The Kähler and Special Lagrangian Calibrations

by

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Waterloo, Ontario, Canada, 2010

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Chapter 1

Introduction

In this paper we will discuss two specific calibrations on $\mathbb{C}^n$ and their corresponding submanifolds.

A calibration on an oriented Riemannian manifold M is a closed k-form φ such that for any orientable submanifold N we have

$$\varphi|_N \leq \mathrm{vol}|_N$$

where vol is the volume form on M. We call a submanifold calibrated if we have equality in the above relation. We will show that calibrated submanifolds are volume-minimizing in their homology class. The calibrations on $\mathbb{C}^n$ that we are interested in are the special Lagrangian calibration and the Kähler calibration.

The Kähler calibration is given by $\omega^k/k!$ where ω is the standard Kähler form on $\mathbb{C}^n$. It is a $2k$-form so the submanifolds it calibrates are $2k$-dimensional. In fact we will show that the submanifolds of $\mathbb{C}^n$ calibrated by the Kähler calibration are precisely the complex submanifolds of $\mathbb{C}^n$. We can derive the Cauchy-Riemann equations on $\mathbb{C}^n$ from the relation defining a calibrated submanifold and we will do this derivation for $\mathbb{C}^2$.

The special Lagrangian calibration is $\alpha = \mathrm{Re}(dz^1 \wedge \cdots \wedge dz^n)$ and the submanifolds of $\mathbb{C}^n$ it calibrates are the special Lagrangian submanifolds. We will show that being calibrated by α is essentially equivalent to a submanifold N satisfying

1. $\omega|_N \equiv 0$ and
2. $\mathrm{Im}(dz^1 \wedge \cdots \wedge dz^n)|_N \equiv 0$.

Note that submanifolds satisfying condition (1) are generally called Lagrangian and so our submanifolds are special Lagrangian. We can also change the phase of the form α defined above and still have a calibration, and the submanifolds it calibrates will still be called special Lagrangian. We do this by taking $\alpha_\theta = \mathrm{Re}(e^{i\theta} dz^1 \wedge \cdots \wedge dz^n)$.

We will also derive partial differential equations defining the special Lagrangian in two special cases: when the submanifold is the graph of a function and when the submanifold is the level set of functions $f_1, \ldots, f_n$ such that $\nabla f_1, \ldots, \nabla f_n$ are linearly independent. Both of these equations will be very non-linear and hence very difficult to solve in general. In each case we will work out a specific example: in the first case we will work out a specific example of submanifolds that are invariant under the diagonal action of $SO(n)$ and in the second case we will work out a specific example of submanifolds that are invariant under T^{n-1} as a subgroup of $SU(n)$.

We will end the paper by noting a correspondence between complex submanifolds and special Lagrangian submanifolds on $\mathbb{C}^2$ when it is identified with the quaternions $\mathbb{H}$.

Chapter 2

Background

Definition 2.0.1. A *volume form* on a smooth orientable n-manifold M is an n-form σ such that $\sigma(p) \neq 0 \, \forall p \in M$.

The existence of a volume form on a manifold is equivalent to that manifold being orientable [1]. We say that two volume forms σ, ρ are equivalent if $\sigma = f\rho$ for $f \in C^\infty(M)$ with $f(p) > 0 \, \forall p \in M$.

Definition 2.0.2. An *orientation* on an orientable manifold is a choice of equivalence class of volume forms.

If M is orientable and connected there are always two choices of orientation. To see this, let $\sigma = s(x)dx^1 \wedge \cdots \wedge dx^n$ for some $s \in C^\infty(M)$ be a local coordinate representation of a volume form on an orientable n-manifold M. Then $-\sigma$ is also a volume form and it is clearly not in the equivalence class of σ. Let $\rho = r(x)dx^1 \wedge \cdots \wedge dx^n$ for some $r \in C^\infty(M)$ be another volume form on M. Suppose, without loss of generality, that $s(x) > 0$ and let $f(x) = r(x)/s(x)$. If $r(x) > 0$ then $f(x) > 0$ and $\rho = f(x)\sigma$ which implies that $\rho \in [\sigma]$. Otherwise if $r(x) < 0$ then $-f(x) > 0$ and $\rho = (-f(x))(-\sigma)$ and so $\rho \in [-\sigma]$. Since ρ was arbitrary, every volume form is either in $[\sigma]$ or $[-\sigma]$, so there are only two equivalence classes of volume forms and hence two choices of orientation.

On a oriented Riemannian n-manifold (M, g) we can choose a canonical volume form, vol, for an orientation by requiring, for a local orthonormal frame $e_1, \ldots, e_n$ of TM, that $\mathrm{vol}(e_1, \ldots, e_n) = 1$. We will show that in coordinates this form becomes $\mathrm{vol} = \sqrt{\det(g)}dx^1 \wedge \cdots \wedge dx^n$.

Given an oriented local orthonormal frame $e_1, \ldots, e_n$ of TM we have the associated oriented coframe $e^1, \ldots, e^n$. Then we must have $\mathrm{vol} = e^1 \wedge \cdots \wedge e^n$. Let $dx^1, \ldots, dx^n$ be a local coordinate coframe for T^*M and $e^i = A^i_j dx^j$. Then

$$\begin{aligned} \mathrm{vol} &= e^1 \wedge \cdots \wedge e^n \\ &= A^1_{i_1} dx^{i_1} \wedge \cdots \wedge A^n_{i_n} dx^{i_n} \\ &= A^1_{i_1} \cdots A^n_{i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \\ &= \det(A) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Now,

$$\begin{aligned} g &= \delta_{kl} e^k e^l \\ &= \delta_{kl} A^k_i dx^i A^l_j dx^j \\ g_{ij} dx^i dx^j &= \delta_{kl} A^k_i A^l_j dx^i dx^j, \end{aligned}$$

so $g_{ij} = \delta_{kl} A^k_i A^l_j$ or in matrix form $g = AA^t$. This gives us $\det(g) = \det(AA^t)$ or $\det(A) = \sqrt{\det(g)}$. So we have

$$\mathrm{vol} = \sqrt{\det(g)} dx^1 \wedge \cdots \wedge dx^n.$$

It is easy to show that this local coordinate description of the volume form associated to a Riemannian metric g and a choice of orientation is a well-defined global n-form.

Definition 2.0.3. The *volume* of a compact oriented submanifold N of a Riemannian manifold (M, g) is given by

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N$$

where vol_N is the volume form on N associated to the induced Riemannian metric $g|_N$ and the given orientation.

We will also need Stokes' theorem to prove a later result so we will state it now.

Theorem 2.0.4 (Stokes' Theorem)**.** *Let M be a oriented n-manifold with boundary ∂M, and let ω be a (compactly supported) smooth $(n-1)$-form. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

For a proof of Stokes' theorem see [1].

Chapter 3

Calibrations

Definition 3.0.5. A k-form φ on a Riemannian n-manifold (M, g), is called a *calibration* if

1. φ is closed, i.e. $d\varphi = 0$
2. for every k-dimensional orientable submanifold N of M, if vol_N is the induced volume form on N then $\varphi|_N \leq \mathrm{vol}_N$.

Condition (2) in the above definition is equivalent to the condition that for every set of k orthonormal tangent vector fields $X_1, \dots, X_k$ we have $\varphi(X_1, \dots, X_k) \leq 1$. We can define an object called the comass on forms and condition (2) is also called the comass one condition.

Definition 3.0.6. If φ is a calibration on (M, g) then an orientable submanifold N of M is called a *calibrated submanifold* or *φ-submanifold* if $\varphi|_N = \mathrm{vol}_N$.

Likewise, definition 3.0.6 is equivalent to: for any $p \in N$ and any oriented orthonormal basis $X_1, \dots, X_k$ of T_pN we have that $\varphi(X_1, \dots, X_k) = 1$.

Now we will give a very rough idea of what it means for two orientable manifolds to be in the same homology class. It will be enough for our purposes.

Definition 3.0.7. Two orientable manifolds (N_1, vol_1), (N_2, vol_2) with volume forms vol_1 and vol_2 respectively are in the *same homology class* if there exists a manifold with boundary L such that $\partial L = (N_1, \mathrm{vol}_1) \cup (N_2, -\mathrm{vol}_2)$. See figure 3.1.

Theorem 3.0.8 (Fundamental Theorem of Calibrations)**.** *Let (M, g) be a Riemannian manifold, φ a calibration on M, and N an orientable compact φ-submanifold of M. Then N is volume-minimizing in its homology class.*

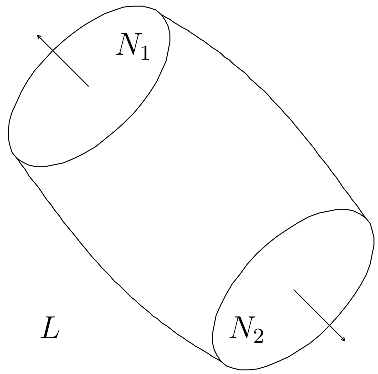

Figure 3.1: The manifold with boundary L such that $\partial L = (N_1, \mathrm{vol}_1) \cup (N_2, -\mathrm{vol}_2)$ with arrows indicating the corresponding orientations.

Proof. Let N' be another compact oriented submanifold of M in the homology class of N. Let vol_N and $\mathrm{vol}_{N'}$ be the volume forms of N and N' respectively. Let L be a submanifold of M such that $\partial L = (N, \mathrm{vol}_N) \cup (N', -\mathrm{vol}_{N'})$. Then

$$\begin{aligned} \int_N \varphi|_N - \int_{N'} \varphi|_{N'} &= \int_{\partial L} \varphi \\ &= \int_L d\varphi \\ &= 0 \end{aligned}$$

and so

$$\int_N \varphi|_N = \int_{N'} \varphi|_{N'}.$$

Now we have

$$\begin{aligned} \mathrm{Vol}(N) &= \int_N \mathrm{vol}_N \\ &= \int_N \varphi|_N && \text{because } N \text{ is calibrated} \\ &= \int_{N'} \varphi|_{N'} \\ &\leq \int_{N'} \mathrm{vol}_{N'} && \text{by condition (2) in definition 3.0.5} \\ &= \mathrm{Vol}(N'). \end{aligned}$$

So $\mathrm{Vol}(N) \leq \mathrm{Vol}(N')$ and so N is volume-minimizing in its homology class. $\square$

The idea for this proof is from a book by Gross, Huybrechts and Joyce, [2].

Chapter 4

Structures on $\mathbb{C}^n$

For the rest of the paper the manifold we are working over will always be $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ or $(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ where $z^j = x^j + iy^j$. The metric is the standard Euclidean metric on $\mathbb{R}^{2n}$ denoted $\langle \cdot, \cdot \rangle$. There is also a Kähler form $\omega = \sum_j (i/2) dz^j \wedge d\bar{z}^j = \sum_j dx^j \wedge dy^j$. The complex structure on $\mathbb{C}^n$ is given by a bundle map $J : T(\mathbb{C}^n) \to T(\mathbb{C}^n)$ defined by $J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}$ and $J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}$.

Lemma 4.0.9. *For all $V, W \in T(\mathbb{C}^n)$, $\langle JV, W \rangle = \omega(V, W)$.*

Proof. Let $V, W \in T(\mathbb{C}^n)$ with $V = V_1^j \frac{\partial}{\partial x^j} + V_2^j \frac{\partial}{\partial y^j}$ and $W = W_1^j \frac{\partial}{\partial x^j} + W_2^j \frac{\partial}{\partial y^j}$. Then

$$\begin{aligned} \langle JV, W \rangle &= \left\langle V_1^j \frac{\partial}{\partial y^j} - V_2^j \frac{\partial}{\partial x^j}, W_1^j \frac{\partial}{\partial x^j} + W_2^j \frac{\partial}{\partial y^j} \right\rangle \\ &= \sum_j \left(V_1^j W_2^j - V_2^j W_1^j \right) \end{aligned}$$

and

$$\begin{aligned} \omega(V, W) &= \omega\left(V_1^j \frac{\partial}{\partial x^j} + V_2^j \frac{\partial}{\partial y^j}, W_1^j \frac{\partial}{\partial x^j} + W_2^j \frac{\partial}{\partial y^j} \right) \\ &= \sum_j \left(V_1^j W_2^j - V_2^j W_1^j \right). \end{aligned}$$

Therefore $\langle JV, W \rangle = \omega(V, W)$. □

Chapter 5

The Kähler calibration

In this chapter we will show that the $2k$-form $\omega^k/k!$ is a calibration and that the submanifolds that it calibrates are the complex submanifolds.

Definition 5.0.10. An oriented $2k$-submanifold N is *complex* if every tangent space is invariant under multiplication by i, i.e. for all $p \in N$ we have $J(T_pN) = T_pN$ and the orientation is given by the ordered local basis $v_1, Jv_1, v_2, Jv_2, \ldots, v_k, Jv_k$ where $v_1, \ldots, v_k$ is a linearly independent set of local vector fields.

Since ω is closed, so is $\omega^k/k!$. To show that $\omega^k/k!$ is a calibration it remains to show that it satisfies the comass one condition.

Theorem 5.0.11 (Wirtinger's Inequality)**.** *Let N be an oriented $2k$ real dimensional submanifold and let $\{X_1, \ldots, X_{2k}\}$ be a local frame for TN. Then*

$$\omega^k(X_1, \ldots, X_{2k}) \leq k!\mathrm{vol}_N(X_1, \ldots, X_{2k})$$

with equality if and only if N is a complex submanifold.

Proof. First note that a change of frame will multiply both sides of the inequality by $\det B$ where B is the change of frame matrix. So we can choose any frame for TN.

Let $p \in N$ be arbitrary.

Suppose $k = 1$. Let $X, Y \in T_pN$ be an orthonormal basis. Then we have $\omega(X, Y) = \langle JX, Y\rangle$ and by the Cauchy-Schwartz inequality $|\langle JX, Y\rangle| \leq \|JX\|\|Y\| = 1$ with equality if and only if JX and Y are linearly dependent. So $\omega(X, Y) \leq 1$ with equality if and only if $JX = Y$ ie N is complex.

Let k be arbitrary. Define a map $A : T_pN \to T_pN$ by $\langle AX, Y\rangle = \omega(X,Y)\ \forall Y \in T_pN$. Note that $AX \perp X$ for all $X \neq 0 \in T_pN$ since $\langle AX, X\rangle = \omega(X,X) = 0$. We have

$$\begin{aligned}\langle AX, AY\rangle &= \omega(X, AY)\\ &= \langle JX, AY\rangle\\ &= \langle AY, JX\rangle\\ &= \omega(Y, JX)\\ &= \langle JY, JX\rangle\\ &= \langle X, Y\rangle,\end{aligned}$$

so A is an orthogonal matrix, i.e. $A^T = A^{-1}$. Also

$$\begin{aligned}\langle Y, A^2X\rangle &= \langle A^2X, Y\rangle\\ &= \omega(AX, Y)\\ &= -\omega(Y, AX)\\ &= -\langle AY, AX\rangle\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}Y^TA^2X &= -(AY)^TAX\\ Y^TA^2X &= -Y^TA^TAX\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}A^2 &= -A^TA\\ &= -A^{-1}A\\ &= -I. \qquad (5.1)\end{aligned}$$

Suppose that for some $X \neq 0 \in T_pN$ we have that $AX = 0$. Then since $0 = \langle AX, Y\rangle = \omega(X,Y) = \langle JX, Y\rangle$ for all $Y \in T_pN$ we have that $JX \perp T_pN$ and so T_pN cannot be complex. Also, if we take any frame for T_pN that includes X, say $\{X, Y_2, \ldots, Y_{2k}\}$, then

$$\omega^k(X, Y_2, \ldots, Y_{2k}) = 0 \leq k!\mathrm{vol}_n(X, Y_2, \ldots, Y_{2k}).$$

So, Wirtinger's inequality is satisfied trivially.

Suppose that $AX \neq 0$ for all $X \neq 0 \in T_pN$. Let X_1 be a unit vector in T_pN. Let $X_2 = AX_1/\|AX_1\|$. Then $AX_1 = a_1X_2$ where $a_1 = \|AX_1\|$ and $X_2 \perp X_1$, $\|X_2\| = 1$.

Choose $X_3 \in T_pN$ so that $X_3 \perp X_1, X_2$ and $\|X_3\| = 1$. Then we can find $a_2 \in \mathbb{R}$ and $X_4 \in T_pN$ so that $AX_3 = a_2X_4$, $X_4 \perp X_3$ and $\|X_4\| = 1$. Continuing this process we get orthonormal vectors $X_1, \ldots, X_{2k}$ and real numbers $a_1, \ldots, a_k$ so that $AX_{2j-1} = a_jX_{2j}$. Now

$$\begin{aligned} & \langle AX_{2j}, X_{2j-1}\rangle \\ = \; & \omega(X_{2j}, X_{2j-1}) \\ = \; & -\omega(X_{2j-1}, X_{2j}) \\ = \; & -\langle AX_{2j-1}, X_{2j}\rangle \\ = \; & -\langle a_jX_{2j}, X_{2j}\rangle \\ = \; & -a_j, \end{aligned}$$

and for $1 \leq l \leq 2k$, $l \neq 2j-1$

$$\begin{aligned} \langle AX_{2j}, X_l\rangle &= \langle A(\frac{1}{a_j}AX_{2j-1}, X_l\rangle \\ &= \frac{1}{a_j}\langle A^2X_{2j-1}, X_l\rangle \\ &= \frac{1}{a_j}\langle -X_{2j-1}, X_l\rangle && \text{by (5.1)} \\ &= \frac{-1}{a_j}\langle X_{2j-1}, X_l\rangle \\ &= 0 && \text{as } l \neq 2j-1. \end{aligned}$$

So $AX_{2j} = -a_jX_{2j-1}$. Then we have

$$\begin{aligned} a_j &= \langle AX_{2j}, X_{2j-1}\rangle \\ &= \omega(X_{2j}, X_{2j-1}) \\ &= \langle JX_{2j}, X_{2j-1}\rangle \\ &\leq \|JX_{2j}\|\|X_{2j-1}\| = 1 \end{aligned}$$

with equality if and only if $JX_{2j} = X_{2j-1}$ by the Cauchy-Schwartz inequality.

Note that

$$\omega(X_i, X_j) = \begin{cases} a_l & i = j-1 \text{ and } 2l = i \text{ or } j \\ 0 & \text{otherwise} \end{cases} . \tag{5.2}$$

So

$$\begin{aligned} \omega^k(X_1, \ldots, X_{2k}) &= (\omega \wedge \cdots \wedge \omega)(X_1, \ldots, X_{2k})| \\ &= \sum_{\sigma \in S_{2k}} \omega(X_{\sigma(1)}, X_{\sigma(2)})| \cdots |\omega(X_{\sigma(2k-1)}, X_{\sigma(2k)}). \end{aligned}$$

We can rewrite this using equation 5.2 to get

$$\begin{aligned}
\omega^k(X_1,\ldots,X_{2k}) &= \sum_{\sigma\in S_k}\omega(X_{\sigma(1)},X_{\sigma(1)+1})|\cdots|\omega(X_{\sigma(k)},X_{\sigma(k)+1}) \\
&= \sum_{\sigma\in S_k}\langle AX_{\sigma(1)},X_{\sigma(1)+1}\rangle\cdots\langle AX_{\sigma(k)},X_{\sigma(k)+1}\rangle \\
&= \sum_{\sigma\in S_k}a_{\sigma(1)}\cdots a_{\sigma(k)} \\
&= k!a_1\cdots a_k \\
&\leq k!
\end{aligned}$$

with equality if and only if $X_{2j-1} = JX_{2j}$, ie N is complex. □

The idea for this proof is from a book by Balmann [3]. So $\omega^k/k!$ is a calibration on $\mathbb{C}^n$ and from Wirtinger's inequality we see that the calibrated submanifolds are exactly the complex submanifolds.

5.1 Cauchy-Riemann Equations in $\mathbb{C}^2$

Let L be a ω-calibrated submanifold of $\mathbb{C}^2$ which is a graph. Then $L = \{(x,y,u(x,y),v(x,y)) : x,y\in\mathbb{R}\}$ for some smooth maps $u,v:\mathbb{R}^2\to\mathbb{R}$. Let $p\in L$. Then $T_pL = \mathrm{span}\{(1,0,u_x,v_x),(0,1,u_y,v_y)\}$.

In this chapter we will derive the Cauchy-Riemann equations using the fact that L is calibrated by ω. Recall that the calibration condition is $\omega = \mathrm{vol}|_L$ and that the canonical volume form in coordinates on L is $\sqrt{\det g}dx\wedge dy$. So we will need to compute ω and g in coordinates on T_pL.

On T_pL we have that

$$\begin{aligned}
\omega &= dx^1\wedge dy^1 + dx^2\wedge dy^2 \\
&= dx\wedge dy + du\wedge dv \\
&= dx\wedge dy + \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\wedge\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= (1+u_xv_y - u_yv_x)dx\wedge dy,
\end{aligned}$$

and

$$g = \begin{bmatrix} 1+u_x^2+v_x^2 & u_xu_y+v_xv_y \\ u_xu_y+v_xv_y & 1+u_y^2+v_y^2 \end{bmatrix}.$$

Now,

$$
\begin{aligned}
& \omega = \sqrt{\det g} dx \wedge dy \\
\Leftrightarrow\quad & (1 + u_x v_y - u_y v_x)^2 = (1 + u_x^2 + v_x^2)(1 + u_y^2 + v_y^2) - (u_x u_y + v_x v_y)^2 \\
\Leftrightarrow\quad & 1 + 2u_x v_y - 2u_y v_x + u_x^2 v_y^2 - 2u_x v_y u_y v_x + u_y^2 v_x^2 \\
& = 1 + u_x^2 + u_y^2 + v_x^2 + v_y^2 + u_x^2 v_y^2 + v_x^2 u_y^2 - 2u_x u_y v_x v_y \\
\Leftrightarrow\quad & u_x^2 - 2u_x v_y + v_y^2 + u_y^2 + 2u_y v_x + v_x^2 = 0 \\
\Leftrightarrow\quad & (u_x - v_y)^2 + (u_y + v_x)^2 = 0,
\end{aligned}
$$

which is equivalent to $u_x = v_y$ and $u_y = -v_x$ which are indeed the Cauchy-Riemann equations.

Chapter 6

Special Lagrangian submanifolds of $\mathbb{C}^n$

Definition 6.0.1. A submanifold L of $\mathbb{C}^{2n}$ is called *Lagrangian* if $\omega|_L \equiv 0$.

The form ω is in fact a symplectic form and Lagrangian submanifolds are important in symplectic geometry [4].

Let α, β be n-forms such that $dz^1 \wedge \cdots \wedge dz^n = \alpha + i\beta$. In this chapter we will show that α is a calibration and that submanifolds calibrated by α are in fact Lagrangian with an extra condition. To do this will will first need to prove Hadamard's inequality.

Lemma 6.0.2 (Hadamard's Inequality)**.** *Let A be an $n \times n$ matrix with column vectors $a_1, \ldots, a_n$. Then*

$$|\det A| \leq \prod_{i=1}^{n} \|a_i\|$$

with equality if and only if the a_i's are orthogonal.

Proof. The inequality in trivial if the columns of A are linearly dependent. Assume the columns of A are linearly independent. Then using Gram-Schmidt find an orthonormal basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that

$$\mathrm{span}_{\mathbb{R}}\{a_1, \ldots, a_i\} = \mathrm{span}_{\mathbb{R}}\{b_1, \ldots, b_i\} \, \forall 1 \leq i \leq n. \tag{6.1}$$

Let $B = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$. Now for any $v \in \mathbb{R}^n$ we have

$$v = \sum_{i=1}^{n} \langle v, b_i \rangle b_i \tag{6.2}$$

by orthonormality, and

$$\|v\|^2 = \sum_{i=1}^{n} \langle v, b_i \rangle^2. \tag{6.3}$$

From equations (6.1) and (6.2) we get

$$a_i = \sum_{j=1}^{i} \langle a_i, b_j \rangle b_j. \tag{6.4}$$

Let $C = [c_{kl}]$ be an upper-triangular $n \times n$ matrix such that

$$\begin{array}{ll} c_{kl} = \langle a_l, b_k \rangle & 1 \leq k \leq l \leq n \\ c_{kl} = 0 & l < k \leq n. \end{array}$$

Then equation (6.4) is equivalent to $A = BC$. So

$$\begin{array}{rcll} (\det A)^2 & = & \det(A^T A) & \\ & = & \det(C^T B^T BC) & \\ & = & \det(C^T C) & \text{by orthogonality of } B \\ & = & (\det C)^2 & \\ & = & \displaystyle\prod_{i=1}^{n} \langle a_i, b_i \rangle^2 & \text{by properties of upper-triangular matrices} \\ & \leq & \displaystyle\prod_{i=1}^{n} \left(\sum_{j=1}^{i} \langle a_i, b_j \rangle^2 \right) & \\ & = & \displaystyle\prod_{i=1}^{n} \|a_i\|^2 & \text{by equations (6.3) and (6.4).} \end{array}$$

Equality occurs if and only if $\langle a_i, b_j \rangle = 0$ for all $j \neq i$, which occurs if and only if the a_i's are orthogonal. Then equation (6.4) becomes $a_i = \langle a_i, b_i \rangle b_i$ and so the a_i's are orthogonal.

If the a_i's are orthogonal then the Gram-Schmidt process will give $a_i = \langle a_i, b_i \rangle b_i$ which will give equality in the above equations. $\square$

Recall that for a matrix A with column vectors $a_1, \ldots, a_n$ that $|\det A| = |a_1 \wedge \cdots \wedge a_n|$.

Proposition 6.0.3. *Let N be an n-dimensional submanifold of $\mathbb{C}^n$ over some open set U and let $\varepsilon_1, \ldots, \varepsilon_n$ be a local orthonormal frame for TN. Then $|\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n| \leq 1$ with equality if and only if N is Lagrangian over U.*

Proof. Lemma 6.0.2 states that $|\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n| \leq |\varepsilon_1| \cdots |\varepsilon_n||J\varepsilon_1| \cdots |J\varepsilon_n| = 1$ with equality if and only if $\varepsilon_1, \ldots, \varepsilon_n, J\varepsilon_1, \ldots, J\varepsilon_n$ are orthogonal. Since as a block matrix J is

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

it is clearly orthogonal and therefore angle-preserving. The vectors $\varepsilon_1, \ldots, \varepsilon_n, J\varepsilon_1, \ldots, J\varepsilon_n$ are orthogonal if and only if $\langle J\varepsilon_i, \varepsilon_k \rangle = 0$. This is true if and only if $\omega(\varepsilon_i, \varepsilon_k) = 0$ by lemma 4.0.9. This gives us $\omega|_N \equiv 0$ which is the definition of Lagrangian. $\square$

For the next lemma we will need to use two different notions of the determinant of a complex matrix. The first is the complex determinant denoted $\det_{\mathbb{C}}$ and it is what we would ordinarily think of as the determinant of a complex matrix. The second is the real determinant denoted $\det_{\mathbb{R}}$, to find this determinant we think of a complex matrix $A = B + iC$ as a block matrix

$$A_{\mathbb{R}} = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

and take its determinant. We then get that $\det_{\mathbb{R}} A = (\det B)^2 + (\det C)^2 = \mathrm{Re}(\det_{\mathbb{C}} A)^2 + \mathrm{Im}(\det_{\mathbb{C}} A)^2$.

Proposition 6.0.4. *For a local orthonormal tangent frame $\varepsilon_1, \ldots, \varepsilon_n$ of an n-dimensional submanifold N of $\mathbb{C}^n$,*

$$\alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 + \beta(\varepsilon_1, \ldots, \varepsilon_n)^2 = |\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n|.$$

Proof. Let $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ be an oriented basis for $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$. Let A be a linear map defined by $A(e_i) = \varepsilon_i$, $A(Je_i) = J\varepsilon_i$. Note that A is $\mathbb{C}$-linear by construction. Now

$$\begin{aligned} & dz^1 \wedge \cdots \wedge dz^n(\varepsilon_1, \cdots, \varepsilon_n) \\ = \;& dz^1 \wedge \cdots \wedge dz^n(Ae_1, \cdots, Ae_n) \\ = \;& \det_{\mathbb{C}} A, \end{aligned}$$

by the definition of $\det_{\mathbb{C}}$. Then

$$\alpha(\varepsilon_1, \ldots, \varepsilon_n) = \mathrm{Re}(\det_{\mathbb{C}} A)$$

$$\beta(\varepsilon_1, \ldots, \varepsilon_n) = \mathrm{Im}(\det_{\mathbb{C}} A)$$

so

$$\begin{aligned}
& \alpha(\varepsilon_1,\dots,\varepsilon_n)^2+\beta(\varepsilon_1,\dots,\varepsilon_n)^2 \\
= \; & \mathrm{Re}(\det_{\mathbb{C}}A)^2+\mathrm{Im}(\det_{\mathbb{C}}A)^2 \\
= \; & \det_{\mathbb{R}}A \\
= \; & |\det_{\mathbb{R}}A| && \text{since } \det_{\mathbb{R}}A \text{ is positive} \\
= \; & |Ae_1\wedge AJe_1\wedge\cdots\wedge Ae_n\wedge AJe_n| \\
= \; & |\varepsilon_1\wedge\cdots\wedge\varepsilon_n\wedge J\varepsilon_1\wedge\cdots\wedge J\varepsilon_n|
\end{aligned}$$

□

Theorem 6.0.5. *The n-form α is a calibration.*

Proof. Let $\varepsilon_1,\dots,\varepsilon_n$ be orthonormal vector fields. By propositions 6.0.3 and 6.0.4 we have

$$\alpha(\varepsilon_1,\dots,\varepsilon_n)^2+\beta(\varepsilon_1,\dots,\varepsilon_n)^2=|\varepsilon_1\wedge\cdots\wedge\varepsilon_n\wedge J\varepsilon_1\wedge\cdots\wedge J\varepsilon_n|\leq 1$$

and so

$$\alpha(\varepsilon_1,\dots,\varepsilon_n)\leq\alpha(\varepsilon_1,\dots,\varepsilon_n)^2+\beta(\varepsilon_1,\dots,\varepsilon_n)^2\leq 1.$$

Also,

$$\begin{aligned}
d\alpha &= d(\mathrm{Re}(dz^1\wedge\cdots\wedge dz^n)) \\
&= \mathrm{Re}(d(dz^1\wedge\cdots\wedge dz^n)) \\
&= 0.
\end{aligned}$$

So α is a calibration. □

Definition 6.0.6. An oriented n-submanifold L is called *special Lagrangian* if it is calibrated by α.

Theorem 6.0.7. *Let L be an oriented n-dimensional submanifold and let $\varepsilon_1,\dots,\varepsilon_n$ be a local orthonormal frame for TL. Then L is special Lagrangian (up to a change of orientation) if and only if*

(1) $\omega|_L\equiv 0$

(2) $\beta|_L\equiv 0.$

Moreover, if A is a complex linear map such that $Ae_j=\varepsilon_j$, then $\beta(\varepsilon_1,\dots,\varepsilon_n)=\mathrm{Im}\det_{\mathbb{C}}A$.

Proof. Suppose L is special Lagrangian, then $\alpha(\varepsilon_1, \ldots, \varepsilon_n) = 1$. Then since

$$1 = \alpha(\varepsilon_1, \ldots, \varepsilon_n) \leq \alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 + \beta(\varepsilon_1, \ldots, \varepsilon_n)^2 \leq 1$$

we have

$$\alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 + \beta(\varepsilon_1, \ldots, \varepsilon_n)^2 = 1$$

but $\alpha(\varepsilon_1, \ldots, \varepsilon_n) = 1$ so

$$\begin{aligned} & 1 + \beta(\varepsilon_1, \ldots, \varepsilon_n)^2 = 1 \\ \Leftrightarrow & \\ & \beta(\varepsilon_1, \ldots, \varepsilon_n) = 0. \end{aligned}$$

Also,

$$1 = \alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 + \beta(\varepsilon_1, \ldots, \varepsilon_n)^2 = |\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n|$$

and on any open neighborhood of L, and by proposition 6.0.3 this equality happens if and only if L is Lagrangian and by the definition of Lagrangian $\omega|_L \equiv 0$.

Conversely, suppose that $\omega|_L \equiv 0$ and $\beta|_L \equiv 0$. Then L is Lagrangian and by proposition 6.0.4

$$\alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 + \beta^2(\varepsilon_1, \ldots, \varepsilon_n) = 1.$$

Since $\beta|_L \equiv 0$ the above equation becomes

$$\begin{aligned} & \alpha(\varepsilon_1, \ldots, \varepsilon_n)^2 = 1 \\ \Leftrightarrow & \\ & \alpha(\varepsilon_1, \ldots, \varepsilon_n) = \pm 1. \end{aligned}$$

So L is calibrated by α up to change of orientation.

Let A be a complex linear map such that $Ae_j = \varepsilon_j$, then $\beta(\varepsilon_1, \ldots, \varepsilon_n) = \operatorname{Im} \det_{\mathbb{C}} A$ as shown in the proof of proposition 6.0.4. □

6.1 Phases of the Special Lagrangian Calibration

We can "rotate" the form α from above and still get a calibration. We do this by taking $\alpha_\theta = \mathrm{Re}(e^{i\theta} dz^1 \wedge \cdots \wedge dz^n)$ for $0 \leq \theta < 2\pi$. Then we have $\alpha = \alpha_0$.

To see that α_θ is a calibration first note that $d\alpha_\theta = d\mathrm{Re}(e^{i\theta} dz^1 \wedge \cdots \wedge dz^n) = 0$. For the comass 1 condition we need to make the following change in the proof of proposition

6.0.4. We let A be a linear map defined by $A(e^{\frac{-i\theta}{n}}e_i) = \varepsilon_i$ and $A(Je^{\frac{-i\theta}{n}}e_i) = J\varepsilon_i$. Then we get

$$\begin{aligned}
& e^{i\theta}dz^1 \wedge \cdots \wedge dz^n(\varepsilon_1, \cdots, \varepsilon_n) \\
= \; & e^{i\theta}dz^1 \wedge \cdots \wedge dz^n(Ae^{\frac{-i\theta}{n}}e_1, \cdots, Ae^{\frac{-i\theta}{n}}e_n) \\
= \; & e^{i\theta}(e^{\frac{-i\theta}{n}})^n dz^1 \wedge \cdots \wedge dz^n(Ae_1, \cdots, Ae_n) \\
= \; & \det_{\mathbb{C}} A,
\end{aligned}$$

and

$$\begin{aligned}
|\det_{\mathbb{R}} A| &= |Ae_1 \wedge AJe_1 \wedge \cdots \wedge Ae_n \wedge AJe_n| \\
&= |e^{\frac{i\theta}{n}}\varepsilon_1 \wedge \cdots \wedge e^{\frac{i\theta}{n}}\varepsilon_n \wedge e^{\frac{i\theta}{n}}J\varepsilon_1 \wedge \cdots \wedge e^{\frac{i\theta}{n}}J\varepsilon_n| \\
&= |(e^{\frac{i\theta}{n}})^n||\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n| \\
&= |\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \wedge J\varepsilon_1 \wedge \cdots \wedge J\varepsilon_n|.
\end{aligned}$$

So proposition 6.0.4 remains true.

Definition 6.1.1. If L is an n-dimensional submanifold that is calibrated by α_θ then we say that L is *special Lagrangian of phase θ.*

However this change will also change the statement of theorem 6.0.7. Let $\beta_\theta = \mathrm{Im}(e^{i\theta}dz^1 \wedge \cdots \wedge dz^n)$.

Theorem 6.1.2. *Let L be an oriented n-dimensional submanifold and let $\varepsilon_1, \dots, \varepsilon_n$ be a local orthonormal frame for TL. Then L is special Lagrangian of phase θ(up to a change of orientation) if and only if*

(1) $\omega|_L \equiv 0$

(2) $\beta_\theta|_L \equiv 0$.

Moreover, if A is a complex linear map such that $Ae^{\frac{-i\theta}{n}}e_j = \varepsilon_j$, then $\beta(\varepsilon_1, \dots, \varepsilon_n) = \mathrm{Im}\det_{\mathbb{C}} A$.

Chapter 7

The Special Lagrangian Differential Equation

Let L be an n-dimensional submanifold of $\mathbb{C}^n$. We may consider L to be given as the graph of a function, i.e. $L = \{(x, f(x)) : x \in U\}$ where $f : U \to \mathbb{R}^n$. We can definitely do this locally.

Proposition 7.0.3. *Suppose $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is a smooth map. Let L be the graph of f in $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. Then L is Lagrangian if and only if the Jacobian of f is symmetric. In particular if U is simply connected, then L is Lagrangian if and only if $f = \nabla F$ for some $F \in C^\infty(U)$.*

Proof. Let $p \in L$ and let U be an open neighborhood of p. Let $\varphi : U \to L$ be defined by $\varphi(x) = (x, f(x)) = x + if(x)$. Then the pushforward φ_* will map $\frac{\partial}{\partial x^j}$ to

$$\varphi_*\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial \varphi}{\partial x^j}\frac{\partial}{\partial x^j} = \sum_{k=1}^{n}\left(\delta_j^k + i\frac{\partial f^k}{\partial x^j}\right)\frac{\partial}{\partial x^j}.$$

So $T_pL = \{v + i\mathrm{Jac}(f)v : v \in \mathbb{R}^n\}$. Let $X, Y \in T_pL$, then $X = v + i\mathrm{Jac}f(p)v$ and

$Y = w + i\mathrm{Jac}f(p)w$ for some v and some w in $\mathbb{R}^n$. Then L is Lagrangian if and only if

$$\begin{aligned}
\omega(X, Y) &= 0 \\
\langle JX, Y\rangle &= 0 \\
\langle -\mathrm{Jac}f(p)v + iv, w + i\mathrm{Jac}f(p)w\rangle &= 0 \\
-\langle \mathrm{Jac}f(p)v, w\rangle + \langle v, \mathrm{Jac}f(p)w\rangle &= 0 \\
\langle v, \mathrm{Jac}f(p)w\rangle &= \langle \mathrm{Jac}f(p)v, w\rangle \\
v^T\mathrm{Jac}f(p)w &= (\mathrm{Jac}f(p)v)^T w \\
v^T\mathrm{Jac}f(p)w &= v^T(\mathrm{Jac}f(p))^T w \\
\mathrm{Jac}f(p) &= (\mathrm{Jac}f(p))^T,
\end{aligned}$$

i.e., $\mathrm{Jac}f(p)$ is symmetric for all $p \in L$ or $\mathrm{Jac}f$ is symmetric. The Poincaré lemma states that on a simply connected domain any smooth closed k-form is exact [1]. Suppose U is simply connected. Now if $\mathrm{Jac}f$ is symmetric then we have that

$$\begin{aligned}
\mathrm{Jac}f &= (\mathrm{Jac}f)^T \\
[\mathrm{Jac}f]_{ij} &= [\mathrm{Jac}f]_{ji} \\
\frac{\partial f_i}{\partial x^j} &= \frac{\partial f_j}{\partial x^i} \qquad (7.1)
\end{aligned}$$

Using the Euclidean metric on $\mathbb{R}^n$ we can regard f as a 1-form on U by taking $f = f_i dx^i$. So we have

$$\begin{aligned}
df &= \sum_{i,j} \frac{\partial f_i}{\partial x^j} dx^i \wedge dx^j \\
&= \sum_{i<j} \left(\frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i}\right) dx^i \wedge dx^j \\
&= 0,
\end{aligned}$$

by equation (7.1). By the Poincaré lemma if U is simply connected then f is exact which is equivalent to the existence of a potential function $F \in C^\infty(U)$ such that $dF = f$. We can then use the metric to get a vector field ∇F from dF to get $\nabla F = f$. $\square$

Definition 7.0.4. Let $U \subseteq \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}$. The *Hessian* of F is an $n \times n$ matrix defined by

$$\mathrm{Hess}F = \left[\frac{\partial^2 F}{\partial x^j \partial x^k}\right].$$

For a matrix A let $\sigma_j(A)$ denote the jth elementary symmetric function of the eigenvalues of A.

Theorem 7.0.5. *Let $U \subseteq \mathbb{R}^n$ be open and $F \in C^\infty(U)$. Let $f = \nabla F$ be the gradient of F and let L be the graph of f in $\mathbb{C}^n$. Then L is special Lagrangian (up to a change of orientation) if and only if*

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \sigma_{2k+1}(\mathrm{Hess}F) = 0, \tag{7.2}$$

or, equivalently

$$\mathrm{Im}\left[\det_{\mathbb{C}}(I + i\mathrm{Hess}F)\right] = 0. \tag{7.3}$$

Proof. By proposition 7.0.3 L is Lagrangian if and only if the matrix $\mathrm{Jac}f (= \mathrm{Hess}F)$ is symmetric. Let $A = I + i\mathrm{Jac}f$; then A is clearly complex linear (it is real linear and preserves multiplication by i) and by definition A sends the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ to the basis of $T_{(x,f(x))}L$. By theorem 6.0.7 it follows that L is special Lagrangian if and only if (7.3) holds. It remains to prove the equivalence of (7.2) and (7.3). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathrm{Hess}F$. Since $\mathrm{Hess}F$ is symmetric there exists a matrix P such that

$$P^{-1}(\mathrm{Hess}F)P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = D,$$

so $\mathrm{Hess}F = PDP^{-1}$. Then

$$\begin{aligned} \mathrm{Im}(\det_{\mathbb{C}}(I + i\mathrm{Hess}F)) &= \mathrm{Im}(\det_{\mathbb{C}}(PP^{-1} + iPDP^{-1})) \\ &= \mathrm{Im}(\det_{\mathbb{C}}(P(I + iD)P^{-1})) \\ &= \mathrm{Im}(\det_{\mathbb{C}}(I + iD)) \\ &= \mathrm{Im}\left(\prod_{j=1}^{n}(1 + i\lambda_j)\right) \\ &= \mathrm{Im}\left(1 + \sum_{j=1}^{n} i^j \sigma_j(\mathrm{Hess}F)\right). \end{aligned}$$

Expanding the first few terms we get

$$\mathrm{Im}\left(1 + \sum_{j=1}^{n} i^j \sigma_j(\mathrm{Hess}F)\right) = \mathrm{Im}\left(1 + i\sigma_1(\mathrm{Hess}F) - \sigma_2(\mathrm{Hess}F) - i\sigma_3(\mathrm{Hess}F) + \cdots\right),$$

so taking the imaginary part results in only odd elementary symmetric functions with alternating sign. Therefore

$$\mathrm{Im}(\det_{\mathbb{C}}(I + i\mathrm{Hess}F)) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \sigma_{2k+1}(\mathrm{Hess}F)$$

as desired. $\square$

For $n = 1, 2, 3$ we get very simple equations from equation 7.3. Let λ_j be the j^{th} eigenvalue of Hess$F$.

Case n=1: We have

$$\mathrm{Hess}F = \left(\frac{d^2F}{dx^2}\right)$$

and $\lambda_1 = \frac{d^2F}{dx^2}$. So

$$\begin{aligned} & \sum_{k=0}^{0} (-1)^k \sigma_{2k+1}(\mathrm{Hess}F) = 0 \\ \Leftrightarrow \quad & \sigma_1(\mathrm{Hess}F) = 0 \\ \Leftrightarrow \quad & \frac{d^2F}{dx^2} = 0 \\ \Leftrightarrow \quad & F(x) = ax + b. \end{aligned}$$

Therefore $f = \nabla F = a$ and $L = \{(x, a)\}$ is a horizontal line.

Case n=2: We have

$$\mathrm{Hess}F = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix}.$$

So

$$\begin{aligned} & \sum_{k=0}^{0} (-1)^k \sigma_{2k+1}(\mathrm{Hess}F) = 0 \\ \Leftrightarrow \quad & \sigma_1(\mathrm{Hess}F) = 0 \\ \Leftrightarrow \quad & \lambda_1 + \lambda_2 = 0 \\ \Leftrightarrow \quad & \mathrm{Tr}(\mathrm{Hess}F) = 0 \\ \Leftrightarrow \quad & F_{xx} + F_{yy} = 0 \\ \Leftrightarrow \quad & \Delta F = 0. \end{aligned}$$

Case n=3: We have

$$\mathrm{Hess}F = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{xz} & F_{yz} & F_{zz} \end{pmatrix}.$$

So

$$
\begin{aligned}
& \sum_{k=0}^{1}(-1)^k \sigma_{2k+1}(\mathrm{Hess}F) = 0 \\
\Leftrightarrow \quad & \sigma_1(\mathrm{Hess}F) - \sigma_3(\mathrm{Hess}F) = 0 \\
\Leftrightarrow \quad & \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1\lambda_2\lambda_3 \\
\Leftrightarrow \quad & \mathrm{Tr}(\mathrm{Hess}F) = \det(\mathrm{Hess}F) \\
\Leftrightarrow \quad & \Delta F = \det(\mathrm{Hess}F).
\end{aligned}
$$

Note that the above is equation is equivalent to

$$
F_{xx} + F_{yy} + F_{zz} = F_{xx}F_{yy}F_{zz} - F_{xx}F_{yz}^2 - F_{yy}F_{xz}^2 - F_{zz}F_{xy}^2 + 2F_{xy}F_{yz}F_{xz}
$$

which a non-linear second order partial differential equation and thus very hard to solve.

7.1 Phases of the Special Lagrangian Differential equation

Let L is an n-dimensional Lagrangian submanifold that is the graph of a function f, so we have $f = \nabla F$. Let $A = I + i\mathrm{Hess}F$ and let $\tilde{A} = e^{\frac{i\theta}{n}}A$. Then by theorem 6.1.2 L is special Lagrangian of phase θ if and only if

$$
\begin{aligned}
\mathrm{Im}(\det_{\mathbb{C}}\tilde{A}) &= 0 \\
\mathrm{Im}(\det_{\mathbb{C}}(e^{\frac{i\theta}{n}}(I + i\mathrm{Hess}F)))) &= 0 \\
\mathrm{Im}(e^{i\theta}\det_{\mathbb{C}}(I + i\mathrm{Hess}F))) &= 0 \\
\mathrm{Im}\left(e^{i\theta}\left(1 + \sum_{j=1}^{n} i^j \sigma_j(\mathrm{Hess}F)\right)\right) &= 0.
\end{aligned}
$$

For phase $\theta = \frac{\pi}{2}$ this becomes

$$
\mathrm{Im}\left(i + \sum_{j=1}^{n} i^{j+1}\sigma_j(\mathrm{Hess}F)\right) = 0, \tag{7.4}
$$

and for $n = 1, 2, 3$ we can explicitly write out this differential equation.

Case $n = 1$: Equation (7.4) becomes

$$
0 = \mathrm{Im}(i + i^2\sigma_1(\mathrm{Hess}F)) = 1.
$$

This is a contradiction, so there are no special Lagrangian submanifolds of phase $\frac{\pi}{2}$ that are graphs of functions. Since the 1-dimensional special Lagrangian submanifolds of phase 0 where horizontal lines and changing phase is a rotation we would expect the 1-dimensional submanifolds of phase $\frac{\pi}{2}$ to be vertical lines which are not graphs of functions.

Case $n = 2$: Equation (7.4) becomes

$$\begin{aligned} 0 &= \mathrm{Im}(i + i^2\sigma_1(\mathrm{Hess}F) + i^3\sigma_2(\mathrm{Hess}F)) \\ &= 1 - \sigma_2(\mathrm{Hess}F) \\ &= 1 - \det(\mathrm{Hess}F), \end{aligned}$$

or $\det(\mathrm{Hess}F) = 1$.

Case $n = 3$: Equation (7.4) becomes

$$\begin{aligned} 0 &= \mathrm{Im}(i + i^2\sigma_1(\mathrm{Hess}F) + i^3\sigma_2(\mathrm{Hess}F) + i^4\sigma_3(\mathrm{Hess}F)) \\ &= 1 - \sigma_2(\mathrm{Hess}F), \end{aligned}$$

or $\sigma_2(\mathrm{Hess}F) = 1$. If we expand this in terms of the partials of F we get

$$F_{xx}F_{yy} + F_{yy}F_{zz} + F_{zz}F_{xx} - F_{xy}^2 - F_{yz}^2 - F_{zx}^2 = 1.$$

7.2 A Specific Example

We look for an L that is invariant under the diagonal action of $SO(n)$. That is L is the orbit of a curve $\alpha : U \subseteq \mathbb{R} \to \mathbb{C}$ with $\mathrm{Re}(\alpha(t)) > 0)$ and $\mathrm{Im}(\alpha(t)) > 0$. Since $SO(n)$ acts transitively on the unit sphere, we will have $L = \{(x, y) \in \mathbb{C}^n : |x|y = |y|x \text{ and } (|x|, |y|) \in \alpha(U)\}$. Let $r = |x|$ and $\rho = |y|$ be a function of r. Choose $P(r)$ to be an antiderivative of $\rho(r)$. Then $\nabla P = \rho(r)\nabla r = \rho(r)(x/r)$ and since $y = |y|x/|x| = \rho(r)x/r$ by definition of L, we see that L is the graph of ∇P and hence by proposition 7.0.3 L is Lagrangian.

Theorem 7.2.1. *Let*

$$L_c = \{(x, y) \in \mathbb{C}^n : ry = \rho x \text{ and } \mathrm{Im}(r + i\rho)^n = c\}$$

for $c \in \mathbb{R}$. Then L_c (up to orientation) is a special Lagrangian submanifold of $\mathbb{C}^n$.

Proof. From the arguments above L_c is Lagrangian and is the graph of $f(x) = \rho(r)(x/r)$. The Jacobian of this map is given by the matrix with entries

$$\begin{aligned} & \frac{\partial}{\partial x^i}\left(\frac{\rho(r)}{r}x^j\right) \\ =\; & \frac{\rho(r)}{r}\delta_{ij} + \frac{d}{dr}\left(\frac{\rho(r)}{r}\right)\frac{x^i x^j}{r}. \end{aligned}$$

Claim 7.2.1.1. The matrix Jac$f(x)$ has x as an eigenvector with eigenvalue $\frac{d\rho}{dr}$. In addition, the hyperplane orthogonal to x is an eigenspace with eigenvalue $\frac{\rho(r)}{r}$ of multiplicity $n-1$.

Proof of claim 7.2.1.1. Let $a = \rho(r)/r$ and $b = \frac{d}{dr}(\rho(r)/r)(1/r)$. Then

$$\begin{aligned}
\mathrm{Jac}f(x)x &= \begin{pmatrix} a+bx^1x^1 & bx^1x^2 & \cdots & bx^1x^n \\ bx^2x^1 & a+bx^2x^2 & \cdots & bx^2x^n \\ \vdots & \vdots & \ddots & \vdots \\ bx^nx^1 & bx^nx^2 & \cdots & a+bx^nx^n \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \\
&= \begin{pmatrix} ax^1 + bx^1(x^1)^2 + bx^1(x^2)^2 + \cdots + bx^1(x^n)^2 \\ ax^2 + bx^2|x|^2 \\ \vdots \\ ax^n + bx^n|x|^2 \end{pmatrix} \\
&= (a+br^2) \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \\
&= \left(\frac{\rho(r)}{r} + r\frac{d}{dr}\left(\frac{\rho(r)}{r}\right)\right) x,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho(r)}{r} + r\frac{d}{dr}\left(\frac{\rho(r)}{r}\right) &= \frac{\rho(r)}{r} + r\left(\frac{\frac{d\rho}{dr}r - \rho}{r^2}\right) \\
&= \frac{\rho(r)}{r} + \frac{d\rho}{dr} - \frac{\rho(r)}{r} \\
&= \frac{d\rho}{dr}.
\end{aligned}$$

So $d\rho/dr$ is an eigenvalue of Jac$f(x)$ with eigenvector x.

Let y be orthogonal to x then

$$\begin{aligned}
\mathrm{Jac}f(x)y &= \begin{pmatrix} a+bx^1x^1 & bx^1x^2 & \cdots & bx^1x^n \\ bx^2x^1 & a+bx^2x^2 & \cdots & bx^2x^n \\ \vdots & \vdots & \ddots & \vdots \\ bx^nx^1 & bx^nx^2 & \cdots & a+bx^nx^n \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix} \\
&= \begin{pmatrix} ay^1 + bx^1(x^1y^1) + bx^1(x^2y^2) + \cdots + bx^1(x^ny^n) \\ ay^2 + bx^2\langle x, y\rangle \\ \vdots \\ ay^n + bx^n\langle x, y\rangle \end{pmatrix} \\
&= \begin{pmatrix} ay^1 \\ ay^2 \\ \vdots \\ ay^n \end{pmatrix} \text{ (since } x \text{ and } y \text{ are orthogonal)} \\
&= \frac{\rho(r)}{r}y.
\end{aligned}$$

So $\rho(r)/r$ is an eigenvalue of $\mathrm{Jac}f(x)$ with eigenspace the hyperplane orthogonal to x. Since the hyperplane orthogonal to x has dimension $n-1$, the associated eigenvalue, $\rho(r)/r$, has multiplicity $n-1$. $\square$

From theorem 7.0.5 we know that L is special Lagrangian if and only if

$$\mathrm{Im}[\det_{\mathbb{C}}(I + i\mathrm{Jac}f)] = 0$$

and since $\mathrm{Jac}f$ is symmetric we can diagonalize it so the eigenvalues of $I + i\mathrm{Jac}f$ are $1 + i\rho(r)/r$ (with multiplicity $n-1$) and $1 + id\rho/dr$. So the above equation becomes

$$\mathrm{Im}\left[\left(1 + i\frac{\rho(r)}{r}\right)^{n-1}\left(1 + i\frac{d\rho}{dr}\right)\right] = 0.$$

Now

$$\mathrm{Im}\left[\det_{\mathbb{C}}(I + i\mathrm{Jac}f)dr\right] = \frac{1}{r^{n-1}}\mathrm{Im}\left[(r + i\rho)^{n-1}(dr + id\rho)\right] = 0$$

is equivalent to

$$\mathrm{Im}\left[(r + i\rho)^{n-1}(dr + id\rho)\right] = 0.$$

This is an exact differential equation with solutions

$$\mathrm{Im}((r + i\rho)^n) = c$$

as desired. $\square$

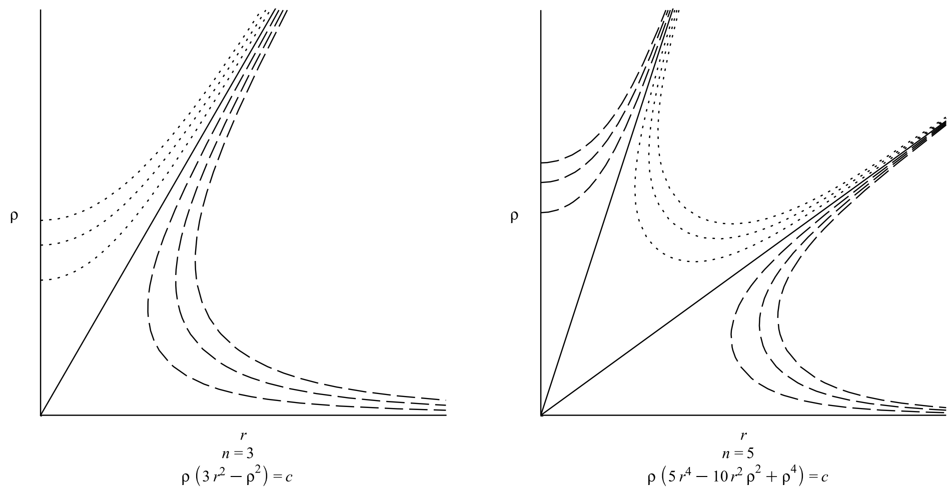

Figure 7.1: The dotted lines are when $c < 0$, the plain lines are when $c = 0$ and the dashed lines are when $c > 0$.

The graphs of ρ versus r for $n = 3$ and $n = 5$ are displayed below (see figure 7.1).

Note that for $n = 3$ the submanifold L_c will always have one component and for $c = 0$ the submanifold will be a hyperplane without the origin. For $n = 5$ the manifold L_c has one component for negative c and two components for positive c but when c is zero L_c is the union of two hyperplanes without the origin which is not a manifold (see figure 7.2).

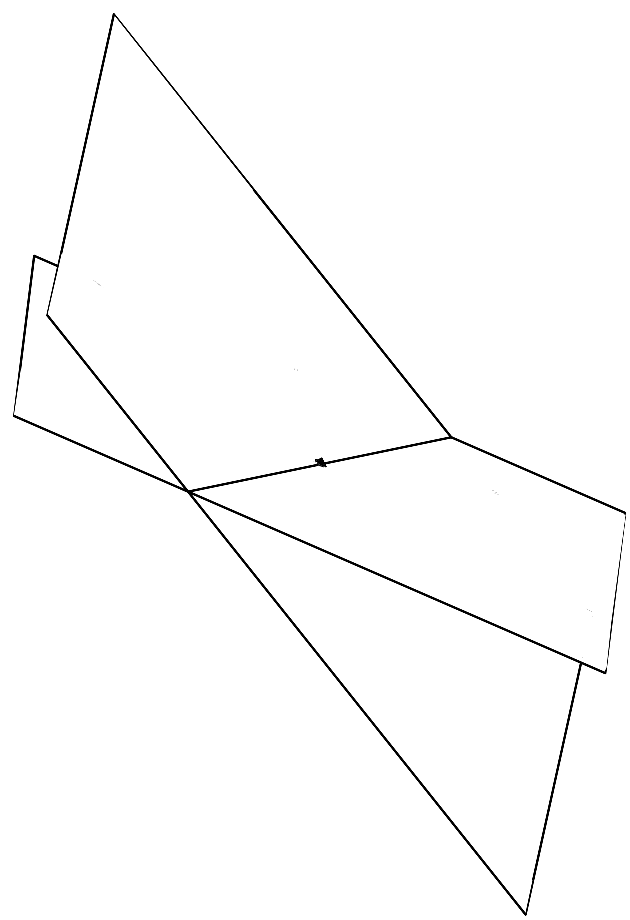

Figure 7.2: L_0 when $n = 5$

Chapter 8

The Implicit Formulation

Lemma 8.0.2. *If L is a real n-dimensional subspace of $\mathbb{C}^n$ then L is Lagrangian if and only if $L^\perp$ is Lagrangian.*

Proof. Let L be an real n-dimensional subspace of $\mathbb{C}^n$. Suppose that L is Lagrangian then $JL \perp L$ and so we have

$$L \oplus JL = \mathbb{C}^n = L \oplus L^\perp.$$

Since JL and $L^\perp$ both have real dimension n and both decompositions are orthogonal, the direct sum must be unique. So,

$$\begin{aligned} & JL = L^\perp \\ \Rightarrow\ & L = JL^\perp \\ \Rightarrow\ & JL^\perp \perp L^\perp \end{aligned}$$

and therefore $L^\perp$ is Lagrangian. The other direction is similar. $\square$

Lemma 8.0.3. *Let $U \subseteq \mathbb{C}^n$ be open and $f_1, \ldots, f_n \in C^\infty(U)$ such that $\nabla f_1, \ldots, \nabla f_n$ are linearly independent on the submanifold $M = \{z \in U : f_1(z) = \cdots = f_n(z) = 0\}$. Then M is Lagrangian if and only if all of the Poisson brackets*

$$\{f_j, f_k\} = \sum_{l=1}^{n} \left(\frac{\partial f_j}{\partial x^l} \frac{\partial f_k}{\partial y^l} - \frac{\partial f_j}{\partial y^l} \frac{\partial f_k}{\partial x^l} \right) = 2i \sum_{l=1}^{n} \left(\frac{\partial f_j}{\partial \bar{z}^l} \frac{\partial f_k}{\partial z^l} - \frac{\partial f_j}{\partial z^l} \frac{\partial f_k}{\partial \bar{z}^l} \right)$$

vanish on M.

Proof. Let $\alpha(t)$ be a curve on M. Then for all $1 \leq j \leq n$

$$\begin{aligned} & f_j(\alpha(t)) = 0 \\ \Rightarrow \quad & \nabla f_j|_{\alpha(t)} \cdot \alpha'(t) = 0 \\ \Rightarrow \quad & \nabla f_j \perp TM. \end{aligned}$$

Since $\nabla f_1, \ldots, \nabla f_n$ are linearly independent, $NM = \mathrm{span}_{\mathbb{R}}\{\nabla f_1, \ldots, \nabla f_n\}$. Note that

$$\begin{aligned} \omega(\nabla f_j, \nabla f_k) &= \left(\sum_{l=1}^n dx^l \wedge dy^l\right)\left(\sum_{s=1}^n \frac{\partial f_j}{\partial x^s}\frac{\partial}{\partial x^s} + \frac{\partial f_j}{\partial y^s}\frac{\partial}{\partial y^s}, \sum_{r=1}^n \frac{\partial f_k}{\partial x^r}\frac{\partial}{\partial x^r} + \frac{\partial f_k}{\partial y^r}\frac{\partial}{\partial y^r}\right) \\ &= \sum_{l=1}^n \left(\frac{\partial f_j}{\partial x^l}\frac{\partial f_k}{\partial y^l} - \frac{\partial f_j}{\partial y^l}\frac{\partial f_k}{\partial x^l}\right) \\ &= \{f_j, f_k\}. \end{aligned}$$

So $\omega|_{NM} \equiv 0 \Leftrightarrow \{f_j, f_k\} = 0$. Therefore NM is Lagrangian if and only if all of the Poisson brackets $\{f_j, f_k\}$ are zero. Since $TM = (NM)^{\perp}$,we see by lemma 8.0.2 that M is Lagrangian if and only if all of the Poisson brackets are zero. $\square$

Theorem 8.0.4. *Suppose that M is the Lagrangian submanifold described in the lemma above. Then M is special Lagrangian if and only if*

$$(1) \quad \mathrm{Im}\left(\det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) = 0 \text{ on } M \text{ for } n \text{ even}$$

or

$$(2) \quad \mathrm{Re}\left(\det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) = 0 \text{ on } M \text{ for } n \text{ odd.}$$

Proof. From the proof of the above lemma, $NM = \mathrm{span}_{\mathbb{R}}\{\nabla f_1, \ldots, \nabla f_n\}$ and since M is Lagrangian $TM = J(NM)$. Therefore

$$\begin{aligned} TM &= \mathrm{span}_{\mathbb{R}}\{J\nabla f_1, \ldots, J\nabla f_n\} \\ &= \mathrm{span}_{\mathbb{R}}\left\{\sum_{j=1}^n \left(-\frac{\partial f_1}{\partial y^j} + i\frac{\partial f_1}{\partial x^j}\right)e_j, \cdots, \sum_{j=1}^n \left(-\frac{\partial f_n}{\partial y^j} + i\frac{\partial f_n}{\partial x^j}\right)e_j\right\} \\ &= \mathrm{span}_{\mathbb{R}}\left\{\sum_{j=1}^n 2i \cdot \frac{1}{2}\left(\frac{\partial f_1}{\partial x^j} + i\frac{\partial f_1}{\partial y_j}\right)e_j, \cdots, \sum_{j=1}^n 2i \cdot \frac{1}{2}\left(\frac{\partial f_n}{\partial x^j} + i\frac{\partial f_n}{\partial y_j}\right)e_j\right\} \\ &= \mathrm{span}_{\mathbb{R}}\left\{\sum_{j=1}^n 2i\frac{\partial f_1}{\partial \bar{z}^j}e_j, \cdots, \sum_{j=1}^n 2i\frac{\partial f_n}{\partial \bar{z}^j}e_j\right\}. \end{aligned}$$

Let $A = [2i\partial f_j/\partial \bar{z}^k]$. Then $Ae_j = \sum_{k=1}^n 2i\partial f_k/\partial \bar{z}^j e_k$. So from theorem 6.0.7 M is special Lagrangian if and only if $\mathrm{Im}(\det_{\mathbb{C}} A) = 0$. Now,

$$\begin{aligned}
\mathrm{Im}(\det_{\mathbb{C}} A) &= \mathrm{Im}\left(\det_{\mathbb{C}}\left[2i\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) \\
&= \mathrm{Im}\left(2^n i^n \det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) \\
&= \begin{cases} \pm\mathrm{Im}\left(2^n \det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) & \text{for } n \text{ even} \\ \pm\mathrm{Re}\left(2^n \det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) & \text{for } n \text{ odd.} \end{cases}
\end{aligned}$$

So

$$\mathrm{Im}(\det_{\mathbb{C}} A) = 0 \Leftrightarrow \begin{cases} \mathrm{Im}\left(\det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) = 0 & \text{for } n \text{ even} \\ \mathrm{Re}\left(\det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right]\right) = 0 & \text{for } n \text{ odd.} \end{cases}$$

as desired. □

8.1 A Specific Example

We will consider a family of submanifolds that are invariant under

$$T^{n-1} = \left\{ \begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix} : \theta_1 + \cdots + \theta_n = 0 \right\} \subseteq SU(n).$$

Let M_c be the set of solutions of the following equations:

$$|z^1|^2 - |z^j|^2 = c_j, \ j = 2, \ldots, n, \tag{8.1}$$

$$\begin{cases} \mathrm{Re}(z^1 \cdots z^n) = c_1 & \text{if } n \text{ is even} \\ \mathrm{Im}(z^1 \cdots z^n) = c_1 & \text{if } n \text{ is odd.} \end{cases} \tag{8.2}$$

Lemma 8.1.1. *Then M_c as defined above is invariant under T^{n-1}.*

Proof. Let $z = (z^1, \cdots, z^n) \in M_c$. Let $A \in T^{n-1}$. Then

$$\begin{aligned}
Az &= \begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix} \begin{bmatrix} z^1 \\ \vdots \\ z^n \end{bmatrix} \\
&= \begin{bmatrix} e^{i\theta_1} z^1 \\ \vdots \\ e^{i\theta_n} z^n \end{bmatrix}.
\end{aligned}$$

Equation (8.1) becomes

$$\begin{aligned}
& |e^{i\theta_1}z^1|^2 - |e^{i\theta_j}z^j|^2 \\
= \;& |e^{i\theta_1}|^2|z^1|^2 - |e^{i\theta_j}|^2|z^j|^2 \\
= \;& |z^1|^2 - |z^j|^2 \\
= \;& c_j.
\end{aligned}$$

Equation (8.2) becomes

$$\begin{aligned}
& \begin{cases} \mathrm{Re}(e^{i\theta_1}z^1 \cdots e^{i\theta_n}z^n) = c_1 & \text{if } n \text{ is even} \\ \mathrm{Im}(e^{i\theta_1}z^1 \cdots e^{i\theta_n}z^n) = c_1 & \text{if } n \text{ is odd} \end{cases} \\
= \;& \begin{cases} \mathrm{Re}(e^{i(\theta_1+\cdots+\theta_n)}z^1 \cdots z^n) = c_1 & \text{if } n \text{ is even} \\ \mathrm{Im}(e^{i(\theta_1+\cdots+\theta_n)}z^1 \cdots z^n) = c_1 & \text{if } n \text{ is odd} \end{cases} \\
= \;& \begin{cases} \mathrm{Re}(z^1 \cdots z^n) = c_1 & \text{if } n \text{ is even} \\ \mathrm{Im}(z^1 \cdots z^n) = c_1 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

So M_c is invariant under T^{n-1}. □

Lemma 8.1.2. *Let* $f_j(z) = |z^1|^2 - |z_j|^2 - c_j$ *for* $j = 2, \ldots, n$ *and let*

$$f_1(z) = \begin{cases} \mathrm{Re}(z^1 \cdots z^n) - c_1 & \text{if } n \text{ is even} \\ \mathrm{Im}(z^1 \cdots z^n) - c_1 & \text{if } n \text{ is odd} \end{cases}.$$

Then the Poisson bracket of f_i *and* f_j *for any* $1 \leq i < j \leq n$ *is zero.*

Proof. For $j = 2, \ldots, n$ we have

$$\begin{aligned}
\frac{\partial f_j}{\partial z^k} &= \frac{\partial}{\partial z^k}(z^1\bar{z}^1 - z^j\bar{z}^j - c_j) \\
&= \bar{z}^1\delta_k^1 - \bar{z}^j\delta_k^j
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f_j}{\partial \bar{z}^k} &= \frac{\partial}{\partial \bar{z}^k}(z^1\bar{z}^1 - z^j\bar{z}^j - c_j) \\
&= z^1\delta_k^1 - z^j\delta_k^j.
\end{aligned}$$

We also have

$$
\begin{aligned}
\frac{\partial f_1}{\partial z^k} &= \begin{cases} \frac{\partial \mathrm{Re}(z^1\cdots z^n)}{\partial z^k} & \text{for } n \text{ even} \\ \frac{\partial \mathrm{Im}(z^1\cdots z^n)}{\partial z^k} & \text{for } n \text{ odd} \end{cases} \\
&= \begin{cases} \frac{\partial}{\partial z^k}\left(\frac{z^1\cdots z^n+\bar{z}^1\cdots\bar{z}^n}{2}\right) & \text{for } n \text{ even} \\ \frac{\partial}{\partial z^k}\left(\frac{z^1\cdots z^n-\bar{z}^1\cdots\bar{z}^n}{2i}\right) & \text{for } n \text{ odd} \end{cases} \\
&= \begin{cases} \frac{z^1\cdots z^n}{2z^k} & \text{for } n \text{ even} \\ \frac{-iz^1\cdots z^n}{2z^k} & \text{for } n \text{ odd.} \end{cases}
\end{aligned}
$$

Similarly we get

$$
\frac{\partial f_1}{\partial \bar{z}^k} = \begin{cases} \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^k} & \text{for } n \text{ even} \\ \\ \frac{i\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^k} & \text{for } n \text{ odd.} \end{cases}
$$

Then for $j = 1, \ldots, n$ and n even

$$
\begin{aligned}
\{f_1, f_j\} &= 2i\sum_{k=1}^{n}\left(\frac{\partial f_1}{\partial \bar{z}^k}\frac{\partial f_j}{\partial z^k} - \frac{\partial f_1}{\partial z^k}\frac{\partial f_j}{\partial \bar{z}^k}\right) \\
&= 2i\sum_{k=1}^{n}\left(\frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^k}(\bar{z}^1\delta_k^1 - \bar{z}^j\delta_k^j) - \frac{z^1\cdots z^n}{2z^k}(z^1\delta_k^1 - z^k\delta_k^j)\right) \\
&= 2i\left(\frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^1}\bar{z}^1 - \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^j}\bar{z}^j - \frac{z^1\cdots z^n}{2z^1}z^1 + \frac{z^1\cdots z^n}{2z^j}z^j\right) \\
&= 0.
\end{aligned}
$$

The calculation for n odd is similar. For $2 \leq j < l \leq n$ we have

$$
\begin{aligned}
\{f_j, f_l\} &= 2i\sum_{k=1}^{n}\left(\frac{\partial f_j}{\partial \bar{z}^k}\frac{\partial f_l}{\partial z^k} - \frac{\partial f_j}{\partial z^k}\frac{\partial f_l}{\partial \bar{z}^k}\right) \\
&= 2i\sum_{k=1}^{n}\left((z^1\delta_k^1 - z^j\delta_k^j)(\bar{z}^1\delta_k^1 - \bar{z}^l\delta_k^l) - (\bar{z}^1\delta_k^1 - \bar{z}^j\delta_k^j)(z^1\delta_k^1 - z^l\delta_k^l)\right) \\
&= 2i(z^1\bar{z}^1 - \bar{z}^1 z^1 + z^j\bar{z}^l\delta^{jl} - \bar{z}^j z^l\delta^{jl}) \\
&= 0.
\end{aligned}
$$

$\square$

Theorem 8.1.3. *Let M_c be as above. Then M_c, with the correct orientation, is a special Lagrangian submanifold of $\mathbb{C}^n$.*

Proof. By lemma 8.0.3 and lemma 8.1.2, M_c is Lagrangian.

If n is even then,

$$\left[\frac{\partial f_j}{\partial \bar{z}^k}\right] = \begin{bmatrix} \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^1} & \cdots & \cdots & \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^n} \\ \\ z^1 & -z^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ z^1 & 0 & \cdots & -z^n \end{bmatrix}.$$

Expanding $\det_{\mathbb{C}}[\partial f_j/\partial \bar{z}^k]$ around the first row gives

$$\begin{aligned} \det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right] &= \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^1}(-1)^{n-1}z^2\cdots z^n - \frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^2}(-1)^{n-2}\frac{z^1\cdots z^n}{z^2} \\ &\quad + \cdots + (-1)^{n-1}\frac{\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^n}(-1)^{n-2}\frac{z^1\cdots z^n}{z^n} \\ &= -\frac{|z^1\cdots z^n|^2}{2|z^1|^2} + \sum_{j=2}^{n}(-1)^{j-1}\frac{|z^1\cdots z^n|^2}{2|z^j|^2}. \end{aligned}$$

So $\operatorname{Im}(\det_{\mathbb{C}}[\partial f_j/\partial \bar{z}^k]) = 0$ as desired and M_c is special Lagrangian by theorem 8.0.4.

If n is odd then,

$$\left[\frac{\partial f_j}{\partial \bar{z}^k}\right] = \begin{bmatrix} \frac{i\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^1} & \cdots & \cdots & \frac{i\bar{z}^1\cdots\bar{z}^n}{2\bar{z}^n} \\ \\ z^1 & -z^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ z^1 & 0 & \cdots & -z^n \end{bmatrix}.$$

Similarly to n even, expanding $\det_{\mathbb{C}}[\partial f_j/\partial \bar{z}^k]$ around the first row gives

$$\det_{\mathbb{C}}\left[\frac{\partial f_j}{\partial \bar{z}^k}\right] = -\frac{i|z^1\cdots z^n|^2}{2|z^1|^2} + \sum_{j=2}^{n}(-1)^{j-1}\frac{i|z^1\cdots z^n|^2}{2|z^j|^2}.$$

So $\operatorname{Re}(\det_{\mathbb{C}}[\partial f_j/\partial \bar{z}^k]) = 0$ as desired and M_c is special Lagrangian by theorem 8.0.4. $\square$

Chapter 9

Calibrated Submanifolds of $\mathbb{H}$

In this chapter we will consider calibrated submanifolds of $\mathbb{H} = \mathbb{C}^2$. On $\mathbb{C}^2$ the Kähler form ω and the special Lagrangian calibration α are both 2-forms and so we want to establish a relationship between complex submanifolds and special Lagrangian ones.

Now, we can identify $\mathbb{H}$ with $\mathbb{C}^2$ in three different ways by choosing the complex structure of $\mathbb{C}$ to be i, j or k. For a point $p \in \mathbb{H}$ we have

$$\begin{aligned} p &= x^0 1 + x^1 i + x^2 j + x^3 k \\ &= (x^0 + ix^1)1 + (x^2 + ix^3)j \\ &= (x^0 + jx^2)1 + (x^3 + jx^1)k \\ &= (x^0 + kx^3)1 + (x^1 + kx^2)i. \end{aligned}$$

Each complex structure gives rise to different complex coordinates: $\{z^1 = x^0 + ix^1, z^2 = x^2 + ix^3\}$, $\{w^1 = x^0 + jx^2, w^2 = x^3 + jx^1\}$ or $\{u^1 = x^0 + kx^3, u^2 = x^1 + kx^2\}$. The corresponding Kähler forms are:

$$\omega_1 = dx^0 \wedge dx^1 + dx^2 \wedge dx^3,$$

$$\omega_2 = dx^0 \wedge dx^2 + dx^3 \wedge dx^1,$$

$$\omega_3 = dx^0 \wedge dx^3 + dx^1 \wedge dx^2.$$

The volume forms become

$$\begin{aligned} \Omega_1 &= dz^1 \wedge dz^2 \\ &= (dx^0 + idx^1) \wedge (dx^2 + idx^3) \\ &= dx^0 \wedge dx^2 + dx^3 \wedge dx^1 + i(dx^0 \wedge dx^3 + dx^1 \wedge dx^2) \\ &= \omega_2 + i\omega_3, \end{aligned}$$

$$
\begin{aligned}
\Omega_2 &= dw^1 \wedge dw^2 \\
&= (dx^0 + j dx^2) \wedge (dx^3 + j dx^1) \\
&= dx^0 \wedge dx^3 + dx^1 \wedge dx^2 + j(dx^0 \wedge dx^1 + dx^2 \wedge dx^3) \\
&= \omega_3 + j\omega_1,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_3 &= du^1 \wedge du^2 \\
&= (dx^0 + k dx^3) \wedge (dx^1 + k dx^2) \\
&= dx^0 \wedge dx^1 + dx^2 \wedge dx^3 + k(dx^0 \wedge dx^2 + dx^3 \wedge dx^1) \\
&= \omega_1 + k\omega_2.
\end{aligned}
$$

Then the special Lagrangian calibration $\alpha_l = \mathrm{Re}(\Omega_l)$ with respect one set of complex coordinates is the Kähler calibration with respect to another set of complex coordinates. This is very nice since we have seen that special Lagrangian submanifolds are hard to find in general, but complex submanifolds are in fact easy to find. For example, level sets of holomorphic functions will be complex submanifolds when they are smooth.

To summarize, we have:

complex structure	Kähler form	volume form	special Lagrangian calibration	corresponding form β_l
i	ω_1	$\omega_2 + i\omega_3$	$\alpha_1 = \omega_2$	$\beta_1 = \omega_3$
j	ω_2	$\omega_3 + j\omega_1$	$\alpha_2 = \omega_3$	$\beta_2 = \omega_1$
k	ω_3	$\omega_1 + k\omega_2$	$\alpha_3 = \omega_1$	$\beta_3 = \omega_2$

Chapter 10

Conclusion

In the paper we have looked at two special calibrations, the Kähler calibration and the special Lagrangian calibration, on $\mathbb{C}^n$ and briefly demonstrated the connection between complex and special Lagrangian submanifolds on $\mathbb{H}$. There is a lot more that one can do with calibrations in general and even with just the Kähler and special Lagrangian calibrations. There are more complicated relations between complex submanifolds and special Lagrangian submanifolds on $\mathbb{H}^n$ for $n > 1$ as well. It is possible to define the Kähler and the special Lagrangian calibrations on more general manifolds. For the Kähler calibration to be defined we need the manifold to be Kähler and then the calibration will be $\omega^k/k!$ where ω in the Kähler form. For the special Lagrangian calibration we need the manifold to be Calabi-Yau. We can also define other types of calibrations, for example associative and coassociative calibrations on $\mathbb{R}^7$ or more generally G_2-manifolds. See the paper by Harvey and Lawson [5] for this and other examples.

We can even define other kinds of calibration by replacing the comass 1 condition with something different. For example we can use calibrations to find connections whose curvature forms minimize the Yang-Mills functional. See the paper by Conan Leung for more details [6].

Bibliography

[1] John M Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2003.

[2] Gross, M., Huybrechts, D., Joyce, D., *Calabi-Yau manifolds and related geometries.*, Springer-Verlag, Berlin, 2003.

[3] Werner Ballmann, *Lectures on Kähler Manifolds*, ESI Lectures in Mathematics and Physics, 2006.

[4] McDuff, Dusa, Salamon, Dietmar, *Introduction to symplectic topology*, Oxford: Clarendon Press, 1995.

[5] Reese Harvey, H. Blaine Lawson, Jr. *Calibrated Geometries*, Acta Math. 148 (1982), 47157.

[6] Naichung Conan Leung. *Riemannian Geometry over different Normed Division Algebras*, J. Differential Geom. 61 (2002), no. 2, 289-333.