. \blacksquare$$

Proposition 2.8. *The relation $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$.*

Proof. It is not yet clear that $\sim_0$ is well-defined on $\mathcal{P}_\infty(A)$, since $\mathcal{P}_\infty(A)$ is a direct limit, where $p \in \mathcal{P}_n(A)$ can also be represented by $p \oplus 0_k$ in $\mathcal{P}_\infty(A)$, for any $k \geq 0$. We will show that $\sim_0$ is an equivalence relation on $\bigsqcup_{r=1}^{\infty} \mathcal{P}_r(A)$, and also satisfies $p \sim_0 p \oplus 0_k$ for $p \in \mathcal{P}_n(A)$, $n \geq 1$ and $k \geq 0$. Then for any $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$ and $k, k' \geq 0$, have $p \sim_0 q$ if and only if

$$p \oplus 0_k \sim_0 p \sim_0 q \sim_0 q \oplus 0_{k'}.$$

So $\sim_0$ descends to an equivalence relation on $\mathcal{P}_\infty(A)$. To this end, let $p \in \mathcal{P}_n(A)$, $q \in \mathcal{P}_m(A)$ and $r \in \mathcal{P}_l(A)$ for some $l, m, n \geq 1$.

To show $p \sim_0 p \oplus 0_k$, let $v = \begin{bmatrix} p & 0_{n\times k} \end{bmatrix}$, then $v^*v = p$ and $vv^* = p \oplus 0_k$. The special case with $k = 0$ verifies reflexivity.

Suppose there exists $v \in M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$. Let $w = v^* \in M_{n,m}(A)$. We have

$$w^*w = q \text{ and } ww^* = p.$$

So $\sim_0$ is symmetric.

Suppose $p \sim_0 q$ and $q \sim_0 r$. Then there exists some $v \in M_{m,n}(A)$ and $u \in M_{l,m}(A)$ for which

$$v^*v = p, \quad vv^* = q, \quad u^*u = q \quad \text{and} \quad uu^* = r$$

hold. Let $z = uv$. Using Lemma 2.7, the following computations hold.

$$z^*z = v^*u^*uv = v^*qv = v^*v = p,$$

$$zz^* = uvv^*u^* = uqu^* = r.$$

Thus $p \sim_0 r$, which proves transitivity. $\blacksquare$

Definition 2.9. Let A be a C*-algebra and p, q projections in $\mathcal{P}_\infty(A)$. We say that p and q are **mutually orthogonal** if $pq = 0$, written $p \perp q$.

Remark 2.10. If $p \perp q$ then

$$qp = q^*p^* = (pq)^* = 0^* = 0,$$

so $q \perp p$. And also,

$$(p+q)^* = p^* + q^* = p + q$$

$$(p+q)(p+q) = pp + pq + qp + qq = pp + qq = p + q.$$

So $p + q$ is also a projection in A.

In the special case that $A = B(H)$ for some Hilbert space H and $P, Q \in B(H)$ are projections, we have $P \perp Q$ if and only if ran $P \perp$ ran Q.

Proposition 2.11. *Let $p, p', q, q' \in \mathcal{P}_\infty(A)$. Then*

1. $p \oplus q \sim_0 q \oplus p$.
2. $p \sim_0 p'$ *and* $q \sim_0 q'$ *implies* $p \oplus q \sim_0 p' \oplus q'$.
3. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.
4. *Suppose p and q are represented by matrices of the same size, and $p \perp q$, then $p + q \sim_0 p \oplus q$.*

Proof. 1. Suppose p is $n \times n$ and q is $m \times m$. Let $v = \begin{bmatrix} 0_{n\times m} & p \\ q & 0_{m\times n} \end{bmatrix}$. Then

$$v^*v = \begin{bmatrix} 0_{m\times n} & q^* \\ p^* & 0_{n\times m} \end{bmatrix} \begin{bmatrix} 0_{n\times m} & p \\ q & 0_{m\times n} \end{bmatrix} = \begin{bmatrix} q^*q & 0_{m\times n} \\ 0_{n\times m} & p^*p \end{bmatrix} = q \oplus p;$$

$$vv^* = \begin{bmatrix} 0_{n\times m} & p \\ q & 0_{m\times n} \end{bmatrix} \begin{bmatrix} 0_{m\times n} & q^* \\ p^* & 0_{n\times m} \end{bmatrix} = \begin{bmatrix} pp^* & 0_{n\times m} \\ 0_{m\times n} & qq^* \end{bmatrix} = q \oplus p.$$

So $q \oplus p \sim_0 p \oplus q$.

2. Suppose $v^*v = p$, $vv^* = p'$, $w^*w = q$ and $ww^* = q'$, then

$$(v \oplus w)^*(v \oplus w) = p \oplus q$$

and

$$(v \oplus w)(v \oplus w)^* = p' \oplus q'.$$

So $p \oplus q \sim_0 p' \oplus q'$.

3. This is by definition.
4. Suppose p and q are of the same size and $pq = 0$. Let $v = \begin{bmatrix} p & q \end{bmatrix}$, then

$$vv^* = \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = pp + qq = p + q,$$

$$v^*v = \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} pp & pq \\ qp & qq \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = p \oplus q.$$

So $p + q \sim_0 p \oplus q$. ■

Definition 2.12. Let A be a C*-algebra. Define $\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0$. The equivalence class of p in $\mathcal{D}(A)$ is written $[p]_\mathcal{D}$. Equip $\mathcal{D}(A)$ with an operation $+$ by $[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}$.

Proposition 2.13. *$(\mathcal{D}(A), +)$ is an abelian monoid.*

Proof. This is mostly a consequence of Proposition 2.11. Point 2 implies that the operation $+$ is well-defined after quotienting by $\sim_0$. Point 3 implies that $+$ is associative. Point 1 implies that it is commutative. So $(\mathcal{D}(A), +)$ is an abelian semigroup. Now we claim that $[0_1]_\mathcal{D}$ is the identity element (note that $0_n \sim_0 0_m$ for all $n, m \in \mathbb{N}$ by Proposition 2.8). To this end, take any $p \in \mathcal{P}_\infty(A)$. By point 1 of Proposition 2.11 and Proposition 2.8,

$$0_1 \oplus p \sim_0 p \oplus 0_1 \sim_0 p,$$

so

$$[0_1]_\mathcal{D} + [p]_\mathcal{D} = [p]_\mathcal{D} + [0_1]_\mathcal{D} = [p]_\mathcal{D}. \blacksquare$$

From the abelian monoid $\mathcal{D}(A)$ we will construct an abelian group, by a construction called the **Grothendieck completion**.

Definition 2.14. Let $(S, +)$ be an abelian semigroup, then $S \times S$ is also naturally a semigroup. Let $\sim$ be a relation on $S \times S$ given by $(a_1, b_1) \sim (a_2, b_2)$ if there exists $x \in S$ so that

$$a_1 + b_2 + x = a_2 + b_1 + x.$$

Define $G(S) = (S \times S)/\sim$, and equip it with the operation $+$ by

$$[(a, b)] + [(c, d)] = [(a + c, b + d)].$$

Proposition 2.15. *The above construction is well-defined, and $G(S)$ is an abelian group. Furthermore, if S is an abelian monoid with identity element* 0*, then $\varphi : S \to G(S)$ by $\varphi(s) = [(s, 0)]$ is a monoid homomorphism.*

Proof. It is easy to see that $\sim$ is an equivalence relation on $S \times S$. To see that $+$ is well-defined on $G(S)$, let $a_i, b_i, c_i, d_i \in S$ for $i = 1, 2$, and suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Then there exists $x, y \in S$ such that

$$a_1 + b_2 + x = a_2 + b_1 + x \qquad \text{and} \qquad c_1 + d_2 + y = c_2 + d_1 + y.$$

Then

$$(a_1 + c_1) + (b_2 + d_2) + (x + y) = (a_2 + c_2) + (b_2 + d_2) + (x + y),$$

so $[(a_1 + c_1, b_1 + d_1)] = [(a_2 + c_2, b_2 + d_2)]$.

Since $+$ is associative and commutative on S, the addition induced on $G(S)$ is associative and commutative as well. For $a, b, c, d \in S$, it is clear that $[(a, a)] = [(b, b)]$. Furthermore

$$[(c, d)] + [(a, a)] = [(c + a, d + a)] = [(c, d)].$$

So (a, a) is the identity element of $G(S)$. Also,

$$[(a, b)] + [(b, a)] = [(a + b, a + b)],$$

so $[(b, a)]$ is the inverse of $[(a, b)]$. Hence $G(S)$ is indeed an abelian group.

Now suppose that S is an abelian monoid with 0, and $\varphi : S \to G(S)$ by $\varphi(s) = [(s, 0)]$. Then it is clear that $\varphi(a + b) = \varphi(a) + \varphi(b)$ and that $\varphi(0)$ is the identity element of $G(S)$. ∎

It is convenient to think of $[(a, b)] \in G(S)$ as "$a - b$".

Example 2.16. 1. $S = \mathbb{N}$. Then $G(\mathbb{N}) = \mathbb{Z}$. This is the standard construction of $\mathbb{Z}$.

2. $S = \mathbb{N} \cup \{\infty\}$. For any $a, b, c, d \in \mathbb{N} \cup \{\infty\}$,

$$a + c + \infty = \infty = b + d + \infty,$$

so $[(a, b)] = [(c, d)]$. Hence $G(S) \cong \{0\}$. This example demonstrates why we required the x in defining $\sim$ in Definition 2.14, where $(a_1, b_1) \sim (a_2, b_2)$

if and only if there exists x for which $a_1 + b_2 + x = a_2 + b_1 + x$. Suppose for instance we define another relation $\sim_{\text{bad}}$ on S by $(a_1, b_1) \sim_{\text{bad}} (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. For any $a, b \in S$, we have

$$\infty + a = \infty = b + \infty,$$

so $(\infty, \infty) \sim_{\text{bad}} (a, b)$. In particular, $(1, 1) \sim_{\text{bad}} (\infty, \infty) \sim_{\text{bad}} (1, 2)$, but clearly $(1, 1) \not\sim_{\text{bad}} (1, 2)$, which shows that $\sim_{\text{bad}}$ is not an equivalence relation! This is the same problem that one runs into when asking "Surely $\infty + \infty = \infty$, but what is $\infty - \infty$?"

Now we are ready to give the definition of the K_0 group of a unital C*-algebra.

Definition 2.17. Let A be a *unital* C*-algebra. Define $K_0(A) = G(\mathcal{D}(A))$. Define the map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by $[p]_0 = \varphi([p]_{\mathcal{D}})$ where $\varphi : \mathcal{D}(A) \to G(\mathcal{D}(A))$ is the monoid homomorphism defined in Proposition 2.15.

Example 2.18. 1. Let $A = \mathbb{C}$. All projections in $\mathcal{P}_\infty(\mathbb{C})$ are projection matrices. Take $p, q \in \mathcal{P}_\infty(\mathbb{C})$. We may assume that p and q are both $n \times n$. Suppose p and q have the same rank $k \leq n$, and let $\{z_1, \ldots, z_k\}$ be an orthonormal basis of $\operatorname{ran} p$ and extend it to an orthonormal basis $\{z_1, \ldots, z_n\}$ of $\mathbb{C}^n$; let $\{w_1, \ldots, w_k\}$ be an orthonormal basis of $\operatorname{ran} q$. Let $v \in M_n(\mathbb{C})$ be the matrix that takes z_j to w_j for $j = 1, \ldots, k$, and takes z_j to 0 for all $j = k + 1, \ldots, n$. Then

$$v^* v z_j = \begin{cases} v^* w_j = z_j & : j = 1, \ldots, k \\ v^* 0 = 0 & : j = k + 1, \ldots, n \end{cases}.$$

So v^*v is the projection onto ran p, hence $v^*v = p$. Similarly, $vv^* = q$, so $p \sim_0 q$.

Conversely suppose $p \sim_0 q$. Then there exits a matrix v for which $v^*v = p$ and $vv^* = q$. Since row rank and column rank coincide, we have

$$\operatorname{rank} p = \operatorname{rank} v^*v = \operatorname{rank} vv^* = \operatorname{rank} q.$$

Hence $p \sim_0 q$ if and only if p and q have the same rank. Furthermore it is clear that $\operatorname{rank} p + \operatorname{rank} q = \operatorname{rank}(p \oplus q)$. Thus $\mathcal{D}(\mathbb{C}) \cong \mathbb{N}$. Therefore

$K_0(\mathbb{C}) \cong G(\mathbb{N}) = \mathbb{Z}$.

2. Let $A = M_m(\mathbb{C})$ for some $m \in \mathbb{N}$. Then for $n \in \mathbb{N}$, the C*-algebra $M_n(A)$ is naturally a subalgebra of $M_{mn}(\mathbb{C})$, and the rank argument from above works just as well. Hence $K_0(M_m(\mathbb{C})) \cong \mathbb{Z}$.

3. Let $A = \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ an infinite dimensional Hilbert space. The same rank argument works since every two Hilbert spaces of the same dimension are isometric. So projections in $\mathcal{P}_\infty(A)$ are once again determined up to Murray - von Neumann equivalence by their dimensions, and $\mathcal{D}(A) \cong \{\dim p : p \in \mathcal{P}_\infty(A)\}$. Since $\mathcal{H}$ is infinite dimensional, $\mathcal{D}(A)$ has a largest element $\alpha_0 = \dim \mathcal{H}$ since $\dim(\mathcal{H}^n) = \dim \mathcal{H}$ for all finite n, and $\alpha_0 + \alpha = \alpha_0$ for all $\alpha \in \mathcal{D}(A)$. So by the same argument in part 2 of Example 2.16, have $K_0(\mathcal{B}(\mathcal{H})) = G(\mathcal{D}(\mathcal{B}(\mathcal{H}))) = 0$.

To summarize,

$$K_0(\mathcal{B}(\mathcal{H})) \cong \begin{cases} \mathbb{Z} & : \dim \mathcal{H} < \aleph_0 \\ 0 & : \dim \mathcal{H} \geq \aleph_0 \end{cases}$$

3 Unitaries and projections

In this section we develop some properties of unitary and projection elements in a C*-algebra. These will be necessary for exploring meaningful properties of the K_0-group of C*-algebras.

From here on $\widetilde{A}$ denotes the unitization of the C*-algebra A. For more information on unitization, see [2].

Definition 3.1. Let X be a topological space and $x, y \in X$. Say x and y are **homotopy equivalent** in X, written $x \sim_h y$, if there exists a continuous path $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Definition 3.2. Let A be a C*-algebra, and $a, b \in A$. We say a is **unitarily equivalent** to b, written $a \sim_u b$, if there exists a unitary $u \in \widetilde{A}$ such that $uau^* = b$. It is clear that these are equivalence relations.

Definition 3.3. Let A be a unital C*-algebra, define $\mathcal{U}(A)$ to be the group of unitary elements in A, and define $\mathcal{U}_0(A)$ to be all $u \in \mathcal{U}(A)$ such that $u \sim_h 1$. That is, $\mathcal{U}_0(A)$ is the path-connected component of 1 in $\mathcal{U}(A)$.

Definition 3.4. Let A be a unital C*-algebra and let $a \in A$. The spectrum $\sigma(a)$ of a is defined to be

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A\}.$$

The general theory of spectrum and of continuous functional calculus can be found in [2].

Lemma 3.5. *Let A be a unital C*-algebra and $u \in \mathcal{U}(A)$. If $\sigma(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$.*

Proof. Suppose $\sigma(u) \neq \mathbb{T}$. Let $w \in \mathbb{T} \setminus \sigma(u)$ and let $\log_w : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ be the branch of logarithm that avoids the ray containing w. Then $\exp(\log_w(z)) = z$ for all $z \in \mathbb{T} \setminus \{w\} \supseteq \sigma(u)$, so $\exp(\log_w(u)) = u$. Let $h = \log_w(u)$, then

$$\sigma(h) \subseteq \log_w(\mathbb{T} \setminus w) \subseteq i\mathbb{R}.$$

For $t \in [0, 1]$, let $h_t = th$. Clearly $\sigma(th) \subseteq i\mathbb{R}$ for all $t \in [0, 1]$, so $\sigma(\exp(th)) \subseteq \mathbb{T}$ for all $t \in [0, 1]$, which implies that $\exp(th)$ is unitary for any $t \in [0, 1]$. Furthermore the map $\beta : [0, 1] \to \mathcal{U}(A)$ mapping $\beta(t) = \exp(th)$ is a continuous path of unitaries from $1_A \in A$ to $u \in A$. Hence $u \in \mathcal{U}_0(A)$. ∎

Lemma 3.6 (Whitehead)**.** *Let A be a unital C*-algebra and let $u, v \in \mathcal{U}(A)$. Then*

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix} \sim_h \begin{bmatrix} vu & 0 \\ 0 & 1 \end{bmatrix} \sim_h \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} \text{ in } \mathcal{U}(M_2(A)).$$

Proof. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has spectrum $\{\pm 1\}$, by Lemma 3.5 have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim_h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\alpha : [0,1] \to \mathcal{U}_0(M_2(A))$ be a path from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define $\beta : [0,1] \to \mathcal{M}_2(A)$ by

$$\beta(t) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \alpha(t) \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \alpha(t).$$

Since for all $t \in [0,1]$, $\beta(t)$ is the product of four unitaries, so β is in fact a path in $\mathcal{U}(M_2(A))$. Further,

$$\begin{aligned} \beta(0) &= \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & u \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \end{aligned}$$

and

$$\beta(1) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix}.$$

So

$$\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \sim_h \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix}.$$

By symmetry and transitivity, it is only left to prove that

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}.$$

This can be accomplished by defining the path

$$\gamma(t) = \alpha(t) \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \alpha(t). \blacksquare$$

Corollary 3.7. *Let A be a unital C^*-algebra, $u \in \mathcal{U}(A)$, then* $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in \mathcal{U}_0(M_2(A))$.

Proof. By Lemma 3.6,

$$\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \sim_h \begin{bmatrix} uu^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \blacksquare$$

Lemma 3.8. *Let A be a unital C^*-algebra and $u \in \mathcal{U}(A)$. If $\|u - 1\| < 2$ then $u = \exp(ih)$ for some self-adjoint element $h \in A$.*

Proof. If $\|u - 1\| < 2$ then $\sigma(u - 1) \subseteq B_2(0)$, in particular $-2 \notin \sigma(u - 1)$, so $-1 \notin \sigma(u)$. Since $\sigma(u) \neq \mathbb{T}$, by the proof of Lemma 3.5, $u = \exp(s)$ for some $s \in A$ with $\sigma(s) \in i\mathbb{R}$. Let $h = -is$, then h is self-adjoint and $\exp(ih) = \exp(s) = u$. ∎

Proposition 3.9. *Let A be a unital C^*-algebra. Then*

$$\mathcal{U}_0(A) = \{\exp(ih_1) \dots \exp(ih_l) : l \in \mathbb{N}, h_j \in A \text{ self-adjoint}\}.$$

Proof. Let $u \in \mathcal{U}_0(A)$. A continuous path from u to 1 can be partitioned into segments

$$u = u_0 \sim_h u_1 \sim_h \cdots \sim_h u_k = 1$$

where $\|u_{j-1} - u_j\| < 2$ for $j = 1, \dots, k$. Now apply induction on k. For $k = 1$, $\|u - 1\| < 2$, and the result follows Lemma 3.8. Suppose the result is true for $k = n - 1$, and the inductive step for n has been completed. Then $u_1 = \exp(ih_1) \dots \exp(ih_l)$ for some $l \in \mathbb{N}$ and h_j self-adjoint. Because $\|u - u_1\| < 2$, so

$$\|uu_1^* - 1\| = \|(u - u_1)u_1^*\| = \|u - u_1\| < 2.$$

By Lemma 3.8, there exists a self-adjoint element $h_0 \in A$ such that $uu_1^* = \exp(ih_0)$. Then

$$u = \exp(ih_0)u_1 = \exp(ih_0)\exp(ih_1) \dots \exp(ih_l).$$

This completes the induction.

Conversely if h is self-adjoint, the proof of Lemma 3.5 implies that $\exp(ih) \in \mathcal{U}_0(A)$. The product of such unitaries is also homotopic to the identity. Thus all elements in $\mathcal{U}_0(A)$ are indeed equal to finite products as in the claim. ∎

Proposition 3.10. *Let A, B be unital C^*-algebras, $\varphi : A \to B$ a surjective $*$-homomorphism. Then*

1. $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$

2. *For any $u \in \mathcal{U}(B)$, there exists $v \in \mathcal{U}_0(M_2(A))$ such that*

$$\varphi(v) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$$

Proof. 1. Since φ takes unitaries to unitaries, $\varphi(\mathcal{U}_0(A)) \subseteq \mathcal{U}_0(B)$. The converse requires some work. Let $u \in \mathcal{U}_0(B)$. By Proposition 3.9, there exists hermitian elements $h_1, \dots, h_l \in B$ such that

$$u = \exp(ih_1)\exp(ih_2)\dots\exp(ih_l).$$

Let $t_1, \dots, t_l \in A$ such that $\varphi(t_j) = h_j$ for $j = 1, \dots, l$, and let $\widetilde{t_j} = \frac{1}{2}(t_j + t_j^*)$ for $j = 1, \dots, l$. Then $\widetilde{t_j}$ are self-adjoint, and

$$\varphi(\widetilde{t_j}) = \frac{1}{2}(\varphi(t_j) + \varphi(t_j)^*) = \frac{1}{2}(h_j + h_j) = h_j.$$

Let

$$v = \exp(i\widetilde{t_1})\dots\exp(i\widetilde{t_l}).$$

The proof of Lemma 3.5 implies that $v \in \mathcal{U}_0(A)$. And happily, $\varphi(v) = u$.

2. Let $u \in \mathcal{U}(B)$. By Corollary 3.7 $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in \mathcal{U}_0(M_2(B))$. Then by part 1 there exists some $v \in \mathcal{U}_0(M_2(A))$ such that $\varphi(v) = u \oplus u^*$. ■

Definition 3.11. Let A be a unital C*-algebra and $a \in A$. Then $\sigma(a^*a) \subseteq \mathbb{R}_{\geq 0}$, where the square root function is defined. So we may define $|a| = (a^*a)^{1/2}$.

Proposition 3.12. *Let A be a unital C*-algebra.*

1. *If $z \in GL(A)$, then $|z| \in GL(A)$, and $w(z) := z|z|^{-1} \in \mathcal{U}(A)$.*

2. *The map $w : GL(A) \to \mathcal{U}(A)$ defined in 1. is continuous. And $w(u) = u$ for all $u \in \mathcal{U}(A)$.*

3. *If $a, b \in GL(A)$ with $a \sim_h b$ in $GL(A)$, then $w(a) \sim_h w(b)$ in $\mathcal{U}(A)$.*

Proof. 1. Suppose z is invertible. Then z^* is also invertible, so $z^*z \in GL(A)$. It follows that

$$\sigma(|z|) = \sigma((z^*z)^{1/2}) = \{t^{1/2} : t \in \sigma(z^*z)\} \not\ni 0.$$

Thus $|z|$ is invertible.

Furthermore,

$$\begin{aligned} w(z)w(z)^* &= z|z|^{-1}(z|z|^{-1})^* = z|z|^{-1}|z|^{-1}z^* \\ &= z(z^*z)^{-1}z^* = zz^{-1}(z^*)^{-1}z^* = 1, \end{aligned}$$

and similarly $w(z)^*w(z) = 1$. So $w(z) \in \mathcal{U}(A)$.

2. The map $a \mapsto a^*a$ is continuous. Also inversion and multiplication are continuous in $GL(A)$. So to prove the claim it is sufficient to prove that $a \mapsto a^{1/2}$ is continuous on $A_{\geq 0}$, where $A_{\geq 0}$ is the set of normal elements in A with spectrum contained in $[0, \infty)$.

Suppose we fix $a \in A_{\geq 0}$ and let U be a bounded open neighbourhood containing $\sigma(a)$. The upper-semicontinuity of spectra [5] implies that there is some $d > 0$ such that if $b \in A$ and $\|b - a\| < d$ then $\sigma(b) \subseteq U$. Thus the problem reduces to proving that the square root map is continuous on $\Omega_r \subseteq A_{\geq 0}$ where

$$\Omega_r = \{a \in A : a^*a = aa^*,\ \sigma(a) \subseteq [0, r]\}.$$

Let f denote the square root function and let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem, there exists a complex polynomial g such that $\|g - f\|_\infty < \varepsilon/3$ on $[0, r]$. For $c \in \Omega_t$,

$$\begin{aligned} \|f(c) - g(c)\| &= \|(f - g)(c)\| \\ &= \sup\{|(f - g)(z)| : z \in \sigma(c)\} \\ &\leq \|f - g\|_\infty < \varepsilon/3. \end{aligned}$$

Therefore g is continuous on Ω_t since $a \mapsto a^n$ is continuous. So there exists $\delta > 0$ such that $\|g(a) - g(b)\| < \varepsilon/3$ whenever $a, b \in A$ with $\|a - b\| < \delta$. Thus when $a, b \in \Omega_t$ with $\|a - b\| < \delta$, have $\|f(a) - f(b)\| < \varepsilon$.

3. Let $\alpha : [0, 1] \to GL(A)$ be a continuous path from a to b. Then by part 2, $w \circ \alpha : [0, 1] \to \mathcal{U}(A)$ is a continuous path from $w(a)$ to $w(b)$. ■

For an element $z \in A$, the form $z = w(z)|z|$ is called the **polar decomposition** of z.

Definition 3.13. The relations $\sim_u$ and $\sim_h$ induce equivalence relations on $\mathcal{P}_\infty(A)$ as follows: $p \sim_u q$, if by representing p and q both as $n \times n$ matrices for some $n \in \mathbb{N}$, there exists a unitary element $u \in \widetilde{M_n(A)}$ such that $u^*pu = q$. We say that $p \sim_h q$ if by representing p and q both as $n \times n$ matrices for some $n \in \mathbb{N}$, there exists a path $\alpha(t)$ in $\mathcal{P}_n(A)$ such that $\alpha(0) = p$ and $\alpha(1) = q$.

Proposition 3.14. *Let A be a unital C^*-algebra, $a, b \in A$ self-adjoint elements, $z \in GL(A)$ and $z = u|z|$ the polar decomposition of z. If $za = bz$ then $ua = bu$.*

Proof. Since a and b are self-adjoint, take the adjoint of the equality to have $az^* = z^*b$. Then

$$|z|^2 a = z^*za = z^*bz = az^*z = a|z|^2.$$

So a commutes with $|z|^2$. Consequently a commutes with $g(|z|^2)$ for all complex polynomials g. By Stone-Weierstrass theorem, the element $|z|^{-1} = ((|z|^2)^{1/2})^{-1}$ is the limit of a sequence of polynomials in $|z|^2$. Hence a commutes with $|z|^{-1}$. It follows that

$$uau^* = z|z|^{-1}au^* = za|z|^{-1}u^* = bz|z|^{-1}u^* = buu^* = b. \blacksquare$$

Proposition 3.15. *Let $n \in \mathbb{N}_{\geq 1}$, and $p, q \in \mathcal{P}_n(A)$. Then*

1. *$p \sim_h q$ implies $p \sim_u q$.*
2. *$p \sim_u q$ implies $p \sim_0 q$.*
3. *$p \sim_0 q$ implies $p \oplus 0_n \sim_u q \oplus 0_n$.*
4. *$p \sim_u q$ implies $p \oplus 0_n \sim_h q \oplus 0_n$.*

Proof. 1. Let $\alpha(t)$ be a path in $\mathcal{P}_n(A)$ that connects p to q, then we can partition the path into segments of length less than $1/2$. It is now sufficient to prove that if $\|p - q\| < 1/2$ then $p \sim_u q$. Let $z = pq + (I - p)(I - q) \in \widetilde{A}$, and $pz = pq = zq$. Also

$$\begin{aligned}
\|z - I\| &= \|pq + (I - p)(I - q) - I\| \\
&= \|pq + (I - p)(I - q) - p - (I - p)\| \\
&= \|p(q - p) + (I - p)\left((I - q) - (I - p)\right)\| \\
&= \|p(q - p) + (I - p)(p - q)\| \\
&\leq \|p\|\|(q - p)\| + \|I - p\|\|p - q\| \\
&\leq 2\|p - q\| < 1.
\end{aligned}$$

Hence $z \in GL(A)$. Let $z = u|z|$ be the polar decomposition of z. By Proposition 3.14, $pu = uq$.

2. Suppose $p \sim_u q$. Then there exists some unitary $u \in \widetilde{M_n(A)}$ such that $u^*pu = q$. Let $v = u^*p$, then $vv^* = u^*ppu = q$ and $v^*v = puu^*p = pp = p$. Also note that $v = u^*p \in M_n(A)$ since $M_n(A)$ is an ideal in $\widetilde{M_n(A)}$. Hence $p \sim_0 q$.

3. Suppose there exists $v \in M_n(A)$ such that $vv^* = q$ and $v^*v = p$. Define

$$u = \begin{bmatrix} v & 1-q \\ 1-p & v^* \end{bmatrix} \text{ and } w = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix}.$$

Then

$$\begin{aligned} u^*u &= \begin{bmatrix} v & I_n - q \\ I_n - p & v^* \end{bmatrix} \begin{bmatrix} v^* & I_n - p \\ I_n - q & v \end{bmatrix} \\ &= \begin{bmatrix} vv^* + (I_n - q) & v - vp + v - qv \\ v^* - pv^* + v^* - v^*q & (I_n - p) + v^*v \end{bmatrix} \\ &= \begin{bmatrix} I_n + q - q & v - v + vv^*v - vv^*v \\ v^* - v^* + v^*vv^* - v^*vv^* & I_n - v^*v + v^*v \end{bmatrix} \\ &= I_{2n} \end{aligned}$$

Lemma 2.7 is used to equate the second line to the third in the above equation. Similar computations show that $uu^* = w^*w = ww^* = I_{2n}$. So $u, w, wu \in \mathcal{U}_{2n}(\widetilde{A})$. And

$$\begin{aligned} wu &= \begin{bmatrix} q & I - q \\ I - q & q \end{bmatrix} \begin{bmatrix} v & I - q \\ I - p & v^* \end{bmatrix} \\ &= \begin{bmatrix} qv + (I-q)(I-p) & q - qq + v^* - qv^* \\ v - qv + q - qp & (I-q)(I-q) + qv^* \end{bmatrix} \\ &= \begin{bmatrix} v + (I-q)(I-p) & (I-q)v^* \\ q(I-p) & (I-q) + qv^* \end{bmatrix} \end{aligned}$$

is an element of $\widetilde{M_{2n}(A)}$. Now,

$$\begin{aligned} &wu(p \oplus 0_n)(wu)^* \\ &\quad = \begin{bmatrix} v+(I-q)(I-p) & 0 \\ q-qp & I-q+v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^*+(I-p)(I-q) & q-pq \\ 0 & I-q+v \end{bmatrix} \\ &\quad = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^*+(I-p)(I-q) & q-pq \\ 0 & I-q+v \end{bmatrix} \\ &\quad = \begin{bmatrix} vv^*+v(I-p)(I-q) & vq-vpq \\ 0 & 0 \end{bmatrix} = q \oplus 0_n \end{aligned}$$

noting that

$$v(I-p)(I-q) = (v - vv^*v)(I-q) = 0$$

and

$$vq - vpq = vvv^* - (vv^*v)vv^* = vvv^* - vvv^* = 0$$

by Lemma 2.7.

4. Suppose $p \sim_u q$. Then there exists unitary $u \in \widetilde{M_n(A)}$ such that $upu^* = q$. By Lemma 3.6 there exists a path $t \mapsto w_t$ in $\mathcal{U}(M_{2n}(\widetilde{A}))$ such that

$$w_0 = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \text{ and } w_1 = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}.$$

Let $p_t = w_t \mathrm{Diag}(p, 0_n) w_t^*$. Then $p_t \in \mathcal{P}_{2n}(A)$ for each $t \in [0,1]$. Furthermore,

$$p_0 = \mathrm{Diag}(p, 0_n) \text{ and } p_1 = \begin{bmatrix} upu^* & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{Diag}(q, 0_n).$$

Therefore $p \oplus 0_n \sim_h q \oplus 0_n$. ∎

4 K_0 as a functor

We will see that K_0 is a contravariant functor from the category of C*-algebras to the category of abelian groups, and that it enjoys many useful properties. Before starting the functoriality, we will first need a way to induce group homomorphisms from semigroups homomorphisms in the Grothendieck completion.

Proposition 4.1. *Let S be an abelian semigroup. For any abelian group H and any semigroup homomorphism $\rho : S \to H$, the map $\rho_G : G(S) \to H$ given by $\rho_G([(s,t)]_G) = \rho(s) - \rho(t)$ for all $(s,t) \in S \times S$ is a well-defined group homomorphism.*

Proof. Let ρ_G be as defined above and let $s_1, s_2, t_1, t_2 \in S$. To see that ρ_G is well-defined, suppose that $[(s_1,t_1)]_0 = [(s_2,t_2)]_0$. Then there exists $r \in S$ such that $s_1 + t_2 + r = s_2 + t_1 + r$, which implies that

$$\rho(s_1) + \rho(t_2) + \rho(r) = \rho(s_2) + \rho(t_1) + \rho(r).$$

But H is a group, where all elements are invertible. So

$$\rho_G([(s_1,t_1)]_G) = \rho(s_1) - \rho(t_1) = \rho(s_2) - \rho(t_2) = \rho_G([(s_2,t_2)]_G).$$

Hence ρ_G is well-defined. Now to check that ρ_G is a homomorphism:

$$\begin{aligned}\rho_G([(s_1,t_1)]_G + [(s_2,t_2)]_G) &= \rho_G([(s_1+s_2, t_1+t_2)]_G) \\ &= \rho(s_1+s_2) - \rho(t_1+t_2) \\ &= (\rho(s_1) - \rho(t_1)) + (\rho(s_2) - \rho(t_2)) \\ &= \rho_G([(s_1,t_1)]_0) + \rho_G([s_2,t_2)]_0) \ \blacksquare\end{aligned}$$

If A and B are C*-algebras, with $\varphi : A \to B$ a continuous $*$-homorphism, then φ extends naturally to a $*$-homomorphism $M_n(A) \to M_n(B)$ for all $n \in \mathbb{N}$ by applying φ entry-wise to matrix entries, i.e. $\varphi(T)_{ij} = \varphi(T_{ij})$. This map clearly respects matrix multiplication and involution. In the same way, φ extends entry-wise to $\mathcal{P}_\infty(A)$ and respects direct sum, and is thus a monoid homomorphism $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$. Let $\pi : \mathcal{P}_\infty(B) \to \mathcal{P}_\infty(B)/\sim_0$ be the quotient map. Then $\pi \circ \varphi$ is a monoid homomorphism $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)/\sim_0$. If $p, q \in \mathcal{P}_\infty(A)$ with $p \sim_0 q$, there exists some matrix v with entries in A such that $vv^* = p$ and $v^*v = q$. Hence

$$\begin{aligned}\pi \circ \varphi(p) &= \pi(\varphi(vv^*)) = \pi(\varphi(v)\varphi(v^*)) \\ &= \pi(\varphi(v^*)\varphi(v)) = \pi(\varphi(v^*v)) \\ &= \pi \circ \varphi(q)\end{aligned}$$

So $\pi \circ \varphi(p)$ factors into a monoid homomorphism $\widetilde{\varphi} : \mathcal{P}_\infty(A)/\sim_0 \to \mathcal{P}_\infty(B)/\sim_0$ by $\widetilde{\varphi}([p]) = \pi \circ \varphi(p) (= [\varphi(p)])$.

Proposition 4.2. *Let A and B be C^*-algebras and $\varphi : A \to B$ a continuous $*$-homomorphism. Then there exists a group homomorphism $K_0(\varphi) : A \to B$ satisfying $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ for all $p \in \mathcal{P}_\infty(A)$.*

Proof. Recall that $K_0(A) = G(\mathcal{P}_\infty(A)/\sim_0)$, where there is a monoid homomorpism $[\cdot]_0 : A \to K_0(A)$. By the previous paragraph, we have a monoid homomorphism

$$\widetilde{\varphi} : \mathcal{P}_\infty(A)/\sim_0 \to \mathcal{P}_\infty(B)/\sim_0 .$$

By Proposition 4.1, let $K_0 = \widetilde{\varphi}_G$, and let ι_A, ι_B be the "inclusion" from $\mathcal{D}(A) \to K_0(A)$ and $\mathcal{D}(B) \to K_0(B)$ respectively, as in Proposition 2.14. Then

$$\begin{aligned} K_0(\varphi)([p]_0) &= K_0(\varphi)(\iota_A([p]_\mathcal{D})) = \widetilde{\varphi}_G([([p]_\mathcal{D}, [0]_\mathcal{D})]_G) \\ &= [(\widetilde{\varphi}([p]_\mathcal{D}), \widetilde{\varphi}([0]_\mathcal{D})]_G = \iota_B \circ \widetilde{\varphi}([p]_\mathcal{D}) \\ &= \iota_B([\varphi(p)]_\mathcal{D}) = [\varphi(p)]_0 \; \blacksquare \end{aligned}$$

Proposition 4.3. *Let A be a unital C^*-algebra, then $K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$, and $[0]_0 = 0$.*

Proof. Every element of $K_0(A)$ can be written as $[([p]_\mathcal{D}, [q]_\mathcal{D})]_G$ for some $p, q \in \mathcal{P}_\infty(A)$, and

$$\begin{aligned} [([p]_\mathcal{D}, [q]_\mathcal{D})]_G &= [([p]_\mathcal{D}, 0)]_G + [(0, [q]_\mathcal{D})]_G \\ &= [([p]_\mathcal{D}, 0)]_G - [([q]_\mathcal{D}, 0)]_G. \end{aligned}$$

Also,

$$[0]_0 = [([0]_\mathcal{D}, 0)]_G = [(0,0)]_G = 0. \; \blacksquare$$

Proposition 4.4. *Let A, B and C be C^*-algebras, let $\varphi : A \to B$ and $\psi : B \to C$ be continuous $*$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, let 0 denote the zero map between any two C^*-algebras, then $K_0(0) = 0$, the zero group map.*

Proof. By Proposition 4.3, every element in $K_0(A)$ is of the form $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Computing using Proposition 4.2,

$$\begin{aligned} K_0(\psi) \circ K_0(\varphi)([p]_0 - [q]_0) &= K_0(\psi)\left(K_0(\varphi)([p]_0) - K_0(\varphi)([q]_0)\right) \\ &= K_0(\psi)\left([\varphi(p)]_0 - [\varphi(q)]_0\right) \\ &= [\psi \circ \varphi(p)]_0 - [\psi \circ \varphi(q)]_0 \\ &= K_0(\psi \circ \varphi)([p]_0 - [q]_0). \end{aligned}$$

Moreover,

$$K_0(0)([p]_0 - [q]_0) = [0(p)]_0 - [0(q)]_0 = 0 - 0 = 0. \blacksquare$$

Corollary 4.5. *The map K_0 is a (covariant) functor, with K_0 on C*-algebras defined as in Definition 2.17 and K_0 on continuous $*$-morphisms defined as in Proposition 4.2.*

Proof. Simply collect the results from Propositions 4.2 and 4.4. ■

5 K_0 of general C*-algebras

Let A be a C*-algebra, possibly non-unital. Let $\widetilde{A}$ denote the unitization of A. Then $\widetilde{A} = A \oplus \mathbb{C}I$ as a vector space, and A is an ideal in $\widetilde{A}$. Let ι_I, ι_A be the inclusion maps from $\mathbb{C}I$ and A into $\widetilde{A}$ respectively, and let π_I and π_A be the natural quotient maps from $\widetilde{A}$ onto $\mathbb{C}I$ and A respectively. Both $\widetilde{A}$ and $\mathbb{C}I$ are unital C*-algebras. Their K_0 groups are defined as in the first section. Also, the inclusion ι_I induces a group homomorphism $K_0(\iota_I) : K_0(\mathbb{C}I) = \mathbb{Z} \to \widetilde{A}$.

Definition 5.1. Let A be a C*-algebra. Define $\overline{K}_0(A) = \ker K_0(\pi_I)$.

Proposition 5.2. *Let A be a C*-algebra. Then*

$$\begin{aligned}\overline{K_0}(A) &= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(\widetilde{A}),\ \pi_I(p) \sim_0 \pi_I(q)\} =: S_1 \\ &= \{([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) : p, q \in \mathcal{P}_\infty(\widetilde{A})\} =: S_2 \\ &= \{[p]_0 - [\pi_I(p)]_0 : p \in \mathcal{P}_\infty(\widetilde{A})\} =: S_3\end{aligned}$$

Proof. Let $g \in K_0(\widetilde{A})$ and $g \in \ker K_0(\pi_I)$. Then there exists some $n \in \mathbb{N}$ and $p, q \in \mathcal{P}_n(\widetilde{A})$ such that $g = [p]_0 - [q]_0$, and that

$$0 = K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.$$

So $\pi_I(p) \sim_0 \pi_I(q)$. Conversely suppose $\pi_I(p) \sim_0 \pi_I(q)$, then

$$K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.$$

This proves the first equality.

With the first equality in mind, suppose $\pi_I(p) \sim_0 \pi_I(q)$. Then

$$[p]_0 - [q]_0 = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.$$

So $\overline{K_0}(A) = S_1 \subseteq S_2$. And

$$\begin{aligned} &K_0(\pi_I)\left(([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0)\right) \\ &= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I \circ \pi_I(p)]_0 - [\pi_I \circ \pi_I(q)]_0) \\ &= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \\ &= 0 \end{aligned}$$

So $S_2 \subseteq \overline{K}_0(A)$, this proves the second equality.

Clearly $S_3 \subseteq S_2$. Take

$$g = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.$$

Suppose q is $n \times n$, and let $p' = p \oplus (I_n - q)$. Then

$$[p']_0 = [p]_0 - [q_0] + [I_n]_0.$$

Also

$$\pi_I(p') = \pi_I(p) \oplus (I_n - \pi_I(q)),$$

so

$$[\pi_I(p')]_0 = [\pi_I(p)]_0 - [\pi_I(q)]_0 + [I_n]_n.$$

Thus $[p']_0 - [\pi_I(p)]_0 = g$, this proves $S_2 = S_3$. ∎

The above gives a definition for the K_0 group of non-unital C*-algebras, and defines another abelian group for a unital C*-algebra. We need to verify that it coincides with the previous definition for the unital case.

Lemma 5.3. *Let A be a unital C*-algebra. Let 1_A denote the identity of A, and let $\widetilde{A} = A \oplus \mathbb{C}I$ as vector space. Then $\widetilde{A} \cong A \oplus \mathbb{C}J$. The C*-algebra $A \oplus \mathbb{C}J$ is defined with norm $\|a + zJ\| = \max(\|a\|, |z|)$ and involution $(a + zJ)^* = a^* + \overline{z}J$.*

Proof. Define $\tau : A \oplus \mathbb{C}J \to \widetilde{A}$ by $a \oplus zJ \mapsto a + z(I - 1_A)$. This is clear a vector space isomorphism and respects the involution. Lastly,

$$\begin{aligned}
&\tau(a \oplus zJ)\tau(b \oplus wJ) \\
&= (a + z(I - 1_A))(b + w(I - 1_A)) \\
&= ab + w(aI - a1_A) + z(Ib - 1_A b) + zw(II - I1_A - 1_A I + 1_A 1_A) \\
&= ab + w(a - a) + z(b - b) + zw(I - 1_A - 1_A + 1_A) \\
&= ab + zw(I - 1_A) \\
&= \tau(ab \oplus zwJ).
\end{aligned}$$

So τ is an isomorphism. ■

Remark 5.4. To gain an intuitive idea of the above lemma, consider the case of where $A = C(X)$ is the set of continuous functions from a compact Haudorff space X into the complex numbers. The unitization $\widetilde{C(X)}$ is isomorphic to $C(X \sqcup \{*\})$ (see Proposition 9.9). Let 1_A denote the function that is constantly 1 on X and zero on $*$. Let 1_* be the function that is 1 on $*$ and constantly zero on X. Then we have

$$C(X \sqcup \{*\}) \cong C(X) \oplus C(\{*\}) \cong C(X) \oplus \mathbb{C}1_*,$$

where $1_* = 1 - 1_A$. The proof of the lemma imitates this idea to prove it in the non-commutative case.

Proposition 5.5. *Let A be a unital C*-algebra, then $\overline{K}_0(A) \cong K_0(A)$.*

Proof. By the lemma above, $\widetilde{A} \cong A \oplus \mathbb{C}J$. Let $\iota_A : A \to A \oplus \mathbb{C}J$ be the natural inclusion map and $\pi_A : A \oplus \mathbb{C}J \to A$ the quotient map. The map $\tau : A \oplus \mathbb{C}J \to \widetilde{A}$ is defined in the previous proof. Define $\alpha : K_0(A) \to K_0(\widetilde{A})$ by

$$[p]_0 - [q]_0 \mapsto [\tau(\iota_A(p))]_0 - [\tau(\iota_A(q))]_0.$$

In other words, $\alpha = K_0(\tau \circ \iota_A)$. Since $\pi_I(\tau(\iota_A(p))) = 0 = \pi_I(\tau(\iota_A(q)))$, the image of α is indeed in $\overline{K}_0(A)$. Let $\beta = K_0(\pi_A \circ \tau^{-1}) : \overline{K}_0(A) \to K_0(A)$. Then,

$$\beta \circ \alpha = K_0(\pi_A \circ \tau^{-1}\tau \circ \iota_A) = K_0(\pi_A \circ \iota_A) = K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}.$$

For $\widetilde{p}, \widetilde{q} \in \mathcal{P}_\infty(\widetilde{A})$ with $\pi_I(\widetilde{p}) = \pi_I(\widetilde{q})$, let $p_1 = \tau \circ \iota_A \circ \pi_A \circ \tau^{-1}(\widetilde{p})$ and $p_2 = \widetilde{p} - p_1$. Then $p_1 + p_2 = \widetilde{p}$ and p_1, p_2 are orthogonal projections. Write $\widetilde{q} = q_1 + q_2$ in the same way. Since $\pi_I(\widetilde{p}) = \pi_I(\widetilde{q})$, by the way that τ is defined, we have that $p_2 = q_2$. So

$$[\widetilde{p}]_0 - [\widetilde{q}]_0 = ([p_1]_0 + [p_2]_0) - ([q_1]_0 + [q_2]_0) = [p_1]_0 - [q_1]_0,$$

and

$$\begin{aligned}(\alpha \circ \beta)([\widetilde{p}]_0 - [\widetilde{q}]_0) &= K_0(\tau \circ \iota_A \circ \pi_A \circ \tau^{-1})([\widetilde{p}]_0 - [\widetilde{q}]_0) \\ &= [p_1]_0 - [q_1]_0 = [\widetilde{p}]_0 - [\widetilde{q}]_0.\end{aligned}$$

Hence α and β are mutual inverses. ∎

Definition 5.6. Let A be a non-unital C*-algebra. Define $K_0(A) := \overline{K}_0(A)$.

Remark 5.7. By Proposition 5.5, we can safely write $K_0(A) = \overline{K}_0(A)$ for any unital C*-algebras A.

The description S_3 in Proposition 5.2 is the one will be used most often. Next is a discussion of when two elements in such description are equivalent.

Lemma 5.8. *Let A be a C*-algebra, $v \in M_{m,n}(A)$ and $w \in M_{n,k}(A)$ for some $k, m, n \in \mathbb{N}$. Then $\pi_I(vw) = \pi_I(v)\pi_I(w)$.*

Proof. We compute $\pi_I(vw)$ to be

$$\pi_I[(v - \pi_I(v))(w - \pi_I(w)) + \pi_I(v)(w - \pi_I(w)) + (v - \pi_I(v))w + \pi_I(v)\pi_I(w)]$$

Since A is an ideal in $\widetilde{A}$, all of $(v - \pi_I(v))(w - \pi_I(w))$, $\pi_I(v)(w - \pi_I(w))$ and $(v - \pi_I(v))w$ have entries in A, which are 0 when they are evaluated under π_I. So

$$\pi_I(vw) = \pi_I(\pi_I(v)\pi_I(w)) = \pi_I(v)\pi_I(w)$$

since $\pi_I(v)\pi_I(w) \in M_{k,l}(\mathbb{C}I)$. ∎

Lemma 5.9. *Let A be a C*-algebra, and let $p, q \in \mathcal{P}_\infty(\widetilde{A})$. Then $p \sim_0 q$ in $\mathcal{P}_\infty(\widetilde{A})$ implies $\pi_I(p) \sim_0 \pi_I(q)$.*

Proof. There exists a matrix v with entries in $\widetilde{A}$ such that $vv^* = p$ and $v^*v = q$. By Lemma 5.8,

$$\pi_I(p) = \pi_I(vv^*) = \pi_I(v)\pi_I(v^*) \sim_0 \pi_I(v^*)\pi_I(v) = \pi_I(v^*v) = \pi_I(q). \blacksquare$$

Proposition 5.10. *Let A be a C^*-algebra, and $p, q \in \mathcal{P}_\infty(\widetilde{A})$. The following are equivalent*

1. $[p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0$
2. *there exists $r_1, r_2 \in \mathcal{P}_\infty(\widetilde{A})$ with $p \oplus r_1 \sim_0 q \oplus r_2$*
3. *there exists $k, l \in \mathbb{N}$ such that $p \oplus I_k \sim_0 q \oplus I_l$ in $\mathcal{P}_\infty(\widetilde{A})$*

Proof. ($1 \implies 2$) The equality $[p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0$ implies that

$$[p \oplus \pi_I(q)]_0 = [p]_0 + [\pi_I(q)]_0 = [q]_0 + [\pi_I(p)]_0 = [q \oplus \pi_I(p)]_0$$

So let $r_1 = \pi_I(q)$ and $r_2 = \pi_I(p)$. This satisfies 2.
($2 \implies 3$) Since $r_i = \pi_I(r_i)$ for $i = 1, 2$, we see that r_1 and r_2 can be considered as matrices in $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$ respectively. Let $k = \operatorname{rank} r_1 \leq n$. Let $\{z_1, \ldots, z_k\}$ be an orthonormal basis of $\mathrm{Ran} r_1 \mathbb{C}^n$, and extend it to an orthonormal basis $\{z_1, \ldots, z_n\}$ of $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, and define $u \in M_n(\mathbb{C})$ by $uz_j = e_j$ for $j = 1, \ldots, n$. Then u is unitary since it takes an orthonormal basis to another one, and

$$ur_1u^*e_j = ur_1z_j = \begin{cases} uz_j = e_j & : j = 1, \ldots, k \\ u0 = 0 & : j = k+1, \ldots, n \end{cases}$$

So

$$r_1 \sim_0 ur_1u^* = I_k \oplus 0_{n-k} \sim_0 I_k.$$

By identifying u as a unitary matrix in $M_k(\mathbb{C}I)$, this also holds true in $\mathcal{P}_\infty(\widetilde{A})$. Similarly, $r_2 \sim_0 I_l$ in $\mathcal{P}_\infty(\widetilde{A})$ for $l = \operatorname{rank} r_2$. So

$$p \oplus I_k \sim_0 p \oplus r_1 \sim_0 q \oplus r_2 \sim_0 q \oplus I_l.$$

($3 \implies 1$) We use Lemma 5.9 here and compute

$$\begin{aligned} [p]_0 - [\pi_I(p)]_0 &= [p]_0 - [\pi_I(p)]_0 + [I_k]_0 - [I_k]_0 \\ &= [p \oplus I_k]_0 - [\pi_I(p) \oplus I_k]_0 \\ &= [p \oplus I_k]_0 - [\pi_I(p \oplus I_k)]_0 \\ &= [q \oplus I_l]_0 - [\pi_I(q \oplus I_l)]_0 \\ &= [q]_0 - [\pi_I(q)]_0. \blacksquare \end{aligned}$$

The next natural step is to extend the functor K_0 to all $*$-homomorphisms on all C*-algebras. Let A, B be C*-algebras. A $*$-homomorphism $\varphi : A \to B$ can be extended to a $*$-homomorphism $\widetilde{A} = A \oplus \mathbb{C}I_A \to \widetilde{B} = B \oplus \mathbb{C}I_B$ by $\widetilde{\varphi}|_A = \varphi$ and $\widetilde{\varphi}(I_A) = I_B$.

Definition 5.11. Let A, B be C*-algebras, $\varphi : A \to B$ a $*$-homomorphism. Define $\overline{K}_0(\varphi) = K_0(\widetilde{\varphi})|_{K_0(A)} : K_0(A) \to K_0(B)$. Then $\overline{K}_0(\varphi)$ is a well-defined group homomorphism.

Proof. Note that $\overline{K}_0(\varphi)$ is the restriction of $K_0(\widetilde{\varphi})$ to $K_0(A)$. So it is a group homomorphism. $\pi_I(\widetilde{\varphi}(p)) = \pi_I(\widetilde{\varphi}(q))$ by the way $\widetilde{\varphi}$ is defined. So the image of $\overline{K}_0(\varphi)$ is in $K_0(B)$. ■

Proposition 5.12. *Let A, B be unital C*-algebras, let $\alpha : K_0(A) \to \overline{K}_0(A)$ be the group isomorphism described in the proof of Proposition 5.5, and similarly let $\beta : K_0(B) \to \overline{K}_0(B)$ be such group isomorphism. Then for any group homomorphism $\varphi : A \to B$, we have*

$$\overline{K}_0(\varphi) \circ \alpha = \beta \circ K_0(\varphi).$$

Proof. We adopt all notation used in Proposition 5.5, where $\alpha = K_0(\tau_A \circ \iota_A)$ and $\beta = K_0(\tau_B \circ \iota_A)$. Then

$$\beta \circ K_0(\varphi) = K_0(\tau_B \circ \iota_A) \circ K_0(\varphi) = K_0(\tau_B \circ \iota_B \circ \varphi)$$

and

$$\overline{K}_0(\varphi) \circ \alpha = K_0(\widetilde{\varphi})|_{\overline{K}_0(A)} \circ K_0(\tau_A \circ \iota_A) = K_0(\widetilde{\varphi} \circ \tau_A \circ \iota_A).$$

For $a \in A$,

$$\tau_B \circ \iota_B \circ \varphi(a) = \varphi(a) \oplus 0I_B = \widetilde{\varphi} \circ \tau_A \circ \iota_A(a).$$

So $\tau_B \circ \iota_B \circ \varphi = \widetilde{\varphi} \circ \tau_A \circ \iota_A$ as maps $A \to \widetilde{B}$, so applying K_0 they are the same as maps from $K_0(A)$ to $K_0(\widetilde{B})$ whose image lie in $\overline{K}_0(B)$. This concludes the proof. ■

Remark 5.13. By the above proposition and Proposition 5.5, we can safely write $\overline{K}_0(\varphi) = K_0(\varphi)$ for any $*$-homomorphism φ.

Proposition 5.14. *Let A, B, C be C*-algebras, and let $\varphi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ and $K_0(0) = 0$ for 0 any zero map.*

Proof. We compute:

$$\begin{aligned} K_0(\psi) \circ K_0(\varphi) &= K_0(\widetilde{\psi})|_{K_0(B)} \circ K_0(\widetilde{\varphi})|_{K_0(A)} \\ &= K_0(\widetilde{\psi} \circ \widetilde{\varphi})|_{K_0(A)} \\ &= K_0(\widetilde{\psi \circ \varphi})|_{K_0(A)} \\ &= K_0(\psi \circ \varphi). \end{aligned}$$

Similarly,

$$\begin{aligned} K_0(\mathrm{id}_A) &= K_0(\widetilde{\mathrm{id}_A})|_{K_0(A)} \\ &= K_0(\mathrm{id}_{\widetilde{A}})|_{K_0(A)} \\ &= \mathrm{id}_{K_0(\widetilde{A})}|_{K_0(A)} \\ &= \mathrm{id}_{K_0(A)}. \end{aligned}$$

Finally,

$$K_0(0) = K_0(\widetilde{0})|_{K_0(A)} = K_0(\pi_I)|_{K_0(A)}.$$

But $K_0(A)$ is exactly $\ker K_0(\pi_I)$, so $K_0(0) = 0$. ∎

Now we have a functor K_0 from the category of C*-algebras to the category of abelian groups.

6 Functorial properties of K_0

The K_0-group of a C*-algebra can be difficult to compute even for most C*-algebras. With the functoriality of K_0 in hand, some useful properties of the functor K_0 will aid calculation. One might say this is similar to how exact sequences help the computation of cohomology groups. In fact, K_0 is an extraordinary cohomology functor, but this will not be discussed here. In short summary, the most basic and important properties of the functor K_0 are homotopy invariance, half exactness and split exactness. Also, K_0 is a continuous functor, meaning that the inductive limit K_0-group is isomorphic to the K_0-group of inductive limits. Other useful tools for computing the K_0-groups include the higher K-groups, Bott periodicity, and the 6-term exact sequence. In this paper we will only prove the three basic functorial properties of K_0.

Definition 6.1. Let A and B be C*-algebras and $\varphi, \psi : A \to B$ be $*$-homomorphisms. We say φ is **homotopic** to ψ, written $\varphi \sim_h \psi$, if there exists a family of continuous $*$- homomorphisms $\varphi_t : A \to B$ for $t \in [0,1]$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$, and that for each $a \in A$, $t \mapsto \varphi_t(a)$ is a continuous map $[0,1] \to B$. The family φ_t is called a homotopy from φ to ψ.

Let A and B be C*-algebras. We say A is **homotopic** to B, written $A \sim_h B$, if there exists $\varphi : A \to B$ and $\psi : B \to A$ continuous $*$-homomorphisms such that $\varphi \circ \psi \sim_h \mathrm{id}_A$ and $\psi \circ \varphi \sim_h \mathrm{id}_B$.

6.1 Homotopy invariance

Proposition 6.2. *Let A and B be C*-algebras, $\varphi, \psi : A \to B$ be continuous $*$-homomorphisms with $\varphi \sim_h \psi$, then $K_0(\varphi) = K_0(\psi)$. If $A \sim_h B$, then $K_0(A) \cong K_0(B)$.*

Proof. Once again, a typical element in $K_0(A)$ is $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Hence it is sufficient to show that $K_0(\varphi)(p) = K_0(\psi)(p)$ for all $p \in \mathcal{P}_\infty$. Let φ_t be a homotopy from φ to ψ. The family φ_t extends to a homotopy from φ to ψ on $M_n(A)$. The map $[0,1] \to M_n(B)$ given by $t \mapsto \varphi_t(p)$ is continuous, and since each φ_t is a $*$-homomorphism, $\varphi_t(p) \in \mathcal{P}_n(B)$, so $t \mapsto \varphi_t(p)$ is a homotopy of

$$\varphi(p) = \varphi_0(p) \sim_h \varphi_1(p) = \psi(p).$$

But we know homotopic projections are equivalent in $\mathcal{D}(A)$, so

$$K_0(\varphi)(p) = [\varphi(p)]_0 = [\psi(p)]_0 = K_0(\psi)(p).$$

Hence $K_0(\varphi) = K_0(\psi)$.

Suppose $A \sim_h B$. There exists continuous homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\alpha \circ \beta \sim_h \mathrm{id}_A$ and $\beta \circ \alpha \sim_h \mathrm{id}_B$. Then using Proposition 4.4 and the first half of this proof,

$$K_0(\alpha) \circ K_0(\beta) = K_0(\alpha \circ \beta) = K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)},$$

$$K_0(\beta) \circ K_0(\alpha) = K_0(\beta \circ \alpha) = K_0(\mathrm{id}_B) = \mathrm{id}_{K_0(B)}.$$

Hence $K_0(\alpha) : K_0(A) \to K_0(B)$ is a group isomorphism, whose inverse is $K_0(\beta)$. ∎

6.2 Half- and split-exactness

Definition 6.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a functor.

1. $\mathscr{F}$ is exact if whenever

   $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   is a short exact sequence in $\mathscr{C}$, then

   $$0 \longrightarrow \mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(C) \longrightarrow 0$$

   is exact in $\mathscr{D}$.

2. $\mathscr{F}$ is half exact if whenever

   $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   is a short sequence in $\mathscr{C}$, then

   $$\mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(C)$$

   is sequence in $\mathscr{D}$ that is exact at $\mathscr{F}(B)$.

3. $\mathscr{F}$ is split exact if whenever

   $$0 \longrightarrow A \xrightarrow{f} B \underset{h}{\overset{g}{\rightleftarrows}} C \longrightarrow 0$$

   is a split exact sequence in $\mathscr{C}$, then

   $$0 \longrightarrow \mathscr{F}(A) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(B) \underset{\mathscr{F}(h)}{\overset{\mathscr{F}(g)}{\rightleftarrows}} \mathscr{F}(C) \longrightarrow 0$$

   is a split exact sequence in $\mathscr{D}$.

Clearly an exact functor would be half-exact. In this section we will show that the functor K_0 is half-exact and split-exact. However, K_0 is not a exact functor. We will see a counterexample in a later section when we have developed more machinery.

Lemma 6.4. *Let*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

be a short exact sequence of C-algebras, and let* $n \in \mathbb{N}$*. Let* $\widetilde{\varphi} : M_n(\widetilde{A}) \to M_n(\widetilde{B})$ *and* $\widetilde{\psi} : M_n(\widetilde{B}) \to M_n(\widetilde{C})$ *be the unital* $*$*-homomorphisms induced by* φ *and* ψ*, respectively. Then,*

1. The map $\widetilde{\varphi} : M_n(\widetilde{A}) \to M_n(\widetilde{B})$ *is injective.*

2. An element $a \in M_n(\widetilde{B})$ *belongs to the image of* $\widetilde{\varphi}$ *if and only if* $\widetilde{\psi}(a) = \pi_I(\widetilde{\psi}(a))$*.*

Proof. 1. The map $\widetilde{\varphi} : A \oplus \mathbb{C}I_A \to B \oplus \mathbb{C}I_B$ is injective on both A and $\mathbb{C}I_A$. Therefore it is injective $\widetilde{A} \to \widetilde{B}$, and also the induced map $\widetilde{\varphi} : M_n(\widetilde{A}) \to M_t(\widetilde{B})$ is continuous.

2. For $a \in A$ and $z \in \mathbb{C}$,

$$\begin{aligned}\widetilde{\psi} \circ \widetilde{\varphi}(a + zI_A) &= \widetilde{\psi}(\varphi(a) + zI_B) = \psi \circ \varphi(a) + zI_C = zI_C \\ &= \pi_I(\widetilde{\psi} \circ \widetilde{\varphi}(a + zI_A)).\end{aligned}$$

Conversely, suppose $b \in B$ and $z \in \mathbb{C}$ with

$$\psi(b) + zI_C = \widetilde{\psi}(b + zI_B) = \pi_I(\widetilde{\psi}(b + zI_B)) = zI_C.$$

Then $\psi(b) = 0$. By exactness there exists $a \in A$ such that $\varphi(a) = b$, then $b + zI_B = \widetilde{\varphi}(a + zI_A)$. ∎

Proposition 6.5. K_0 *is half-exact.*

Proof. Let A, B and C be C*-algebras with $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$, where φ is injective, ψ is surjective, and $\text{im}(\varphi) = \ker(\psi)$.

A typical element in $K_0(A)$ is $[p]_0 - [\pi_I(p)]_0$ for some $p \in \mathcal{P}_\infty(\widetilde{A})$. By Lemma 6.4 the equation

$$\widetilde{\psi} \circ \widetilde{\varphi}(p) = \pi_I(\widetilde{\psi} \circ \widetilde{\varphi}(p)) = \widetilde{\psi} \circ \widetilde{\varphi}(\pi_I(p))$$

holds. So

$$K_0(\psi) \circ K_0(\varphi)([p]_0 - [\pi(p)]_0) = [\widetilde{\psi} \circ \widetilde{\varphi}(p)]_0 - [\widetilde{\psi} \circ \widetilde{\varphi}(\pi_I(p))]_0 = 0.$$

So $\mathrm{im}(K_0(\varphi)) \subseteq \ker(K_0(\psi))$.

Conversely, let $[p]_0 - [\pi_I(p)]_0 \in K_0(B)$ be in the kernel of $K_0(\psi)$. Since $\widetilde{\psi}(p) \sim_0 \widetilde{\psi}(\pi_I(p))$ in $\mathcal{P}_n(C)$ for some $n \in \mathbb{N}$, by Proposition 3.15 there exists a unitary element $u \in M_{2n}(C)$ such that

$$u(\widetilde{\psi}(p) \oplus 0_n)u^* = \widetilde{\psi}(\pi_I(p)) \oplus 0_n.$$

By Lemma 3.10 there exists a unitary $v \in M_{4n}(B)$ such that $\widetilde{\psi}(v) = u \oplus u^*$. Let $p_1 = v(p \oplus 0_{3n})v^*$. Then

$$p \sim_0 p \oplus 0_{3n} \sim_0 p_1,$$

and similarly $\pi_I(p) \sim_0 \pi_I(p_1)$. Also,

$$\begin{aligned}
\widetilde{\psi}(p_1) &= \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \widetilde{\psi}(p) \oplus 0_n & 0 \\ 0 & 0_{2n} \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u \end{bmatrix} \\
&= \begin{bmatrix} u(\widetilde{\psi}(p) \oplus 0_n)u^* & 0 \\ 0 & 0_{2n} \end{bmatrix} \\
&= \pi_I(\widetilde{\psi}(p)) \oplus 0_{3n}.
\end{aligned}$$

It follows that $\widetilde{\psi}(p_1) = \pi_I(\widetilde{\psi}(p_1))$. By Lemma 6.4 there exists $e \in M_{3n}$ such that $\widetilde{\varphi}(e) = p_1$. Also,

$$\widetilde{\varphi}(ee) = \widetilde{\varphi}(e)\widetilde{\varphi}(e) = p_1 p_1 = p_1,$$

$$\widetilde{\varphi}(e^*) = p_1^* = p_1.$$

By Lemma 6.4, $\widetilde{\varphi} : M_{4n}(\widetilde{A}) \to M_{4n}(\widetilde{B})$ is injective, which implies $e = ee = e^*$, and hence e is a projection. Now

$$K_0(\varphi)([e]_0 - [\pi_I(e)]_0) = [p_1]_0 - [\pi_I(p_1)]_0 = [p]_0 - [\pi_I(p)]_0.$$

This shows that $\ker K_0(\psi) \subseteq \mathrm{im}\, K_0(\varphi)$. Therefore $\ker K_0(\psi) = \mathrm{im}\, K_0(\varphi)$. ■

Proposition 6.6. *The functor K_0 is split-exact.*

Proof. Suppose

$$0 \longrightarrow A \xrightarrow{\ \varphi\ } B \underset{\lambda}{\overset{\psi}{\rightleftarrows}} C \longrightarrow 0$$

is a split exact sequence of C*-algebras. By the half-exactness just proved, the sequence

$$K_0(A) \xrightarrow{K_0(\varphi)} K_0(B) \xrightarrow{K_0(\psi)} K_0(C)$$

is exact. Also, since K_0 is a functor, we have

$$K_0(\psi) \circ K_0(\lambda) = K_0(\psi \circ \lambda) = K_0(\mathrm{id}_C) = \mathrm{id}_{K_0(C)},$$

so the sequence is also exact at $K_0(C)$. It is left to show that $K_0(\varphi)$ is injective.

Let $g \in K_0(A)$ be in the kernel of $K_0(\varphi)$. By the proof of Proposition 6.5, there exits some $n \in \mathbb{N}$, $p \in \mathcal{P}_n(\widetilde{A})$ and some unitary $u \in M_n(\widetilde{B})$ such that $g = [p]_0 - [\pi_I(p)]_0$ and $u\widetilde{\varphi}(p)u^* = \pi_I(\widetilde{\varphi}(p))$. Let $v = (\widetilde{\lambda} \circ \widetilde{\psi})(u^*)u$. Then

$$v^*v = u^*(\widetilde{\lambda} \circ \widetilde{\psi}(u))(\widetilde{\lambda} \circ \widetilde{\psi}(u^*))u = u^* I_n u = I_n,$$

$$vv^* = (\widetilde{\lambda} \circ \widetilde{\psi}(u^*))uu^*(\widetilde{\lambda} \circ \widetilde{\psi}(u)) = I_n,$$

and

$$\widetilde{\psi}(v) = (\widetilde{\psi} \circ \widetilde{\lambda} \circ \widetilde{\psi}(u^*))(\widetilde{\psi}(u)) = \widetilde{\psi}(u^*)\widetilde{\psi}(u) = \widetilde{\psi}(I_n) = I_n.$$

Since $\widetilde{\psi}(v) = \pi_I(\widetilde{\psi}(v))$, by Lemma 6.4, there exists $w \in M_n(\widetilde{A})$ such that $\widetilde{\varphi}(w) = v$. Since $\widetilde{\varphi}$ is injective and $\widetilde{\varphi}(w^*w) = I_n = \widetilde{\varphi}(ww^*)$, have $ww^* = I_n = w^*w$, so w is unitary. Moreover,

$$\begin{aligned}
\widetilde{\varphi}(wpw^*) &= v\widetilde{\varphi}(p)v^* = (\widetilde{\lambda} \circ \widetilde{\psi})(u^*)u\widetilde{\varphi}(p)u^*(\widetilde{\lambda} \circ \widetilde{\psi})(u) \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})(u^*)\pi_I(\widetilde{\varphi}(p))(\widetilde{\lambda} \circ \widetilde{\psi})(u) \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})(u^*\pi_I(\widetilde{\varphi}(p))u) \\
&= (\widetilde{\lambda} \circ \widetilde{\psi})(\widetilde{\varphi}(p)) = \widetilde{\lambda}((\widetilde{\psi} \circ \widetilde{\varphi})(p)) \\
&= \widetilde{\lambda}((\widetilde{\psi} \circ \widetilde{\varphi})(\pi_I(p))) \\
&= \widetilde{\varphi}(\pi_I(p)).
\end{aligned}$$

By the injectivity of $\widetilde{\varphi}$ we can conclude that $\pi_I(p) = wpw^*$. Hence $p \sim_0 \pi_I(p)$ in $\mathcal{P}_n(\widetilde{A})$. Therefore $g = 0$. ∎

Corollary 6.7. *Let A and B be C*-algebras. Then $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.*

Proof. The sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \rightleftarrows B \longrightarrow 0$$

is split-exact. Hence by the split-exactness of K_0, we have a split-exact sequence of abelian groups:

$$0 \longrightarrow K_0(A) \longrightarrow K_0(A \oplus B) \rightleftarrows K_0(B) \longrightarrow 0.$$

Therefore $K_0(A) \oplus K_0(B) \cong K_0(A \oplus B)$. ∎

7 K-theory of compact Hausdorff spaces

Definition 7.1. Let X be a Hausdorff topological space, V and W topological vector bundles over X. Define the map $\pi_V : V \to X$ by $\pi_V(v) = x$ if $v \in V_x$. We write $\pi = \pi_V$, when it is understood that π has domain V. A map $\varphi : V \to W$ is a bundle homomorphism if φ is continuous, $\varphi(v) \in \pi_W^{-1}(\pi_V(v))$ for all $v \in V$, and that $\varphi_x = \varphi|_{V_x} : V_x \to W_x$ is a linear homomorphism for all $x \in X$. We say V is isomorphic to W if there exists $\varphi : V \to W$ and $\psi : W \to V$ bundle homomorphisms such that $\varphi \circ \psi = \mathrm{id}_V$ and $\psi \circ \varphi = \mathrm{id}_W$.

Definition 7.2. Let X be a Hausdorff space and let $n \in \mathbb{N}$. Define $\Theta^n(X)$ to be the rank-n trivial bundle over X; specifically, $\Theta^n(X) = X \times \mathbb{C}^n$.

Definition 7.3. For X a Hausdorff space, define $\mathrm{Vect}(X)$ to be the set of all isomorphism classes of topological vector bundles on X.

Definition 7.4. Let X be a Hausdorff space, define $C(X)$ to be the set of all continuous functions from X to $\mathbb{C}$. If X is compact, then $C(X)$ can be equipped with the sup-norm as the norm and with pointwise conjugation as its involution. This gives $C(X)$ a C*-algebra structure.

Remark 7.5. Let $\mathscr{C}$ be the category of compact Hausdorff spaces and let $\mathscr{A}$ be the category of unital C*-algebras. Define a contravariant functor $\mathscr{F} : \mathscr{C} \to \mathscr{A}$ as follows. If X is a compact Hausdorff space, then $\mathscr{F}(X) = C(X)$. If X, Y are compact Hausdorff spaces and $\varphi \in \mathrm{Hom}(X, Y)$, then $\mathscr{F}(\varphi) = \varphi^* \in \mathrm{Hom}(C(Y), C(X))$ where $\varphi^* f(x) = f(\varphi(x))$ for all $f \in C(Y)$ and $x \in X$, where $\mathrm{Hom}(X, Y)$ is the set of continuous functions from X to Y, and $\mathrm{Hom}(C(Y), C(X))$ is the set of $*$-homomorphisms from $C(Y)$ to $C(X)$.

If X is a Hausdorff space, not necessarily compact, then $C(X)$ is not necessarily a C*-algebra since the sup-norm cannot be defined. However $C(X)$ is a ring, so for $m, n \in \mathbb{N}$, it makes sense to consider $M_{m,n}(C(X))$, all m by n matrices with entries in $C(X)$. Note that $M_{m,n}(C(X))$ is naturally isomorphic to $C(X, M_{m,n}(\mathbb{C}))$, by taking a matrix $F \in M_{m,n}(C(X))$ to $f \in C(X, M_{m,n}(\mathbb{C}))$, where $[f(x)]_{ij} = F_{ij}(x)$ for all $x \in X$.

Lemma 7.6. *Let X be a Hausdorff space, and let $m, n \in \mathbb{N}$. For every $f \in C(X, M_{m,n}(\mathbb{C}))$, define a bundle homomorphism $\Gamma(f) : \Theta^n(X) \to \Theta^m(X)$ by $\Gamma(f)(x, v) = (x, f(x)v)$. Then $\Gamma : f \mapsto \Gamma(f)$ is a bijection from $C(X, M_{m,n}(\mathbb{C}))$ to Hom$(\Theta^n(X), \Theta^m(X))$. In other words, we have a one-to-one correspondence between Hom$(\Theta^n(X), \Theta^m(X))$ and $C(X, M_{m,n}(\mathbb{C})) = M_{m,n}(C(X))$.*

Proof. Suppose $f, g \in M_{m,n}(C(X))$ with $f \neq g$. Pick $x \in X$ for which $f(x) \neq g(x)$. Then there exists $v \in \mathbb{C}^n$ for which $g(x)v \neq f(x)v$, which shows that Γ is injective. It is left to show that Γ is surjective.

Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be equipped with their standard inner products. Define $p : \Theta^n(X) \to \mathbb{C}^n$ by $p(x, w) = w$. Suppose $\varphi : \Theta^n(X) \to \Theta^m(X)$ is a bundle homomorphism. Define $f : X \to M_{m,n}(\mathbb{C})$ so that

$$f(x)_{ij} = \langle p(\varphi(x, e_j)), e_i \rangle$$

for all $x \in X$. Clearly f is continuous. Moreover,

$$\begin{aligned}
\Gamma(f)(x, v) &= (x, f(x)v) \\
&= (x, \sum_{i=1}^{m} \sum_{j=1}^{n} f(x)_{ij} v_j e_i) \\
&= (x, \sum_{i=1}^{m} \sum_{j=1}^{n} \langle p(\varphi(x, e_j)), e_i \rangle v_j e_i) \\
&= (x, \sum_{i=1}^{m} \sum_{j=1}^{n} \langle p(\varphi(x, v_j e_j)), e_i \rangle e_i) \\
&= (x, \sum_{i=1}^{m} \langle p(\varphi(x, v)), e_i \rangle e_i) \\
&= (x, \varphi(x, v))
\end{aligned}$$

for all $(x, v) \in \Theta^n(X)$. Thus $\Gamma(f) = \varphi$, and we conclude that Γ is surjective. ∎

Lemma 7.7. *Let V and W be vector bundles over a compact Hausdorff space X, and suppose that $\varphi : V \to W$ is a bundle homomorphism such that φ_x is a vector space isomorphism for every $x \in X$. Then φ is a bundle isomorphism.*

Proof. Let $X_1, \dots X_k$ be the connected components of X, let $V_j = V|_{X_j}$ and $W_j = W|_{X_j}$ for $j = 1, \dots, k$. If $\varphi : V \to W$ is a bundle homomorphism such that $\varphi|_{V_j}$ is an isomorphism from V_j onto W_j, then φ is an isomorphism from V onto W. Thus for the rest of the proof we may assume that X is connected.

By hypothesis φ is a bijection, so φ^{-1} is defined, with $\varphi^{-1}|_x$ a vector space isomorphism. We need to check that φ^{-1} is continuous. Choose an open cover $\{U_1, \dots, U_l\}$ for which $V|_{U_k}$ and $W|_{U_k}$ are trivial for $k = 1, \dots, l$. For each k, let $\varphi_k = \varphi|_{V|_{U_k}}$. Then it is sufficient to show that φ_k^{-1} is continuous.

Let n be the rank of V and W. We can identify $V|_{U_k}$ and $W|_{U_k}$ with $\Theta^n(U_k)$, and can consider φ_k to be a bundle isomorphism from $\Theta^n(U_k)$ to itself. Apply Lemma 7.6 to obtain a continuous function $f_k : U_k \to M_n(\mathbb{C})$ such that $\varphi_k(x, v) = (x, f_k(x)v)$ for all $(x, v) \in \Theta^n(U_k)$. Since $\varphi_k(x)$ is an isomorphism for all $x \in U_k$, have $f_k(x) \in GL_n(\mathbb{C})$ for all $x \in U_k$.

Each f_k is an element of $C(U_k, M_n(\mathbb{C}))$. The matrix $f_k(x)$ is invertible for every $x \in U_k$, since inversion is continuous, we have that $f^{-1}(x) \in C(U_k, M_n(\mathbb{C}))$. Apply the lemma again have φ_k^{-1} is continuous. ∎

Proposition 7.8. *Let V be a vector bundle over a compact Hausdorff space X. Then V is isomorphic to a subbundle of the trivial bundle $\Theta^N(X)$ for some $N \in \mathbb{N}$.*

Proof. Let $X_1, \dots, X_m$ be the distinct connected components of X. If $V|_{X_k}$ is a subbundle of $\Theta^{N_k}(X_k)$ for some $N_k \in \mathbb{N}$, then let $N = N_1 + N_2 + \cdots + N_m$, and V is itself a subbundle of $\Theta^N(X)$. So for the rest of the proof we may assume that X is connected.

Since V is locally trivial, let $\mathcal{U} = \{U_1, \dots, U_l\}$ be an open cover of X such that $V|_{U_k} \cong \Theta^M(U_k)$ for some $M \in \mathbb{N}$. (Note that this M is the same for all k since X is connected.) Let $\varphi_k : V|_{U_k} \to \Theta^M(U_k)$ be a bundle isomorphism. Define $q_k : \Theta^M(U_k) \to \mathbb{C}^M$ by $q_k(x, w) = w$ for $x \in U_k$ and $w \in \mathbb{C}^M$; also let $\pi : V \to X$ be projection onto the point in X that an element $v \in V$ lies above. Choose a partition of unity $\{f_1, \dots, f_l\}$ subordinate to the cover $\mathcal{U}$, and let $N = M \cdot l$. Then define $\Phi : V \to \bigoplus_{k=1}^{l} \mathbb{C}^M$ by

$$\Phi(v) = (f_1(\pi(v))q_1(\varphi_1(v)) \oplus \cdots \oplus f_l(\pi(v))q_l(\varphi_l(v))).$$

Then $\varphi(v) = (\pi(v), \Phi(v))$ defines a bundle homomorphism $V \to \Theta^N(X)$. Since φ is injective, this is a bijective homomorphism onto a subbundle of $\Theta^N(X)$. By Lemma 7.7 this is indeed an isomorphism. ∎

Corollary 7.9. *Every vector bundle over a compact Hausdorff space admits a Hermitian metric.*

Proof. It is clear that every trivial bundle naturally has a Hermitian metric, and since every bundle over a compact Hausdorff space is a subbundle of some trivial bundle, then it inherits the restriction of the Hermitian metric. ∎

Definition 7.10. Let X be a Hausdorff space, and let $[V], [W] \in \text{Vect}(X)$. Define $[V \oplus W]$ to be the isomorphism class of bundles as follows. There exists $n, m \in \mathbb{N}$ such that V is a subbundle of $\Theta^n(X)$ and W is a subbundle of $\Theta^m(X)$. Let Q be the subbundle of $\Theta^{n+m}(X)$ such that $Q_x = V_x \oplus W_x \subseteq \mathbb{C}^n \oplus \mathbb{C}^m$ for all $x \in X$. Define $[V \oplus W]$ to be $[Q]$.

Proposition 7.11. *Let X be a compact Hausdorff space, and let V, W be vector bundles over X. Then $[V \oplus W]$ is well-defined and it is a vector bundle.*

Proof. The proof is easy and is left as an exercise for the reader. ∎

Remark 7.12. The vector bundle $V \oplus W$ is called the Whitney sum of V and W. The general construction is more abstract and it may take some work to check the bundle definitions. Proposition 7.8 allows for a concrete description of the class $[V \oplus W]$. Also, in K-theory it is more helpful to think of a vector bundle as a subbundle of some trivial bundle, as we will see when we relate the topological K-theory to the C*-algebra K-theory.

Proposition 7.13. *Let X be a compact Hausdorff space. The set Vect(X) equipped with the operation $[V] + [W] = [V \oplus W]$, is an abelian monoid.*

Proof. The only non-trivial part is to verify that $[V] + [W] = [W] + [V]$. Suppose V is a subbundle of $\Theta^n(X)$ and W is a subbundle of $\Theta^m(X)$. We'll write $V \oplus W$ and $W \oplus V$ as the corresponding subbundles of $\Theta^{n+m}(X)$. Let $\rho : V \oplus W \to W \oplus V$ be such that

$$\rho(x, v \oplus w) = \rho(x, w \oplus v)$$

for all $x \in X$ and $v \in V_x$, $w \in W_x$. Clearly $\rho|_x$ is a vector space isomorphism for all $x \in X$, so by Lemma 7.7 it is left to show that ρ is continuous. For any $x \in X$, take an open neighbourhood U of x for which both $V|_U$ and $W|_U$ are trivial. There exists $k \leq n$ and $l \leq m$ for which there exists bundle isomorphisms

$$\varphi : V|_U \xrightarrow{\cong} \Theta^k(U); \quad \psi : W|_U \xrightarrow{\cong} \Theta^l(U).$$

Definition 7.14. Let X be a compact Hausdorff space. Define $K^0(X) = G(\mathrm{Vect}(X))$, where $G(\cdot)$ is the Grothendieck completion.

The following is a lemma that helps with computation of K^0-groups.

Lemma 7.15. *Let X be a compact Hausdorff space and let I denote the closed interval $[0,1]$. If V is a vector bundle over $X \times I$, then $V|_{X\times\{0\}} \cong V|_{X\times\{1\}}$.*

Proof. First we show that a bundle V over $X \times [a,b]$ is trivial if there exists some $c \in (a,b)$ such that $V|_{X\times[a,c]}$ and $V|_{X\times[c,b]}$ are trivial. To see this, let $\varphi : V|_{X\times[a,c]} \to \Theta^n(X \times [a,c])$ and $\psi : V|_{X\times[c,b]} \to \Theta^n(X \times [c,b])$ be bundle isomorphisms for some $n \in \mathbb{N}$. There exists a function $h : X \to GL_n(\mathbb{C})$ such that $\varphi(v) = h(\pi(v))\psi(v)$ for all $v \in V|_x$. Then the map $\Phi : V \to \Theta^n(X \times [a,b])$ defined by

$$\Phi(v) = \begin{cases} \varphi(v) & : a \leq t \leq c \\ h(\pi(v))\psi(v) & : c < t \leq b \end{cases}$$

is a bundle isomorphism.

Next, for every $x \in X$ and $t \in [0,1]$ there exists some $U_{x,t} \subseteq X$ a neighbourhood of x and some $\delta_t > 0$ such that V is trivial over

$$U_{x,t} \times (t - \delta_t, t + \delta_t).$$

Because $[0,1]$ is compact, there exists a finite collection $\{t_0, \ldots, t_k\} \subseteq [0,1]$ such that

$$\bigcup_{i=0}^{k} (t_i - \delta_{t_i}, t_i + \delta_{t_i}) \supseteq [0,1].$$

Let $U_x = \bigcap_{i=0}^{k} U_{x,t_i}$. Then V is trivial over $U_x \times (t_i - \delta_{t_i}, t_i + \delta_{t_i})$ for all $i = 0, \ldots, k$. Hence by observation from the previous paragraph, we see that $V|_{U_x \times I}$ is trivial. Thus, since X is compact, there exists a finite cover $\{U_1, \ldots, U_r\}$ of X such that $V|_{U_j \times I}$ is trivial for all $j = 1, \ldots, r$.

Let $\{f_1, \ldots, f_r\}$ be a partition of unity subordinate to the cover $\{U_1, \ldots, U_r\}$. For $j = 0, \ldots, r$ let

$$F_j = f_1 + \cdots + f_j.$$

In particular $F_0 = 0$ and $F_r = 1$. Also define

$$X_0 = \{(x, F_j(x)) : x \in X\}$$

for $j = 1, \ldots, r$. Because $V|_{U_j \times I}$ is trivial, there exists a bundle isomorphism $\Phi_j : V|_{U_j \times I} \to \Theta^n(U_j \times I)$. Define $\Psi_j : V|_{X_{j-1}} \to V|_{X_j}$ by

$$\Psi_j(v) = \begin{cases} v & : \pi(v) \notin U_j \times I \\ \Phi_j^{-1}(w) & : \pi(v) \in U_j \times I \end{cases}$$

where $w = ((x, f_j(x)), u)$ if $\Phi_j(v) = ((x, f_{j-1}(x)), u)$. Then Ψ_j is a bundle isomorphism. Thus we have

$$V|_{X \times \{0\}} = V|_{X_0} \cong V|_{X_1} \cong \ldots \cong V|_{X_r} = V|_{X \times \{1\}}. \blacksquare$$

Corollary 7.16. *Every vector bundle over a contractible compact Hausdorff space is trivial.*

Proof. Let X be a contractible compact Hausdorff space. There exists a fixed point $x_0 \in X$ and a continuous function $\varphi : X \times [0, 1] \to X$ satisfying $\varphi|_{X \times \{0\}}(x) = x$ for all $x \in X$ and $\varphi|_{X \times \{1\}}(x) = x_0$ for all $x \in X$. Suppose V is a vector bundle over X. Then $\varphi^*(V)$ is a bundle over $X \times [0, 1]$ with

$$V \cong \varphi^*(V)|_{X \times \{0\}} \cong \varphi^*(V)|_{X \times \{1\}} \cong \Theta^{\operatorname{rank} V}(X)$$

by Lemma 7.15. $\blacksquare$

Example 7.17. Consider the compact Hausdorff space $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $A = \{e^{i\theta} : 0 \leq \theta \leq \pi\}$ be the closed upper half of S^1 and let $B = \{e^{i\theta} : \pi \leq \theta \leq 2\pi\}$ be the lower half of S^1. Fix a rank n complex vector bundle V over S^1. Because A and B are both contractible, by Corollary 7.16 $V|_A$ and $V|_B$ are trivial bundles. Let $\varphi : V|_A \to \Theta^n(A)$ and $\psi : V|_B \to \Theta^n(B)$ be bundle isomorphisms. Let $g \in GL_n(\mathbb{C})$ be the matrix that represents $\varphi \circ \psi^{-1}$ at 1, and let h be the matrix that represents $\varphi \circ \psi^{-1}$ at -1. The group $GL_n(\mathbb{C})$ is path connected, so let g_t and h_t be continuous paths from A and B respectively to the identity matrix.

Define a rank n bundle W over $S^1 \times I$ as follows. The bundle W is trivial over $A \times I$ and $B \times I$, with trivializations $\Phi : W|_{A\times I} \to \Theta^n(A \times I)$ and $\Psi : W|_{B\times I} \to \Theta^n(B \times I)$. Furthermore, the transition function is defined to be

$$\Psi^{-1}((1,t),u) = \Phi^{-1}((1,t),g_t u) \text{ and } \Psi^{-1}((-1,t),u) = \Phi^{-1}((-1,t),h_t u)$$

for $\pm 1 \in S^1, t \in [0,1]$ and $u \in \mathbb{C}^n$. Finally, Lemma 7.15 implies that

$$V \cong W|_{S^1\times\{0\}} \cong W|_{S^1\times\{1\}} \cong \Theta^n(S^1).$$

Therefore equivalence classes of vector bundles over S^1 are characterized by ranks, and $K^0(S^1) \cong G(\mathbb{N}) \cong \mathbb{Z}$.

8 $K^0(X) \cong K_0(C(X))$

The main result of this section is the proof of the equivalence of K-theories. When X is compact Hausdorff, then $C(X)$ is a unital C*-algebra, and it makes sense to ask if the two definitions of K-theories agree.

Theorem 8.1. *Let X be compact Hausdorff. Then $K_0(C(X)) \cong K^0(X)$ as abelian groups.*

Now we will develop some results necessary to prove this theorem.

Definition 8.2. Let X be a compact Hausdorff space. For $E \in \mathcal{P}_\infty(C(X))$, and $x \in X$, let $\operatorname{Ran} E(x)$ be the image of $E(x)$. That is, if E is $n \times n$, then $\operatorname{Ran} E(x) = E(x)\mathbb{C}^n$. Define $\operatorname{Ran} E = \bigcup_{x\in X} \bigcup_{v \in \operatorname{Ran} E(x)} (x,v)$.

Proposition 8.3. *Let X be a compact Hausdorff space, $n \in \mathbb{N}$ and $E \in \mathcal{P}_\infty(C(X))$. Then $\operatorname{Ran} E$ is a vector bundle over X.*

Proof. Fix $x_0 \in X$ and let

$$U = \{x \in X : \|E(x_0) - E(x)\|_{op} < 1\}$$

As E and the operator norm are both continuous, the set U is the pull back of $(-\infty, 1)$ through a continuous function, and is hence open. Observe that for any $x_1 \in X$, the element $I_n + E(x_0) - E(x_1)$ is within distance 1 from I_n, and as such is an invertible matrix. Also, for any $v \in \mathbb{C}^n$, we have

$$\begin{aligned}(I_n + E(x_0) - E(x_1))E(x_1)v &= E(x_1)v + E(x_0)E(x_1)v - E(x_1)E(x_1)v \\ &= E(x_1)v + E(x_0)E(x_1)v - E(x_1)v \\ &= E(x_0)E(x_1)v\end{aligned}$$

So $I_n + E(x_0) - E(x_1)$ maps $\operatorname{Ran} E(x_1)$ into $\operatorname{Ran} E(x_0)$, and since this is an invertible matrix, we have that $\dim \operatorname{Ran} E(x_0) \geq \dim \operatorname{Ran} E(x_1)$. A similar calculation shows that

$$(I_n - E(x_0) + E(x_1))(\operatorname{Ran} E(x_0)) \subseteq \operatorname{Ran} E(x_1))$$

Thus we see that $\operatorname{Ran} E(x_0)$ and $\operatorname{Ran} E(x_1)$ have the same dimension, and $I_n + E(x_0) - E(x_1)$ maps $\operatorname{Ran} E(x_1)$ to $\operatorname{Ran} E(x_0)$ isomorphically. Thus, the map

$$\begin{aligned}\varphi : \operatorname{Ran} E|_U &\to U \times \operatorname{Ran} E(x_0) \\ (x, v) &\mapsto (x, (I_n + E(x_0) - E(x_1))v)\end{aligned}$$

is a bundle isomorphism. So $\operatorname{Ran} E$ is locally trivial, thus is a vector bundle. ∎

Proposition 8.4. *Let X be a compact Hausdorff space, and let $E, F \in \mathcal{P}_\infty(C(X))$. Then $\operatorname{Ran} E \cong \operatorname{Ran} F$ as bundles if and only if $E \sim_u F$.*

Proof. Since $\operatorname{Ran} Q \cong \operatorname{Ran}(\operatorname{diag}(Q, 0_r))$ for any $Q \in \mathcal{P}_\infty(C(X))$ and $r \in \mathbb{N}$, we can take some $n \in \mathbb{N}$ large enough so that E and F are both in $M_n(C(X))$.

Suppose that $E \sim_u F$. Then we can find $U \in \mathcal{U}_n(C(X))$ such that $UEU^* = F$. Define $\gamma : \operatorname{Ran} E \to \operatorname{Ran} F$ by

$$\gamma(x, E(x)v) = (x, U(x)E(x)v) = (x, F(x)U(x)v) \in \operatorname{Ran} F(x),$$

for $x \in X$ and $v \in \mathbb{C}^n$. It has the inverse map

$$\gamma^{-1}(x, F(x)v) = (x, U^*(x)F(x)v) = (x, E(x)U^*(x)v).$$

So γ is a bundle isomorphism between $\operatorname{Ran} E$ and $\operatorname{Ran} F$.

Conversely, suppose that $\operatorname{Ran} E$ and $\operatorname{Ran} F$ are isomorphic vector bundles. Let $\varphi : \operatorname{Ran} E \to \operatorname{Ran} F$ be a bundle isomorphism. We define matrices

$A, B \in M_n(C(X))$ as follows. For $f \in (C(X))^n$, let $Af = \varphi(Ef)$ and $Bf = \varphi^{-1}(Ff)$. Then

$$ABf = A(\varphi^{-1}(Ff)) = \varphi(E(\varphi^{-1}(Ff))).$$

However $\varphi^{-1}(Ff)$ is a continuous section of $\operatorname{Ran} E$, so

$$ABf = \varphi(E(\varphi^{-1}(Ff))) = \varphi(\varphi^{-1}(Ff)) = Ff$$

Which shows that $AB = F$. A similar computation shows that $BA = E$. Also,

$$EBf = E\varphi^{-1}(Ff) = \varphi^{-1}(Ff) = Bf$$

and

$$BFf = \varphi^{-1}(FFf) = \varphi^{-1}(Ff) = Bf.$$

So $EB = B = BF$. Similarly, $FA = A = AE$.

Now define

$$T = \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \in M_{2n}(C(X)).$$

With the observations above it is straightforward to check that T is invertible, with inverse

$$T^{-1} = \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}.$$

Then

$$\begin{aligned} T\operatorname{diag}(E, 0_n)T^{-1} &= \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix} \\ &= \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} = \operatorname{diag}(F, 0_n) \end{aligned}$$

Thus E is similar to F through an invertible matrix T. Since E and F are normal and similar to each other, they are in fact unitarily equivalent by Proposition 3.14. ∎

Corollary 8.5. *Let X be compact Hausdorff. The range map*

$$Ran : \mathcal{P}_\infty(C(X))/\sim_u \to Vect(X)$$

mapping

$$[E] \mapsto [Ran\, E]$$

is well-defined and injective.

Proposition 8.6. *Let X be a compact Hausdorff space, let $N \in \mathbb{N}$, and suppose that V is a subbundle of $\Theta^N(X)$. Let $\Theta^N(X)$ be equipped with the standard Hermitian metric, and for $x \in X$, let $E(x)$ be the orthogonal projection of $\Theta^N(X)|_x$ onto $V|_x$. Then the map $E : x \mapsto E(x)$ defines an idempotent $E \in M_N(C(X))$.*

Proof. By using Lemma 7.7 again, we only need to show that each $x_0 \in X$ has an open neighbourhood for which $E|_U : x \mapsto E(x)$ is continuous on U. Fix x_0 and choose U to be a connected open neighbourhood of x_0 over which V is trivial. Let n be the rank of V, and let $\varphi : \Theta^n(U) \to V|_U$ be a bundle isomorphism. For $k = 1, \ldots, n$, define $s_k : U \to \Theta^n(U)$ by $s_k(x) = (x, e_k)$, the k^{th} standard basis vector lying above x. Then for each $x \in U$, the set

$$\{\varphi(s_1(x)), \varphi(s_2(x)), \ldots, \varphi(s_n(x))\}$$

is a vector space basis for $V|_x$. Let $\langle ., . \rangle$ be the standard Hermitian metric of $\Theta^N(U)$ restricted to V. By the Gram-Schmidt process, we obtain a an orthogonal basis of $V|_x$ by defining inductively

$$s'_k(x) = \varphi(s_k(x)) - \sum_{i=1}^{k-1} \frac{\langle \varphi(s_k(x)), s'_i(x) \rangle}{\langle s'_i(x), s'_i(x) \rangle} s'_i(x)$$

for $k = 1, \ldots, n$. Then the set

$$\left\{ \frac{s'_1(x)}{\|s'_1(x)\|}, \ldots, \frac{s'_n(x)}{\|s'_n(x)\|} \right\}$$

is an orthonormal basis for $V|_x$ equipped with $\langle ., . \rangle$, where $\| \cdot \|$ denotes the norm induced by $\langle ., . \rangle$. Moreover, the map $x \mapsto \frac{s'_1(x)}{\|s'_1(x)\|}$ is continuous. Finally, for E the orthogonal projection as in the statement, we have

$$E(x)w = \sum_{k=1}^{n} \left\langle \varphi(x, w), \frac{s'_k(x)}{\|s'_k(x)\|} \right\rangle \frac{s'_k(x)}{\|s'_k(x)\|}$$

and the above is jointly continuous in $x \in X$ and $w \in \mathbb{C}^n$. Therefore $x \mapsto E(x)$ is continuous. ∎

Corollary 8.7. *Let V be a vector bundle over a compact Hausdorff space X. Then $V \cong Ran\, E$ for some $E \in \mathcal{P}_\infty(C(X))$. Hence the map*

$$Ran : \mathcal{P}_\infty(C(X))/\sim_u \to Vect(X)$$

is surjective.

Proof. There exists $N \in \mathbb{N}$ such that V is isomorphic to a subbundle of $\Theta^N(X)$. So assume that V is embedded in $\Theta^N(X)$, and let $\Theta^N(X)$ be equipped with the canonical metric. For each $x \in X$ let $E(x)$ be the orthogonal projection of $\Theta^N(X)_x$ onto V_x. By Proposition 8.6 , $x \mapsto E(x)$ defines an element in $E \in \mathcal{P}_N(X)$, and $\mathrm{Ran}\, E = V$. ∎

Corollary 8.8. *Let V be a vector bundle over a compact Hausdorff space X. Then there exists another vector bundle $V^\perp$ over X such that $V \oplus V^\perp \cong \Theta^N(X)$ for some $N \in \mathbb{N}$.*

Proof. We know that there exists some $N \in \mathbb{N}$ such that V is isomorphic to a subbundle of $\Theta^N(X)$. For each $x \in X$, let $E(x)$ be the orthogonal projection of $\Theta^N(X)_x$ onto V_x. By Proposition 8.6, this family of projections defines an element $E \in \mathcal{P}_N(C(X))$. Define $V^\perp = \mathrm{Ran}\,(I_N - E)$. Then

$$V \oplus V^\perp \cong \mathrm{Ran}\, E \oplus \mathrm{Ran}\,(I_N - E) = \mathrm{Ran}\, I_N = \Theta^N(X). \blacksquare$$

Theorem 8.9. *Let X be a compact Hausdorff space. Then $\mathcal{P}_\infty(C(X))$ and $Vect(X)$ are isomorphic as abelian monoids.*

Proof. Define $\Psi : \mathcal{P}_\infty(C(X)) \to \mathrm{Vect}(X)$ by $\Psi([E]) = [\mathrm{Ran}\, E]$. By Corollaries 8.5 and 8.7, Ψ is well-defined, injective and surjective. It is left to show that it is a monoid homomorphism, i.e. $\mathrm{Ran}\,(E \oplus F) \cong \mathrm{Ran}\, E \oplus \mathrm{Ran}\, F$. But this is obvious, as they are not just isomorphic, but are in fact equal. ∎

Corollary 8.10. *Let X be a compact Hausdorff space. Then $K^0(X) \cong K_0(C(X))$ as abelian groups.*

Proof. Apply the Grothendieck completion to the isomorphism obtained in Theorem 8.9 to obtain

$$K^0(X) = G(\mathrm{Vect}(X)) \cong G(\mathcal{P}_\infty(C(X))) = K_0(C(X)). \blacksquare$$

For X a compact Hausdorff space and V a topological vector bundle over X, we write $[V]^0$ for the element in $K^0(X)$ that is represented by V.

Proposition 8.11. *Let X be a compact Hausdorff space, then*

$$K^0(X) = \{[V]^0 - [W]^0 : V, W \text{ vector bundles over } X\}.$$

Proof. This follows from Corollary 8.10 and Proposition 4.3. ∎

Now that we've shown that $K^0(X)$ and $K_0(C(X))$ are isomorphic as abelian groups, we will verify that the associated morphisms are preserved by this identification.

Definition 8.12. Let X and Y be compact Hausdorff spaces, let $f : X \to Y$ be a continuous map and let V be a rank r subbundle of some trivial bundle $\Theta^n(Y)$ of Y. (By Proposition 7.8 all vector bundles over Y are isomorphic to a bundle of this form). Define the pull-back of V via f, written $f^*(V)$, to be the rank r subbundle of $\Theta^n(X)$ where the fibre at a point $x \in X$ is $(f^*(V))_x = V_{f(x)}$.

Proposition 8.13. *Let X and Y be compact Hausdorff spaces, $f : X \to Y$ continuous and V is a subbundle of $\Theta^n(Y)$. Then $f^*(V)$ is indeed a vector bundle on X.*

Proof. Take any $x \in X$, let U be an open neighbourhood of $f(x)$ in Y for which $V|_U$ is trivial. Then $f^{-1}(U)$ is an open neighbourhood of x and $f^*(V)|_{f^{-1}(U)} = f^*(V|_U)$ is trivial. ∎

Proposition 8.14. *Let X and Y be compact Hausdorff spaces, let $f : X \to Y$ be continuous, and $E \in \mathcal{P}_\infty(C(Y))$. Then $f^*(E)$ is a projection in $\mathcal{P}_\infty(C(X))$, and $f^*(Ran\, E) = Ran\, f^*(E)$.*

Proof. For $x \in X$,

$$(E \circ f)(x) \cdot (E \circ f)(x) = E(f(x))E(f(x)) = EE(f(x)) = E \circ f(x)$$

and

$$(E \circ f)^*(x) = (E \circ f(x))^* = E^*(f(x)) = E \circ f(x).$$

So $E \circ f$ is a projection. Furthermore, suppose E is $n \times n$. Then

$$f^*(\operatorname{Ran} E)_x = (\operatorname{Ran} E)_{f(x)} = E(f(x))\mathbb{C}^n = (\operatorname{Ran} f^*(E))_x.$$

Therefore $f^*(\operatorname{Ran} E) = \operatorname{Ran} f^*(E)$. ∎

Definition 8.15. Let X and Y be compact Hausdorff spaces and let $f : X \to Y$ be a continuous map. Then f^* is a $*$-homomorphism from $C(Y)$ to $C(X)$. Define $K^0(f) : K^0(Y) \to K^0(X)$ by

$$K^0(f)([V]_0 - [W]_0) = [f^*(V)]_0 - [f^*(W)]_0.$$

Remark 8.16. According to Proposition 8.14, if $f : X \to Y$ is a continuous map then by identifying $K^0(Y)$ with $K_0(C(Y))$ and $K^0(X)$ with $K_0(C(X))$, we conclude that $K^0(f)$ and $K_0(f^*)$ are the same map. To be precise, the diagram

$$\begin{array}{ccc} K^0(Y) & \xrightarrow{K^0(f)} & K^0(X) \\ \downarrow\cong & & \downarrow\cong \\ K_0(C(Y)) & \xleftarrow[K_0(f^*)]{} & K_0(C(X)) \end{array}$$

commutes.

Proposition 8.17. *The map $X \mapsto K^0(X)$ is a covariant functor from the category of compact Hausdorff spaces to the category of abelian groups.*

Proof. Let X, Y, Z be compact Hausdorff spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous. Consider the commutative diagrams

$$\begin{array}{ccccc} K^0(Z) & \xrightarrow{K^0(g)} & K^0(Y) & \xrightarrow{K^0(f)} & K^0(X) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ K_0(C(Z)) & \xleftarrow[K_0(g^*)]{} & K_0(C(Y)) & \xleftarrow[K_0(f^*)]{} & K_0(C(X)) \end{array}$$

and

$$\begin{array}{ccc} K^0(Z) & \xrightarrow{K^0(f \circ g)} & K^0(X) \\ \downarrow\cong & & \downarrow\cong \\ K_0(C(Z)) & \xleftarrow[K_0((f \circ g)^*)]{} & K_0(C(X)) \end{array}$$

Since K_0 is a functor, we have

$$K_0((f \circ g)^*) = K_0(g^* \circ f^*) = K_0(g^*) \circ K_0(f^*).$$

Hence the first rows of the two diagrams imply that $K^0(f \circ g) = K^0(g) \circ K^0(f)$. The fact that $K^0(\mathrm{id}_X) = \mathrm{id}_{K^0(X)}$ also follows from the functoriality of K_0 and Remark 8.16 in a similar way. ∎

Example 8.18. Let $X = \{*\}$ be a point. Then $C(X) \cong \mathbb{C}$. By Example 2.18 and Corollary 8.10, we see that $K^0(X) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$.

9 K-theory of locally compact spaces

The K-theory of locally compact spaces correspond to the K-theory of non-unital C*-algebras.

Definition 9.1. Let X be a topological space. We say X is locally compact if for every $x \in X$ there exists some open neighbourhood $U \subseteq X$ of x such that the closure $\overline{U}$ of U in X is compact.

Definition 9.2. Let X be a locally compact space. Define X^+ to be the set $X \sqcup \{\infty\}$ with the collection of open sets given by

$$\mathcal{T}^+ := \{U \subseteq X : U \text{ open in } X\} \cup \{(X \backslash F) \cup \{\infty\} : F \text{ closed and compact } X\}.$$

Proposition 9.3. *Let X be a topological space, then X^+ is a compact topological space. Moreover, $X^+ \setminus \{\infty\}$ is homeomorphic to X in the obvious way.*

Proof. We first check that the collection of open sets $\mathcal{T}^+$ is a topology on X^+.

1. The empty set $\emptyset$ is open in X, so $\emptyset \in \mathcal{T}^+$. The empty set $\emptyset$ is obviously closed and compact, so $X^+ = (X \setminus \emptyset) \cup \{\infty\} \in \mathcal{T}^+$.

2. Define

$$\mathcal{T}_0 := \{U : U \text{ open in } X\},$$

$$\mathcal{T}_1 := \{(X \backslash F) \cup \{\infty\} : F \text{ closed and compact in } X\}.$$

Clearly $\mathcal{T}_0$ is closed under arbitrary union. Let $\{F_i : i \in I\}$ be an arbitrary collection of closed compact subsets of X. Then $F := \bigcap_{i \in I} F_i$ is clearly closed. Pick any $i_0 \in I$. Then F is a closed subset of the compact set F_{i_0}, thus F is also compact. Then

$$\bigcup_{i\in I}(X \setminus F_i) \cup \{\infty\} = (X \setminus F) \cup \{\infty\} \in \mathcal{T}_1.$$

So $\mathcal{T}_1$ is closed under arbitrary union. Finally, take $U \in \mathcal{T}_0$ and $(X \setminus F) \cup \{\infty\} \in T_1$. We have

$$\begin{aligned} U \cup (X \setminus F) \cup \{\infty\} &= (X \setminus (X \setminus U)) \cup (X \setminus F) \cup \{\infty\} \\ &= (X \setminus ((X \setminus U) \cap F)) \cup \{\infty\} \in \mathcal{T}_1 \end{aligned}$$

because $(X \setminus U) \cap F$ is closed and compact (it is a closed subset of F). Therefore $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ is closed under arbitrary union.

3. Clearly $\mathcal{T}_0$ is closed under finite intersection. A finite union of compact closed sets is also closed and compact, so $\mathcal{T}_1$ is also closed under finite intersection. Lastly, suppose U is open and F is closed and compact, then

$$U \cap ((X\setminus F) \cup \{\infty\}) = U \cap (X\setminus F) \in \mathcal{T}_1.$$

Therefore $\mathcal{T}$ is closed under finite intersection.

The above verifies that $\mathcal{T}$ is a topology on X. The subspace topology on $X^+ \setminus \{\infty\}$ is $\mathcal{T}_0$, which coincides with the topology on X. Hence $X^+ \setminus \{\infty\} \cong X$. Next we check that X^+ is compact.

Let $\{U_i\}_{i\in I}$ be a open cover for X^+. Since this collection covers the point ∞, there exists some $i_0 \in I$ such that $U_{i_0} \in \mathcal{T}_1$. Then $X^+ \setminus U_{i_0}$ is a compact subset of X, hence also a compact subset of X^+, so there exists a finite subset $J \subseteq I$ for which $X^+ \setminus U_{i_0} \subseteq \bigcup_{i\in J} U_i$. Whence $\{U_i : i \in J \cup \{i_0\}\}$ is a finite cover for X^+. Therefore X^+ is compact. ∎

Remark 9.4. The space X^+ is called the one point compactification of X.

Proposition 9.5. *Let X be a locally compact topological space. If X is Hausdorff then X^+ is also Hausdorff.*

Proof. Let $\mathcal{T}_0$ be $\mathcal{T}_1$ be as defined in the proof of Proposition 9.3. By Proposition 9.3 we know that $X^+ \setminus \{\infty\} \cong X$ is Hausdorff. Fix $x \in X^+ \setminus \{\infty\}$ and let U be an open neighbourhood of x where $\overline{U}$ is compact in X. Then $V := X^+ \setminus \overline{U}$ is an open neighbourhood of ∞, and $U \cap V = \emptyset$. Therefore X^+ is Hausdorff. ∎

Proposition 9.6. *Let X be a compact Hausdorff space, and let $x_0 \in X$. The map $f : X \to (X \setminus x_0)^+$ given by*

$$f(x) = \begin{cases} x & : x \neq x_0 \\ \infty & : x = x_0 \end{cases}$$

is a homeomorphism.

Proof. It is clear that f is bijective. It is also clear that for any $S \subseteq X \setminus \{x_0\}$, S is open in X if and only if $f(S)$ is open in $(X \setminus \{x_0\})^+$.

Suppose $U \subseteq X$ is an open neighbourhood of x_0. Let $F = X \setminus U$. Since F is a closed subset of X, it is compact. Also,

$$U = ((X \setminus \{x_0\}) \setminus F) \cup \{x_0\}.$$

On the other hand, suppose $F \subseteq X \setminus \{x_0\}$ is closed and compact, then

$$((X \setminus \{x_0\}) \setminus F) \cup \{\infty\} = X \setminus F$$

is an open neighbourhood of x_0. Hence $x_0 \in X$ and $\infty \in (X \setminus \{x_0\})^+$ have the "same" open neighbourhoods. It follows that a subset $S \subseteq X$ containing x_0 is open if and only if $f(S)$ is open. Therefore f is a homeomorphism. ∎

Definition 9.7. Let X be a locally compact Hausdorff space. Define $C_0(X)$ to be the set of all continuous functions $f \in C(X)$ satisfying the following: for any $\varepsilon > 0$ there exists a compact subset $F \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus F$.

Proposition 9.8. *Let X be a locally compact Hausdorff space and let $f \in C_0(X)$. Define $\widetilde{f}$ on X^+ to be*

$$\widetilde{f} = \begin{cases} f(x) & : x \in X \\ 0 & : x = \infty \end{cases}.$$

Then $\widetilde{f} \in C(X^+)$. If $h \in C(X^+)$ satisfies $h(\infty) = 0$, then $h|_X \in C_0(X)$ and $\widetilde{h|_X} = h$.

Proof. It is clear that $\widetilde{f}$ is continuous on $X^+ \setminus \{\infty\}$, so we only need to check that $\widetilde{f}$ is continuous at ∞. Given any $\varepsilon > 0$, by the definition of $C_0(X)$, there exists a compact subset $F \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus F$. But $U := (X \setminus F) \cup \{\infty\}$ is an open neighbourhood of ∞. We have $|\widetilde{f}(x) - \widetilde{f}(\infty)| = |\widetilde{f}(x)| < \varepsilon$ for all $x \in U$. Therefore $\widetilde{f}$ is continuous.

The second part of the proof follows essentially the same proof. ∎

Proposition 9.9. *Let X be a locally compact Hausdorff space. Let I_X denote the identity element of $\widetilde{C_0(X)}$ and let 1_{X^+} denote the constant function 1 on X^+. Define $\varphi : \widetilde{C_0(X)} \to C(X^+)$ by $\varphi(f) = \widetilde{f}$ for all $f \in C_0(X)$ and $\varphi(I) = \varphi(1_{X^+})$ and extend linearly. Then φ is a C^*-algebra isomorphism.*

Proof. It is easy to see that φ is a $*$-homomorphism. Suppose

$$0 = \varphi(f + zI_X) = \widetilde{f} + z1_{X^+}$$

for some $f \in C_0(X)$ and $z \in \mathbb{C}$. Then

$$z = (\widetilde{f} + z1_{X^+})(\infty) = 0.$$

It then follows that $\widetilde{f}(x) = 0$ for all $x \in X$, so $f = 0$. Hence φ is injective.

Take any $h \in C(X^+)$ and let $z = h(\infty)$. By Proposition 9.8 the function $(h - z1_{X^+})|_X \in C_0(X)$. Also, $\varphi((h - z1_{X^+}) + zI_X) = h$. This shows that φ is surjective. Therefore φ is an isomorphism. ∎

Definition 9.10. Let X be a locally compact Hausdorff space, and let $\iota : \{\infty\} \to X^+$ be the inclusion map. Define $K^0(X) := \ker K^0(\iota) \subseteq K^0(X^+)$.

Remark 9.11. Suppose X is a locally compact Hausdorff space and $\iota : \{\infty\} \to X^+$ is the inclusion map. The induced $*$-homomorphism $\iota^* : C(X^+) \to C(\{\infty\})$ does the following:

$$\iota^*(\widetilde{f}) = \widetilde{f} \circ \iota = 0, \ \forall f \in C_0(X)$$

and

$$\iota^*(1_{X^+}) = 1_{X^+} \circ \iota = 1_{\{\infty\}}.$$

This means that $\iota : C(X^+) \to C(\{\infty\})$ is the projection onto the one dimensional subspace generated by the identity element and $\ker \iota = C_0(X)$. Whence in light of Remark 8.16 and Proposition 9.9, $K^0(X)$ is isomorphic to $K_0(C_0(X))$ in the expected way.

9.1 Relative and reduced K-theory

Definition 9.12. Let X be a compact Hausdorff space, and let A be a compact subset of X. Let $\iota : A \to X$ be the inclusion map. Then $K^0(\iota)$ is a group homomorphism $K^0(X) \to K^0(A)$. Define $K^0(X, A)$ to be $\ker(K^0(\iota))$. The group $K^0(X, A)$ is called the relative K-group of the compact pair (X, A).

Proposition 9.13. *Let X be a locally compact Hausdorff space. Then $K^0(X) \cong K^0(X^+, \infty)$.*

Proof. This is a consequence of Remark 9.11. ∎

Proposition 9.14. *Let X be a compact Hausdorff space and fix $x_0 \in X$. Then $K^0(X) \cong K^0(X, x_0) \oplus \mathbb{Z}$.*

Proof. Let $\iota : \{x_0\} \to X$ be the inclusion map, and let $\lambda : X \to \{x_0\}$ be the only constant map. Consider the sequence

$$0 \longrightarrow K^0(X, x_0) \longrightarrow K^0(X) \underset{K^0(\lambda)}{\overset{K^0(\iota)}{\rightleftarrows}} K^0(\{x_0\}) \longrightarrow 0.$$

By the definition of $K^0(X, x_0)$, this sequence is exact. Furthermore, $\iota \circ \lambda = \mathrm{id}_{\{x_0\}}$, then by the functoriality of K^0 we have that

$$K^0(\lambda) \circ K^0(\iota) = K^0(\iota \circ \lambda) = K^0(\mathrm{id}_{\{x_0\}}) = \mathrm{id}_{K^0(\{x_0\})}.$$

Hence the above is a split exact sequence of abelian groups. Therefore $K^0(X) \cong K^0(X, x_0) \oplus K^0(\{x_0\})$. Lastly, by Example 8.18 we have $K^0(\{x_0\}) \cong \mathbb{Z}$. ∎

Remark 9.15. Let X be a compact Hausdorff space. Let G_0 be the subgroup of $K^0(X)$ generated by $[\Theta^1(X)]_0$. Since

$$[\Theta^n(X)]_0 + [\Theta^m(X)]_0 = [\Theta^n(X) \oplus \Theta^m(X)]_0 = [\Theta^{n+m}(X)]_0,$$

we have that $G_0 = \{\pm[\Theta^n(X)]_0 : n \in \mathbb{N}_{\geq 0}\} \cong \mathbb{Z}$. Fix $x_0 \in X$, and let $\iota_{x_0} : \{x_0\} \to X$ be the inclusion map. Then

$$K^0(\iota_{x_0})([\Theta^n(X)]_0) = [\iota_{x_0}^*(\Theta^n(X))]_0 = [\Theta^n(\{x_0\})]_0,$$

which corresponds to $n \in \mathbb{Z}$ in the isomorphism $\mathbb{Z} \cong K^0(\{x_0\})$. Hence $K^0(\iota_{x_0})|_{G_0} \to K^0(\{x_0\})$ is an isomorphism for any $x_0 \in X$. Thus we have that $K^0(X, x_0) \cong K^0(X)/G_0$ for any $x_0 \in X$. More importantly, we have that $K^0(X, x_0) \cong K^0(X, x_1)$ for any $x_0, x_1 \in X$.

Definition 9.16. Let X be a compact Hausdorff space. Define the **reduced K-group** of X, denoted $\widetilde{K}^0(X)$, to be $K^0(X, x_0)$ for any choice of $x_0 \in X$.

Remark 9.17. Let X be a compact Hausdorff space and fix $x_0 \in X$. By Proposition 9.13 we have $\widetilde{K}^0(X) \cong K^0(X, \{x_0\})$. By Remark 9.15, the definition of $\widetilde{K}^0(X)$ is independent of the choice $x_0 \in X$.

10 Functorial properties of K^0

10.1 Homotopy invariance

Definition 10.1. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. We say f is **homotopic** to g if there exists a continuous map $f_\bullet : [0,1] \times X \to Y$ mapping $(t, x) \mapsto f_t(x)$ such that $f_0(x) = f(x)$ and $f_1(x) = g(x)$ for all $x \in X$.

Definition 10.2. Let X and Y be topological spaces. Then X is said to be **homotopic** to Y if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Lemma 10.3. *Let X and Y be compact Hausdorff spaces, and let $\varphi_\bullet : [0,1] \times X \to Y$ mapping $(t, x) \mapsto \varphi_t(x)$ be continuous. Then the map $t \mapsto (\varphi_t)^*(f) = f \circ \varphi_t$ is continuous from $[0,1]$ to $C(X)$ for any $f \in C(Y)$.*

Proof. Let $f \in C(Y)$ and $\varepsilon > 0$ be given. Then $f \circ \varphi_\bullet : [0,1] \times X \to \mathbb{R}$ is a continuous function. By continuity, for any $t \in [0,1]$ and $x \in X$, there exists $\delta_t > 0$ and an open neighbourhood $U_x \subseteq X$ of x such that

$$|f \circ \varphi_s(y) - f \circ \varphi_t(x)| < \varepsilon$$

for every $s \in B_{\delta_t}(t) \cap [0,1]$ and $y \in U_x$. By compactness, X can be covered by a finite collection of open sets of the form $U_{x_1}, \ldots, U_{x_k}$. Let $\delta = \min\{\delta_{t_1}, \ldots, \delta_{t_k}\} > 0$. Then for any $x \in X$,

$$|(\varphi_s)^*(f)(x) - (\varphi_t)^*(f)(x)| = |f \circ \varphi_s(x) - f \circ \varphi_t(x)| < \varepsilon,$$

so $\|(\varphi_s)^*(f) - (\varphi_t)^*(f)\|_\infty < \varepsilon$. ∎

Proposition 10.4. *Let X and Y be compact Hausdorff spaces. Let $f : X \to Y$ and $g : Y \to X$ be a homotopy between X and Y. Then $f^* : C(Y) \to C(X)$ and $g^* : C(X) \to C(Y)$ give a homotopy between $C(X)$ and $C(Y)$.*

Proof. By assumption $g \circ f$ is homotopic to the identity map id_X on X. Hence there exists a continuous family $\varphi_t : X \to X$ for $t \in [0,1]$ satisfying $\varphi_0 = \mathrm{id}_X$ and $\varphi_1 = g \circ f$. By Lemma 10.3, $(\varphi_\bullet)^*$ is a homotopy from $(\varphi_0)^* = (\mathrm{id}_X)^* = \mathrm{id}_{C(X)}$ to $(\varphi_1)^* = (g \circ f)^* = f^* \circ g^*$. Similarly $g^* \circ f^*$ is homotopic to $\mathrm{id}_{C(Y)}$. ∎

Corollary 10.5. *Let X and Y be compact Hausdorff spaces and $f : X \to Y$ be a homotopy. Then $K^0(f) : K^0(Y) \to K^0(X)$ is a group isomorphism.*

Proof. By Proposition 10.4 we see that $f^* : C(Y) \to C(X)$ is a homotopy. It follows by Proposition 6.2 that $K_0(f^*)$ is an isomorphism, whence Remark 8.16 gives us the conclusion that $K^0(f)$ is an isomorphism. ∎

Example 10.6. Let $X = [0,1]$. Then X is homotopic to a point. Hence by Corollary 10.5 and Example 8.18, we have

$$K_0(C([0,1])) \cong K^0([0,1]) \cong K^0(\{*\}) \cong \mathbb{Z}.$$

Remark 10.7. The functor K^0 is not homotopy-invariant for locally compact Hausdorff spaces. In Example 7.17 we saw that $K^0(S^1) \cong \mathbb{Z}$. The unit circle S^1 is homeomorphic to the one point compactification of $\mathbb{R}$, and $\mathbb{R}$ is homotopic to a point. However, Proposition 9.14 says that $K^0(\mathbb{R}) \oplus \mathbb{Z} \cong K^0(S^1)$, which implies that $K^0(\mathbb{R}) \cong 0$. On the other hand, the K^0-group of a point is $\mathbb{Z}$, as shown in Example 10.6, which is not isomorphic to $K^0(\mathbb{R})$.

Example 10.8. We will now exhibit an example that shows K_0 is not an exact functor.

Consider the short exact sequence

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\ \iota\ } C([0,1]) \xrightarrow{\ \pi\ } \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

Where

$$(\iota(f))(t) := \begin{cases} f(t) & : t \in (0,1) \\ 0 & : t \in \{0,1\} \end{cases}$$

for any $f \in C_0((0,1))$ and $t \in [0,1]$, and

$$\pi(g) := (g(0), g(1))$$

for any $g \in C([0,1])$. It is left to the reader to check that this sequence is exact.

Corollary 6.7 and Example 8.18 give us the isomorphism

$$K_0(\mathbb{C} \oplus \mathbb{C}) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

On the other hand $\mathbb{C}([0,1]) \cong \mathbb{Z}$ by Example 10.6. The map $K_0(\pi) : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is not a surjection, since $\mathbb{Z}$ is generated by one element but $\mathbb{Z} \oplus \mathbb{Z}$ cannot be generated by one element. Therefore the functor K_0 does not take the short exact sequence in consideration to a short exact sequence of abelian groups.

10.2 Half-exactness of $\widetilde{K}^0$

Proposition 10.9. *Let X be a compact Hausdorff space and let A be a closed subset of X. Define $I(A)$ to be all the continuous functions $f \in C(X)$ that vanish on A, i.e. $f(A) = \{0\}$. Then the following are true*

1. *$I(A)$ is a closed ideal of $C(X)$.*
2. *$I(A) \cong C_0(X \setminus A)$.*
3. *Let $[A]$ denote the point corresponding to A in the quotient X/A. Then $(X/A) \setminus \{[A]\} \cong X \setminus A$ as locally compact Hausdorff spaces.*
4. *$I(A) \cong C_0((X/A) \setminus \{[A]\})$.*
5. *$C(X)/I(A) \cong C(A)$.*

Proof. 1. Let $f \in I(A)$ and $g \in C(X)$, then

$$(f \cdot g)(a) = f(a)g(a) = 0g(a) = 0$$

for all $a \in A$, so $f \cdot g \in I(A)$. Clearly if a convergent sequence of functions vanish on A then so does the limit. Hence $I(A)$ is a closed ideal in $C(X)$.

2. Let $\varphi : C_0(X \setminus A) \to C(X)$ be defined by

$$\varphi(f)(x) = \begin{cases} f(x) & : x \in X \setminus A \\ 0 & : x \in A \end{cases}$$

for all $f \in C_0(X \setminus A)$ and $x \in X$. For each $\varepsilon > 0$, there exists an open neighbourhood $U \subseteq X$ with $A \subseteq U$ satisfying $|\varphi(f)(x)| < \varepsilon$ for all $x \in U$. Hence we see that $\varphi(f) \in C(X)$ for all $f \in C_0(X \setminus A)$. It is also clear from definition that the image of φ is contained in $I(A)$. We also define a map $\psi : I(A) \to C(X \setminus A)$ by

$$\psi(g)(x) = g(x)$$

for all $g \in I(A)$ and $x \in X \setminus A$. Since $g(A) = \{0\}$, then for every $\varepsilon > 0$ there exists an open neighbourhood $U \supseteq A$ satisfying $|g(x)| < \varepsilon$ for all $x \in U$. Hence $\psi(g) \in C_0(X \setminus A)$. It is easy to check that φ and ψ are mutual inverses. Therefore $C_0(X \setminus A) \cong I(A)$.

3. This is obvious.

4. This is a consequence of 2 and 3.

5. Define $\varphi : C(X)/I(A) \to C(A)$ by letting $\varphi([f]) = f|_A$. If $[f] = [g]$, then $(f - g)|_A = 0$, so $\varphi([f]) = \varphi([g])$. Hence φ is well-defined.

Define $\psi : C(A) \to C(X)/I(A)$ as follows. Fix $h \in C(A)$, by Tietze's extension theorem [7] the function h extends to a continuous function $\widetilde{h} \in C(X)$. Let $\psi(h) = [\widetilde{h}]$. It is easy to check that φ and ψ are mutual inverses. Therefore

$$C(A) \cong C(X)/I(A). \blacksquare$$

Corollary 10.10. *Let X be a compact Hausdorff space and let A be a closed subset of X. Under the identifications $I(A) \cong C_0(X \setminus A)$ and $C(A) \cong C(X)/I(A)$, the following sequence is exact:*

$$K_0(C_0((X/A) \setminus \{[A]\})) \longrightarrow K_0(C(X)) \longrightarrow K_0(C(A))$$

Proof. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(A) & \longrightarrow & C(X) & \longrightarrow & C(X)/I(A) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow = & & \downarrow \cong & & \\ 0 & \longrightarrow & C_0((X/A) \setminus \{[A]\}) & \longrightarrow & C(X) & \longrightarrow & C(A) & \longrightarrow & 0 \end{array}$$

The upper row is clearly exact. The isomorphisms from the upper row to the lower row are given by Lemma 10.9. By the half-exactness of the functor K_0 6.5, we obtain the exactness of the K_0-groups. ∎

Corollary 10.11. *Let X be a compact Hausdorff space and let A be a closed subset of X. Let $\iota : A \to X$ be the inclusion map and let $\pi : X \to X/A$ be the projection map. The following sequence is exact:*

$$\widetilde{K}^0(X/A) \xrightarrow{K^0(\pi)} K^0(X) \xrightarrow{K^0(\iota)} K^0(A).$$

Proof. By Corollary 8.10, we know $K^0(X) \cong K_0(C(X))$ and $K^0(A) \cong K_0(C(A))$. By Remark 9.17 and Remark 9.11, we have that $K_0((X/A) \setminus \{[A]\}) \cong K^0((X/A) \setminus \{[A]\}) \cong \widetilde{K}^0(X/A)$. To see that $K^0(\pi)$ and $K^0(\iota)$ are the maps in this exact sequence, one can take π and ι and chase through the proofs in this section. ∎

Remark 10.12. The functor K^0 is not half-exact. If A is a compact subset of a compact Hausdorff space X and we take the quotient X/A, the subspace A is contracted to a point rather than deleted, and this point is not present in the corresponding C*-algebra quotient. The point in X/A representing A detects the rank of the bundles, so we take the reduced $\widetilde{K}^0$ to delete this extra information and make the sequence exact.

Proposition 10.13. *Let X and Y be locally compact Hausdorff spaces. Then $K^0(X) \oplus K^0(Y) \cong K^0(X \sqcup Y)$.*

Proof. It can be easily verified that $C(X) \oplus C(Y) \cong C(X \sqcup Y)$. By Corollaries 6.7 and 8.10 we have

$$K^0(X) \oplus K^0(Y) \cong K_0(C(X)) \oplus K_0(C(Y)) \cong K_0(C(X) \oplus C(Y)) \cong K^0(X \sqcup Y). \blacksquare$$

11 What's next

Computing the K_0 or K^0 group can be very difficult even with the machinery we have developed. The next step is to define the higher K-groups by $K_{n+1}(A) := K_n(SA)$ or $K^{n-1}(X) := K^n(SX)$, were S denotes the suspension of the C*-algebra or the topological space. The isomorphism $K_n(C(X)) \cong K^{-n}(X)$ holds for all n. For a C*-algebra and a closed ideal I, there exist connecting maps for which the long sequence

$$\ldots \to K_2(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I) \to K_0(A) \to K_0(A/I)$$

is exact. The corresponding sequence is exact for the reduced topological K-theory, with arrows pointed in the opposite direction.

The celebrated Bott Periodicity theorem says that $K_n(A) \cong K_{n+2}(A)$ (or $K^n(X) \cong K^{n+2}(X)$) for all n. This reduces the above sequence to a sequence with six elements. It also implies that if we know the K_0- and K_1-group of a C*-algebra then we can read off the K-groups of its suspensions. For example, to find the K-groups of spheres of any dimension, one only needs to compute K^0 and K^1 for the two pointed space S^0. The interested readers are referred to [1] and [4] for more details.

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