The moduli space of ASD connections on compact 4-manifolds

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0 Introduction

Consider a compact Riemannian manifold X. It is common to introduce new structures on it and study them to get information about the underlying manifold X. In this paper, we consider the anti-self-dual (ASD) connections on a principal bundle over a 4-manifold X, i.e., connections which satisfy the ASD equation. However, the space of all such objects may be too big, infinite-dimensional. Hence, we quotient it by the gauge group, which is the group of symmetries of the bundle. The obtained moduli space turns out to carry a natural geometric and topological structure, except for maybe some singular points corresponding to so-called irreducible connections.

We start with section 1, by going through the basic definitions and facts, assuming the familiarity of the reader with notions such as differentiable manifolds, Riemannian structures, vector bundles, Hodge Theory, etc. Knowing the theory of Banach, Sobolev spaces, elliptic theory is recommended, but not required. We will use the results listed in the Appendix, the proofs of which can be found in the corresponding sources.

In Section 2 we study the ASD connections in the presence of a holomorphic sctructure on the manifold X. It turns out that part of the conditions for being ASD can be rewritten as an integrablity condition.

Section 3 establishes a local "sequential-compactness" result for moduli spaces. This leads to the process of compactification and the Removable Singularities Theorem which states that connections with finite action have no point singularities. We will not cover these two topics, although an interested reader can see the original papers by K. Uhlenbeck [11] and [10], along with the book by S. K. Donaldson and P. B. Kroheimer [2] for more information.
We finish with section 4, by showing that locally moduli spaces are smooth manifolds, under a series of nice assumptions, using the Fredholm Theory.
This naturally leads to the Donaldson Theory, whose goal is to distinguish smooth 4-manifolds which have the same classical invariants, by introducing new ones using the ASD moduli spaces. It turns out that these new invariants depend not only on the topology of X, but on its smooth structure. One of the results of Donaldson Theory is the existence of exotic smooth structures on $\mathbb{R}^4$. One can see [2] and [4] if interested.

1 Connections and curvature

Definition 1.1. Let G be a Lie group. A principal G-bundle P over a smooth manifold X is a manifold with a smooth (right) G action $P \times G \to P$, which we write as $(p, g) \mapsto pg = R_g p$, and orbit space $P/G = X$. Also, this action is locally equivalent to the obvious action on $U \times G$, where U is an open set in X. Hence, we obtain a fibration $\pi : P \to X$. We say P has structure group G. ▲

Remark 1.2. We write $\mathfrak{g}_E$ for the bundle of Lie algebras associated to the adjoint represenation ad_g, i.e. $\mathfrak{g}_E$ is a real subbundle of End $E = E \otimes E^*$.

Definition 1.3. A connection on a principal G-bundle P over X can be defined in any of the following equivalent ways:

- As a field of horizontal subspaces $H \subset TP$ transverse to the fibres of π. This means, for each point $p \in P$, we have $TP_p = H_p \oplus T(\pi^{-1}(x))$, where $\pi(p) = x$. This field of subspaces is required to be preserved by the action of G on P, i.e., $(R_g)_* H_p = H_{pg}$, for $g \in G$.
- As a 1-form A on P with values in the Lie algebra $\mathfrak{g}$ of G, i.e. a section of the bundle $T^*P \otimes \mathfrak{g}$ over P. Again, we require this to be invariant under G, i.e., $(R_g)^* A = ad_{g^{-1}} \circ A$, for $g \in G$.

▲

Remark 1.4. Note that a connection on the frame bundle can be defined by a covariant derivative on E, which is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$ that satisfies the Leibnitz rule. Observe that ∇ is a local operator. Then we say that a local section σ of the frame bundle (i.e. a collection of local sections $s_1, \dots s_n$ of E, where $n = \text{rank}(E)$) is horizontal at x in X if all the ∇s_i vanish at x. Finally, we define H_p to be the tangent space to a horizontal section σ through p, regarded as a submanifold of P. This construction can easily be inverted, giving us the desired equivalence. We will write ∇_A for a covariant derivative, using A to denote a connection.

Definition 1.5. We quickly recall the concept of holonomy. This will not be needed until Section 4, where it will be used to get rid of connections which cause problems. Assume we have a principal G-bundle P over a manifold X, and a connection A. For $p, q \in P$, we write $p \sim q$ if there exists a piece-wise smooth horizontal curve in P joining p to q. Clearly, $\sim$ is an equivalence relation. Fix $p \in P$ and and define the holonomy group of (P, A) based at p to be $\text{Hol}_p(P, A) = \{g \in G : p \sim gp\}$. Then the holonomy group $\text{Hol}_p(P, A)$ depends on the base point $p \in P$ only up to conjugation in G. Thus, we can regard it as an equivalence class of subgroups of G under conjugation, which is independent of p, and is written as $\text{Hol}(P, D)$. See Chapter 2 of [6] for more details. ▲

Remark 1.6. The space of connections $\mathcal{A}(E)$ on a vector bundle E can be described as the affine space $\{A + a | a \in \Omega^1(\text{End } E)\}$, where A is any connection on E. This can easily be obtained by a direct calculation that the difference of two connections is linear over $C^\infty(X)$.

Definition 1.7. Let E be a vector bundle. Let ∇_A denote a covariant derivative on E. The gauge group $\mathcal{G}$ is the group of all automorphisms $u : E \to E$. It acts on the set of connections by the rule $\nabla_{u(A)} s = u \nabla_A (u^{-1} s)$. ▲

Remark 1.8. If we regard u as a section of the vector bundle End E, then we can write $\nabla_{u(A)}$ as $\nabla_A - (\nabla_A u) u^{-1}$, so we get $u(A) = A - (\nabla_A u) u^{-1}$.
Now, let's look at how u acts on trivializations τ. We get that $u\tau$ is a new trivialization, and $A^{u\tau} = uA^\tau u^{-1} - (du)u^{-1}$. Hence, suppressing the superscript τ, we get the action $A \to uAu^{-1} - (du)u^{-1}$.

Definition 1.9. Let $\Omega^p_X(E) = \Gamma(\bigwedge^p T^* X \otimes E)$.
We have exterior covariant derivatives $d_A : \Omega^p_X(E) \to \Omega^{p+1}_X(E)$ which are uniquely determined by the properties:

- $d_A = \nabla_A$ on $\Omega^0_X(E)$.
- $d_A(w \wedge \theta) = (dw) \wedge \theta + (-1)^p w \wedge (d_A \theta)$, for $w \in \Omega^p_X, \theta \in \Omega^q_X(E)$.

▲

Remark 1.10. The operator d_A extends on $a \in \Omega^p(\text{End } E)$ in the following way:

$$d_A a = da + A \wedge a - (-1)^p a \wedge A = da + [A, a]$$

Proof. First, note that we identify End E with $E \otimes E^*$. So, we need to find what the induced d_A on the dual space E^* and on the tensor product is locally.
For the dual space E^*, the exterior covariant derivative $d_A : \Gamma(E^*) \to \Omega^1(E^*)$ is determined by the following formula: $d_A(t)(s) = d_A(t(s)) - t(d_A(s))$, where t is a section of E^* and s is a section of E. This map can easily be shown to satisfy the Leibniz rule.
We can rewrite this as $d_A(t(s)) = d_A(t)(s) + t(d_A(s))$. Assume that locally $d_A(e^j) = B^j_i \otimes e^i$, where $e_1, ..., e_n$ is the basis of TX and $e^1, ..., e^n$ is its dual basis. So, after applying the above equation to $t = e^j$, $s = e_i$, we get:

$$\begin{aligned} 0 = d_A(e^j(e_i)) &= (B^j_k \otimes e^k)(e_i) + e^j(A^k_i \otimes e_k) \\ &= B^j_k \delta_{ki} + A^j_i \delta_{jk} \\ &= B^j_i + A^j_i \end{aligned}$$

Hence, we get that $d_A(e^j) = -A^j_i \otimes e^i$ locally, and so $d_A(s_i e^i) = (ds_i - s_j A^j_i) \otimes e^i$.
Next, the operators d_A on E and E^* naturally induce d_A on $E \otimes E^*$ by $d_A(s \otimes t) = d_A(s) \otimes t + s \otimes d_A(t)$, for t a section of E^* and s a section of E.
Now, we are ready to compute what this d_A is locally. Using the second bullet point of Definition 1.9,

for $a \in \Omega^p$, we have:

$$\begin{aligned}
d_A(a^i_j \otimes e_i \otimes e^j) &= da^i_j \otimes e_i \otimes e^j + (-1)^p a^i_j \wedge d_A(e_i \otimes e^j) \\
&= da^i_j \otimes e_i \otimes e^j + (-1)^p a^i_j \wedge (A^k_i \otimes e_k \otimes e^j - A^j_k \otimes e_i \otimes e^k) \\
&= da^i_j \otimes e_i \otimes e^j + (-1)^p a^i_j \wedge A^k_i \otimes e_k \otimes e^j + (-1)^{p+1} a^i_j \wedge A^j_k \otimes e_i \otimes e^k \\
&= da^i_j \otimes e_i \otimes e^j + (-1)^p (-1)^p A^k_i \wedge a^i_j \otimes e_k \otimes e^j - (-1)^p a^i_j \wedge A^j_k \otimes e_i \otimes e^k \\
&= da^i_j \otimes e_i \otimes e^j + A^i_k \wedge a^k_j \otimes e_i \otimes e^j - (-1)^p a^i_k \wedge A^k_j \otimes e_i \otimes e^j \\
&= (da^i_j + [A, a]^i_j) \otimes e_i \otimes e^j
\end{aligned}$$

And hence, using matrix notation we get the desired $d_A a = da + [A, a]$. □

Definition 1.11. The curvature of the connection is $F_A = d_A^2 \in \Omega^2_X(\mathfrak{g}_E)$. One can easily show that it transforms in the following way under bundle automorphisms: $F_{u(A)} = uF_A u^{-1}$. We call a connection flat if its curvature is zero. ▲

Remark 1.12. For a smooth manifold X, consider the trivial bundle $\underline{\mathbb{R}^n} = \mathbb{R}^n \times X$ or $\underline{\mathbb{C}^n} = \mathbb{C}^n \times X$. Then this bundle admits a product connection whose covariant derivative is just ordinary differentiation of vector-valued functions, i.e. $\nabla_A = d$. Clearly this connection is flat.

Proposition 1.13. *Given a connection A and $a \in \Omega^1(\text{End } E)$, we have that $F_{A+a} = F_A + d_A a + a \wedge a$.*

Proof. We will use the fact that $F_A = dA + A \wedge A$. So,

$$\begin{aligned}
F_{A+a} &= d(A + a) + (A + a) \wedge (A + a) \\
&= dA + A \wedge A + (da + [A, a]) + a \wedge a \\
&= F_A + d_A a + a \wedge a
\end{aligned}$$

□

Proposition 1.14. *The Bianchi identity: $d_A F_A = 0$.*

Proof.

$$\begin{aligned}
d_A(F_A) &= d(F_A) + [A, F_A] \\
&= d(dA + A \wedge A) + [A, F_A] \\
&= d^2 A + d(A \wedge A)) + [A, F_A] \\
&= dA \wedge A - A \wedge dA + [A, F_A] \\
&= (F_A - A^2) \wedge A - A \wedge (F_A - A^2) + [A, F_A] \\
&= F_A \wedge A - A^3 - A \wedge F_A + A^3 + [A, F_A] \\
&= F_A \wedge A - A \wedge F_A + [A, F_A] \\
&= 0
\end{aligned}$$

□

Definition 1.15. Using the Hodge star $*$, we define the L^2 inner product $(a, b)_{L^2} = \int_X tr(a \wedge *b) d\mu$, for $a, b \in \Omega^p(\mathfrak{g}_E)$. Then the induced operator d_A on $\Omega^p(\mathfrak{g})$ has a formal adjoint d_A^* characterized by the equation $(d_A a, b)_{L^2} = (a, d_A^* b)_{L^2}$. ▲

Definition 1.16. From now on, we let X be a 4-dimensional, oriented and Riemannian manifold. Then we have the orthogonal decomposition of the 2-forms on X into the self-dual part, which is the $+1$ eigenspace of the Hodge star $*$, and the anti-self-dual part, the -1 eigenspace:

$$\Omega^2_X = \Omega^+_X \oplus \Omega^-_X$$

This splitting extends to bundle-valued 2-forms and in particular to the curvature tensor F_A of a connection on a bundle E over X as

$$F_A = F_A^+ \oplus F_A^- \in \Omega^+_X(\mathfrak{g}_E) \oplus \Omega^-_X(\mathfrak{g}_E)$$

where $\Omega^\pm_X(\mathfrak{g}_E) = \Gamma(\Lambda^\pm_X \otimes \mathfrak{g}_E)$. We say a connection is anti-self-dual (ASD) if $F_A^+ = 0$ and is self-dual if $F_A^- = 0$. ▲

Proposition 1.17. *The differential operator $d_A^* + d_A^+ : \Omega^1 \to \Omega^0 \oplus \Omega^-$ is elliptic. See section 5.2 of the Appendix for the definition of ellipticity.*

Proof. Let $a \in \Omega^1$. Then $(d_A^* + d_A^+)(a) = (d_A^* a, d_A^+ a) = (- * d_A * a, \frac{d_A a + * d_A a}{2})$.
In order to check for ellipticity, we need to calculate the symbol, which is obtainded by replacing all the occurences of d_A by $\zeta \wedge$, where ζ is a non-zero 1-form.
So, we obtain a function $a \mapsto (- * (\zeta \wedge *a), \frac{\zeta\wedge a + *(\zeta \wedge a)}{2}) = (-\zeta^\sharp \lrcorner a, \frac{\zeta\wedge a + *(\zeta \wedge a)}{2})$. Now we need to prove it is invertible. Note that the dimensions of the domain and the codomain are both equal to 4, so it is enough to show injectivity, i.e. if $\zeta^\sharp \lrcorner a = 0$ and $\zeta \wedge a + *(\zeta \wedge a) = 0$, then $a = 0$. So, assume this is the case. Then:

$$\begin{aligned}
0 &= \zeta^\sharp \lrcorner (\zeta \wedge a + *(\zeta \wedge a)) \\
&= \zeta^\sharp \lrcorner (\zeta \wedge a - \zeta^\sharp \lrcorner * a)) \\
&= \zeta^\sharp \lrcorner (\zeta \wedge a) - \zeta^\sharp \lrcorner (\zeta^\sharp \lrcorner * a) \\
&= (\zeta^\sharp \lrcorner \zeta) \wedge a - \zeta \wedge (\zeta^\sharp \lrcorner a) - 0 \\
&= |\zeta|^2 a - \zeta \wedge 0 \\
&= |\zeta|^2 a
\end{aligned}$$

So, since ζ is non-zero, we obtain $a = 0$, as required.
Similarly, one can show that the operator $d^* + d : \bigoplus_i \Omega^{2i+1} \to \bigoplus_i \Omega^{2i}$ is also elliptic. □

Definition 1.18. The Yang-Mills functional on the space of all connections on E is defined to be:

$$\|F_A\|^2 = \int_X |F_A|^2 d\mu = \int_X |F_A^-|^2 d\mu + \int_X |F_A^+|^2 d\mu,$$

for a connection A. ▲

Proposition 1.19. *The critical points of the Yang-Mills functional satisfy $d_A^* F_A = 0$.*

Proof. We know that the space of connections can be described as $\{A + a : a \in \Omega^1(\text{End } E)\}$ and that $F_{A+a} = F_A + d_A a + a \wedge a$.
Suppose A is a critical point of the Yang-Mills functional. Consider a one-parameter family of connections on E given by $A_\epsilon = A + \epsilon a$, with $\epsilon \in (-t_0, t_0)$, for some small t_0 and $a \in \Omega^1(\text{End } E)$. Then,

$$F_{A+\epsilon a} = F_A + d_A(\epsilon a) + (\epsilon a) \wedge (\epsilon a) = F_A + \epsilon d_A a + \epsilon^2 a \wedge a$$

and

$$
\begin{aligned}
\|F_{A+\epsilon a}\|_{L^2}^2 &= (F_{A+\epsilon a}, F_{A+\epsilon a})_{L^2} \\
&= (F_A + \epsilon d_A a + \epsilon^2 a \wedge a, F_A + \epsilon d_A a + \epsilon^2 a \wedge a)_{L^2} \\
&= (F_A, F_A)_{L^2} + 2\epsilon (F_A, d_A a)_{L^2} + \epsilon^2(\cdots)
\end{aligned}
$$

Hence,

$$
\frac{d}{d\epsilon}(\|F_{A+\epsilon a}\|_{L^2}^2)|_{\epsilon=0} = 2(F_A, d_A a)_{L^2} = 2(a, d_A^* F_A)_{L^2}.
$$

Since A is a critical point, $0 = 2(a, d_A^* F_A)_{L^2}$, for any a. Thus, $d_A^* F_A = 0$. □

Corollary 1.20. *Combining with the Bianchi dentity $d_A F_A = 0$, we get that the critical points of the Yang-Mills functional are solutions to the Yang-Mills equations:*

$$
\begin{cases} d_A F_A = 0 \\ d_A^* F_A = 0 \end{cases}
$$

Remark 1.21. Here, we will use some facts about Chern classes, which can be found in [8].
Let L be a Hermitian line bundle over a manifold X. The curvature of a connection A on L is a purely imaginary 2-form which we write as $-2\pi i\phi$, where ϕ is a real 2-form, which is closed by the Bianchi identity. It therefore defines a de Rham cohomology class $[\phi] \in H^2(X; \mathbb{R})$. Consider a second connection $A^{'} = A + a$. Then $F^{'} = F + da$, as $a \wedge a = 0$ for rank 1. Thus, $[\phi^{'}] = [\phi]$. Hence, we obtain a cohomology class which is independent of the choice of connection, and thus depends only on the bundle L. This class is called the first Chern class $c_1(L)$ which classifies L. More generally, for any complex vector bundle E, with a connection A, the first Chern class $c_1(E)$ is represented by $\frac{i}{2\pi}Tr(F_A)$. See the appendix of [8] for more details.
Now, consider the 4-form $Tr(F_A^2)$ definded by a connection on a Hermitian bundle E. Again, this is a closed form whose de Rham cohomology class depends only on E, not on the particular choice of the connection.
In fact, for a complex vector bundle we have $[\frac{1}{8\pi^2}Tr(F_A^2)] = c_2(E) - \frac{1}{2}c_1(E)^2 \in H^4(X)$, where c_1, c_2 are Chern classes.
When bundles have the structure group $SU(2)$, then $Tr(F_A) = c_1(E) = 0$. In this case, for a compact, oriented manifold, we identify H^4 with integers and write $c_2(E) = \frac{1}{8\pi^2}\int_X Tr(F_A^2) \in \mathbb{Z}$.

Remark 1.22. For the Lie algebra $\mathfrak{u}(n)$ of skew adjoint matrices $Tr(\zeta^2) = -|\zeta|^2$.
Also, we have that for any 2-form β, $\beta = \beta_+ + \beta_-$, and so $\beta^2 = \beta_+^2 + 2\beta_+ \wedge \beta_- + \beta_-^2 = \beta_+^2 + \beta_-^2$, since ASD and SD spaces are orthogonal. Next, using the fact that $\beta_\pm = \pm * \beta_\pm$, we get that $\beta^2 = \beta_+ \wedge *\beta_+ - \beta_- \wedge *\beta_- = |\beta_+|^2 d\mu - |\beta_-|^2 d\mu$, where $d\mu$ is the Riemannian volume form. Hence, $Tr(F_A^2) = -(|F_A^+|^2 - |F_A^-|^2)d\mu$. Thus, a connection is ASD if and only if $Tr(F_A^2) = |F_A^-|^2 d\mu$.
Also,

$$
8\pi^2 c_2(E) = \int_X Tr(F_A^2) d\mu = \int_X |F_A^-|^2 d\mu - \int_X |F_A^+|^2 d\mu,
$$

while for the Yang-Mills functional

$$
\|F_A\|^2 = \int_X |F_A|^2 d\mu = \int_X |F_A^-|^2 d\mu + \int_X |F_A^+|^2 d\mu.
$$

So,

$$
\|F_A\|^2 = 8\pi^2 c_2(E) + 2\int_X |F_A^+|^2 d\mu.
$$

Thus, when c_2 is nonnegative we get that A is ASD $\Rightarrow$ A is an absolute minimizer for the Yang-Mills functional.
Clearly, in this case, A is ASD iff $\|F_A\|^2 = 8\pi^2 c_2(E)$.

Similarly, if $c_2(E)$ is nonpositive, we obtain that A is $SD \Rightarrow A$ is an absolute minimizer, and A is SD iff $\|F_A\|^2 = -8\pi^2 c_2(E)$.

Definition 1.23. The standard 4-dimensional characteristic class for a real $O(n)$ bundle is the Pontryagin class $p_1(V) = -c_2(V \otimes \mathbb{C}) \in H^4(X, \mathbb{Z})$. Such a bundle also has a Stiefel-Whitney class $w_2(V) \in H^2(X, \mathbb{Z}_2)$ such that $w_2(V)^2 = p_1(V)$ *mod* 4. See [9] for more details. ▲

Definition 1.24. We make the following convention for vector bundles over a compact oriented 4-manifold:

$$\begin{aligned} \kappa(E) &= c_2(E), && \text{for } SU(2) \text{ bundles } E, \\ &= c_2(E) - \frac{1}{2}c_1(E)^2, && \text{for } U(r) \text{ bundles } E, \\ &= -\frac{1}{4}p_1(V), && \text{for } SO(r) \text{ bundles } V. \end{aligned}$$

We then have the Chern-Weil formula $\kappa(E) = \frac{1}{8\pi^2}\int_X Tr(F_A^2)$. ▲

Proposition 1.25. *If a bundle E over a compact, oriented Riemannian 4-manifold admits an ASD connection, then $\kappa(E) \geq 0$, and if $\kappa(E) = 0$, any ASD connection is flat.*

Proof. This follows from Remark 1.22. □

2 Holomorphic bundles

When the base space X admits a complex structure, we can rewrite the ASD condition in two pieces, one of which has a simple geometric interpretation as an integrability condition.
For simplicity, consider the base to be the Euclidean space $\mathbb{R}^4$, with connection matrices A_i. Then the ASD condition $F_A^+ = 0$ can be represented as a system of partial differential equations:

$$\begin{aligned} F_{12} + F_{34} = 0 \\ F_{14} + F_{23} = 0 \\ F_{13} + F_{42} = 0 \end{aligned}$$

where $F_{ij} = [\nabla_i, \nabla_j] = \frac{\partial A_j}{x_i} - \frac{\partial A_i}{x_j} - [A_i, A_j]$.
Now, when you think about the base space as $\mathbb{C}^2$ equipped with the flat Euclidean metric, these conditions can be rewritten as:

$$\begin{aligned} [\nabla_1 + i\nabla_2, \nabla_3 + i\nabla_4] = 0 \text{ (the integrability condition)} \\ [\nabla_1, \nabla_2] + [\nabla_3, \nabla_4] = 0 \text{ (the condition that } \hat{F}_A = 0\text{, from Proposition 2.11)} \end{aligned}$$

Definition 2.1. Let Z be a complex manifold. A holomorphic vector bundle $\mathcal{E}$ over Z is a complex manifold with a holomorphic projection map $\pi : \mathcal{E} \to Z$ and a complex vector space structure on each fibre $\mathcal{E}_z = \pi^{-1}(z)$, such that the data is locally holomorphically equivalent to the standard product bundle. ▲

Definition 2.2. The complexified de Rham complex (Ω_Z^*, d) splits into a double complex $(\Omega_Z^{p,q}, \partial, \bar{\partial})$, with $d = \partial + \bar{\partial}$, and $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$.
Now, for any complex vector bundle E over Z, we write $\Omega_Z^{p,q}(E)$ for E-valued (p,q)-forms.
Given a holomorphic structure $\mathcal{E}$ on E, there is a linear operator $\bar{\partial}_{\mathcal{E}} : \Omega_Z^{0,q}(E) \to \Omega_Z^{0,q+1}(E)$ uniquely determined by:

- $\bar{\partial}_{\mathcal{E}}(fs) = (\bar{\partial} f)s + f(\bar{\partial}_{\mathcal{E}} s)$,
- $\bar{\partial}_{\mathcal{E}} s$ vanishes on an open subset $U \subset Z$ if and only if s is holomorphic ($\bar{\partial} s = 0$) over U.

▲

Remark 2.3. The operators $\bar{\partial}_{\mathcal{E}}$ satisfy $\bar{\partial}_{\mathcal{E}}^2 = 0$, hence we obtain the Dolbeault cohomology groups $H^*(\mathcal{E}) = \ker \bar{\partial}_{\mathcal{E}} / \operatorname{im} \bar{\partial}_{\mathcal{E}}$.

Theorem 2.4. *On a C^∞ complex vector bundle E over a complex manifold Z, a partial connection $\bar{\partial}_\alpha$, i.e. an operator $\Omega_Z^0(E) \to \Omega_Z^{0,1}(E)$ which satisfies the Leibniz rule, is integrable, meaning that local holomorphic trivializations exist, if and only if $\bar{\partial}_\alpha^2 \in \Omega_Z^{0,2}(End\ E)$ is zero.*

Proof. See page 282 of [5] for a proof. □

Corollary 2.5. *Decompose $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$. Then $F_A^{0,2} = \bar{\partial}_A^2$, where $\bar{\partial}_A$ is the partial connection induced by A. Then the connection is compatible with a holomorphic structure, i.e. $\bar{\partial}_A = \bar{\partial}_{\mathcal{E}}$, if and only if $F_A^{0,2} = 0$.*

Proof. If $\bar{\partial}_A$ is the $\bar{\partial}$ operator for some holomorphic structure, then clearly $F_A^{0,2} = \bar{\partial}_A^2 = 0$.
For the converse, we just apply Theorem 2.4. □

Definition 2.6. A connection A on a complex vector bundle E over Z with a Hermitian metric is called unitary if $d(s,t) = (d_A s, t) + (s, d_A t)$, for any two sections s, t of E.
Note that a unitary connection has a skew-Hermitian matrix of coefficients in any unitary local trivialization, i.e., $A^\tau = -(A^\tau)^*$, where A^τ is a matrix of 1-forms s.t. $\nabla_A = d + A^\tau$.
This is because if e_i forms a local orthonormal frame for E, then $\delta_{ij} = (e_i, e_j)$. So, after differentiating, we get:

$$\begin{aligned} 0 &= (d_A e_i, e_j) + (e_i, d_A e_j) \\ &= (A_{ik} e_k, e_j) + (e_i, A_{jk} e_k) \\ &= A_{ik}\delta_{kj} + \overline{A_{jk}}\delta_{ik} \\ &= A_{ij} + \overline{A_{ji}} \end{aligned}$$

▲

Lemma 2.7. *If E is a complex vector bundle over Z with a Hermitian metric on the fibres, then for each partial connection $\bar{\partial}_\alpha$ on E there is a unique unitary connection A such that $\bar{\partial}_A = \bar{\partial}_\alpha$.*

Proof. Consider local unitary trivializations, where the partial connection $\bar{\partial}_\alpha$ is represented by a^τ, a matrix of $(0,1)$-forms. Then A^τ is uniquely determined as $a^\tau - (a^\tau)^*$, because it satisfies the unitary condition $A^\tau = -(A^\tau)^*$ and the compatability connection $(A^\tau)^{0,1} = a^\tau$, as $(a^\tau)^*$ is a matrix of $(1,0)$-forms. □

Proposition 2.8. *A unitary connection on a Hermitian complex vector bundle over Z is compatible with a holomorphic structure if and only if it has curvature of type $(1,1)$, and in this case the connection is uniquely determined by the metric and holomorphic structure.*

Proof. Note that the curvature of a unitary connection is skew-adjoint, hence $F^{0,2} = -(F^{0,2})^*$. Thus, having a curvature of type $(1,1)$ is equivalent to $F^{0,2} = 0$. So, we can just apply Corollary 2.5 and Lemma 2.7. □

Definition 2.9. Let Z be a complex manifold of complex dimension 2 with a Hermitian metric on its tangent bundle. Then it is also an oriented Riemannian 4-manifold. We have two orthogonal decompositions of the complexified 2-forms on Z:

$$\Omega^2 = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$$
$$\Omega^2 = \Omega^+ \oplus \Omega^-$$

The complex structure and metric together define a $(1,1)$-form ω by the rule $\omega(\zeta, \eta) = (\zeta, i\eta)$. Hence, $\Omega^{1,1} = \Omega_o^{1,1} \oplus \Omega^0 \cdot \omega$, where $\Omega_o^{1,1}$ is forms pointwise orthogonal to ω. ▲

Lemma 2.10. *The complexified self-dual 2-forms over Z are*

$$\Omega^+ = \Omega^{2,0} \oplus \Omega^0\omega \oplus \Omega^{0,2}$$

and the complexified anti-self-dual forms are

$$\Omega^- = \Omega_o^{1,1}$$

Proof. It is enough to prove this result in $\mathbb{C}^2$, with complex coordinates $z_1 = x_1 + ix_2$,$z_2 = x_3 + ix_4$. For the standard Hermitian metric, where dx_1, dx_2, dx_3, dx_4 is the orthonormal basis, we know that:

$$\Omega^+ = span\{dx_1 \wedge dx_2 + dx_3 \wedge dx_4, dx_1 \wedge dx_3 - dx_2 \wedge dx_4, dx_1 \wedge dx_4 + dx_2 \wedge dx_3\}$$

$$\Omega^- = span\{dx_1 \wedge dx_2 - dx_3 \wedge dx_4, dx_1 \wedge dx_3 + dx_2 \wedge dx_4, dx_1 \wedge dx_4 - dx_2 \wedge dx_3\}$$

Now, $\Omega^{2,0}$ and $\Omega^{0,2}$ are spanned by $dz_1 \wedge dz_2$ and $d\bar{z}_1 \wedge d\bar{z}_2$ respectively, i.e., $\Omega^{2,0} \oplus \Omega^{0,2}$ is spanned by real and imaginary parts of

$$dz_1 \wedge dz_2 = d(x_1 + ix_2) \wedge d(x_3 + ix_4) = (dx_1 \wedge dx_3 - dx_2 \wedge dx_4) + i(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)$$

Since $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, we get that $\Omega^+ = \Omega^{2,0} \oplus \Omega^0\omega \oplus \Omega^{0,2}$.
Note that to finish the proof, it is enough to show $\Omega^- \subseteq \Omega_o^{1,1}$.
We have:

$$\begin{aligned}
(\omega, dx_1 \wedge dx_2 - dx_3 \wedge dx_4) =&(dx_1 \wedge dx_2 + dx_3 \wedge dx_4, dx_1 \wedge dx_2 - dx_3 \wedge dx_4) \\
=&(dx_1 \wedge dx_2, dx_1 \wedge dx_2) - (dx_1 \wedge dx_2, dx_3 \wedge dx_4) \\
&+ (dx_3 \wedge dx_4, dx_1 \wedge dx_2) - (dx_3 \wedge dx_4, dx_3 \wedge dx_4) \\
=&1 - 0 + 0 - 1 \\
=&0
\end{aligned}$$

The other two calculations are similar. □

Proposition 2.11. *If A is an ASD connection on a complex vector bundle E over the Hermitian complex surface Z, then the operator $\bar{\partial}_A$ defines a holomorphic structure on E. Conversely if $\mathcal{E}$ is a holomorphic structure on E, and A is a compatible unitary connection, then A is ASD if and only if $\hat{F_A} := (F_A, \omega) = 0$.*

Proof. Assume A is ASD, i.e., $F_A^+ = 0$. Then by Lemma 2.10, $F^{0,2} = 0$, so we just apply Corollary 2.5. For the second part, if A is ASD, then $(F_A, \omega) = (F_A^+ + F_A^-, \omega) = (F_A^-, \omega) = 0$, because $\Omega^- = \Omega_o^{1,1}$. Conversely, since A is a compatible unitary connection, its curvature has type $(1,1)$, meaning that the $(0,2)$ and $(2,0)$ parts are 0. So, $F_A = F_A^o + F_A^\omega$, where F_A^o is the $\Omega_o^{1,1}$ part, and F_A^ω is the $\Omega^0\omega$ part. Since, $0 = (F_A, \omega) = (F_A^o + F_A^\omega, \omega) = (F_A^\omega, \omega)$, we get that $F_A^\omega = 0$, implying that $F^+ = 0$. □

3 Uhlenbeck theorem

One can easily show that a flat connection can be locally represented by a zero connection matrix in a suitable gauge. Therefore, it is natural to ask whether a connection with small curvature can be represented by another small connection matrix in some gauge. This leads to the concept of "gauge fixing", which gives us the optimal connection matrix. The main result of this section is Corollary 3.8 which is a "sequential compactness" results of ASD connections, modulo gauge transformations, on a small ball.

3.1 Main Theorems

Remark 3.1. We will work in the framework of Sobolev spaces. See section 5.1 of the Appendix for more information.

Definition 3.2. Suppose A_0 is a connection on a unitary bundle $E \to X$ over a Riemannian manifold X, and consider the gauge equivalence class of another connection A on E:

$$\mathcal{H} = \{u(A) | u \in \mathcal{G}\} \subset \mathcal{A}.$$

We say that a point B in $\mathcal{H}$ is in Coulomb gauge relative to A_0 if $d^*_{A_0}(B - A_0) = 0$. ▲

Remark 3.3. The definition of Coulomb gauge relative to A_0 is motivated by the desire to minimize the L^2 norm of $B - A_0$ over the equivalence class $\mathcal{H}$.
For example, consider a one-parameter family of gauge transformations $\exp(t\chi)$, denoted as $e^{t\chi}$, where $\chi \in \Omega^0(\mathfrak{g}_E)$ has compact support. Then:

$$\frac{d}{dt}\|e^{t\chi}(B) - A_0\|^2\Big|_{t=0} = \frac{d}{dt}\|e^{t\chi}Be^{-t\chi} - d(e^{t\chi})e^{-t\chi} - A_0\|^2\Big|_{t=0}$$

Note that for a fucntion $H(t)$, we have $\frac{d}{dt}\|H(t)\|^2 = \frac{d}{dt}(H(t), H(t)) = 2(H(t), H'(t))$. So, using also Remark 1.8, we get:

$$\begin{aligned}
\frac{d}{dt}\|e^{t\chi}(B) - A_0\|^2\Big|_{t=0} &= 2(e^{t\chi}Be^{-t\chi} - d(e^{t\chi})e^{-t\chi} - A_0, \frac{d}{dt}(e^{t\chi}Be^{-t\chi} - d(e^{t\chi})e^{-t\chi} - A_0))\Big|_{t=0} \\
&= 2(B - A_0, (\chi e^{t\chi}Be^{-t\chi} - e^{t\chi}B\chi e^{-t\chi}) - (d(\chi e^{t\chi})e^{-t\chi} + d(e^{t\chi})(-\chi e^{t\chi})))\Big|_{t=0} \\
&= 2(B - A_0, (\chi B - B\chi) - d\chi) \\
&= 2(B - A_0, [\chi, B] - d\chi) \\
&= 2(B - A_0, -d_B(\chi)) \\
&= -2(d^*_B(B - A_0), \chi)
\end{aligned}$$

Proposition 3.4. *Let X be a compact Riemannian 4-manifold and A be a connection on a unitary bundle E over X. There is a constant $c(A)$ such that if B is another connection on E and if $a = B - A$ satisfies $\|\nabla_A\nabla_A a\|^2 + \|a\|^2 < c(A)$, then there is a gauge transformation u such that $u(B)$ is in Coulomb gauge relative to A.*

Proof. We have:

$$\begin{aligned} u(B) &= u(A+a) \\ &= A + a - (d_{A+a}u)u^{-1} \\ &= A + a - (d_A u + [a,u])u^{-1} \\ &= A + a - (d_A u)u^{-1} - (au - ua)u^{-1} \\ &= A - (d_A u)u^{-1} + uau^{-1} \end{aligned}$$

We need u such that $d_A^*(A - u(B)) = 0$, i.e. $d_A^*((d_A u)u^{-1} - uau^{-1}) = 0$ We will look for u of the form $\exp(\chi)$, where $\chi \in \Gamma(\mathfrak{g}_E)$, which we will denote by e^χ. To do this, we will apply the Implicit function theorem to the function

$$G(a, \chi) = d_A^*((d_A e^\chi)e^{-\chi} - e^\chi a e^{-\chi})$$

First, we will think of G as a function on 1-forms a in L_2^2 and sections χ in L_3^2. Also, note that G now has values in L_1^2. Next, we need to calculate the derivative at $a = 0, \chi = 0$:

$$\begin{aligned} DG(0,0)(b,\zeta) &= \frac{d}{dt}\Big|_{t=0} G(t\zeta, tb) \\ &= \frac{d}{dt}\Big|_{t=0} d_A^*((d_A e^{t\zeta})e^{-t\zeta} - e^{t\zeta} tb e^{-t\zeta}) \\ &= d_A^*((d_A e^{t\zeta})(-\zeta e^{-t\zeta}) + (d_A(\zeta e^{t\zeta})e^{-t\zeta}) - (e^{t\zeta} b e^{-t\zeta} + t(\dots)))\Big|_{t=0} \\ &= d_A^*(d_A \zeta - b) \end{aligned}$$

In order to use the Implicit function theorem (Theorem 5.8 from the Appendix), we need the map $\zeta \mapsto DG(0,0)(0,\zeta) = d_A^* d_A \zeta$ to be surjective. This follows from the fact that the image of the Laplace operator $d_A^* d_A$ is the image of d_A^*, see [12] for this Hodge-theoretic fact. Also, note that $\|\nabla_A \nabla_A a\|^2 + \|a\|^2$ is a norm on L_2^2, so the Implicit function theorem gives us a small solution χ if a is small, i.e. $\|\nabla_A \nabla_A a\|^2 + \|a\|^2 < c(A)$, for some constant $c(A)$. □

Definition 3.5. Let B^4 be the unit ball in $\mathbb{R}^4$ and $m : \mathbb{R}^4 \to S^4$ be the standard stereographic projection map, which is a conformal diffeomorphism from $\mathbb{R}^4$ to S^4 minus a point.

▲

Now, we state two main theorems of this section along with their corollary. Their proofs will come in section 3.2 after proving some preliminary results.

Theorem 3.6. *There are constants $\epsilon_1, M > 0$ such that any connection on the trivial bundle over $\overline{B}^4$ (it means it is smooth up to the boundary) with $\|F_A\|_{L^2} < \epsilon_1$ is gauge equivalent to a connection $\widetilde{A}$ over B^4 with*

$$d^* \widetilde{A} = 0$$

$$\|\widetilde{A}\|_{L_1^2} \leq M \|F_{\widetilde{A}}\|_{L^2}$$

Proof. Given in Section 3.2. □

Theorem 3.7. *There is a constant $\epsilon_2 > 0$ such that if $\widetilde{A}$ is any ASD connection on the trivial bundle over B^4 which satisfies the Coulomb gauge condition $d^* \widetilde{A} = 0$ and $\|\widetilde{A}\|_{L^4} \leq \epsilon_2$, then for any interior domain $D \subset B^4$ (i.e. $\bar{D} \subset B^4$) and any $l \geq 1$ we have*

$$\|\widetilde{A}\|_{L^2_l(D)} \leq M_{l,D}\,\|F_A\|_{L^2(B^4)}$$

for a constant $M_{l,D}$ depending only on l and D.

Proof. Given in Section 3.2. □

Corollary 3.8. *There exists a constant $\epsilon > 0$, such that for any sequence of ASD connections A_α, over B^4 with $\|F(A_\alpha)\|_{L^2} \leq \epsilon$ there is a subsequence α' and gauge equivalent connections $\bar{A}_{\alpha'}$ which converge in C^∞ on the open ball. Note that this is a "sequential compactness" property for ASD connections modulo gauge transformations and restricted to a small ball.*

Proof. Let $\epsilon = \min(\epsilon_1, \frac{\epsilon_2}{CM})$, where C is the Sobolev constant from the Sobolev inequality 5.2 $\|A\|_{L^4} \leq C\|A\|_{L^2_1}$.
Then, if A_α is a sequence of ASD connections over B^4, with $\|F(A_\alpha)\|_{L^2} \leq \epsilon \leq \epsilon_1$, we can apply Theorem 3.6 to get a gauge equivalent sequence $\widetilde{A}_\alpha$, with all the conditions for $\widetilde{A}_\alpha$'s from that theorem. Then, note that for all α, $\|\widetilde{A}_\alpha\|_{L^4} \leq C\|\widetilde{A}_\alpha\|_{L^2_1} \leq CM\|F_{\widetilde{A}}\|_{L^2} \leq CM\frac{\epsilon_2}{CM} = \epsilon_2$. Thus, we can now apply Theorem 3.7 for each of the $\widetilde{A}_\alpha$'s. We obtain a uniformly bounded sequence, which we call again as $\widetilde{A}_\alpha$ with uniformly bounded derivatives. Therefore, we can use Arzelà–Ascoli theorem to extract a subsequence which converges uniformly in C^∞ on the open ball. □

3.2 Proofs of Theorems 3.6 and 3.7

Lemma 3.9. *Let B be a connection on the trivial bundle over S^4 in Coulomb gauge relative to the product connection (i.e. with $d^*B = 0$). There are constants $N, \eta > 0$ such that if $\|B\|_{L^4} < \eta$ then $\|B\|_{L^2_1} < N\,\|F_B\|_{L^2}$.*

Proof. Since $H^1(S^4) = 0$ and $d^*B = 0$, we can use Remark 5.2 to get an inequality:

$$\|B\|_{L^2_1} \leq c_1\|dB\|_{L^2}$$

for some $c_1 > 0$.
Next,

$$\|B \wedge B\|_{L^2} \leq \|B\|^2_{L^4}$$

Using Proposition 5.2 from the Appendix, we get

$$\|B\|_{L^4} \leq c_2\|B\|_{L^2_1}$$

for some $c_2 > 0$.
Hence, since $dB = F_B - B \wedge B$, we get that

$$\|B\|_{L^2_1} \leq c_1\|dB\|_{L^2} \leq c_1(\|F_B\|_{L^2} + c_2\|B\|_{L^4}\|B\|_{L^2_1})$$

and thus,

$$\|B\|_{L^2_1}(1 - c_1c_2\|B\|_{L^4}) \leq c_1\|F_B\|_{L^2}.$$

So, take $\eta = \frac{1}{2c_1c_2}$. We get that if $\|B\|_{L^4} \leq \frac{1}{2c_1c_2}$, then $\|B\|_{L^2_1}(1 - c_1c_2\frac{1}{2c_1c_2}) \leq c_1\|F_B\|_{L^2}$ and so, $\|B\|_{L^2_1} \leq 2c_1\|F_B\|_{L^2}$. Therefore, the required $N = 2c_1$.

□

Lemma 3.10. *There is a constant $\eta' > 0$ such that if the connection B of Lemma 3.9 has $\|B\|_{L^2} < \eta'$ then for each $l \geq 1$, a bound $\|B\|_{L^2_{l+1}} \leq f_l(Q_l(B))$ holds, for a universal continuous function f_l, independent of B, with $f_l(0) = 0$, where $Q_l(B) = \|F_B\|_{L^\infty} + \sum_{i=1}^{l}\|\nabla_B^{(i)}F_B\|_{L^2}$, for $\nabla_B^{(i)} = \nabla_B \cdots \nabla_B$.*

Proof. The proof is quite technical, it uses mostly elliptic theory, Sobolev and Hölder inequalities. We therefore choose to omit it. It can be found at p. 61 of [2]. $\square$

Proof of Theorem 3.7. Let $\widetilde{A}$ be an ASD connection on the trivial bundle over B^4 such that $d^*\widetilde{A} = 0$. Consider the differential operator $\delta = d^* + d^+$. Similarly to Proposition 1.17, it can be shown to be elliptic. Since $\widetilde{A}$ is ASD, we have that $(d\widetilde{A} + \widetilde{A} \wedge \widetilde{A})^+ = 0$, i.e., $d^+\widetilde{A} + (\widetilde{A} \wedge \widetilde{A})^+ = 0$. Combining with the fact that $d^*\widetilde{A} = 0$, we get that $\delta\widetilde{A} + (\widetilde{A} \wedge \widetilde{A})^+ = (d^+ + d^*)\widetilde{A} + (\widetilde{A} \wedge \widetilde{A})^+ = 0$.
Assume that the base manifold is S^4. Also, let B^4 be contained in S^4 using the stereographic map. Note that the flat metric on B^4 is conformal to the round metric on S^4 and the L^2 norm on 2-forms in 4-dimensions is conformally invariant.
For any interior domain D of B^4, let ϕ be the map with support in B^4 and 1 on D. Then, extending by 0, we get another connection matrix $\alpha \equiv \phi\widetilde{A}$ defined on all of S^4. Now,

$$\begin{aligned}
\delta(\phi\widetilde{A}) &= d^+(\phi\widetilde{A}) + d^*(\phi\widetilde{A}) \\
&= (d(\phi\widetilde{A}))^+ - {*d*}(\phi\widetilde{A}) \\
&= (d\phi \wedge \widetilde{A} + \phi d\widetilde{A})^+ - {*d}(\phi * \widetilde{A}) \\
&= (d\phi \wedge \widetilde{A})^+ + \phi d^+(\widetilde{A}) - *(d\phi \wedge *\widetilde{A} + \phi d(*\widetilde{A})) \\
&= (d\phi \wedge \widetilde{A})^+ - \phi(\widetilde{A} \wedge \widetilde{A})^+ - *(d\phi \wedge *\widetilde{A}) + \phi d^*\widetilde{A} \\
&= (d\phi \wedge \widetilde{A})^+ - (\alpha \wedge \widetilde{A})^+ - (d\phi)^\sharp \lrcorner \widetilde{A}
\end{aligned}$$

Now, we apply the estimates for δ. Referring to Remark 5.2, we have $\|\alpha\|_{L^2_2} \leq const\|\delta\alpha\|_{L^2_1}$.
From the calculations above, we get $\|\delta\alpha\|_{L^2_1} \leq \|d\phi \wedge \widetilde{A}\|_{L^2_1} + \|\alpha \wedge \widetilde{A}\|_{L^2_1} + \|(d\phi)^\sharp \lrcorner \widetilde{A}\|_{L^2_1}$.
Now, we evaluate a part of $\|\alpha \wedge \widetilde{A}\|_{L^2_1}$:

$$\begin{aligned}
\|\nabla(\phi\widetilde{A} \otimes \widetilde{A})\|_{L^2} &= \|\nabla(\phi\widetilde{A}) \otimes \widetilde{A} + \phi\widetilde{A} \otimes \nabla\widetilde{A}\|_{L^2} \\
&= \|\nabla(\phi\widetilde{A}) \otimes \widetilde{A} + \widetilde{A} \otimes \phi\nabla\widetilde{A}\|_{L^2} \\
&= \|\nabla(\phi\widetilde{A}) \otimes \widetilde{A} + \widetilde{A} \otimes \nabla(\phi\widetilde{A}) - \widetilde{A} \otimes \nabla\phi \otimes \widetilde{A}\|_{L^2} \\
&\leq \|\nabla(\phi\widetilde{A}) \otimes \widetilde{A}\|_{L^2} + \|\widetilde{A} \otimes \nabla(\phi\widetilde{A})\|_{L^2} + \|\widetilde{A} \otimes \nabla\phi \otimes \widetilde{A}\|_{L^2} \\
&\leq const(\|\nabla(\phi\widetilde{A})\|_{L^4}\|\widetilde{A}\|_{L^4} + \|\widetilde{A}\|^2_{L^4})
\end{aligned}$$

for some constant depending only on ϕ.
Next, $\|d\phi \wedge \widetilde{A}\|_{L^2_1}$ is bounded by some constant, again depending on ϕ, times $\|\widetilde{A}\|_{L^2_1}$. Same for $\|(d\phi)^\sharp \lrcorner \widetilde{A}\|_{L^2_1}$.
So, combining all the inequalities, and the Sobolev embedding theorem, we get

$$\|\alpha\|_{L^2_2} \leq const\|\delta\alpha\|_{L^2_1} \leq const(\|\widetilde{A}\|_{L^2_1}\|\alpha\|_{L^2_2} + \|\widetilde{A}\|_{L^2_1} + \|\widetilde{A}\|^2_{L^2_1}).$$

This, can be rearranged to get an upper bound on $\|\alpha\|_{L^2_2}$ in terms of $\|\widetilde{A}\|_{L^2_1}$, if $\|\widetilde{A}\|_{L^2_1}$ is small enough. Then, when $\phi = 1$, $\alpha = \widetilde{A}$, so we went from an L^2_1 bound over B to an L^2_2 bound over D. This argument can be iterated to get estimates on all the higher derivatives over domains containing D.

$\square$

Proposition 3.11. *If A_i, B_i are sequences of connections on a unitary bundle over a manifold X, all of whose derivatives are bounded, and if A_i is gauge equivalent to B_i for each i, then there are subsequences converging to limiting connections A_∞, B_∞, such that A_∞ is gauge equivalent to B_∞.*

Proof. Since all derivatives of A_i, B_i's are bounded, by using Ascoli-Arzela theorem, by possibly passing to a sunsequence, we can assume that these sequences converge.
Now, let $B_i = u_i(A_i)$, for all i, for some u_i's. Then we have

$$B_i = u_i(A_i) = u_i A_i u_i^{-1} - du_i u_i^{-1}$$

and hence

$$du_i = (u_i A_i u_i^{-1} - B_i)u_i = u_i A_i - B_i u_i$$

Now, both sequences A_i, B_i converge, hence all their derivatives are uniformly bounded. We will now show using induction, that all the derivatives of u_i's are also uniformly bounded. Since $U(n)$ is compact, we have that u_i's are in C^0. Now, assume that u_i is bounded in C^r, then by the formula above, du_i is bounded in C^r, hence u_i is bounded in C^{r+1}. So, we can apply Ascoli-Arzela theorem to obtain a subsequence which converges in C^∞ to some limit u_∞. Note that the gauge relation between A_i's and B_i's is preserved in the limit, finishing the proof of the theorem.

□

Proposition 3.12. *There is a constant $\zeta > 0$ such that if $B_t^{'}$ $(t \in [0,1])$ is a one-parameter family of connections on the trivial bundle over S^4 with $\|F_{B_t}^{'}\|_{L^2} < \zeta$ for all t, and with $B_0^{'}$ the product connection, then for each t there exists a gauge transformation u_t such that $u_t(B_t^{'}) = B_t$ satisfies*

$$d^* B_t = 0$$

$$\|B_t\|_{L_1^2} \leq 2N \|F_{B_t}\|_{L^2}$$

where N is the constant of Lemma 3.9.

Proof. We will use the continuity method. Consider the set $S \subseteq [0,1]$ for which such a gauge transformation u_t exists. We will show that S is open and closed. Combining with the obvious fact that $0 \in S$, we then obtain that $S = [0,1]$.
First, we show that S is closed. We consider a sequence s_i in S with a limit s and we need to show that $s \in S$.

Note that for each $t = s_i$, $\|B_t\|_{L^4} \leq C\|B_t\|_{L_1^2} \leq 2CN \|F_{B_t}\|_{L^2} < 2CN\zeta$. So, pick ζ such that $2CN\zeta < \eta, \eta'$ from Lemmas 3.9 and 3.10.
Lemma 3.9 gives us that $\|B_t\|_{L_1^2} < N \|F_{B_t}\|_{L^2}$. Lemma 3.10 gives us that $\|B_t\|_{L_{l+1}^2} \leq f_l(Q_l(B_t))$, for all $l \geq 1$, for some continuous functions f_l. So since all the derivatives of $\nabla_{B_t}^{(j)} F_{B_t}$ are bounded, and covariant derivatives of curvature are gauge invariant, we obtain bounds on all the derivatives of B_t.
Now, from Proposition 3.11, we can note that if we have two sequences all of whose derivatives are bounded, then by possibly passing to a subsequence, we can assume that the sequences are convergent. So consider two sequences B_{t_i} and B'_{t_i} and apply Proposition 3.11 to get subsequences which converge in C^∞ to gauge equivalent limits B_s and B'_s respectively. Note that we will have $d^* B_s = 0$ and $\|B_s\|_{L_1^2} \leq 2N \|F_{B_s}\|_{L_1^2}$. Thus, $s \in S$, so S is closed.

Next, we show that S is open. Consider any point $t_0 \in S$. We will prove that there exists an interval around t_0 contained in S. WLOG assume $B_{t_0} = B'_{t_0}$ and call this connection B. Let $B'_{t_0+\delta} = B + b_\delta$. We want to show that for small b_δ there exists a required gauge transformation $u_{t_0+\delta}$.
As in the proof of Proposition 3.4, we will look for solutions of the form e^{χ_δ} and will try to solve the equation $H(\chi_\delta, b_\delta) = 0$, where $H(\chi, b) = d^*(e^\chi (B + b)e^{-\chi} - d(e^\chi)e^{-\chi})$.
So, we can think of H, as of a function on L_l^2 sections of $\mathfrak{g}_E$ and L_{l-1}^2 1-forms with values in $\mathfrak{g}_E$. Also, H has values in L_{l-2}^2 sections of $\mathfrak{g}_E$. In order to use the Implicit function theorem, we need the map $(D_1 H)_0 : \Gamma(\mathfrak{g}_E) \to \Gamma(\mathfrak{g}_E)$ to be surjective. Then, we will have a small solution χ to $H(\chi, b) = 0$, if b is also small, implying that there will be interval around t_0 which is in S.
So, assume for contradiction that it is not surjective. Similarly as before, $(D_1 H)_0 \chi = d^* d_B \chi$. Then there exists a non-zero smooth η such that $(d^* d_B \chi, \eta)_{L^2} = 0$, for all χ.

Choose $\chi = \eta$. Then $0 = (d^* d_B \eta, \eta)_{L^2} = (d_B\eta, d\eta)_{L^2} = (d\eta + [B,\eta], d\eta)_{L^2} = \|d\eta\|^2_{L^2} + ([B,\eta], d\eta)_{L^2}$. Now,

$$\begin{aligned}\|d\eta\|^2_{L^2} \leq& \|([B,\eta], d\eta)\|_{L^2}\\ \leq& \|[B,\eta]\|_{L^2}\|d\eta\|_{L^2}\\ \leq& const\|B\|_{L^4}\|\eta\|_{L^4}\|d\eta\|_{L^2}\end{aligned}$$

We know from Hodge Theory that $\Omega^k = \ker(\triangle_{\Omega^k}) \oplus \mathrm{im}(d^*) \oplus \mathrm{im}(d)$. In our case, for $k = 0$, since $\mathrm{im}(d) \cap \Omega^0 = \{0\}$, we have that $\Omega^0 = \ker(\triangle_{\Omega^0}) \oplus \mathrm{im}(d^*)$, where this decomposition is L^2-orthogonal. For X compact, connected we also have that $\ker(\triangle_{\Omega^0})$ is the constant functions and hence $\mathrm{im}(d^*)$ is functions orthogonal to constants. Note that these functions have integral zero: $\int_X (d^*\alpha)vol = \pm\int_X d(\alpha^\sharp \lrcorner vol) = 0$, by Stokes' theorem. Thus, $\| \triangle_{\Omega^0}\eta\|^2_{L^2_{l-2}} = \|d\eta\|^2_{L^2_{l-1}} + \|d^*\eta\|^2_{L^2_{l-1}} = \|d\eta\|^2_{L^2_{l-1}}$.

Consider Theorem 5.5 from the Appendix. Since $\eta \perp \ker(\triangle_{\Omega^0})$ by assumption, we get an elliptic inequality: $\|\eta\|_{L^2_l} \leq const\| \triangle_{\Omega^0}\eta\|_{L^2_{l-2}} = const\|d\eta\|_{L^2_{l-1}}$. Hence, $\|\eta\|_{L^2_1} \leq const\|d\eta\|_{L^2}$.

Next, recall the Sobolev inequality 5.2 and use it twice on B and η to get $\|B\|_{L^4} \leq const\|B\|_{L^2_1}$ and $\|\eta\|_{L^4} \leq const\|\eta\|_{L^2_1}$. Hence, we obtain: $\|d\eta\|^2_{L^2} \leq const\|B\|_{L^2_1}\|d\eta\|^2_{L^2}$ and therefore, $const \leq \|B\|_{L^2_1}$, for some constant.
However, we can choose $\zeta > 0$, from the statement of the proposition, small enough to have the L^2_1 norm of B_t as small as we want, for all $t \in S$. So, we obtain a contradiction, which concludes the proof.

□

Proof of Theorem 3.6. First, we observe that there exists a path from connection A on the trivial bundle over the ball to the product connection. Consider $\delta_t : \mathbb{R}^4 \to \mathbb{R}^4, x \mapsto tx$, for $t \in (0,1)$. For such t's, let $A_t = \delta_t^*(A)$ be a connection matrix over B^4. Note that $A_0 = 0$ and $A_1 = A$. Now,

$$\int_{B^4} \|F(A_t)\|d\mu = \int_{\|x\|\leq 1} \|F(A)\|d\mu \leq \int_{B^4} \|F(A)\|d\mu$$

which follows from the conformal invariance of the L^2 norm of the curvature in 4-dimensions.
Identify B^4 with a hemisphere of S^4. Let $r : S^4 \to S^4$ be the reflection map, which is equal to the identity on the equatorial S^3 sphere and flips the two hemispheres. Also, let $p : S^4 \to B^4$ be the projection map, which is the identity on B^4 and r on the other hemisphere. Note that p is not differentiable on the equator, but it is not really an issue. For the clean way to deal with this problem, see page 68 of [2].
Let α be a connection matrix on B^4. Set $\beta = p^*(\alpha)$, $F_\beta = p^*(F_\alpha)$. These are well-defined L^∞ 1-form and 2-form over S^4 correspondingly. Then $\int_{S^4} \|F_\beta\|^2 d\mu = 2\int_{B^4} \|F_\alpha\|^2 d\mu$.
So, if we have a connection A over $\overline{B}^4$, we can use the construction above to make a path to 0 without increasing the L^2 norm. Let $B_t = p^*(A_t)$, i.e., we have a path of connections over S^4 with the curvatures being $\sqrt{2}$ times bigger than the ones of A_t. Pick ϵ_1 so that $\sqrt{2}\epsilon_1 < \zeta$ from Proposition 3.12 and apply it. We get a sequence of gauge equivalent connections B'_t with $d^*(B'_t) = 0$ and $\|B'_t\|_{L^2_1} \leq 2N\|F_{B'_t}\|_{L^2}$. Then our desired $\widetilde{A}$ is B'_1 restricted to B^4.

□

4 ASD moduli spaces

As before, let E be a bundle over a compact, Riemannian 4-manifold. In this section we define the moduli space M_E to be the set of ASD connections on E modulo gauge transformations. This set carries a natural quotient topology induced from the one on the space on connections $\mathcal{A}$. Although the space $\mathcal{A}$ may be infinite-dimensional, M_E is finite dimensional. Also, we will see that it has a structure of a real analytic space. In the case of $SU(2)$ or $SO(3)$ bundles over some simple manifolds, such as, for example, S^4, $\mathbb{CP}^2$, or $S^2 \times S^2$, the ASD equations can be solved, giving us explicit examples of moduli spaces. See pp. 126–129 of [2] for more information.

4.1 Basic theory

Remark 4.1. The definition of the moduli space is the solutions to the ASD equation, divided by the gauge group. However, we will consider this problem in the reverse order. First, we consider the set of all connections modulo the gauge group and then we describe the moduli space inside it.

Definition 4.2. We continue to work in the framework of Sobolev spaces L_l^2. We will stay in the range $l > 2$ because the Sobolev embedding theorem in 4-dimensions tells that for any such l, the space L_l^2 consists of continuous functions. So, for any $l > 2$, we define $\mathcal{A}$ to be the space of L_{l-1}^2 connections on a bundle E, $\mathcal{G}$ to be the group of L_l^2 gauge transformations.
Hence, set $\mathcal{B} = \mathcal{A}/\mathcal{G}$ with the quotient topology. We write $[A]$ for the equivalence class of a connection A in $\mathcal{B}$.
The moduli space $\mathcal{M}_E$ is the set of gauge equivalence classes of ASD connections on E.
In fact, the description of the moduli spaces we will get, is independent of the choice of l. This follows from the fact that the natural inclusion of $M(l+1)$ in $M(l)$ is a homeomorphism, where $M(l)$ is L_{l-1}^2 ASD connections modulo L_l^2 gauge transformations. ▲

Proposition 4.3. *The function d on $\mathcal{B} \times \mathcal{B}$, defined by*

$$d([A],[B]) = \inf_{g \in \mathcal{G}} \|A - g(B)\|$$

is a metric, where $\|A - B\| = \left(\int_X |A - B|^2 d\mu \right)^{\frac{1}{2}}$ is the L^2 metric on $\mathcal{A}$.

Proof. The properties that $d([A],[B]) = d([B],[A])$, $d([A],[B]) \leq d([A],[C]) + d([C],[B])$, and $d([A],[A]) = 0$ follow immediately. So, it is only left to show that $d([A],[B]) = 0$ implies $[A] = [B]$.
Hence, assume $d([A],[B]) = 0$. Then we have a sequence g_α such that $g_\alpha(B)$ converges to A. We need to show that B and A are gauge equivalent. Let $B_\alpha = g_\alpha(B)$. Then we have $d_B g_\alpha = (B - B_\alpha) g_\alpha$.
Note that since G is compact, g_α's are uniformly bounded and hence $d_B g_\alpha$'s are also bounded in L^2. Thus, we can take a subsequence, which we continue to call g_α, with a limit g, so that we have weak convergence in L_1^2, and strong convergence in L^2.
Then, $d_B g = (B - A) g$.
Now, if ϕ is any smooth section of End E, as g_α's, we have:

$$(d_B g, \phi) = \lim(d_B g_\alpha, \phi) = \lim((B - B_\alpha) g_\alpha, phi) = ((B - A) g, \phi)$$

because $B_\alpha g_\alpha$ converges to Ag in L^1.
Thus, we have an elliptic equation for g with coefficients in L_{l-1}^2 and so from elliptic regularity (Proposition 5.7 of the Appendix), g is in L_l^2.

In particular, $\mathcal{B}$ is Hausdorff in the quotient L_{l-1}^2 topology. □

Definition 4.4. For $A \in \mathcal{A}$ and $\epsilon > 0$, we set $T_{A,\epsilon} = \{a \in \Omega^1(\mathfrak{g}_E) : d_A^* a = 0, \|a\|_{L_{l-1}^2} < \epsilon\}$. We will see that a neighbourhood of $[A]$ in $\mathcal{B}$ can be described as a quotient of $T_{A,\epsilon}$ for small ϵ. ▲

Definition 4.5. A connection A on a G-bundle E is reducible if for each point $x \in X$ the holonomy maps T_γ of all loops γ based at x lie in some proper subgroup of the automorphisms group Aut $E_x \cong G$. ▲

Definition 4.6. The isotropy group Γ_A of a connection A in the gauge group $\mathcal{G}$ is

$$\Gamma_A = \{u \in \mathcal{G} | u(A) = A\}$$

▲

Remark 4.7. Note that if the base manifold X is connected, then Γ_A is isomorphic to the centralizer of the holonomy group H_A in $\mathcal{G}$. Also, Γ_A, as a closed subgroup of $\mathcal{G}$, is also a Lie group. Its elements are the covariant constant sections of the bundle $\mathrm{Aut}\, E$, and hence the Lie algebra of Γ_A is kernel of the covariant derivative d_A in $\Omega^0_X(\mathfrak{g}_E)$. The group Γ_A acts on $\Omega^1_X(\mathfrak{g}_E)$ and on $T_{A,\epsilon}$.

Proposition 4.8. *For small ϵ the projection map from $\mathcal{A}$ to $\mathcal{B}$ induces a homeomorphism h from the quotient $T_{A,\epsilon}/\Gamma_A$ to a neighbourhood of $[A]$ in $\mathcal{B}$. For a in $T_{A,\epsilon}$, the isotropy group of a in Γ_A is naturally isomorphic to that of $h(a)$ in $\mathcal{G}$.*

Proof. This is a common argument in differential geometry, however, the proof is highly non-trivial. For a similar statement and its proof, one can consult [3]. □

Definition 4.9. Let $\mathcal{A}^* = \{A \in \mathcal{A} | \Gamma_A = C(G)\}$, where $C(G)$ is the centre of G. Note that $C(G)$ is the smallest isotropy group that can be, because the elements of $C(G)$ correspond to the constant gauge transformations u, and so $u(A) = uAu^{-1} - (du)u^{-1} = A$. Now, let $\mathcal{B}^*$ in $\mathcal{B}$ be the quotient of $\mathcal{A}^*$. ▲

Remark 4.10. For simplicity, we only consider the irreducible connections, i.e., when the stabilizer Γ_A is trivial. Proposition 4.8 says that $\mathcal{B}^*$ is modelled locally on the balls $T_{A,\epsilon}$ in the Hilbert spaces $\ker d_A^* \subset L^2_{l-1}(\Omega^1(\mathfrak{g}_E))$. However, the description of $\mathcal{B} \setminus \mathcal{B}^*$ is more complicated. For example, when working with $SU(2)$ connections, there is a result saying that a connection is reducible if there is a decomposition $E = L \oplus L^{-1}$, where L is a complex line bundle over X. This, in turn is equivalent to the condition $c_2(E) = -c_1(L)$. See pp. 132–134 of [2] for more information. Thus, we always make sure to stay in the irreducible case.

Remark 4.11. We obtain local models for M within the local models for the orbit $\mathcal{B}$ space. Let A be an ASD connection and define: $\psi : T_{A,\epsilon} \to \Omega^+(\mathfrak{g}_E)$, $\psi(a) = F^+(A + a) = d_A^+ a + (a \wedge a)^+$.
Let $Z(\psi) \in T_{A,\epsilon}$ be the zero set of ψ. The map h from 4.8 induces a homeomorphism from the quotient $Z(\psi)/\Gamma_A$ to a neighbourhood of $[A]$ in M.

4.2 Fredholm theory

Definition 4.12. A bounded linear map $L : U \to V$ between Banach spaces is Fredholm if it has finite-dimensional kernel, cokernel and closed image. The last condition is actually redundant.
Hence, the kernel and the image of L are closed and admit topological complements, so we can write $U = U_0 \oplus F$, $V = V_0 \oplus G$, where F and G are finite-dimensional and L is a linear isomorphism from U_0 to V_0. The index of L is the difference of the dimensions: $\mathrm{ind}(L) = \dim(\ker L) - \dim(\mathrm{coker}\, L) = \dim(F) - \dim(G)$. ▲

Definition 4.13. Let N be a connected open neighbourhood of 0 in the Banach space U. A smooth map $\phi : N \to V$ is called Fredholm if for each point x in N the derivative $(D\phi)_x : U \to V$ is a linear Fredholm operator. In this case the index of $(D\phi)_x$ is independent of x and is referred to as the index of ϕ. ▲

Remark 4.14. Let ϕ be such a Fredholm operator with index r and $\phi(0) = 0$. For the purposes of this section, we regard such two maps being equal if they are equal on an arbitrary small neighbourhood of 0. Suppose that $L = (D\phi)_0$ is surjective, so the index is the dimension of the kernel of L. The implicit function theorem in Banach spaces the says that there is then a diffeomorphism f from one neighbourhood of 0 in U to another, such that $\phi \circ f = L$. We will just say that ϕ is right equivalent to the map L if they agree under composition on the right with a local diffeomorphism.

Proposition 4.15. *A Fredholm map ϕ from a neighbourhood of 0 is locally right equivalent to a map of the form $\widetilde{\phi} : U_0 \times F \to V_0 \times G$, $\widetilde{\phi}(\zeta, \eta) = (L(\zeta), \alpha(\zeta, \eta))$, where L is a linear isomorphism from U_0 to V_0, F and G are finite-dimensional, $\dim(F) - \dim(G) = \mathrm{ind}(\phi)$, and the derivative of α vanishes at 0.*

Proof. Let $\phi' = pr_{V_0} \circ \phi : N \to V_0$. Then the derivative of ϕ' at 0 is surjective by construction. So, using the previous remark, we can apply a diffeomorphism f between some small neighbourhoods of 0, to get the required map $\widetilde{\phi}$. $\square$

Corollary 4.16. *We obtain a finite dimensional model for neighbourhood of 0 in the zero set $Z(\phi)$. Under a diffeomorphism of U this is taken to the zero set of the smooth map: $f : F \to G$ between finite-dimensional vector spaces given by $f(y) = \alpha(0, y)$.*

Proof. We use the previous Proposition 4.15 to get the right equivalent map $\widetilde{\phi}$. So, we need to find the zero set of $\widetilde{\phi}(\zeta, \eta) = (L(\zeta), \alpha(\zeta, \eta))$. Since L is an isomorphism, ζ has to be equal zero. So, the zero set of $\widetilde{\phi}$ can be identified with the zero set of the function $F \to G : \eta \mapsto \alpha(0, \eta)$, which we call f. $\square$

4.3 Local models for the moduli space

Proposition 4.17. *If A is an ASD connection over X, a neighbourhood of $[A]$ in M is modelled on a quotient $f^{-1}(0)/\Gamma_A$, where $f : \ker(\delta_A) \to \mathrm{coker}(d_A^+)$ is a Γ_A-equivariant map.*

Proof. We have the map $\psi : T_{A,\epsilon} \subset \ker(d_A^*) \to \Omega_X^+(\mathfrak{g}_E), a \mapsto d_A^+ a + (a \wedge a)^+$.
Hence,

$$(D\psi)_0(b) = \frac{d}{dt}\Big|_{t=0} (d_A^+(tb) + (tb \wedge tb)^+) = d_A^+(b).$$

We already know that the operator $\delta_A \equiv d_A^* + d_A^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$ is elliptic. This is equivalent to saying that $d_A^+ : \ker(d_A^*) \to \Omega_X^+(\mathfrak{g}_E)$ is elliptic, which (because X is compact) implies that it is Fredholm.
Consider the the following sequence of operators:

$$0 \longrightarrow \Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A^+} \Omega^+ \longrightarrow 0$$

Note that since A is ASD, it is a complex. So, take the associated symbol complex:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\zeta\wedge} V \xrightarrow{\pi_+(\zeta\wedge)} \Lambda_+^2(V) \longrightarrow 0$$

where $\zeta \neq 0$ and π_+ is the projection on the self-dual part. It can be easily shown that this complex is exact, and so the original complex is elliptic. Hence, the standard elliptic theory tells us that in the original complex the images of the operators are closed subspaces and that the cohomology groups are finite dimensional. For more information on this argument, see Lemma 5.2.5 on p. 97 of [4].
So, in particular, $d_A(\Omega^0) = \mathrm{im}(d_A)$ is closed in Ω^1, which implies that $\Omega^1 = \ker(d_A^*) \oplus \mathrm{im}(d_A)$. Thus, we have:

$$\mathrm{im}(\delta_A) = \mathrm{im}(d_A^+|_{\ker(d_A^*)}) = d_A^+(\ker(d_A^*)) = d_A^+(\ker(d_A^*) \oplus \mathrm{im}(d_A)) = d_A^+(\Omega^1) = \mathrm{im}(d_A^+)$$

where the equality $d_A^+(\ker(d_A^*)) = d_A^+(\ker(d_A^*) \oplus \operatorname{im}(d_A))$ follows from the fact that $d_A^+(\operatorname{im}(d_A)) = 0$, as A is ASD. Hence, $\operatorname{coker}(\delta_A) = \operatorname{coker}(d_A^+)$.

Thus, apply Corollary 4.16 to get that $Z(\psi)$ is modelled as the zero set of some smooth map

$$f : \ker(\delta_A) \to \operatorname{coker}(\delta_A) = \operatorname{coker}(d_A^+)$$

and combined with Remark 4.11, we get the desired result.
Note that in the complex above, $\ker(\delta_A)$ is the cohomology group at Ω_0 and $\operatorname{coker}(d_A^+)$ is the cohomology at d_A^+, which, as mentioned before, are finite dimensional. Therefore, we can summarize this proposition as the following theorem: $\square$

Theorem 4.18. *Locally, the moduli space is the zero set of a smooth map $f : F \to G$, for some finite dimensional spaces F, G. Hence, if $G = 0$, it is smooth. This is why G is called the "obstruction space" and when $G = 0$, we say that the moduli space is "unobstructed".*

5 Appendix

5.1 Sobolev spaces

Definition 5.1. Consider a compact manifold X with a vector bundle V. After choosing a metric on X, a fibre metric and compatible connection A on V, we define the Sobolev space L^p_k to be the completion of smooth sections of V with respect to the norm

$$\|s\|_{L^p_k} = \Big(\sum_{i=0}^{k} \int_X \|\nabla_A^{(i)} s\|^p dvol\Big)^{\frac{1}{p}}$$

For more information on Sobolev spaces and the proof of the following theorem, one can consult [1].

▲

Proposition 5.2. *Sobolev embedding theorem: There exists a constant $C > 0$ such that for any connection A, we have:*

$$\|A\|_{L^4} \leq C\|A\|_{l^2_1}$$

5.2 Elliptic inequalities

Definition 5.3. Let E, F be two vector bundles over a compact manifold X and $P : \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order k. That means that we can write P in index notation as

$$Pv = A^{i_1\ldots i_k}\nabla_{i_1\ldots i_k} v + B^{i_1\ldots i_{k-1}}\nabla_{i_1\ldots i_{k-1}} v + \cdots + K^{i_1}\nabla_{i_1} v + Lv$$

for a section v of E, where $A^{i_1\ldots i_k}, B^{i_1\ldots i_{k-1}}$ are tensors taking values in $E^* \otimes F$.
Now, for each point $x \in X$, and each $\zeta \in T^*_x(X)$, define $\sigma_\zeta(P, x) = A^{i_1,\ldots,i_k}\zeta_{i_1}\ldots\zeta_{i_k}$. Then $\sigma_\zeta(P, x)$ is a linear map from E_x to F_x. Let $\sigma(P) : T^*(X) \times E \to F$ be the bundle map defined by $\sigma(P)(\zeta, v) = \sigma_\zeta(P, x)v \in F_x$, whenever $x \in X$, $\zeta \in T^*_x(X)$ and $v \in E_x$. Then $\sigma(P)$ is called the symbol of P and $\sigma(P)(\zeta, v)$ is homogeneus of degree k in ζ and linear in v. ▲

Definition 5.4. Let $P : \Gamma(E) \to \Gamma(F)$ be as before. We say P is an elliptic operator if for each $x \in X$ and each nonzero $\zeta \in T^*_x(X)$, the linear map $\sigma_\zeta(P, x) : E_x \to F_x$ is invertible. ▲

Theorem 5.5. *Let $P : \Gamma(E) \to \Gamma(F)$ be a smooth linear elliptic operator of order k, for E, F vector bundles over a compact Riemannian manifold X. Let $l \geq 0$ and $p > 1$ be integers. Then there exists a constant D such that if $v \in L^p_{k+l}(E)$ and $v \perp \ker(P)$, then $\|v\|_{L^p_{k+l}} \leq D\|Pv\|_{L^p_l}$.*

Proof. See Proposition 1.5.2 on page 17 of [6]. □

Remark 5.6. Consider the operator

$$d^* + d : \bigoplus_i \Omega_X^{2i+1} \to \bigoplus_i \Omega_X^{2i}$$

Similarly to Proposition 1.17 it can be shown to be elliptic.
If we suppose $H^1(X) = 0$, all the 1-forms are orthogonal to the kernel. For example, in 4-dimensions, if α is any odd degree mixed form in the kernel of $d+d^*$, it is in the kernel of $(d+d^*)^2 = d^2+d^*d+dd^*+(d^*)^2 = d^*d + dd^*$, which is the Laplacian. So, both the 1-form and 3-form parts are harmonic. However, since $H^1(X) = H^3(X) = 0$, by the Hodge theorem, there are no nonzero harmonic 1-forms or 3-form. So, we get that $\alpha = 0$, and so the kernel of $d + d^*$ is zero.
Hence, Theorem 5.5 gives the inequality

$$\|A\|_{L^2_k} \leq c(\|d^*A\|_{L^2_{k-1}} + \|dA\|_{L^2_{k-1}})$$

for some $c > 0$.
In particular, if we suppose $d^*A = 0$, we obtain:

$$\|A\|_{L^2_k} \leq c\|dA\|_{L^2_{k-1}}$$

Similarly,

$$\delta \equiv d^* + d^+ : \Omega^1_X \to \Omega^0_X \oplus \Omega^+_X$$

is an elliptic operator, which has kernel zero on Ω^1_X, if $H^1(X) = 0$.
So we can obtain

$$\|\alpha\|_{L^2_2} \leq c'\|\delta\alpha\|_{L^2_1}$$

for some $c' > 0$.

Proposition 5.7. *Elliptic regularity: Suppose that $P : \Gamma(E) \to \Gamma(F)$ is as in the setting of Theorem 5.5. Let $p > 1, l \geq 0$ be integers. Assume $P(v) = w$ holds weakly with $v \in L^1(V)$, $w \in L^1(W)$. Then if $w \in L^p_l(W)$, we have $v \in L^p_{k+l}(V)$.*

Proof. This is Theorem 1.4.1 on page 13 of [6]. □

5.3 Implicit function theorem

Theorem 5.8. *Suppose E_1, E_2 and F are Banach spaces and $f : E_1 \times E_2 \to F$ is a smooth map. Write $(D_1)f, (D_2)f$ for the partial derivatives. Then if $(D_2)f : E_2 \to F$ is surjective and admits a bounded right inverse, then for all η_1 near ζ_1, there exists a unique solution η_2 to the equation $f(\eta_1, \eta_2) = f(\zeta_1, \zeta_2)$.*

Proof. See Theorem 6.2.1 of [7]. □

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