A Review of Whitehead's Asphericity Conjecture

by

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I hereby declare that I am the sole author of this paper. This is a true copy of the paper, including any required final revisions, as accepted by my examiners.

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Abstract

This paper summarizes some of the work done to date on Whitehead's question about the asphericity of subcomplexes of an aspherical 2-complex. We start with a review of the theory of higher homotopy groups. Next, we study some of their particular properties for 2-complexes; including their translation into an algebraic structure called crossed modules. The next section includes a translation of Whitehead's conjecture using properties of crossed modules.

We also review a different approach using homotopy of finite spaces; we include a short summary of the main definitions and results of that theory, and the implications for Whitehead's conjecture. We finish the paper by considering some interesting questions that arise from the above mentioned translations.

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Dedication

To Malbekian people.

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1 Introduction

In a 1941 paper studying some properties of homotopy groups [27], J.H.C. Whitehead formulated a very short question that has not been solved so far, although many different approaches have been taken. The author formulated the statement as a question, but for historical reasons the affirmative answer to his question has become known as Whitehead's Asphericity Conjecture. The original question is:

"Is any subcomplex of an aspherical 2-dimensional complex itself aspherical?"

Conjecture 1.1 (Whitehead's Asphericity Conjecture) *Any subcomplex of an aspherical 2-dimensional (CW-) complex is itself aspherical.*

In the current paper, we summarize the background needed to understand the question and the work done to date to find an answer to it.

Section 2 provides a review of the topological foundations required to understand Whitehead's Conjecture, namely, homotopy groups and their properties. Section 3 describes some results for homotopy groups of 2-complexes and a translation of their exact sequence in homotopy to an algebraic structure called crossed module. After describing the main tools that will let us understand its statement, in section 4 we promote the study of Whitehead's Conjecture by summarizing some related problems; we finish this section with some classical results. Section 5 describes the results obtained for Whitehead's question using crossed modules and their equivalent category, $\mathcal{C}\mathrm{at}^1$-groups.

Another approach to Whitehead's question using discrete homotopy is reviewed in Section 6. This section includes a short summary of the background required for this approach as well as yet another partial translation of Whitehead's question, in this case to a question in Universal Algebra. Finally, as a way of conclusion, we include in section 7 a very brief description of other approaches that have been taken to answer the question of our interest, and leave some open questions to the reader that arise naturally from the translation of Whitehead's conjecture to discrete homotopy and crossed modules.

We assume the reader has a reasonable, although not too deep, knowledge of algebraic topology including the notions of the fundamental group of a topological space, covering spaces, homology groups, and the definition of CW-complexes, and the notions of skeleta and dimension for them.

2 Higher Homotopy Groups

Let's start going through the question. It concerns aspherical complexes which are defined as follows:

Definition 2.1 *A topological space is called aspherical if its universal cover is contractible.*

Equivalently, considering higher homotopy groups, we can say a space is aspherical if $\pi_k(X) = 0, \forall k > 1$. Furthermore, for the particular case of 2-complexes that we will study, by Whitehead's theorem [28] a connected 2-complex X is aspherical if and only if $\pi_2(X) = 0$, as will be explained in section 3.

In this section, we will understand what that characterization means by understanding the meaning of the second homotopy group of a topological space (i.e $\pi_2(X)$), as well as other higher homotopy groups. This can be found in many textboooks on algebraic topology, such as [20], [9], [14].

We will start by defining the homotopy groups of a space K, and prove some properties of them. Next, we will consider relative homotopy and construct a long exact sequence similar to the one that is known for homology groups. Finally, we will see some methods of computation of homotopy groups by considering the mentioned long exact sequence and also using an homomorphism from homotopy to homology.

Let (X, x_0) be a based topological space. The homotopy groups $\pi_k(X, x_0)$, for $k \geq 1$ are a generalization of the notion of the fundamental group $\pi_1(X, x_0)$.

Recall that an element of $\pi_1(X, x_0)$ is the equivalence class with respect to based homotopy at x_0 of a loop $f : I \to X$, whose initial and end points are x_0. An element of $\pi_k(X; x_0)$ is defined as an homotopy class of maps:

$f : (I^k, \partial I^k) \to (X, x_0)$, where I^k denotes the n-th dimensional unit cube and ∂I^k denotes its boundary.

In other words, as in the 1-dimensional case, we have classes of maps from the k-th dimensional cube to the space which send its boundary to the point x_0.

Remark 2.2 *Since we know I^k is homeomorphic to D^k and $D^k \sqcup \{p\} / (p \sim \partial D^k)$ is a cell decomposition for S^k, we can equivalently define an element of $\pi_k(X, x_0)$ as an homotopy class of maps $(S^k, t) \to (X, x_0)$, where we have fixed a point $t = [p]$ in the cell decomposition of S^k.*

Similarly to what has been done for the fundamental group, we define the group operation by pasting two cubes together and then contracting them:

$$fg(t_1,\dots,t_k) := \begin{cases} f(2t_1,t_2,\dots,t_k), & t_1 \le 1/2; \\ g(2t_1-1,t_2,\dots,t_k), & t_1 \ge 1/2. \end{cases}$$

Remark 2.3 *The choice of the first coordinate for "gluing" the maps appears to be arbitrary, and it is. Nevertheless, the homotopy class remains the same, no matter which coordinate we choose. This will be seen clearer after the proof of Proposition 2.6.*

Using the same arguments as for π_1, we can check that $\pi_k(X,x_0)$ is a group, where the identity is the class of the constant map $C_{x_0} : I^k \mapsto x_0$.

As in the case of the fundamental group, there is a functorial relation between based spaces and higher homotopy groups. Namely, a map $\phi : (X,x_0) \to (Y,y_0)$ induces a map $\phi_* : \pi_k(X,x_0) \to \pi_k(Y,y_0)$, by $[f] \mapsto [\phi \circ f]$.

Moreover, the map ϕ_*, by its definition, is invariant under based homotopy, and in the case that ϕ defines an homotopy equivalence, ϕ_* will be an isomorphism.

Example 2.4 *If X is a contractible space, then $\pi_n(X,x_0) = 0$ for every $n > 0$.*

Proof. If X is contractible, there is a deformation retract $\phi : X \to x_0$. Then, for every $n > 0$, $\phi_*(\pi_n(X,x_0)) = \pi_n(x_0,x_0) = 0$. ■

Remark 2.5 *In a similar way, the 0-th homotopy set, $\pi_0(X)$, can be defined to be the set of path components of X. This is simply a set and has no group structure though.*

Higher homotopy groups have several properties; we describe some of them in the rest of the section.

Proposition 2.6 *Let (X,x_0) be a based space. Then for $k > 1$, the k-th homotopy group, $\pi_k(X,x_0)$ is abelian.*

Sketch of the proof. Let $[f],[g] \in \pi_k(X,x_0)$, we will prove that fg is homotopy equivalent to gf. Remember that an homotopy can be regarded as a continuous film (a scene), where each frame is a continuous map. With this idea in mind, Figure 2 shows some of the frames that let us find such an homotopy:

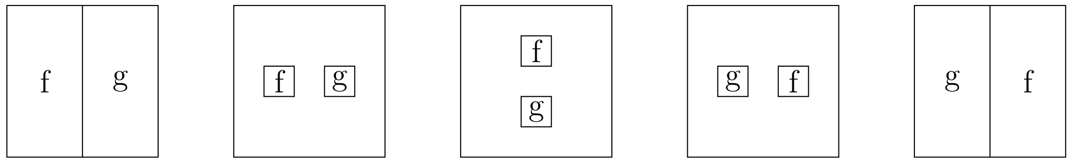

Figure 1: Homotopy between fg and gf

We can also see, that a similar argument lets us realize, as stated before, that the coordinate in which the sum is defined can be arbitrary. ■

As an example, we can see that the higher homotopy groups $\pi_k(S^1)$ are trivial for $k > 1$. Indeed, any map f from S^k to S^1 can be lifted to a map $\widetilde{f}$ from S^k to $\mathbb{R}$ since S^k is simply connected. Since $\mathbb{R}$ is contractible, $\widetilde{f}$ is homotopic to a constant map. Projecting back to S^1 we get an homotopy between f and a constant map, proving the required result.

This means, that we have:

$\pi_0(S^1, 1) = 0$, $\pi_1(S^1, 1) = \mathbb{Z}$,

$\pi_n\ (S^1, 1) = 0$, if $n > 1$.

The above argument can actually be generalized by replacing S^1 by any space whose universal cover is trivial.

The same kind of argument will be generalized at the end of the next subsection as a method to compute π_k for some spaces in which we have a fibration.

One can also prove that if $m < n$; every map from S^m to S^n is nullhomotopic, which implies:

$\pi_m(S^n, s_0) = 0$, if $m < n$.

To prove this, one can consider triangulations of S^m and S^n; then, without loss of generallity, one can consider a map $f : S^m \to S^n$ to be simplicial (mapping simplices to simplices). Since $m < n$, the induced map on the corresponding simplices is not surjective, and therefore $f(S^m) \subsetneqq S^n$; this implies that $f(S^m)$ is contractible. Therefore f is homotopically equivalent to a trivial map.

Considering S^n to be the boundary of the $(n+1)$-simplex one can verify that the homotopy class of a map $f : S^n \to S^n$ is completely determined by its degree, and therefore:

$$\pi_n(S^n, s_0) = \mathbb{Z}.$$

A more rigourous proof for this fact can be obtained using Theorem 2.18.

Remark 2.7 *In general, higher homotopy groups are hard to compute. Even $\pi_k(S^2)$ is not known for high values of k. Nevertheless, in the next section we will summarize some results which lead us to computations of higher homotopy groups for some special cases.*

2.1 Relative Homotopy

As remarked before, the axiomatic definition of the higher homotopy groups does not give us a tool for effectively computing these groups. In the current section, we will review some methods for doing such computations in particular cases.

Remark 2.8 *Consider the space F^n of maps $f:(I^n, I^{n-1}) \to (X, x_0)$. We can divide them into homotopy classes. Define the set $\pi_n(X, x_0)$ as the set of such classes. Topologizing F^n by the compact-open topology we get that equivalently, $\pi_n(X, x_0)$ becomes the set of all path-components of the space F^n.*

Remark 2.9 *So far, we have a nice closed formula for computing low dimensional homotopy spaces of spheres (low meaning lower than the sphere's dimension). However, to compute higher-dimensional homotopy groups, as well as to compute these groups for other spaces, we will need other results, which will be described below.*

As a generalization of the homotopy groups, one can also define the so-called relative homotopy groups as follows.

Definition 2.10 *Given a triplet (X, A, x_0), where A is a closed subspace of X, and $x_0 \in A$; for $n > 0$, define the n-th relative homotopy set $\pi_n(X, A, x_0)$ by considering the n-cube I^n. Let I^{n-1} be the initial $(n-1)$-face of I^n (the face where $t_n = 0$). Denoting the union of all remaining $(n-1)$-faces of I^n by J^{n-1}. we have:*

$$\partial I^n = I^{n-1} \cup J^{n-1}, \quad \partial I^{n-1} = I^{n-1} \cap J^{n-1}.$$

Remark 2.11 *A map $f:(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ is a continuous function from I^n to X which sends I^{n-1} into A and J^{n-1} into x_0. In particular, it sends ∂I^n into A and ∂J^{n-1} into x_0. Denote by $F_A^n = F_A^n(X, A, x_0)$ the set of all such maps. We can divide them into homotopy classes. Define the set $\pi_n(X, A, x_0)$ as the set of such classes. Topologizing F_A^n by the compact-open topology we get that equivalently, $\pi_n(X, A, x_0)$ becomes the set of all path-components of the space F_A^n.*

Similarly, as it was done before, one can define an operation on the set $\pi_n(X, A, x_0)$, for $n > 1$ (unlike in the case of the "absolute" homotopy groups, the operation now defined is not commutative in general).

As in the case of the homotopy groups, for $n > 2$, we can mimic the proof in Proposition 2.6 and prove that $\pi_n(X, A, x_0)$ is abelian for $n > 2$.

Also, in a similar fashion as was described in Remark 2.2 (pinching J^{n-1} to a single point s_0), one might equally well define an element of $\pi_n(X, A, x_0)$ as an homotopy class of maps from (E^n, S^{n-1}, s_0) into (X, A, x_0).

A property that will be useful, is stated as follows (to see the proof read, for example, [13, Prop.3.4]).

Given a topological space X and a closed subspace A such that X can be obtained from A by adjoining a single n-cell in the same way that is done for CW-complexes, we call the pair (X, A) a relative n-cell.

Proposition 2.12 *If (X, A, x_0) is a triplet such that (X, A) is a relative n-cell, then we have $\pi_m(X, A, x_0) = 0$ for every m satisfying $0 < m < n$.*

2.2 Some Results for Homotopy Groups

Analogous to the case of the fundamental group, one can obtain simple expressions for the computations of higher homotopy groups of some particular kinds of spaces. For example, we have that the homotopy group of the product of two spaces is just the direct sum of their homotopy groups. More formally, we have the following:

Proposition 2.13 *Let $(X, x_0), (Y, y_0)$ be two given based spaces. Consider the product $(Z, z_0) = (X \times Y, (x_0, y_0))$.*

Then, for every $n > 0$, we have:

$$\pi_n(Z, z_0) = \pi_n(X, x_0) \oplus \pi_n(Y, y_0).$$

Proof. The proof is exactly the same as the one for the fundamental group. Namely, taking a function $f : (I^n, \partial I^n) \to (Z, z_0)$, we consider the corresponding projections $f_X : (I^n, \partial I^n) \to (X, x_0)$ and $f_Y : (I^n, \partial I^n) \to (Y, y_0)$. Then we can observe that the map which sends $[f] \to ([f_X], [f_Y])$ is an isomorphism. ■

In a similar way [13, Thm 3.1], one can get a result involving the one point union of spaces by considering them embedded in the product space, as follows:

Proposition 2.14 *Let* $(X, x_0), (Y, y_0)$ *be two given based spaces. Consider the one point sum* $(X \vee Y, u_0) = (X \sqcup Y / x_0 \sim y_0, [x_0])$. *For every* $n > 1$*, we have:*

$$\pi_n(X \vee Y, u_0) = \pi_n(X, x_0) \oplus \pi_n(Y, y_0) \oplus \pi_{n+1}(X \times Y, X \vee Y, (x_0, y_0)).$$

In particular, using the usual CW-decomposition of spheres, we see that $(S^p \times S^q, S^p \vee S^q)$ is a relative $p+q$-cell, getting:

Proposition 2.15 *For every* $p > 0, q > 0$ *and* $n < p + q - 1$,

$$\pi_n(S^p \vee S^q) \approx \pi_n(S^p) \oplus \pi_n(S^q).$$

Proof. Follows immediately from Propositions 2.14 and 2.12 for the pair $(S^p \times S^q, S^p \vee S^q)$.
■

2.3 Long Exact Sequence in Homotopy

Similarly to homology, we can obtain a long exact sequence in homotopy by considering the relative homotopy and a boundary operator defined as follows:

Let (X, A, x_0) be a triplet, (i.e we consider both spaces based at $x_0 \in A \subset X$). For every $n > 0$, we define $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ by restricting the function f to I^{n-1}.

An element α of $\pi_n(X, A, x_0)$ is an homotopy class which can be represented by a map $f{:}(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$.

If $n = 1$, $f(I^{n-1})$ is a point of A which determines a path-component $\beta \in \pi_0(A, x_0)$ of A. For $n > 1$, the restriction $f|_{I^{n-1}}$ is simply a map from $(I^{n-1}, \partial I^{n-1})$ into (A, x_0) and hence, by definition, it represents an element of $\pi_{n-1}(A, x_0)$.

We can observe that the choice of such a β is independent of the choice of $f \in \alpha = [f]$. Therefore, we can define the boundary operator ∂, where $\partial(\alpha) = \beta$.

By definition of this boundary operator, we have a couple of simple properties that will be useful later.

Proposition 2.16 *Given* $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ *as above,* $\partial(0) = 0$.

Proposition 2.17 *For $n > 1$, ∂ is a group homomorphism. [Hu, thm 5.2].*

Now, by the functoriality of homotopy (which can be checked by the reader in a similar way as it is done for $\pi_1(\cdot)$), every function from a based space to another (and in general from a pair to another, as well) induces transformations of the corresponding homotopies, which in the cases where homotopies are groups, are also homomorphisms.

To build a similar long exact sequence as the one obtained for homology, we will work with the following inclusion maps:

$$\begin{aligned} i &: (A, x_0) \to (X, x_0), \\ j &: (X, x_0) \to (X, A, x_0). \end{aligned}$$

These maps induce the transformations:

$$\begin{aligned} i_* &: \pi_n(A, x_0) \to \pi_n(X, x_0), \ n \geq 0, \\ j_* &: \pi_n(X, x_0) \to \pi_n(X, A, x_0), \ n \geq 0, \end{aligned}$$

and such transformations are homomorphisms for $n \geq 1$.

The transformations i_*, j_*, ∂ define a long exact sequence in homotopy as follows:

$$\cdots \xrightarrow{i_*} \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{j_*} \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \cdots \xrightarrow{j_*} \pi_0(X, x_0).$$

The proof of the exactness of the sequence above can be seen, for example, in [13, Ch 4, S 7, Prop4].

2.4 Hurewicz Homomorphism

We have already found a similarity between homotopy and homology of a topological space above by considering the long exact sequences that can be obtained in both cases. Furthermore, we can define a natural homomorphism from an homotopy group $\pi_n(X)$ to its corresponding homology group $H_n(X)$, for all $n \geq 1$.

Given a triplet (X, A, x_0), consider for $n > 1$ the homotopy set $\pi_n(X, A, x_0)$.

For an element $\alpha \in \pi_n(X, A, x_0)$, we can represent α by a map $\phi : (E^n, S^{n-1}, s_0) \to (X, A, x_0)$. Without loss of generality, we can consider $s_0 = (1, 0, \cdots, 0)$. The natural coordinate system in E^n determines an orientation of E^n and therefore, it also defines a generator ξ_n of the free cyclic homology group $H_n(E^n, S^{n-1})$. We know that a map ϕ from (E^n, S^{n-1}) into (X, A) induces a homomorphism:

$\phi_* : H_n(E^n, S^{n-1}) \to H_n(X, A)$.

Since homology is invariant under homotopy of maps, ϕ_* only depends on $\alpha \in \pi_n(X, A, x_0)$. This means that we have just defined a transformation:

$$\psi_n : \pi_n(X, A, x_0) \to H_n(X, A),$$

where $\psi_n(\alpha) = \phi_*(\xi_n)$. This transformation, for $n > 1$, (or for $A = x_0$) defines a group morphism, which will be called the natural (Hurewicz) homomorphism of $\pi_n(X, A, x_0)$ to $H_n(X, A)$.

It can be checked that ψ_n commutes with any map $f{:}(X, A, x_0) \to (Y, B, y_0)$ and also commutes with the boundary operator defined above. In particular, this gives us a map $h_n : \pi_n(X, x_0) \to H_n(X)$, which makes the following infinite homotopy-homology ladder commutative:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & \pi_{n+1}(X, A, x_0) & \xrightarrow{j_*} & \pi_n(A, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) & \xrightarrow{\partial} & \pi_n(X, A, x_0) & \xrightarrow{j_*} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & H_{n+1}(X, A, x_0) & \xrightarrow{j_*} & H_n(A, x_0) & \xrightarrow{i_*} & H_n(X, x_0) & \xrightarrow{\partial} & H_n(X, A, x_0) & \xrightarrow{j_*} & \cdots
\end{array}$$

Theorem 2.18 (Hurewicz Theorem) *If X is an $(n-1)$-connected finite simplicical complex with $n > 1$ (i.e. if $\pi_k(X) = 0$, for $k < n$), then the natural homomorphism h_n is an isomorphism.*

Proof. We will consider two cases, first the case when X is n-dimensional, and then the case when it has a higher dimension.

In the first case, we can define $Y = X/X^{n-1}$, by identifying its $(n-1)$-skeleton to a single point, y_0 (we can contract it this way since X is $(n-1)$-connected). More formally, we have the projection:

$$p : \left(X, X^{n-1}\right) \to (Y, y_0).$$

This projection is, furthermore, an homotopy equivalence of pairs, which implies both the induced map in homotopy $p_* : \pi_k\left(X, X^{n-1}\right) \to \pi_k(Y, y_0)$ and the induced map in homology $p_\# : H_k\left(X, X^{n-1}\right) \to H_k(Y, y_0)$ are isomorphisms.

This implies that calling h_n, and k_n the corresponding Hurewicz morphisms for (X, X_0) and (Y, Y_0), we have the following commutative diagram:

$$\begin{array}{ccc} \pi_n\left(X, X^{n-1}\right) & \cong & \pi_n(Y, y_0) \\ \downarrow h_n & & \downarrow k_n \\ H_n\left(X, X^{n-1}\right) & \cong & H_n(Y, y_0). \end{array}$$

Now, clearly the pair (Y, y_0) is homeomorphic to a one point union of several n-spheres, and using (inductively) Proposition 2.15 , we obtain:

$$\pi_n\left(Y, y_0\right) = \pi_n\left(\bigvee_i \left(S_i^n\right)\right) \cong \bigoplus_i \pi_n(S_i^n) \cong H_n\left(\bigvee_i \left(S_i^n\right)\right).$$

This implies that k_n is an isomorphism, and therefore, by the commutativity of the diagram above, h_n has to be an isomorphism as well.

Now, when considering the case when $X^n \subsetneq X$, we can consider the pair $\left(X^{n+1}, X^n\right)$, and a piece of the homotopy ladder, namely,

$$\begin{array}{ccccccc} \pi_{n+1}\left(X^{n+1}, X^n, x_0\right) & \stackrel{\partial}{\to} & \pi_n\left(X^n, x_0\right) & \stackrel{i_*}{\to} & \pi_n\left(X^{n+1}, x_0\right) & \stackrel{j_*}{\to} & 0 \\ \downarrow \chi_{n+1} & & \downarrow k_n & & \downarrow h_n & & \\ H_{n+1}\left(X^{n+1}, X^n, x_0\right) & \stackrel{\partial}{\to} & H_n\left(X^n, x_0\right) & \stackrel{i_\#}{\to} & H_n\left(X^{n+1}, x_0\right) & \stackrel{j_\#}{\to} & 0, \end{array}$$

where $\pi_n\left(X^{n+1}, X^n, x_0\right) = 0$ by Proposition 2.12, and by the previous part, k_n is an isomorphism, which helps us to chase the diagram above to prove that h_n is surjective. In fact, for $\beta \in H_n(X, x_0)$, we can find $\gamma \in H_n(X^n, x_0)$ with $\beta = i_*(\gamma)$, such that $g = i_* k_n^{-1}\left(\gamma\right) \in \pi_n\left(X^{n+1}, x_0\right)$ satisfies $h_n\left(g\right) = \beta$, by the commutativity of the ladder.

It can be proved that χ_{n+1} is surjective, and therefore taking $g \in \pi_n\left(X^{n+1}, x_0\right)$ such that $h_n\left(g\right) = 0$. One can follow the diagram and find an element $\beta \in H_n\left(X^n, x_0\right)$, such that $g = i_* k_n^{-1}\left(\beta\right)$. By exactness of the chain and by the surjectivity of χ_{n+1}, we can finally find $\delta \in \pi_{n+1}\left(X^{n+1}, X^n, x_0\right)$, such that $\beta = \partial\chi_{n+1}\left(\delta\right)$. Therefore $g = i_* k_n^{-1}\partial\chi_{n+1}\left(\delta\right) =$

$i_* k_n^{-1} k_n \partial(\delta) = 0$ by the exactness of the sequence, proving that h_n is injective, and concluding that h_n is an isomorphism.

Finally, we can consider the inclusion $X^{n+1} \xrightarrow{i} X$, and since (X, X^{n+1}) is an $(n+2)$-cel, we have the following commutative ladder:

$$\begin{array}{ccccccc} 0 & \to & \pi_n(X^{n+1}, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) & \to & 0 \\ & & \downarrow h_n & & \downarrow k_n & & \\ 0 & \to & H_n(X^{n+1}, x_0) & \xrightarrow{i_\#} & H_n(X, x_0) & \to & 0 \end{array}$$

Since h_n, i_*, and $i_\#$ are isomorphisms, k_n also is, which finishes the proof. ■

2.5 Homotopy of Fibrations

Let p be a fibering of E over B. Choose a base point $b_0 \in B$ such that the fiber $F = p^{-1}(b_0)$ is not empty. For the fiber F, which we will call a standard fiber, we can pick a base point and obtain a triplet (E, F, e_0).

Since $p(F) = b_0$, the projection $p : (E, e_0) \to (B, b_0)$ defines a map $q : (E, F, e_0) \to (B, b_0)$ and $p = qj$, where $j : (E, e_0) \to (E, F, e_0)$ denotes the inclusion map. Furthermore, in this case, we obtain an isomorphism $q_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ for each $n \geq 1$. For the proof of this fact see, for example [13, III.9.VI].

Let:

$d_* = \partial q_*^{-1} : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)\,,\ n \geq 1.$

By the functoriality of homotopy groups, $p_* = q_* j_*$, and from the long exact sequence in homotopy of the triplet (E, F, e_0), we get the exact sequence:

$$\begin{array}{l} \cdots \xrightarrow{p_*} \pi_{n+1}(B, b_0) \xrightarrow{d_*} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{d_*} \cdots \\ \cdots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{d_*} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0)\,. \end{array}$$

which is called the homotopy sequence of the fibering $p : E \to B$ based at e_0.

From this exact sequence, we get several properties for some fibrations.

Proposition 2.19 *If the standard fiber F is totally pathwise disconnected, then $\pi_n(E, e_0) \cong \pi_n(B, b_0)$, $n \geq 2$; and p_* is a monomorphism for $n = 1$.*

Proof. If F is totally pathwise disconnected, then $\pi_n(F, e_0) = 0$ for $n \geq 1$. Hence, we can split the long exact sequence, obtaining the required isomorphisms and monomorphism. ■

Example 2.20 *We have* $\pi_n(S^2) = \pi_n(S^3)$, $n \geq 3$.

It suffices to consider the Hopf fibration, $p : S^3 \to S^2$, *whose fiber is* S^1. *This result can be seen easily when considering* S^3 *embedded in* $\mathbb{C}^2$ *as* $S^3 = \{(x, y) \in \mathbb{C}^2 : \|(x, y)\| = 1\}$, *and* $S^2 = \{(x, y) \in \mathbb{C}^2 : \|(x, y)\| = 1, y \in \mathbb{R}\}$. *Since we know that* S^1 *has trivial higher homotopy groups, applying the exactness of the sequence in the same way as it was done in the last proposition, the result follows.*

In particular, since a covering space is a fibration, we get the following results:

Proposition 2.21 *For a covering space* E *over a base space* B *relative to a projection* $p : (E, e_0) \to (B, b_0)$, *we have:*

$$p_* : \pi_n(E, e_0) \cong \pi_n(B, b_0),\ n \geq 2,$$

and p_* *is a monomorphism if* $n = 1$.

In particular, for the universal cover, we get:

$$\pi_n(\tilde{B}, \tilde{b}_0) \cong \pi_n(B, b_0),\ n \geq 2,\ \pi_1(\tilde{B}) = 0,$$

and using Theorem 2.18, we get the following result:

Proposition 2.22 *If* B *is a connected, locally pathwise connected and semilocally simply connected space, the second homotopy group* $\pi_2(B)$ *is isomorphic to the second homology group* $H_2(\widetilde{B})$ *of its universal cover.*

3 Homotopy Groups of 2-Complexes

In this section, we will consider some properties of 2-complexes; in particular, we will see that for a 2-complex being aspherical is equivalent to simply having a trivial second homotopy group, as stated before. Such properties will let us study the homotopy of 2-complexes in a different manner.

For example, given a connected 2-complex K^2, we can find a spanning tree T, for K^1, and since it is contractible, we can consider without loss of generality, that the complex K^2 has only one 0-cell.

This observation implies that the complex K^2 can be expressed in the form of a group presentation:

$$\mathcal{P} = \langle \mathbf{x} | \mathbf{r} \rangle \,.$$

Here, the generators, $\mathbf{x}$, correspond to the 1-cells and the relators, $\mathbf{r}$, correspond to the 2-cells of K^2,. The notation $\langle \mathbf{x} | \mathbf{r} \rangle$ stands for $F(\mathbf{x})/N(\mathbf{r})$, the quotient group of the free group generated by $\mathbf{x}$, by the normal closure of the relators $\mathbf{r}$.

Consider the complexes K, L regarded as their corresponding group presentations, $\mathcal{P} = \langle \mathbf{x} | \mathbf{r} \rangle$, $\mathcal{Q} = \langle \mathbf{x} | \mathbf{s} \rangle$, respectively, which have the same generator set and whose normal closures are the same $N(\mathbf{r}) = N(\mathbf{s})$; therefore, actually, $\mathcal{P} = \mathcal{Q} = \pi_1(K) = \pi_1(L)$.

Consider the 2-complexes K, L such that $K^1 = L^1$; we have the following lemmas, whose proofs can be found in [10, Section 2.2]:

Lemma 3.1 *Any map $G : K \to L$ that induces the identity isomorphism on the fundamental group $\pi_1(K) = \pi_1(L)$ is based homotopic to one that is the identity when restricted to the common one skeleton, K^1.*

Lemma 3.2 *Any extension $G : K \to L$ of the identity map on the common 1-skeleton that induces an isomorphism $G_* : \pi_2(K) \to \pi_2(L)$ is an homotopy equivalence.*

The lemmas above let us prove the 2-dimensional version of a theorem proved by Whitehead [28]. For this, we should also consider that given two presentations of a given group, one can expand them trivially in such a way that the new presentations have the same set of generators.

Theorem 3.3 *A map $f : K \to L$ of connected 2-complexes is a based homotopy equivalence if and only if it induces isomorphisms on the first and second homotopy groups.*

Proof. When f is an homotopy equivalence, it follows immediately that the corresponding homotopy groups are isomorphic.

Conversely, K and L may be assumed to be models of group represantations, say, $P = \langle \mathbf{x}|\mathbf{r}\rangle$ and $Q = \langle \mathbf{y}|\mathbf{s}\rangle$, of isomorphic groups G, H , respectively. Let $f_{*1} : \pi_1(K) \to \pi_1(L)$ be the corresponding isomorphism and let g_{*1} be its inverse. We can expand the corresponding presentations by adding trivial terms corresponding to the other presentations, namely:

$P' = \langle \mathbf{x}, \mathbf{y}|\mathbf{r}, g_{*1}(y)y^{-1}\,(y \in \mathbf{y})\rangle$, and $Q' = \langle \mathbf{x}, \mathbf{y}|\mathbf{s}, f_{*1}(x)x^{-1}\,(x \in \mathbf{x})\rangle$.

These new expanded presentations have the same generators and present the same group. Let K' L' be the corresponding topological models for those presentations, which are trivially homotopically equivalent to K and L, respectively. Using the previous result, the corresponding extended map $\widehat{f} : K' \to L'$ induces the identity isomorphism on $\pi_1(K') = \pi_1(L')$, and an isomorphism on π_2. Therefore, using the lemmas, K' and L' are homotopy equivalent, and by transitivity, the same holds for K and L. ■

Let's remember the long exact sequence in homotopy for a pair (X, A) :

$$\cdots \xrightarrow{i_*} \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{j_*} \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \cdots \xrightarrow{j_*} \pi_0(X, x_0)$$

We will see that an homotopy action of $\pi_1(A)$ on the modules appearing in this sequence exists; this action is given by "dragging" the image of the base point backwards along a loop.

More precisely, given a based map $F : (B^{n+1}, S^n) \to (X, A)$ and a loop $\alpha : I \to A$, consider the map $\langle F, \alpha\rangle : (B^{n+1}, S^n) \vee I \to (X, A)$, which is defined by F on (B^{n+1}, S^n) and by α on I. Consider a retraction $R : (B^{n+1}, S^n) \times I \to (B^{n+1}, S^n) \vee I$, then we have found an homotopy $H = \langle F, \alpha\rangle \circ R : (B^{n+1}, S^n) \times I \to (X, A)$ such that $H_0 = F$, the homotopy class of the corresponding function $G = H_1$ is defined to be the action of $[\alpha]$ over $[F]$, and is denoted by:

$[G] = [\alpha] \cdot [F]$.

This gives us, for each $n \geq 2$ a group homomorphism:

$h : \pi_1(A) \to \mathrm{Aut}(\pi_n(X, A))$,

$h_\alpha [F] = [\alpha] \cdot [F]$.

The above homomorphism can be extended linearly on $\pi_n(X, A)$ making it a $\pi_1(A)$ module over the group ring $\mathbb{Z}(\pi_1(A))$.

Similarly, we can find homomorphisms:

$h : \pi_1(X) \to \mathrm{Aut}(\pi_n(X))$,

$h_\alpha [F] = [\alpha] \cdot [F]$.

These properties let us consider $\pi_1(X)$ as a group of operators on $\pi_n(X)$. For $n = 1$, the above action is simply conjugation, and as above, for $n > 1$, this action makes the abelian group $\pi_n(X)$ a left module over $\mathbb{Z}(\pi_1(X))$.

The following proposition summarizes the preoperties that the action of $\pi_1(A)$ on $\pi_2(X, A)$ has. For a proof of it, we recommend the reader to read [10].

Proposition 3.4 *In particular, the action of $\pi_1(A)$ on $\pi_2(X, A)$ has the following properties:*

$$(\partial [G]) [F] = [G] [F] [G]^{-1},$$
$$\partial (\alpha [F]) = \alpha \partial [F] \alpha^{-1}.$$

The second property is known as the Pfeifer identity.

This gives the last part of the sequence a structure known as a crossed module which will be detailed in the next (sub)section. Furthermore, if $X \backslash A$ consists only of open 2-cells, the basis for $\pi_2(X, A)$ as a $\pi_1(A)$ free module has a one-to-one correspondence with the two cells in $X \backslash A$.

Considering now only connected 2-complexes $K = K^2$, we have that $\pi_n(K^1) = 0$ for $n \geq 2$, splitting the long exact sequence into shorter ones

$$0 \to \pi_n\left(K^2\right) \xrightarrow{j_*} \pi_n\left(K^2, K^1\right) \to 0, \text{ for } n \geq 3,$$

and

$$0 \to \pi_2\left(K^2\right) \xrightarrow{j_*} \pi_2\left(K^2, K^1\right) \xrightarrow{\partial} \pi_1\left(K^1\right) \xrightarrow{i_*} \pi_1\left(K^2\right) \to 0. \tag{1}$$

Actually, Whitehead, completely characterized the homotopy classfication of 2-complexes as the classification of free crossed modules whose action groups are free groups [28].

4 Whitehead's Question

Let's go back to the main topic of our paper, Whitehead's Question. Now that we have the required background to understand its statement, we will summarize some of the properties that have been proven and conjectured about low dimensional topology that would make it really tough to “guess” whether Whitehead’s conjecture is true or false and to see why it is an “interesting” question. In the second part of this section, we will summarize some classical results that are known about Whitehead’s question, which are used in further sections which study some approaches to the mentioned question.

4.1 Related Problems

For instance, if for a pair of 2-complexes $K^2 \subset L^2$ there existed an injective homomorphism $\pi_2(K^2) \to \pi_2(L^2)$, the answer to this question would be trivial. But we can see that it is not the case by considering the following example:

Example 4.1 *Consider $K^2 = S^1 \vee S^2$, $L^2 = D^2 \vee S^2$. In this case, we have $\pi_1(K^2) \cong \mathbb{Z}$, where the generator is the homotopy class of the loop α that goes one lap around the S^1 component of K^2. Furthermore, in this case $\pi_2(K^2) \cong \mathbb{Z}(\pi_1(K^2))$ is a module on which $\pi_1(K^2)$ acts, and is generated by the homotopy class of a homomorphism f from S^2 to the S^2-component of K^2(as it was detailed in the previous section). Consider the element $[g] = (\alpha - 1)[f] \in \pi_2(K^2)$. When we consider the same element as a map to K^2 (i.e, composing it with the inclusion of K^2 into L^2), such a map is homotopic to the constant path, $[g] = 0 \in \pi_2(L^2)$.*

We can think of similar problems, like what happens when we allow cells of higher degree in a complex K. Is it possible to annihilate $\pi_2(K) \neq 0$ by adding a 2-cell in this case? (So the new complex would be aspherical and the original would be a non-aspherical subcomplex of the latter).

Example 4.2 (Adams 55 [1]) *Consider a 3-complex K^3 whose 2-skeleton is $K^2 = S^1 \vee S^2$, and has a single 3-cell attached by a map homotopic to $(2\alpha - 1)[f] \in \pi_2(K^2)$. Consider $L^3 \supset K^3$, the cell complex obtained by attaching a 3-cell to $L^2 = D^2 \vee S^2 \supset K^2$ in the same way as it was done for K. Therefore, $L^3 \backslash K^3$ is a 2-cell. Now, by the same argument as in the former example, the attaching map is $(2\alpha - 1)[f] = [f] \in \pi_2(L^3)$, but L^3 and $D^2 \vee D^3$ share the same 2-skeleton, and since the latter is collapsible, so is L^3. Therefore the Whitehead conjecture would be false in the case of general CW-complexes.*

Similarly, we may think of a straightforward generalization of Whitehead’s Asphericity Conjecture, namely, Having $K^n \subset L^n$ with $\pi_n(L^n) = 0$, does this imply that $\pi_n(K^n) = 0$?

It is trivially true in dimension 1, since $\pi_1(K^1) \to \pi_1(L^1)$ is always injective. On the other hand, for dimension $n \geq 3$, it is false. Consider $K = S^{n-1}$ and $L = D^n$. For $n = 3$, the Hopf map gives a non-trivial element of $\pi_3(S^2)$, and similarly it was checked that for $n \geq 4$, $\pi_n(S^{n-1}) \cong \mathbb{Z}_2$, see, for example, [13, Thm 15.1].

So, only the 2-dimensional case, namely Whitehead's Conjecture remains open.

As another motivation for the study of the Whitehead conjecture, one can consider the problem of finding the second homotopy group of a knot complement in S^3.

Problem 4.3 *Given any knot $K \subset S^3$, the space $S^3 \backslash K$ is the knot complement. What is $\pi_2(S^3 \backslash K)$?*

Proposition 4.4 ([25, Thm 1.1]) *If Whitehead's Asphericity Conjecture is true, then knot complements are aspherical.*

Proof. Glue a thickened meridian disk into $S^3 \backslash K$ to get a 3-ball which collapses to an aspherical 2-complex, say L. So if Whitehead's Asphericity Conjecture were true then L has to be aspherical and the asphericity of knot complements would be shown. ∎

This fact would give us an option to prove that Whitehead's Conjecture is false in the case that we could find a knot in S^3 whose complement is not aspherical. Papakyriakopoulos proved the asphericity of knot complements in [24] using other techniques for 3-manifolds, and therefore Whitehead's Conjecture remains open.

4.2 Classical Results

Now, we will summarize some results that have led to partial answers to Whitehead's question, as well as some classical theorems involving asphericity of 2-complexes.

It has been already verified that for particular cases, Whitehead's conjecture holds.

Among the results obtained for the Whitehead's Asphericity Conjecture, we are going to state the following particular cases:

Consider a 2-complex L, $K \subset L^2$, and $\pi_2(L) = 0$:

Proposition 4.5 *[6] Whitehead's Conjecture is true if:*

- *K has at most one 2-cell, or*
- *$\pi_1(L)$ is finite and non-trivial, abelian, or free.*

Theorem 4.6 (Cockroft [6]) *If $f : \pi_1(K) \to \pi_1(L)$ is injective then $\pi_2(K) = 0$.*

Proof. Let $\widetilde{L}$ be the universal cover of L and $p : \widetilde{L} \to L$ the corresponding covering projection. Since K is a subcomplex of L, we can consider a component of $p^{-1}(K)$, say $\overline{K}$.

It is known that $\overline{K}$ is a regular cover of K. Thus, since any loop in $\ker f \subset \pi_1(K)$ can be lifted to one in $\pi_1(\overline{K})$, considering the corresponding induced morphism for the fundamental group, $p_* : \pi_1(\overline{K}) \to \pi_1(K)$, we get:

$$p_* \left(\pi_1(\overline{K})\right) = \ker(f) = 0.$$

Therefore, $\overline{K}$ is the universal cover of K. The fundamental group of the universal cover of a CW-complex is always trivial, so by Theorem 2.18 we have,

$$\pi_2(K) \cong H_2(K) < H_2(\widetilde{L}) \cong \pi_2(L) = 0,$$

proving the required result. ■

Another classical result, is the following:

Theorem 4.7 (Howie 1979 [11]) *If the answer to Whitehead's question is no, then there exists a connected 2-complex L, such that either:*

1. *L is finite and contractible and $L - e$ is not aspherical for some open 2-cell e of L.*
2. *L is the union of an infinite ascending chain of finite connected non-aspherical subcomplexes $K_0 \subset K_1 \subset \dots$ where each inclusion is nullhomotopic.*

Notice that the word 'contractible' can be substituted with 'aspherical' to obtain a weaker version of part 1 of the theorem.

Proof. (of the weaker version) Suppose Whitehead's Conjecture is false for finite complexes, then we may assume that there exists a finite aspherical 2-complex Y, containing a non-aspherical subcomplex X. Let $\{e_i\}$ be the set of 2-cells in $Y - X$ and let m be the minimum i such that $X \cup Y^1 \cup e_1 \cup e_2 \cup \cdots \cup e_i$ is aspherical. Then we can take $L = X \cup Y^1 \cup e_1 \cup e_2 \cup \cdots \cup e_m$ and $e = e_m$ and obtain an example of the form 1.

Now, assume that Whitehead's Conjecture is false in general, but is true for finite connected complexes. Let Y be an aspherical 2-complex, and X be a non-aspherical subcomplex of Y. Let $\overline{X}$ be one connected component of $p^{-1}(X)$ for the universal cover $p : \widetilde{Y} \to Y$. Then, $\overline{X}$ is not aspherical and has a finite connected non-aspherical subcomplex, say, K_0. Now, since $\widetilde{Y}$ is contractible, the inclusion map $K_0 \to \widetilde{Y}$ is nullhomotopic, so, we can extend it over the cone $CK_0 \to \widetilde{Y}$. Since K_0 is finite, its image is a finite subcomplex K_1 of $\widetilde{Y}$. Repeating inductively the argument for K_{i+1} in the place of K_i and defining $L = \cup_i K_i$, we get the example as required in part 2. ■

Furthermore, this result was strengthened by Lüft, getting the following:

Theorem 4.8 (Lüft 1996 [19]) *If Whitehead's Conjecture is false, then there is a counterexample of type* 2 *of Theorem 4.7.*

Several approaches have been taken for this problem using different methods: all of them have provided just partial answers to Whitehead's question and some translations of Whitehead's Conjecture to other setups. The methods include the study of crossed modules, geometric invariants such as the Betti numbers, using group presentations, simple homotopy types and using discrete homotopy defined in the poset of simplical complexes.

The next sections provide a review of two of those methods. Section 5 deals with crossed modules and Section 6 studies the approaches using discrete homotopy.

5 Crossed Modules

In the current section, we will study some of the properties of crossed modules which will allow us to study some properties of aspherical 2-complexes. In particular, we will see that any pair (X, A) where X is a 2-complex, naturally induces a structure of a crossed module over the group $\pi_1(A)$ which has some particular properties and can facilitate the study of 2-complexes.

Definition 5.1 *Consider the triple (C, ∂, G) where G, C are groups. If G acts on C on the left and the morphism $\partial : C \to G$ satisfies the following conditions:*

1. $\partial(gc) = g\partial c g^{-1}$. $\forall c \in C, \forall g \in G$, *and*
2. $\partial(d)\, c = dcd^{-1}$, $\forall c, d \in C$,

then the triple is called a G-crossed module. If the triplet only satisfies the first condition, the structure is called a pre-crossed module.

Notice that $K = \ker \partial$ is contained in the center of C and $N = im\partial$ is a normal subgroup of G. Therefore, we get:

$$0 \to K \to C \to N \to 0.$$

In particular, as it was noticed before, for a pair (X, A), the left action of $\pi_1(A)$ on $\pi_2(X, A)$, together with the boundary homomorphism ∂, form a $\pi_1(A)$-crossed module.

Definition 5.2 *A morphism of crossed modules $(\alpha, \beta) : (C, \partial, G) \to (C', \partial', G')$ is a pair of group homomorphisms $\alpha : C \to C'$, $\beta : G \to G'$ such that $\partial' \circ \alpha = \beta \circ \partial$ and $\alpha (gc) = \beta (g) \alpha (c)$, for all $g \in G$, $c \in C$.*

In the case that $A = X^1$, we note that the crossed module $(\pi_2 (X^2, X^1), \partial, \pi_1 (X^1))$ is free in the category of $\pi_1 (X^1)-$crossed modules, meaning that it satisfies the following universal property: For any indexed minimal generator set $T = \{c_\alpha : \alpha \in A\} \subset C$, and any $G'-$crossed module $\partial' : C' \to G'$, and any homomorphism $\tau : G \to G'$ such that $(\tau \circ \partial)(c_\alpha) = \partial' (c'_\alpha)$ for each $\alpha \in A$, there exists a minimal generator set $T' = \{c'_\alpha : \alpha \in A\} \subset C'$ and a unique homomorphism $\eta : G \to G'$ such that the pair (η, τ) is a morphism of crossed modules, and $\eta (c_\alpha) = c'_\alpha$ for each $\alpha \in A$.

In particular, free modules over a group are an example of free crossed modules.

Now, we will see what that result means in the case of the pair (K^2, K^1).

5.1 Relation Between Crossed Modules and 2-Complexes

As it was stated at the end of section 3, the exact sequence in (1) can be regarded equivalently as a crossed module. In what follows, we will show there exists an equivalence between free crossed modules over a free group and 2-complexes. Furthermore, we also consider another equivalent algebraic structure, namely, the category of $\mathcal{C}\mathrm{at}^1$-groups.

Consider a 2-complex K which is the model for a presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle = F(\mathbf{x})/N(\mathbf{r})$. Define the group $E (\mathcal{P})$ as the free group on the set $F (\mathbf{x}) \times \mathbf{r}$. Then, we can construct a pre-crossed module associated to that representation by considering the action of $F (\mathbf{x})$ on $E (\mathcal{P})$ by $v \cdot (w, r) = (vw, r)$.

Let the action of $F (\mathbf{x})$ on itself be conjugation, and define the boundary homomorphism $\partial : E (\mathcal{P}) \to F (\mathbf{x})$ by defining it on the elements of the basis by $\partial (w, r) = wrw^{-1}$. From this, we can notice that $\operatorname{im}\partial = N(\mathbf{r})$ is the normal closure of $\mathbf{r}$ in $F(\mathbf{x})$. The subgroup $I(\mathcal{P}) = \ker\partial$ is called the group of identities of the presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$.

This operator is a pre-crossed module, since for all $v \in F(\mathbf{x})$ and all $(w, r) \in E(\mathcal{P})$, we have:

$$\partial(v(w,r)) = \partial((vw,r)) = vwrw^{-1}v^{-1} = v\partial(w,r)v^{-1}.$$

We can construct a crossed module by forcing the second property of the definition to hold by factoring $E(\mathcal{P})$ by the normal closure $P(\mathcal{P})$ of the Pfeifer elements, i.e, the elements of the form:

$$(w,r)(v,s)(w,r)^{-1}\left(wrw^{-1}v,s\right)^{-1}.$$

Furthermore, notice that the Pfeiffer elements belong to the group of identities $I(\mathcal{P})$.

Therefore, we define the $F(\mathbf{x})$-crossed module associated with the presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ by considering the induced homomorphism of ∂ (which we will still call ∂) acting on the group $C(\mathcal{P}) = E(\mathcal{P})/P(\mathcal{P})$. For the crossed module just defined, we have $\ker\partial = I(\mathcal{P})/P(\mathcal{P})$ and $\operatorname{im}\partial = N(\mathbf{r})$.

This means we have an analogue to the fundamental sequence for 2-complexes:

$$(\mathcal{P}) : 0 \to I(\mathcal{P})/P(\mathcal{P}) \to C(\mathcal{P}) \to F(\mathbf{x}) \to F(\mathbf{x})/N(\mathbf{r}) \to 0. \tag{2}$$

We can easily see that the last two terms in this sequence are isomorphic to the last two terms of the corresponding sequence (1) for the pair (X^2, X^1), where we will call τ the corresponding homomorphism between $F(\mathbf{x})$ and $\pi_1(X^1)$.

It can be seen, for example in [10, II.Lemma 2.5], that the τ-morphism $\eta : E(\mathcal{P}) \to \pi_2(X, X^1)$ is surjective and its kernel is the group of Pfeiffer identities, $P(\mathcal{P})$. By considering the induced morphism on $C(\mathcal{P})$, we can verify the following theorem:

Theorem 5.3 (Reidemeister) *The $F(\mathbf{x})$-crossed module $\partial : C(\mathcal{P}) \to F(\mathbf{x})$ associated with the presentation $P = \langle \mathbf{x}, \mathbf{r} \rangle$ is isomorphic to the $\pi_1(X^1)$-crossed module $\partial : \pi_2(X, X^1) \to \pi_1(X^1)$, for the model X of the presentation $\mathcal{P}$.*

We actually have the commutative ladder:

$$\begin{array}{ccccccccc}
0 & \to & I(\mathcal{P})/P(\mathcal{P}) & \to & C(\mathcal{P}) & \to & F(\mathbf{x}) & \to & F(\mathbf{x})/N(\mathbf{r}) & \to & 0 \\
 & & \downarrow & & \eta|| & & \tau|| & & || & & \\
0 & \to & \pi_2(X) & \to & \pi_2(X, X^1) & \to & \pi_1(X^1) & \to & \pi_1(X) & \to & 0.
\end{array}$$

So, using the five-lemma, we get that there is also an isomorphism:

$$I\left(\mathcal{P}\right)/P\left(\mathcal{P}\right) \cong \pi_2\left(X\right).$$

And therefore, we have a characterization of asphericity in terms of crossed modules. Namely, given a group presentation $\mathcal{P}$, its topological model X is aspherical if and only if $I\left(\mathcal{P}\right) = P\left(\mathcal{P}\right)$.

Also, this characterization permits us to see that the crossed module $\partial : \pi_2\left(X, X^1\right) \to \pi_1\left(X^1\right)$ is a free $\pi_1\left(X^1\right)$-crossed module. This follows directly from the fact that $F(\mathbf{x})$ is a free group and acts freely on $C\left(\mathcal{P}\right)$.

Similarly, one can define the notion of a *projective crossed module*, as being projective in the corresponding category, meaning that a G-crossed module (M, ∂, G) is called G-projective, if for any epimorphism $f := (f, id) : (M_1, \partial_1, G) \to (M_2, \partial_2, G)$ and any G-homomorphism $h := (h, id) : (M, \partial, G) \to (M_2, \partial_2, G)$, there exists a morphism $q := (q, id) : (M, \partial, G) \to (M_1, \partial_1, G)$, such that $fq = h$.

When we consider the general case of a pair (X, A), the crossed module may not be projective (and therefore not free either). Although, Dyer [8] proved that if X is a 2-complex and $A \subset X$, then the corresponding crossed module associated to the pair (X, A) is projective in the category of crossed modules.

These properties let us translate asphericity in terms of crossed modules.

Similarly, it can also be proven that a crossed module is equivalent to a different type of algebraic structure, namely a $\mathcal{C}at^1$-group.

Definition 5.4 *Let G be a group and let $s, t \in End(G)$. A triple (G, s, t) is called a $\mathcal{C}at^1$-group if the following conditions hold:*

1. $st = t$ *and* $ts = s$,
2. $[ker(s), ker(t)] = 1$.

Proposition 5.5 (Loday [18]) *The category of crossed modules, $\mathcal{CM}$, and the category of $\mathcal{C}at^1$-groups, $\mathcal{C}at^1$, are equivalent.*

Proof. Let (M, ∂, G) be a crossed module. Consider the following functor:

$$S : (M, \partial, G) \mapsto (M \rtimes G, s : (m, g) \mapsto g,\ t : (m, g) \mapsto \partial(m)g)\,,$$

Clearly, the triplet $(M \rtimes G, s, t)$ satisfies the properties of a $\mathcal{C}at^1$-group, since $st(m, g) = s\,(\partial(m)g) = \partial(m)g$, $ts(m, g) = t\,(g) = g$. The inverse functor is given by $M = \ker s$ and $\partial = t|_{\ker s}$, and the action of G on M is conjugation. The Pfeifer identity for a crossed module is equivalent to the second axiom for a $\mathcal{C}at^1$-group, since if $x = (m, 1) \in \ker s$ and $y = (n^{-1}, \partial(n)) \in \ker t$ with m and $n \in M$, we have $xy = (mn^{-1}, \partial(n))$ and $yx = (n^{-1}\partial(n)m, \partial(n))$. Therefore $xy = yx$ is equivalent to $nmn^{-1} = \partial\,(n)\,m$. ■

5.2 Approaches Using Crossed Modules

As seen in section 3, a 2-complex L can be regarded equivalently as a free $\pi_1\,(L^1)$-crossed module $\partial : \pi_2\,(L, L^1) \to \pi_1\,(L^1)$, and in this case a subcomplex K of L would again be a crossed module $\partial' : \pi_2\,(K, K^1) \to \pi_1\,(K^1)$.

Under this assumption, Conduché [7] has obtained a characterization for K being aspherical using the crossed module structure.

Using a morphism between the respective crossed modules, it has been proved that for subcomplexes of aspherical 2-complexes, the second homotopy module can be expressed as the intersection of the lower central series of the corresponding crossed module. Here we will sketch the proof given by Mikhailov in [23], although this result was first proved by Conduché [7], using different arguments.

In what follows, we will call the crossed module $\mathcal{L}_1\,(X)\,, \partial : \pi_2\,(X, X^1) \to \pi_1\,(X^1)$ the fundamental crossed module associated to X, and similarly, $\mathcal{L}^1\,(X) = \pi_2\,(X, X^1) \rtimes \pi_1\,(X^1)$ the fundamental $\mathcal{C}at^1$-group.

Consider the crossed module (M, ∂, G). For $g \in G$, $m \in M$, define $[g, m] := (g \circ m)\, m^{-1}$. Let $\{\gamma_\tau(M, G)\}$ be the lower central series of (M, ∂, G), defined recursively by:

$$\gamma_1(M, G) = M,$$
$$\gamma_{i+1}(M, G) = [\gamma_i(M, G), M] = \langle [g, m] | g \in \gamma_i(M, G), m \in M \rangle\,,$$
$$\gamma_\omega(M, G) = \cap_{i=1}^{\infty} \gamma_i(M, G).$$

A crossed module (M, ∂, G) is called *residually nilpotent* if $\gamma_\omega(M, G) = \{1\}$.

Similarly, for transfinite ordinals, one can define again the transfinite lower central series by $\gamma_\tau(M, G) = \cap_{\alpha<\tau} \gamma_\alpha(M, G)$. A similar definition works for the lower central series of a group G. In particular, define $[x,y] := xyx^{-1}y^{-1}$,

$$\gamma_1(G) = G,$$

$$\gamma_{\tau+1}(G) = [\gamma_\tau(G), G] = \langle [x,y] | x \in \gamma_\tau(G), y \in G \rangle ,$$

$$\gamma_\omega(G) = \cap_{\alpha<\omega}\gamma_\alpha(G).$$

Remark 5.6 *It is a fact that a free group F is residually nilpotent since any non-trivial word in $\gamma_n(F)$ must have length at least n, and therefore the only word belonging to $\gamma_\omega(F) = \cap_{n\in\mathbb{N}}\gamma_n(F)$ is the empty word.*

We will state the result that implies the characterization of asphericity of a subcomplex of an aspherical 2-complex here.

Lemma 5.7 *Let F be a free group and (M, ∂, F) an F-crossed module. Then $\gamma_\omega(M,F) \subset \ker \partial$.*

Proof. If $m \in \gamma_2(M,F)$, then m is presented as the product of the elements of the form $(g \circ s)\, s^{-1}$, $s \in M$, $g \in F$. But $\partial((g \circ s)s^{-1}) = [g, \partial(s)]$ by the Pfeiffer identity. Therefore $\partial(m) \in \gamma_2(M,F)$. We can extend the same argument and inductively prove that $\partial(m) \in \gamma_n(M,F)$, $m \in \gamma_n(M,F)$ for all $n \geq 1$. Since F is a free group, it is residually nilpotent. Then $m \in \ker(\partial)$ for $m \in \gamma_\omega(M,F)$ as required. ■

As can be seen in the proof, we do not really need F to be free, but just residually nilpotent. This requirement is not really a restriction in the case of a pair (K, K^1) since we know from section 3 that the fundamental crossed module $\mathcal{L}_1(K)$ of a 2-complex is a $\pi_1(K^1)$-free crossed module, where $\pi_1(K^1)$ is a free group. Therefore, we have that $\gamma_\omega(\pi_2(K,K^1), \pi_1(K)) \subset \ker \partial = \pi_2(K)$.

The other result relevant for the proof is the following theorem whose proof can be found in [23].

Theorem 5.8 *Let F be a free group and (M, ∂, F) a residually nilpotent non-aspherical projective F-crossed module. Then the group $\operatorname{coker}(\partial)$ is residually nilpotent.*

From this theorem we can get a straightforward corollary.

Corollary 5.9 *Let (M, ∂, F) be a projective F-module with F free and* $\operatorname{coker}(\partial)$ *not residually nilpotent. Then the following conditions are equivalent:*

1. $\ker \partial = 0$;
2. $\gamma_\omega (M, F) = \{1\}$.

Theorem 5.10 *Let K be a subcomplex of a contractible 2-dimensional CW-complex L. Then the following conditions are equivalent:*

1. $\pi_2 (K) = 0$;
2. $\gamma_\omega (\pi_2 (K, K^1), \pi_1 (K^1)) = \{1\}$.

Proof. It is known from the exact sequence (1) that $\operatorname{coker} \partial = \pi_1 (K)$. If we suppose that $\pi_2 (K) \neq 0$, the above mentioned group $\pi_1 (K)$ is not residually nilpotent since it has a nontrivial perfect radical [1]. Applying Lemma 5.7 and Corollary 5.9 to the free crossed module $(\pi_2 (K, K^1), \partial, \pi_1 (K))$, we get the result. ■

Using these results, together with a similar technique to the one used in [4] and the equivalence between $\mathcal{C}at^1$-groups and crossed modules, Mikhailov [22] proved another group theoretical characterization of asphericity.

Theorem 5.11 ([22]) *Let L be an aspherical 2-complex, and K a subcomplex of L. Then the following conditions are equivalent:*

1. *K is aspherical;*
2. *The fundamental Cat^1-group $\mathcal{L}^1(K)$ is residually solvable.*

Furthermore, using the same kind of argument as Bradenburg and Dyer in [4] one can also obtain the following result.

Theorem 5.12 ([23]) *Let K be a two-dimensional complex such that $H_1(K)$ is torsion-free and $H_2(K) = 0$. Then the following conditions are equivalent:*

1. *$\mathcal{L}^1(K)$ is residually solvable;*
2. *K is aspherical.*

6 Homotopy of Finite Spaces

To the knowledge of the author, the most recent approach to Whitehead's question is a discrete approach, which considers the posets associated with simplicial complexes. The first part of the present section introduces the theory of Homotopy of finite spaces. Next, we will review some results for this theory and finally study the applications to Whitehead's Conjecture.

We can first consider that for a finite partially ordered set, there exists a (canonical) way to give it the structure of a topological space, as follows.

Given a preorder $\leq$ in a set X and an element $x \in X$, we set

$$U_x := \{y \in X : y \leq x\}.$$

This gives X a T_0 topology for which the sets U_x form a basis.

Conversely, for a finite topological space, we can define a preorder in the following way. Let X be a topological space, define the order $\leq$, by

$$x \leq y \iff x \in U \text{ for every open set } U \text{ containing } y.$$

In particular, if the space X is T_0, we can find for any pair of points $x \neq y \in X$ an open set U containing only one of them, which makes the preorder above actually a partial order. Furthermore, these definitions give a one-to-one correspondence between finite T_0 topological spaces and finite posets.

Remark 6.1 *Since we are considering finite spaces, for a given poset, the opposite poset will also induce a T_0-space (i.e., defining the basic open sets as $F_x := \{y \in X : y \geq x\}$).*

To represent a poset, we will use its Hasse diagram which represents each element of X as a vertex in the plane and draws a line segment that goes upward from x to y whenever $x < y$ and there is no z such that $x < z < y$. A chain in a poset is a totally ordered subset of it. We define the length of a chain as the number of edges it has in the Hasse diagram, i.e. the number of points in the chain minus one, and the height of a finite poset as the height of its longest chain.

In this context, continuity of a function $f : X \to Y$ between posets translates simply to preserving the order relation for their Hasse diagrams. That is, it preserves edges if seen as a graph.

We define a fence as the following discrete structure:

$X = \mathbb{N}$, and the relation is given by:

$n \preceq n$, $\forall n \in \mathbb{N}$,

$2n \preceq 2n \pm 1$, $\forall n \in \mathbb{N}$.

Given this definition, it is not difficult to see that for a finite topological space X, it is connected if and only if is path connected, which is also equivalent to having a continuous function from the fence into X.

Since we can also define an order on the space of functions Y^X, by defining $f \leq g \iff f(x) \leq g(y), \forall x \leq y$, we can define homotopy equivalences between posets by considering two functions $f, g \in Y^X$ to be homotopic if there is a path between them in Y^X. As usual, we will say that a pair X, Y of spaces are homotopy equivalent if there exists a pair of continuous functions $f \in Y^X$, $g \in X^Y$, such that $f \circ g$ is homotopic to the identity on Y, and $g \circ f$ is homotopic to the identity on X. In particular, a discrete topological space is called contractible if it is homotopy equivalent to a point.

An important feature is that if a space has a maximum, (or a minimum since in this case, its opposite poset has a maximum) it is contractible.

We can also use the usual notion of homotopy groups as defined in the first section and furthermore, the homotopy group for a poset is naturally isomorphic to the usual homotopy group for its realization complex.

Other results let us prove, for example, that two finite posets are homotopy equivalent if and only if they are homeomorphic, see [2].

Applying these definitions to the poset $X(K)$ associated to a simplicial complex K, it can be seen that the homotopy groups defined on the discrete structure $X(K)$ are isomorphic to their continuous counterparts in the realization $|K|$ of K, [2], [17].

Also, given a finite poset X, we can consider each chain on it as a simplex, and obtain a simplicial complex $K(X)$, which has the same homotopy groups. Furthermore, if we consider a simplicial complex K, then $K(X(K))$ is the baricentric subdivision of the simplicial complex K, and therefore, its realization is the same as the one for K.

Another important fact is that using this transformation between simplicial complexes and posets, the length of a chain starting at the bottom is the dimension of the simplex associated to it, and therefore the height of a poset coincides with the dimension of the simplicial complex.

We say that two finite posets are weakly equivalent if they have the same homotopy groups. For instance, two different subdivisions of a 1-sphere will have the same homotopy groups since they are both simplicial models for the same space, although their corresponding posets may not be homotopy equivalent (as shown in Figure 2).

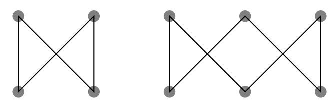

Figure 2: Two different subdivisions of S^1

6.1 Simple Homotopy of Finite Spaces

In what follows, we will consider operations on a finite poset that do not change the homotopy type of it.

A point $x \in X$ is called a down beat point if $\hat{U}_x = \{y \in X : y < x\}$ has a maximum and an up beat point if $\hat{F}_x = \{y \in X : y > x\}$ has a minimum. In both cases the subspace $X \backslash \{x\} \subset X$ is a strong deformation retract. Two finite spaces X and Y have the same homotopy type if and only if Y can be obtained from X by removing and adding beat points. The characterization of homotopy equivalent spaces in terms of beat points is credited to Stong [26]. In particular, if a finite space X has a maximum, all the points immediately below it in its Hasse diagram will be beat points. Hence, we can remove them without changing the homotopy type. Doing this procedure inductively, we end up with a space which contains only one point. In other words, this proves that a finite space with a maximum is contractible.

A point $x \in X$ is called a weak point if $\hat{U}_x$ or $\hat{F}_x$ is a contractible finite space. In this case, the inclusion $X \backslash \{x\} \subset X$ is a weak homotopy equivalence.

Since finite spaces with a maximum or a minimum are contractible, the notion of weak point generalizes that of beat point.

Definition 6.2 *An elementary collapse is the process of removing a beat point. A space X is called collapsible if there is a sequence of elementary collapses starting in X and finishing in a singleton.*

It is also worth noticing that in this approach via finite spaces, the usual notion of simple homotopy type for a simplicial complex coincides with the simple homotopy type of its associated face poset.

Remark 6.3 *In this context, collapsibility coincides with the collapsibility of the associated simplicial complex, but contractibility in finite spaces does not coincide with its analogue for general topological spaces. For instance, while with the usual definition of collapsibility, a collapasible space is contractible, for finite spaces the implication goes the other way. Contractible finite spaces are collapsible and collapsible spaces are weakly homotopically trivial (meaning that its homotopy groups are trivial). None of the converse implications hold [2], [3]. Figure 3 is an example of a collapsible, but not contractible finite space.*

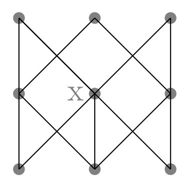

Figure 3: Collapsible and non-contractible poset of height 2

Definition 6.4 *Let X be a finite space of height 2 and let $a, b \in X$ be two maximal points. If $U_a \cup U_b$ is contractible, we say that there is a qc-reduction from X to $Y \setminus \{a, b\}$, where $Y = X \cup \{c\}$ with $a, b < c$. If after several qc-reductions starting in a finite space of height 2, one obtains a space with a maximum, the original space will be called qc-reducible.*

Remark 6.5 *The important feature of qc-reducibility is the fact that it doesn't change the simple homotopy type of a space.*

Given a discrete space X, the following lemma gives us a nice way to find spaces with less points, that are homotopically equivalent spaces to X. We recommend the reader see [2] for their proofs.

Lemma 6.6 ([2]) *Let X be a finite space of height at most 2 such that $H_2(X) = 0$. Let (a, b) be two maximal elements of X. Then the following are equivalent:*

1. *$U_a \cup U_b$ is contractible,*
2. *$U_a \cap U_b$ is nonempty and connected,*
3. *$U_a \cap U_b$ is contractible.*

6.2 Whitehead's Conjecture and Finite Spaces

For a particular class of complexes, it has been proved using finite homotopy theory that Whitehead conjecture holds. In this part of the document, we will review the results obtained in [5] where the authors define quasi-constructible complexes, and verify that for these, an aspherical complex L does not admit a non-aspherical subcomplex of the form $L - e$, where e is a 2-cell. Finally, applying Theorem 4.7, Whitehead's asphericity conjecture will be proved for these complexes.

Definition 6.7 *A finite simplicial complex K of dimension at most 2 is said to be quasi-constructible if K is a simplex or, recursively, if it can be written as $K = K_1 \cup K_2$ in such a way that:*

- *K_1 and K_2 are quasi-constructible, and*
- *$K_1 \cap K_2$ is nonempty and connected, and*
- *No maximal simplex of K_1 is in K_2, and no maximal simplex of K_2 is in K_1.*

Using definition 6.7, the following proposition is proved in [2].

Proposition 6.8 ([2]) *Let K be a finite simplicial complex of dimension at most 2. Then the following are equivalent:*

1. *K is quasi-constructible and $H_2(|K|) = 0$,*
2. *$X(K)$ is qc-reducible,*
3. *K is quasi-constructible and contractible.*

This gives us an equivalent statement to Whitehead's Conjecture by considering the corresponding face poset associated to a triangulation of a given 2-complex.

Conjecture 6.9 *Let X be a homotopically trivial (contractible) finite space of height 2 and let $a \in X$ be a maximal point such that $X \setminus \{a\}$ is connected. Then $X \setminus \{a\}$ is aspherical.*

Proof of equivalence. Suppose that there exists a counterexample of type (1) in Theorem 4.7. It is well known that a finite CW-complex has the same homotopy type of a simplicial complex of the same dimension. It follows that there is a contractible 2-complex L and a subcomplex $K = L \backslash \sigma$ with σ a 2-simplex of L, such that K is connected and non-aspherical. Let $X(L)$ and $X(K)$ be their face posets. Then $X(L)$ is a homotopically trivial space of height 2 and $X(K)$ is a connected, non-aspherical subspace of $X(L)$, obtained by removing the maximal point $\sigma \in X(L)$. The truth of Conjecture 6.9 would imply that there is no counterexample of the first type in Theorem 4.7.

Similarly, suppose that Whitehead's conjecture is true. Given a homotopically trivial finite space X of height 2 and a maximal point $a \in X$ such that $X \backslash a$ is connected, the associated simplicial complex $K(X)$ is contractible and the subcomplex $K(X \backslash a)$ of $K(X)$ is connected. Hence, since we are assuming that Whitehead's conjecture is true, $K(X \setminus a)$ and $X \setminus a$ are both aspherical. This means that, in this case, Conjecture 6.9 also holds. In other words, we have found a translation of the finite case of Whitehead's asphericity conjecture to finite spaces of height 2. ■

Furthermore, in this setup we have another partial answer for Whitehead's Conjecture, namely:

Theorem 6.10 ([5]) *Let X be a finite qc-reducible space of height 2 and let $a \in X$ be a maximal point such that $X \backslash \{a\}$ is connected. Then $X \backslash \{a\}$ is aspherical.*

6.3 Homotopy of Reflexive Structures

Similarly to the above notion, Larose and Cardiff [17] have defined discrete homotopy for reflexive structures, generalizing the notion defined for posets and confirming that for a simplicial complex, the usual homotopy groups agree with the discrete ones that they define. In [17] the authors define the functor σ_k from the category of pointed/based binary reflexive structures (in particular for posets) to the category of groups. They prove an isomorphism between the group $\sigma_k(X, x_0)$, which is a discrete version of the homotopy group and will be defined in the current section, and the homotopy group $\pi_k(K(X))$, where $K(X)$ is the simplicial complex obtained by considering each ascending chain in X as a simplex. The results in [16] relating discrete homotopy and Taylor operations let us identify a property of weak homotopically trivial finite spaces which leads us to another possible approach to Whitehead's Question, this time relating it to a problem in universal algebra. We will briefly describe the above mentioned results and state the possible use of them to study Whitehead's Asphericity Conjecture.

Definition 6.11 *A reflexive structure is a set together with a reflexive binary relation θ. We write the relation $(x,y) \in \theta$ as $x \to y$. A morphism $f : X \to Y$ between reflexive structures is a map which preserves θ.*

Definition 6.12 *A (weak) path from x to y within a space X is a sequence of elements $x = x_0, x_1, \ldots, x_n = y$ in which $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$. This is equivalent to saying that there is a morphism from a fence, F (set of integers such that $2k \to 2k \pm 1$) to the space X sending 0 to x and staying at y after it arrives at some time N.*

The space of paths from x_0 to x_0 is called $F(X, x_0)$. More generally, if we denote by F^k the Cartesian product of F with itself k times, we can define $F^k(X, x_0)$ to be the set of homomorphisms from F^k to X, which after some time N becomes constantly equal to x_0 and are x_0 if any coordinate is 0. (Notice that this is intuitively equivalent to considering the interval $[0,1]$ as the set of steps $\{0, 1, \ldots, n\}$ and therefore the cube I^k as a product $\{0, 1, \ldots, n_1\} \times \{0, 1, \ldots, n_2\} \times \cdots \times \{0, 1, \ldots, n_k\}$). We can define a reflexive relation ϕ in $F(X, x_0)$ by setting $f \to g$ if and only if $f(x) \to g(y), \forall x \to y$. This lets us define a notion of homotopy for finite spaces. Two maps in $F(X, x_0)$ are homotopic if there exists a path between them in $F(X, x_0)$.

The quotient of this space by the discrete homotopy mentioned above is the so-called discrete homotopy group $\sigma_k(X, x_0)$, where the product is given as usual by gluing one map after the other in one of the directions (although no rescaling is needed). It is proved that σ_k is a functor from the category of reflexive structures to the category of groups.

Finally, Larose and Cardiff [17] prove that for a finite reflexive structure X, and for its poset of simplices (ascending chains in X), the groups σ_k coincide, which implies that the (weak)-homotopy type of reflexive structures is completely equivalent to the one for the posets.

In a very similar fashion to the usual notion, as described in Section 1, Witboi [29], defines the relative homotopy for relational structures. In particular, for posets it is also possible to get a long exact sequence in homotopy, which resembles the one described in Section 2.3.

Some results of [17] are summarized in [16], where the author also gives some applications to the case of finite digraphs relating the higher homotopy groups to the existence of polymorphisms satisfying certain conditions.

Definition 6.13 *Let X be a binary reflexive structure. A Taylor operation is an n-ary map $r : X^n \to X$ which satisfies the following properties:*

- *r is idempotent (meaning $r(x, x, \ldots, x) = x, \forall x \in X$),*
- *r is not a projection,*
- *r preserves the relation in X, and satisfies identities of the form*

$$r\left(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots x_n\right) = r\left(x'_1, \ldots, x'_{i-1}, z, x'_{i+1}, \ldots x'_n\right),$$

for some $y, z \in X$ and all $x_1, x_2, \ldots, x_n, x'_1, x'_2, \cdots, x'_n \in X$.

Using these polymorphisms, Larose proves the following theorem.

Proposition 6.14 ([16]) *Let X be a finite connected binary reflexive structure. If it admits a Taylor operation, all of its (discrete) homotopy groups are trivial.*

Another couple of sufficient conditions for a reflexive structure to be idempotent trivial can be found in [17].

Given an aspherical space X, consider a 2-cell e (or just a 2-simplex, if we consider a simplicial model for X). In this case, to prove the finite case of Whitehead's Conjecture, we could find a Taylor operation for the universal cover of $X - e$.

It is worth noticing that for the case of posets of height 1, they admit a Taylor operation if they do not have cycles, and are idempotent trivial if every point belongs to a cycle. Therefore proposition 6.14 implies that for the first case, we have a contractible space agreeing with the fact that a tree is contractible, and in the other case we have a bouquet of circles which is a non-homotopically trivial space. To the knowledge of the author, there are no characterizations of admitting Taylor operations for the case of posets of height 2.

7 Other Approaches and Open Questions

As it was already stated, other different approaches have been taken for the study of Whitehead's Conjecture. Ivanov [15] for example, uses the group presentation to translate an aspherical complex into a group whose presentation is faithful, and describes a group theoretic approach, where Whitehead's conjecture translates into proving that for a faithful presentation, any subpresentation (presentation where the generators and relators are a subset of the ones for the original) remains faithful. Rosebrock [25] uses another graph representation for some particular kinds of group presentations, and studies what he calls

Labelled Oriented Trees (LOTs). These are directed graphs in which each generator x is written in a node of a tree and there is a labeled arc z from x to y if there exist a relator which writes y as the conjugate of x by z, namely $y = z^{-1}xz$.

Considering the approaches that have been described in the current paper, we can ask the following questions:

Problem 7.1 *What is the precise description of the translation of Dyer's result on projectivity of crossed modules to the* $\mathcal{C}at^1$*-groups language?*

Problem 7.2 *Knowing that crossed modules admit a transfinite generalization of the lower central series, is it possible to find a transfinitely nilpotent crossed module, that is not residually nilpotent?*

Problem 7.3 *How does qc-reducibility of finite spaces translate into a group theoretical representation for the corresponding presentation of a* 2*-complex?*

Problem 7.4 *Under which conditions does the converse of Proposition 6.14 hold? This would give us a (-n almost complete) translation from Whitehead's conjecture to a universal algebra setup.*

Problem 7.5 *For posets of height* 2*, when do they admit Taylor operations?*

The reader, as the author did, could realize a very interesting feature in Mathematics (and in other fields of life as well); there could exist many different valid approaches to a given problem. We may not get a complete solution, but a step forward is always worthy.

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