Milnor's Exotic Spheres

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this paper. This is a true copy of the paper, including any required final revisions, as accepted by my examiners.

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Abstract

In 1956, John Milnor surprised the mathematical community by exhibiting examples of smooth manifolds that were homeomorphic to the 7-sphere but not diffeomorphic to it with its standard smooth structure; this was the first example of so-called “exotic” manifolds. This paper concerns itself with John Milnor’s exotic spheres. After establishing some familiar terminology and notation, we will use Morse theoretical methods to provide a means of determining whether a given manifold is homeomorphic to the n-sphere. We shall then use tools from the theory of characteristic classes to define a quantity (Milnor’s invariant) that distinguishes smooth structures on manifolds. We will give Milnor’s original construction of his exotic spheres and show that they are all homeomorphic to the 7-sphere but that they are not all diffeomorphic to the 7-sphere with its standard smooth structure by means of computing Milnor’s invariant for these spaces.

This paper assumes familiarity with elementary smooth manifold theory and Riemannian geometry, including differential forms and integration thereof, familiarity with vector bundles, elements of algebraic topology and quaternion arithmetic. Facts pertaining to these topics are freely used throughout, though many definitions are repeated to establish terminology and notation.

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Contents

1 Prologue: Smooth Structures

In this section we establish some definitions and give examples illustrating the concepts with which we will be working. Recall that an **n-dimensional topological manifold** M is a second-countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$; that is to say, every point $p \in M$ admits a neighbourhood U and a homeomorphism $\phi : U \longrightarrow \phi(U) \subset \mathbb{R}^n$. The pair (U, ϕ) is called a **chart** for M, and a collection of charts whose domains cover M is called an **atlas** for M.

Let (U, ϕ), (V, ψ) be two charts for M with $U \cap V \neq \emptyset$. The transition function $\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$ is then a well-defined homeomorphism. We say the charts are smoothly compatible if the corresponding transition function is a diffeomorphism; that is, a C^∞ function with C^∞ inverse (charts with non-intersecting domains are vacuously smoothly compatible). Then a smooth atlas for M is an atlas all of whose charts are smoothly compatible. Note that if we have a smooth atlas and we add any chart smoothly compatible with every chart in A, we obtain another smooth atlas, so we may speak of a maximal smooth atlas for M, defined by the property that any chart smoothly compatible with every chart in A is already in A. Such an atlas will be called a **smooth structure** on M. Finally, a **smooth manifold** will be a topological manifold with a given smooth structure. It's clear that every smooth atlas is contained in a unique maximal smooth atlas, for if a given atlas A were contained in two maximal atlases, their union would again be a smooth atlas containing A and thus both would be contained in the same maximal smooth atlas containg A. Thus prescribing one smooth atlas for M is enough to determine the smooth structure on M, defined to be the unique maximal smooth atlas containing the prescribed atlas.

Given any mathematical object, one is interested in determining equivalences between those objects; in the category of smooth manifolds and smooth maps between them, the appropriate notion of equivalence is given by diffeomorphism. Let M, N be smooth manifolds, and let $F : M \longrightarrow N$ be a map.

Let (U, ϕ), (V, ψ) be charts on M, N containing p, $F(p)$, respectively and such that $F(U) \subset V$. Then the coordinate representation of F with respect to these charts is given by $\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \longrightarrow \psi(F(U))$. We say F is **smooth** (respectively, a **diffeomorphism**) if its coordinate representation with respect to any appropriate pair of charts is smooth (respectively, a diffeomorphism). Then two smooth manifolds are diffeomorphic if there exists a diffeomorphism between them. It is a matter of course that diffeomorphism is indeed an equivalence relation.

Let's illustrate the preceding notions with two examples; the second will be important in everything that follows.

Example 1.1. (Two smooth structures on the real line.) Consider the smooth atlas for $\mathbb{R}$ consisting of the single chart $(\mathbb{R}, \mathrm{id})$; this atlas determines the standard smooth structure on the real line. Alternatively, we may consider the atlas consisting of the chart $(\mathbb{R}, \psi)$ where $\psi : \mathbb{R} \longrightarrow \mathbb{R}, x \mapsto x^3$. Note that this smooth structure is not smoothly compatible with the standard smooth structure, as the transition function $\mathrm{id} \circ \psi^{-1}(y) = y^{1/3}$ fails to be smooth at the origin. Thus the two atlases allow us to consider $\mathbb{R}$ as a smooth manifold in two distinct ways.

However, this distinction is superficial, for the map $\phi : \mathbb{R} \longrightarrow \mathbb{R}, x \mapsto x^{1/3}$ is a diffeomorphism of these two real lines. Indeed, its coordinate representation $\mathrm{id} \circ \phi \circ \psi^{-1} = \mathrm{id}$ is a diffeomorphism.

In fact, the above case is typical: [1]

Theorem 1.2. *(**Uncountably many smooth structures**.) Let M be a topological manifold. If M admits one smooth structure, then it has uncountably many distinct smooth structures.*

We will now exhibit the sphere as a smooth manifold by prescribing two atlases. These atlases will turn out to be smoothly compatible; the corresponding maximal smooth atlas will be called the standard smooth structure on the sphere.

Example 1.3. (Standard smooth structure on S^n).

Consider S^n as the set of unit vectors in $\mathbb{R}^{n+1}$. Let

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \in S^n \mid x^i > 0\},$$
$$U_i^- = \{(x^1, \ldots, x^{n+1}) \in S^n \mid x^i < 0\},$$

and define functions

$$\phi_i^\pm : U_i^\pm \longrightarrow \phi_i^\pm(U_i^\pm)$$
$$(x^1, ..., x^{n+1}) \mapsto (x^1, \ldots, \hat{x^i}, \ldots, x^{n+1})$$

where a hat denotes omission. Computing the transition functions

$$\phi_i^\pm \circ (\phi_j^\pm)^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, \hat{u^i}, \ldots, \pm\sqrt{1 - ||u||^2}, \ldots, u^n),\ i < j$$
$$\phi_i^\pm \circ (\phi_j^\pm)^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, \pm\sqrt{1 - ||u||^2}, \ldots, \hat{u^i}, \ldots, u^n),\ i > j$$
$$\phi_i^\pm \circ (\phi_i^\pm)^{-1} = \mathrm{id}$$

we see that the $(U_i^\pm, \phi_i^\pm)_i$ constitute a smooth atlas for S^n. The corresponding smooth structure will be called the **standard smooth structure** on S^n.

Later on we will use a more convenient smooth atlas for S^n, that determined by so-called "stereographic projection". Let N denote the "north pole" of S^n i.e. $N = (0, \ldots, 0, 1)$ and let S denote the "south pole", $S = (0, 0, \ldots, -1)$. Define **stereographic projection from the North pole** by

$$\sigma_N : S^n - \{N\} \longrightarrow \mathbb{R}^n$$
$$\sigma(x^1, ..., x^{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}}$$

and **stereographic projection from the South pole** by $\sigma_S(x) = -\sigma(-x)$.

Then

$$\sigma_N^{-1}(u^1, ..., u^n) = \frac{(2u^1, ..., 2u^n, ..., ||u||^2 - 1)}{||u||^2 + 1}$$

and the transition function takes the particularly simple form $\sigma_S \circ \sigma_N^{-1}(u) = u/||u||^2$. Geometrically, the point $\sigma_N(x)$ is the point at which the line from the North pole through x intersects the $x^{n+1} = 0$ subspace. One verifies tediously but straightforwardly that this atlas is compatible with the standard atlas and thus determines the same smooth structure on S^n.

As above, we may get uncountably many distinct smooth structures on the sphere. In light of this, the following results are perhaps surprising [2]:

Theorem 1.4. *Any 1-manifold is diffeomorphic to either* $\mathbb{R}$ *or* S^1 *with their standard smooth structures.*

Theorem 1.5. *For* $n = 2, 3$*, any smooth* n*-manifold has a unique smooth structure, up to diffeomorphism.*

Thus, while we may exhibit uncountably many incompatible smooth structures on S^2, the corresponding manifolds are all diffeomorphic to the standard sphere.

In higher dimensions, things are rather different. This paper concerns itself with John Milnor's discovery of "exotic" spheres; topological seven-spheres not diffeomorphic to S^7 with its standard smooth structure [3]. His discovery was the first example of such "exotic" smooth structures. Much later, Taubes showed that there are uncountably many "exotic" $\mathbb{R}^4$'s, that is, uncountably many $\mathbb{R}^4$'s not diffeomorphic to $\mathbb{R}^4$ with its standard smooth structure, while for $n \neq 4$, it is known that there is only one smooth structure on $\mathbb{R}^n$, up to diffeomorphism. It is not known if there are any exotic S^4's. [2]

We wish to understand Milnor's exotic spheres. To this end, we proceed as follows. First we obtain a convenient characterization of topological spheres using Morse theory, and then define Milnor's space and show it is indeed homeomorphic to S^7. After introducing some heavy machinery in the the theory of characteristic classes, we will define an invariant that "sees" smooth structure, and determine a sufficient condition for a manifold to not be diffeomorphic to the standard 7-sphere. Finally, we compute the invariant for Milnor's space, thereby determining that it is indeed an exotic sphere.

2 Morse Theory and Topological Spheres

2.1 Morse Functions

Morse theory provides a way of gleaning topological data from the properties of certain functions defined on a manifold. Let us first establish some definitions. Let M be a smooth manifold and let $f \in C^\infty(M)$ be a smooth function. A point $p \in M$ is called a **critical point** of f if the differential of f evaluated at p is the zero map i.e. $df(p) = 0$. We will define a symmetric bilinear form on the tangent space to a critical point p, the so-called **Hessian of f at p**; then a critical point p will be called **non-degenerate** if the Hessian of f at p is non-singular, which amounts to invertibility of the matrix of second-order partial derivatives $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$.

Proposition 2.1. *Non-degeneracy of critical points is a coordinate independent phenomenon.*

Proof. The Hessian of f at p may be defined as a bilinear function on T_pM as follows. Let v, w be tangent vectors at p; then we may always locally extend v, w to vector fields $\tilde{v}$, $\tilde{w}$ on some neighbourhood of p. Then the Hessian f_{**} at p is given by $f_{**}(v, w) = \tilde{v}(p)(\tilde{w}(f))$. Note that $\tilde{v}(p)(\tilde{w}f - \tilde{w}(p)(\tilde{v}(f)) = [\tilde{v}, \tilde{w}]_p f = 0$, where the Lie bracket at p acting on f vanishes by virtue of p being a critical point. In particular, this shows that f_{**} is symmetric and defined independently of the extensions $\tilde{v}$, $\tilde{w}$.

Finally, let $\{x^i\}$ be local coordinates, and write $v = a^i \frac{\partial}{\partial x^i} \mid_p$, $w = b^i \frac{\partial}{\partial x^i} \mid_p$ for some constants $\{a^i\}, \{b^i\}$. Then $\tilde{w} = b^i \frac{\partial}{\partial x^i}$ is an appropriate extension of w, and we compute

$$f_{**}(v, w) = v(\tilde{w}f) = a^i b^j \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right). \tag{2.1}$$

That is, the matrix of second-order partial derivatives represents f_{**} with respect to our chosen basis. In particular, a change of coordinates amounts to conjugation of this matrix by invertible matrices, and so non-degeneracy of the critical point p is indeed a coordinate independent fact. $\square$

Now the **index** of the bilinear form f_{**} at p is defined to be the maximal dimension of a subspace of T_pM on which f_{**} is negative definite i.e. those tangent vectors v, w such that $f_{**}(v,w) \leq 0$, with equality iff one of v or w is zero. In what follows, by the **index of** f at a non-degenerate critical point p we will mean the index of f_{**} on T_pM. Then a **Morse function** is a smooth function with only non-degenerate critical points. Perhaps surprisingly, a Morse function is completely determined in a neighbourhood of its critical points by its index at that point.

Lemma 2.2. (Lemma of Morse) Let p be a non-degenerate critical point of f. Then there is a local coordinate system $\{y^i\}$ in a neighbourhood U of p with $y^i(p) = 0$ and such that

$$f(y^1, \ldots, y^n) = f(p) - (y^1)^2 - \ldots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \ldots + (y^n)^2$$

on U, with λ the index of f at p.

Proof. Let us first show that if such coordinates exist, then λ is necessarily the index of f at p; this is the easy part. Suppose such coordinates exist; then

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right) = \begin{cases} -2 & \text{if } i = j \leq \lambda \\ +2 & \text{if } i = j > \lambda \\ 0 & \text{otherwise} \end{cases}$$

In particular, we see that there exists a subspace of dimension λ on which f_{**} is negative definite, and a subspace V of dimension $n - \lambda$ on which f_{**} is positive definite. If λ were not the index of f_{**}, then the maximal subspace on which f_{**} is negative definite would intersect V, giving a contradiction.

To show that such coordinates do exist, we will need the following lemma:

Lemma 2.3. Let f be a smooth function in a convex neighbourhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then $f(x^1, \ldots, x^n) = \sum x^i g_i(x^1, ..., x^n)$ for suitable smooth functions g_i defined on V, with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$.

Proof. let $y^i(t) = tx^i$. Then by convexity of our domain and the fundamental theorem of calculus, we may write

$$f(x^1, ..., x^n) = \int_0^1 \frac{df(tx^i, ..., tx^n)}{dt}\, dt = \sum \int_0^1 \frac{\partial f}{\partial y^i}(y^1(t), ..., y^n(t))x^i\, dt$$

Putting $g_i(x^1, \ldots, x^n) = \int \frac{\partial f}{\partial y^i}(y^1(t), ..., y^n(t))dt$ gives the desired expression for f. □

Returning to the proof of the Morse lemma, let us work in an arbitrary coordinate chart centred at p, so that $f(p) = f(0)$. Then by the above result, we may write $f(x^1, \ldots, x^n) = \sum x^i g_i(x^1, ..., x^n)$, with $g_j(0) = \frac{\partial f}{\partial x^j}(0) = 0$, as p is a critical point. Applying the above lemma again to the g_j's, we find $f(x^1, \ldots, x^n) = \sum x^i x^j h_{ij}(x^1, ..., x^n)$. By symmetrizing, we can assume $h_{ij} = h_{ji}$. Thus finding suitable coordinates amounts of diagonalizing h_{ij} (or the symmetric bilinear form associated with it) in a neighbourhood of p. That this can be done follows from the standard diagonalization procedure together with the inverse function theorem [4]. □

2.2 Topology via Morse theory

To see how Morse functions may yield some topological data about the manifolds on which they're defined, we will prove the following:

Theorem 2.4. ***(Reeb)*** *If M is a compact manifold and f is a Morse function on M with only two critical points, then M is homeomorphic to the sphere.*

We need another fact before getting to the proof of Reeb's charming result. Consider the following:

Theorem 2.5. *Let f be a smooth function on a manifold M. Put $M^a = f^{-1}(-\infty, a]$, suppose $a < b$ and suppose the set $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b.*

Proof. Let g be any Riemannian metric on M, and put $\langle X, Y\rangle = g(X, Y)$ for tangent vectors X, Y. Let $\mathrm{grad} f$ denote the vector field defined by $\langle X, \mathrm{grad} f\rangle = X(f)$. Define a new vector field $X = \rho \mathrm{grad} f$ where ρ is a smooth function equal to $1/||\mathrm{grad} f||^2$ on $f^{-1}[a, b]$ and which vanishes outside a compact neighbourhood of this set. Then X is a compactly-supported smooth vector field on M, and by a standard result of smooth manifold theory, generates a unique 1-parameter family of diffeomorphisms. Letting ϕ_t denote this family, consider the function $t \to f(\phi_t(q))$ for a fixed point q. If $\phi_t(q)$ lands in $f^{-1}[a, b]$, then we have

$$\frac{df(\phi_t(q))}{dt} = \langle \frac{\phi_t(q)}{dt}, \mathrm{grad} f\rangle = \langle \mathrm{grad} f, \mathrm{grad} f\rangle = 1$$

and thus

$$f(\phi_t(q)) = f(q) + t.$$

With this in mind, consider the diffeomorphism ϕ_{b-a}. If $f(q) \leq a$, then $f(\phi_{b-a}(q)) = b - a + f(q) \leq b$, so $\phi_{b-a}(q) \in M^b$, and if $f(q) \leq b$, then $f(\phi_{a-b}(q)) = a - b + f(q) \leq a$, so $\phi_{a-b}(q) \in M^a$. Appealing to the fact that ϕ_{b-a} is a diffeomorphism, we have that M^a is diffeomorphic to M^b. □

Let us get on with the proof of Reeb's result.

Proof. By compactness of M, we must have that the critical points p, q of f are the points at which f attains its maximum and minimum values, say $f(p) = 1$, $f(q) = 0$. Appealing to the lemma of Morse, we may find coordinates about p such that f takes the form

$$f(p) = 1 - (y^1)^2 + ... + (y^n)^2$$

and coordinates about q such that f looks like

$$f(q) = (y^1)^2 + ... + (y^n)^2$$

In particular, for sufficiently small ϵ, the sets $f^{-1}[0, \epsilon]$ and $f^{-1}[1 - \epsilon, 1]$ are closed n-cells. Moreover, by the result just proved, $f^{-1}[0, \epsilon] = M^\epsilon$ is homeomorphic (in fact diffeomorphic) to $M^{1-\epsilon}$. Thus, M is the union of two closed n-cells matched along their common boundary, and thus clearly homeomorphic to the n-sphere. Indeed, the n-sphere may be expressed as the union of the sets

$$\begin{aligned} V^+ &= \{(x^1, \dots, x^{n+1} \in S^n | x^{n+1} \geq 0\} \\ V^- &= \{(x^1, \dots, x^{n+1} \in S^n | x^{n+1} \leq 0\} \end{aligned}$$

Then the homeomorphisms

$$\begin{aligned} &\psi_\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, x^n) \\ &\psi_\pm^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, \pm\sqrt{1 - (x^1)^2 - \dots - (x^n)^2}) \end{aligned}$$

may be glued together to yield a global homeomorphism between the n-sphere and a pair of closed n-cells joined along their boundaries. □

With this preliminary machinery in place, we go on to define Milnor's space and show that it is indeed homeomorphic to the 7-sphere by explicitly defining a Morse function on the manifold and showing that it admits exactly two critical points.

3 Sphere Bundles and Milnor's Space

3.1 Fibre Bundles

Fibre bundles are topological spaces that are locally product spaces, but may have a non-trivial global topology. Though the definition is rather technical, we will see through some examples that many familiar spaces may be characterized as fibre bundles, and Milnor's space itself will turn out to be a particularly simple fibre bundle.

Formally, a **fibre bundle** ξ is a triplet of topological spaces (B, F, E) and a continuous map $\pi : E \to B$ with the following properties. For each point $p \in B$, there exists a neighbourhood U of p and a homeomorphism h such that $h(U \times F) = \pi^{-1}(U)$ and $\pi^{-1}(b) = F$ (h is **fibre-preserving**). The space E is called the **total space**, B the **base space**, and F the **fibre**. The pair (U, h) is called a **local trivialization**, and if U may be chosen to be the entire base space, then ξ is said to be a **trivial** fibre bundle. Note that one may compose local trivializations $(U, h_1), (V, h_2)$ where $U \cap V \neq \emptyset$ to obtain so-called **transition functions** $h_2 \circ h_1^{-1} : h_1(U \cap V \times F) \to h_2(U \cap V \times F)$; by default, these are homeomorphisms but one may choose to constrain the transition functions further (for instance, a smooth fibre bundle is one for which the transition functions are smooth functions). The group of transition functions under consideration is called the **structure group** of the fibre bundle.

Let us consider some examples.

Example 3.1. (Vector bundles)

A real (complex) vector bundle is simply a fibre bundle with fibre $\mathbb{R}^n$ ($\mathbb{C}^n$) with structure group $GL_n(\mathbb{R})$ ($GL_n(\mathbb{C})$). The tangent bundle of a manifold has as base space the manifold itself with fibre equal to the space of tangent vectors, and total space the disjoint union of tangent spaces. Similarly, the alternating k-tensor bundle has as fibre the space of alternating k-tensors at p, with total space given by the disjoint union of the fibres.

Example 3.2. (Fibre bundles over spheres)

Familiar examples include the cylinder (trivial line bundle over the circle), Möbius strip (non-trivial line bundle over the circle), the torus (trivial circle bundle over the circle) and the Klein bottle (non-trivial circle bundle over the circle). The Hopf fibration expresses the 3-sphere as a (non-trivial) circle bundle over the 2-sphere. In fact, one may also express the 7-sphere as a 3-sphere bundle over the 4-sphere and the 15-sphere as an 7-sphere bundle over the 8-sphere. Aside from the trivial expression of the circle as a point-bundle over a circle, these are the only spheres which may be expressed as sphere-bundles over spheres (the so-called **Hopf fibrations**) [5].

Example 3.3. (Induced (pullback) bundles)

let ξ be a fibre bundle over B with base E, fibre F, and let B_1 be another topological space. For any continous map $f : B_1 \longrightarrow B$ one may construct the **induced** (or **pullback**) bundle $f^*\xi$ over B_1 as follows. The total space E_1 of $f^*\xi$ is the subset $E_1 \subset B_1 \times E$ consisting of all pairs (b, e) such that $\pi(e) = f(b)$ where π is the projection map associated with ξ. The projection map π_1 of $f^*\xi$ is then defined to be $\pi_1(b, e) = b$. Local coordinates are obtained as follows: if (U, h) is a local trivialization of ξ, put $U_1 = f^{-1}(U)$ and $h_1 : U_1 \times F \longrightarrow \pi_1^{-1}(U_1)$ by $h_1(b, x) = (b, h(f(b)), x))$. Then (U_1, h_1) is a local trivialization for $f^*\xi$. Essentially, the fibre over $b \in B_1$ is just the fibre over $f(b) \in B$. Later on in the paper, we will have occasion to pullback smooth vector bundles and their sections (in this context, these constructions should be familiar).

Finally, let ξ, ζ be two fibre bundles. A **bundle map** is a continous map $g : E(\xi) \longrightarrow E(\zeta)$ taking each fibre of ξ homeomorphically onto a fibre of ζ. Two fibre bundles are **isomorphic** if there exists a bundle map g that is a homeomorphism of total spaces (we will be interested in the case where the fibre bundles are smooth, in which case we require g to be a diffeomorphism).

3.2 Milnor's space

We will construct Milnor's space as the total space of a certain S^3 bundle over S^4. If we restrict our attention to such fibre bundles with rotation group $SO(4)$ as structural group, then the equivalence classes of such bundles are in one-to-one correspondence with elements of the third homotopy group $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ [5]. The idea is that S^4 is diffeomorphic to two copies of $\mathbb{R}^4$ glued together appropriately (through stereographic projection), and so the bundle restricted to each copy of $\mathbb{R}^4$ will be the trivial bundle $S^3 \times \mathbb{R}^4$, as bundles over homotopic base spaces are isomorphic, $\mathbb{R}^4$ is contractible and every bundle over a point is trivial [5]. Thus we need only prescribe the transition functions where the stereographic charts in the base overlap. But this overlap is homotopic to the equator $S^3 \subset S^4$, so all we need is a map from S^3 into the structure group of our fibre i.e. a map $f \in \pi_3(SO(4))$. That $\pi_3(SO(4))$ takes the form it does is a separate fact.

Explicitly, for each pair of integers (h, j) the map $f_{hj} : S^3 \longrightarrow SO(4)$ defined by $f_{hj}(u)v = u^h v u^j$ (juxtaposition understood to be quaternion multiplication) determines a representative bundle, as we will see [3]. That quaternion multiplication defines an element of $SO(4)$ is clear: such multiplication takes a a unit 3-vector and returns another, by multiplicativity of the quaternion norm, in a linear fashion, and is orientation-preserving by connectivity of S^3 and the fact that the connected component of identity in $O(4)$ is $SO(4)$.

Let ξ_{hj} denote the sphere bundle corresponding to f_{hj}, defined above, and let M_k^7 be the total space of ξ_{hj}, with $k = h - j$, $h + j = 1$. We will se that the first constraint parametrizes diffeomorphism classes of the bundles, while the second constraint ensures the bundles are all topological 7-spheres.

We construct M_k^7 explicitly by gluing together two trivial S^3 bundles over $\mathbb{R}^4$ using an appropriate f_{hj}. Taking stereographic coordinates in the base, we take two copies of $\mathbb{R}^4 \times S^3$ (corresponding to trivial S^3 bundles on $S^4 - \{N\}$, $S^4 - \{S\}$ respectively) and identify the $(\mathbb{R}^4 - \{0\}) \times S^3$ via the diffeomorphism

$$(u, v) \to (u', v') = (u/||u||^2, u^h v u^j/||u||)$$

where the first component is determined by changing stereographic charts in the base, and the second component describes how points in the fibre are identified via f_{hj} on the overlap.

Consider the function defined in one chart by

$$f(u, v) = \frac{Re(v)}{(1 + ||u||^2)^{1/2}}$$

If we define new coordinates $u'' = u'(v')^{-1}$ (quaternion inversion understood), then f takes the form

$$f(u, v) = \frac{Re(u'')}{(1 + ||u''||^2)^{1/2}}$$

It is clear that the double primed coordinates have the same domain as the primed coordinates, so these expressions define f on the entire manifold.

To see that the second expression follows from the first, note that

$$\frac{Re(u'')}{(1 + ||u''||^2)^{1/2}} = \frac{Re(u'')||u||}{(1 + ||u||^2)^{1/2}}$$

so we need to show that $||u||Re(u'') = Re(v)$. This is indeed the case, as we have

$$\begin{aligned}
2Re(u'') &= u'' + ||u''||^2(u'')^{-1} \\
&= u'(v')^{-1} + ||u''||^2(u'(v'^{-1}))^{-1} \\
&= \frac{1}{||u||}(u^{h+j}u^{-j}v^{-1}u^{-h} + u^h v u^j u^{-(j+h)}) \ \ (h+j=1) \\
&= \frac{2}{||u||}Re(\hat{u}^h v \hat{u}^{-h}), \ \hat{u} = u/||u|| \\
&= \frac{2}{||u||}Re(v)
\end{aligned}$$

where the last line follows from the fact that conjugation by a unit quaternion does not change the real part. Note that our assumption that $h + j = 1$ was essential at this stage, though we have not yet used the second assumption $h - j = k$.

Let us determine the critical points of f. Writing $u = (x^1, x^2, x^3, x^4)$, $v = (y^1, y^2, y^3, y^4)$ with $||v||^2 = 1$, we have

$$f(u,v) = \frac{(1 - (y^2)^2 - (y^3)^2 - (y^4)^2)^{1/2}}{(1 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2)^{1/2}}$$

where without loss of generality we take the positive square root (the sign of the square root will not change the critical points). Then we find

$$\begin{aligned}
\frac{\partial f}{\partial y^i} &= \frac{-y^i}{([1 - (y^2)^2 - \ldots - (y^4)^2][1 + (x^1)^2 + \ldots + (x^4)^2])^{1/2}} \\
\frac{\partial f}{\partial x^i} &= \frac{(1 - (y^2)^2 - \ldots (y^4)^2)^{1/2}}{(1 + (x^1)^2 + \ldots + (x^4)^2)^{3/2}}(-x^i)
\end{aligned}$$

It is clear that $df = 0$ if and only if $y^i = 0, i = 2,3,4,\ x^i = 0, i = 1,2,3,4$ yielding two critical points $(u,v) = (0, \pm 1)$.

In the other chart, we have

$$f(u,v) = \frac{x^1}{(1+(x^1)^2+\ldots+(x^4)^2)^{1/2}},\ u'' = (x^1,x^2,x^3,x^4)$$

wherein we calculate

$$df = \frac{(1+(x^2)^2+...+(x^4)^2)dx^1 + (\text{linear combination of } dx^2, dx^3, dx^4)}{(1+(x^1)^2+\ldots+(x^4)^2)^{3/2}}$$

In particular, the coefficient of dx^1 never vanishes, so we get no critical points here.

Finally, let's verify that the two critical points found above are indeed non-degenerate. We have

$$\begin{aligned}
\frac{\partial^2 f}{(y^i)^2}|_{(0,\pm1)} = & \left[\frac{1}{(1+(x^1)^2+...+(x^4)^2)^{1/2}}\right]\times \\
& \left[\frac{-1}{(1-(y^2)^2-...-(y^4)^2)^{1/2}} + \frac{(y^i)^2}{(1-(y^2)^2-...-(y^4)^2)^{3/2}}\right]|_{(0,\pm1)} \\
= & -1
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^j \partial y^i}|_{(0,\pm1)} = & \left[\frac{1}{(1+(x^1)^2+...+(x^4)^2)^{1/2}}\right]\times \\
& \left[\frac{y^i y^j}{(1-(y^2)^2-...-(y^4)^2)^{3/2}}\right]|_{(0,\pm1)} \\
= & 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^j \partial y^i}|_{(0,\pm1)} = & \left[\frac{-y^i}{(1-(y^2)^2-...-(y^4)^2)^{1/2}}\right]\left[\frac{x^j}{(1+(x^1)^2+...+(x^4)^2)^{3/2}}\right]|_{(0,\pm1)} \\
= & \ 0
\end{aligned}$$

Similarly, we find

$$\frac{\partial^2 f}{\partial (x^i)^2}|_{(0,\pm 1)} = 1$$
$$\frac{\partial^2 f}{\partial x^i \partial x^j}|_{(0,\pm 1)} = 0$$

That is, the matrix of second-order partials of f is diagonal with entries ± 1. Thus, our critical points are non-degenerate, so f is a Morse function satisfying the hypotheses of Reeb's thereom, and thus Milnor's manifold is indeed homeomorphic to the 7-sphere. To show that it is not diffeomorphic to the 7-sphere will require considerable more machinery from the theory of characteristic classes, which we consider in the next section.

4 Characteristic Classes

4.1 Differential Forms and de Rham Cohomology

Let us recall some definitions. An alternating k-tensor is a k-linear map totally antisymmetric in its arguments. A differential k-form on a smooth manifold M is then a (smooth) section of the bundle of alternating k-tensors of M, denoted $\Lambda^k(M)$. The wedge product is a graded-commutative operation that takes two forms α, β of degrees k, l respectively and returns a form $\alpha \wedge \beta$ of degree $k + l$. Define the Grassmann algebra of M to be $\Lambda(M) = \bigoplus_k \Lambda^k(M)$. Then the wedge product turns $\Lambda(M)$ into a graded-commutative ring. The exterior derivative $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is an $\mathbb{R}$ (or $\mathbb{C}$)-linear map satisfying the Leibniz rule, determined by its action on functions by $df = \dfrac{\partial f}{\partial x^i} dx^i$ and $d^2 x^i = 0$, in a coordinate basis [1].

A differential form ω is closed if $d\omega = 0$ and exact if $\omega = d\tau$. Every exact form is closed, but not every closed form is exact; the failure of closed forms to be exact is determined by the topology of the underlying manifold and is codified via de Rham cohomology. Define $Z^k(M) = \text{kernel}(d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M))$ to be the group of k-**cocycles**, and $B^k(M) = \text{image}(d : \Lambda^{k-1} \longrightarrow \Lambda^k)$ to be the group of k-**coboundaries**. Then the k**-th de Rham cohomology group** of M is defined to be the group $H^k(M) = Z^k(M)/B^k(M)$ [1].

We will also have need for relative cohomology. Let M be a smooth manifold with boundary ∂M. Let $\Omega^k(M, \partial M)$ denote differential k-forms on M which vanish on ∂M. The exterior derivative preserves this, in the sense that $d(\Omega^k(M, \partial M)) \subset \Omega^{k+1}(M, \partial M)$, since if $\omega \in \Omega^k(M, \partial M)$ and $\iota : \partial M \hookrightarrow M$ denotes inclusion, then $\iota^*(d\omega) = d(\iota^*\omega) = 0$ by properties of d. Thus we may define

$$H^k(M, \partial M) = \frac{\text{kernel}(d : \Omega^k(M, \partial M) \longrightarrow \Omega^{k+1}(M, \partial M))}{\text{image}(d : \Omega^{k-1} \longrightarrow \Omega^k(M, \partial M))},$$

the k**-th relative cohomology group** of M relative to its boundary ∂M.[6]

For appropriate spaces, and certainly the spaces that we will be considering, all cohomology theories are isomorphic, but it will be most convenient to use differential forms rather than singular chains, so we stick with them.

In what follows, we will define special cohomology classes, the so-called "characteristic classes," of our manifold (to be precise, the characteristic classes of the tangent bundle of our manifold). These classes give rise to powerful theorems, which will be of utmost importance to our task.

4.2 Chern-Weil Theory

Let ζ be a complex vector bundle with base M and connection ∇ i.e. a $\mathbb{C}$-linear map $\nabla : C^\infty(\zeta) \times T(M) \to C^\infty(\zeta)$ that is C^∞-linear in the second argument and satisfies the Leibniz rule in its first argument. In a chart, the connection is determined by the connection 1-forms ω_{ij} via $\nabla(s_i) = \sum_j \omega_{ij} s_j$ where the $\{s_j\}$ constitute a local basis of sections. Given a connection, recall that the associated **curvature 2-form** K is an $\mathrm{End}(\zeta)$-valued differential form defined locally in terms of the connection 1-forms via $K(s_i) = \sum_j \Omega_{ij} s_j$ with $\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}$ [7].

We can build our characteristic classes from the curvature form as follows. An **invariant polynomial** is a function $P : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ which can be expressed as a complex polynomial in the entries of the matrix, satisfying $P(XY) = P(YX)$. Familiar examples include the trace and determinant functions. Given an invariant polynomial P, one can build an exterior form $P(K)$ as follows. Choosing a local basis of sections $\{s_i\}$ gives a matrix of 2-forms Ω defining K in that frame. Since even forms commute under the wedge product, we obtain a well-defined exterior form $P(\Omega)$ by evaluating P on the (2-form valued) entries of Ω. Declare $P(K)(x)$ to be $P(\Omega)(x)$. If we change our local basis of sections, we have that Ω changes via conjugation by an invertible matrix. By the defining property of an invariant polynomial, we get a globally well-defined exterior form this way.

We go through all this trouble for a good reason [7]:

Proposition 4.1. *For any invariant polynomial, $P(K)$ is closed; that is, $P(K)$ is a de-Rham cocycle.*

Proof. We work in local coordinates. Writing $P = P(A_{ij})$ for indeterminates A_{ij}, and putting $\frac{\partial P}{\partial \Omega} = \frac{\partial P}{\partial A_{ij}}(\Omega)$, thought of as a matrix, we have

$$\begin{aligned} dP(\Omega) &= \sum_{i,j} \frac{\partial P}{\partial A_{ij}}(\Omega) \wedge d\Omega_{ij} \\ &= \sum_{i,j} \left(\frac{\partial P}{\partial \Omega}^T\right)_{ji} \wedge d\Omega_{ij} \\ &= \operatorname{tr}(\frac{\partial P}{\partial \Omega} \wedge d\Omega). \end{aligned} \tag{4.1}$$

Now by definition, we have

$$\Omega = d\omega - \omega \wedge \omega$$

from which we compute the Bianchi identity

$$\begin{aligned} d\Omega &= -d\omega \wedge \omega + \omega \wedge d\omega \\ &= \omega \wedge (\Omega + \omega \wedge \omega) - (\Omega + \omega \wedge \omega) \wedge \omega \\ &= \omega \wedge \Omega - \Omega \wedge \omega. \end{aligned} \tag{4.2}$$

Now for any invariant polynomial, $\frac{\partial P}{\partial A}^T$ commutes with A. Letting E_{ji} denote the $(j, i) - th$ elementary matrix, we have, by definition of P,

$$P((1 + tE_{ji})A) = P(A(1 + tE_{ji})). \tag{4.3}$$

Taking the time derivative of the LHS and evaluating at $t = 0$, we have

$$\frac{d}{dt}P((1 + tE_{ji})A)|_{t=0} = \sum_{\alpha\beta} \frac{\partial P}{\partial B_{\alpha\beta}}(B(0))\frac{dB_{\alpha\beta}}{dt}|_{t=0}$$

where $B(t) = (1 + tE_{ij})A$. Note that

$$\begin{aligned}(E_{ji}A)_{\alpha\beta} &= \sum_{\gamma} \delta_{j\alpha}\delta_{\gamma i}A_{\gamma\beta} \\ &= \delta_{j\alpha}A_{i\beta}\end{aligned}$$

and thus

$$\begin{aligned}\sum_{\alpha\beta} \frac{\partial P}{\partial B_{\alpha\beta}}(B(0))\frac{dB_{\alpha\beta}}{dt}|_{t=0} &= \sum \frac{\partial P}{\partial B_{\alpha\beta}}(B(0))\delta_{j\alpha}A_{i\beta} \\ &= \sum A_{i\beta}\frac{\partial P}{\partial B_{j\beta}}(B(0)) \\ &= \sum A_{i\beta}\frac{\partial P}{\partial A_{j\beta}}. \qquad (4.4)\end{aligned}$$

Evidently taking $\frac{d}{dt}|_{t=0}$ of both sides of (4.3) yields

$$\sum A_{i\beta}\frac{\partial P}{\partial A_{j\beta}} = \sum \frac{\partial P}{\partial A_{\beta i}}A_{\beta j}$$

and in particular, $\Omega \wedge \dfrac{\partial P^T}{\partial \Omega} = \dfrac{\partial P^T}{\partial \Omega} \wedge \Omega$. Thus we compute

$$
\begin{aligned}
dP(\Omega) &= tr(\frac{\partial P^T}{\partial \Omega} \wedge d\Omega) \\
&= tr(\frac{\partial P^T}{\partial \Omega} \wedge (\omega \wedge \Omega - \Omega \wedge \omega)) \\
&= tr(\frac{\partial P^T}{\partial \Omega} \wedge \omega \wedge \Omega - \Omega \wedge (\frac{\partial P^T}{\partial \Omega} \wedge \omega)) \\
&= \sum (X_{ij} \wedge \Omega_{ji} - \Omega_{ji} \wedge X_{ji}) \\
&= 0
\end{aligned}
$$

because the $X_{ij} = (\dfrac{\partial P^T}{\partial \Omega} \wedge \omega)_{ij}$ commute with the 2-forms Ω_{ji}. $\square$

In fact, the cohomology class of this cocycle is independent of the choice of connection [7].

Proof. Take two connections ∇_0, ∇_1 on ζ, and consider the bundle ζ' on $M \times \mathbb{R}$ induced by the projection $\rho : (x,t) \mapsto x$; that is to say, ζ' is obtained by pulling back ζ by ρ. We get induced connections ∇'_i on ζ' defined by $\nabla'_i(\rho^* s) = \nabla_i(s)$, and a third connection $\nabla = t\nabla'_0 + (1-t)\nabla'_1$, with respect to which $P(K_\nabla)$ is a cocycle on $M \times \mathbb{R}$.

Consider the maps $\rho_\epsilon : x \to (x, \epsilon), \epsilon = 0, 1$ from M to $M \times \mathbb{R}$. Then the induced connections $(\rho_\epsilon)^* \nabla$ on $\rho_\epsilon^* \zeta'$ can be identified with the original connections ∇_ϵ on ζ, and so $\rho_\epsilon^*(P(K_\nabla)) = P(K_{\nabla_\epsilon})$. But ρ_0 is clearly homotopic to ρ_1 via the projection ρ, and so $P(K_{\nabla_0})$ represents the same cohomology class as $P(K_\nabla)$. $\square$

Let $\sigma_k(A)$ denote the k-th elementary symmetric polynomial of the eigenvalues of the matrix A, defined by

$$\det(I + tA) = 1 + t\sigma_1(A) + ... + t^n \sigma_n(A)$$

Then the **k-th Chern class** $c_k(\zeta)$ of ζ is defined to be the cohomology class of $\frac{1}{2\pi i}\sigma_k(K_\nabla)$.

Finally, let ξ be a real vector bundle. Define the **k-th Pontrjagin class** of ξ to be $(-1)^k c_{2k}(\xi \otimes \mathbb{C})$; that is, the Pontrjagin classes of a real vector bundle are gotten from the Chern classes of its complexification. We make this definition for two reasons. First, the odd Chern classes are all zero. This is most easily seen by taking a Riemannian metric on ζ and choosing a connection compatible with this metric. Then the connection 1-forms, and hence the curvature form, are skew-symmetric, and we calculate $\sigma_m(\Omega) = \sigma_m(\Omega^T) = (-1)^m \sigma_m(\Omega)$. Thus σ_m and hence c_m vanishes for m odd [7].

However, the real reason is that the Pontrjagin classes arise in the statement of a powerful theorem, which we consider in the next section.

4.3 Hirzebruch Signature Theorem

Let M be a smooth, compact, oriented manifold. Recall that a manifold is orientable if it admits an atlas of charts all of whose transition functions have positive Jacobian determinant; a choice of such atlas determines an orientation. One then has a well-defined notion of integration of forms on such manifolds. Explicitly, suppose we wish to integrate a top form α over a compact domain $D \subset M$. If D is contained in a single chart U, define the integral of α over D by

$$\int_D \tilde{\alpha} dx^1 \wedge \ldots \wedge dx^n = \int_{\phi(D)} \tilde{\alpha} dx^1 ... dx^n$$

where $\alpha = \tilde{\alpha} dx^1 \wedge \ldots \wedge dx^n$ in the chart U. Otherwise, choose a cover of D by coordinate charts, take a partition of unity $\{\psi_i\}$ subordinate to this cover and define

$$\int_D \alpha = \sum_i \int_{U_i} \psi_i \alpha$$

Then the support of each $\psi_i\alpha$ is contained in a single chart, and the integral reduces to a sum of integrals of the first kind. It may be shown that this definition is independent of choice of cover or partition of unity. The most important fact about integration of forms on manifolds is expressed in the following classic result [1]:

Theorem 4.2. ***(Stokes' theorem)*** *Let M be a smooth manifold with boundary ∂M. If α is an $(n-1)$-form on M, then*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

In particular, if $\partial M = \emptyset$, then the integral over M of any exact form vanishes.

Now suppose M is a manifold whose dimension is divisible by 4. Define a quadratic form I, the so-called **intersection form**, on $H^{2k}(M^{4k})$ by $I([\alpha]) = \int_M \alpha \wedge \alpha$, where α is a representative of its cohomology class. If $\beta = \alpha + d\tau$ is another representative, then $I[\beta] = \int_M \alpha \wedge \alpha + \int_M d(\tau \wedge \alpha + \alpha \wedge \tau + \tau \wedge d\tau) = I[\alpha]$, the second term vanishing by Stokes' theorem on a manifold with no boundary. Thus I is well-defined on cohomology. Define the **signature** $\sigma(M)$ of M to be the signature of this quadratic form; that is, the number of positive eigenvalues minus the number of negative eigenvalues when diagonalized over $\mathbb{R}$.

The Pontrjagin classes may be used to calculate the signature, by means of the Hirzebruch signature theorem. The full statement is rather involved and will not be needed for what follows, so we state the following abbreviated version [7].

Theorem 4.3. ***(Hirzebruch)*** *Let M be a smooth, compact, oriented 8-manifold. Then $\sigma(M) = \int_M \frac{1}{45}(7p_2 - p_1^2)$, where p_i denotes the i-th Pontrjagin class of M.*

The theorem may be proved by direct application of the Atiyah-Singer index theorem [7, 8]. It should be noted that the assumption of smoothness is absolutely essential and will play a key role in our application of the theorem later on.

5 Milnor's Exotic Spheres

5.1 Milnor's Invariant

Let M^7 be a compact, oriented, smooth 7-manifold with cohomology groups $H^3(M) = H^4(M) = 0$. It is known that every compact, smooth, 7-manifold without boundary is the boundary of some smooth 8-manifold B^8 (that is to say, the 7-dimensional smooth oriented cobordism ring is trivial) [9]. We will define an invariant of M^7 in terms of B^8; it is "invariant" in the sense that it will not depend on the choice of manifold B^8.

Define a quadratic form $I' : H^4(B^8, M^7) \longrightarrow \mathbb{R}$ by $[\alpha] \to \int_{B^8} \alpha \wedge \alpha$. Though similar to the intersection form defined above, we need to check that I' is well-defined on relative cohomology. If β is another representative of $[\alpha]$, then $\beta - \alpha = d\sigma$ where $\sigma \in \Omega^3(B^8, M^7)$. Then $I'([\beta]) = \int_{B^8} \alpha + \int_{M^7} \sigma = \int_{B^8} \alpha = I'([\alpha])$ where we again have used Stokes' theorem and the fact that σ vanishes on M^7. Let $\sigma'(B^8)$ be the signature of this quadratic form (cf. intersection form and signature of 8-manifold above).

By the long exact sequence of cohomology [6], our assumption on the vanishing cohomology groups implies that inclusion $\iota : H^4(B^8, M^7) \to H^4(B^8)$ is an isomorphism. Indeed, we have

$$\ldots \longleftarrow 0 \longleftarrow H^4(B^8) \longleftarrow H^4(B^8, M^7) \longleftarrow 0 \longleftarrow \ldots$$

Then $\ker \iota = \operatorname{im} 0 = 0$ gives that ι is a monomorphism, and $\operatorname{im} \iota = \ker 0 = H^4(B^8)$ gives that ι is an epimorphism.

Let $p_1 \in H^4(B^8)$ be the first Pontrjagin class of the tangent bundle of B^8, and put $q'(B^8) = I'([\iota^{-1} p_1]) = \int_{B^8} (\iota^{-1})(p_1) \wedge (\iota^{-1})(p_1)$.

Proposition 5.1. *The equivalence class of $2q(B^8) - \sigma(B^8)$ mod 7 does not depend on the choice of manifold B^8.*

Proof. Let B_1^8, B_2^8 be two manifolds with boundary M^7, and put $C^8 = B_1^8 \bigsqcup_{M_k^7} \bar{B}_2^8$, where a bar denotes the same manifold with boundary orientation opposite that of B_1^8 i.e. C^8 is the smooth manifold without boundary gotten by identifying B_1^8 and B_2^8 along their common boundary M^7. Indeed, by the collar neighbourhood theorem, a neighbourhood of $p \in C^8 \cap \iota(M^7)$ looks like $U \times (-\epsilon, \epsilon)$ for some small ϵ, where U is an open set in M^7; smoothness away from $\iota(M^7)$ is clear.

If $\sigma(C^8)$ denotes the signature of C^8, we have

$$\sigma(C^8) = \int_{C^8} \frac{1}{45}(7p_2(C^8) - p_1 \wedge p_1(C^8))$$

by the Hirzebruch signature theorem, from which we calculate that

$$45\sigma(C^8) + q(C^8) = \int_{C^8} 7p_2(C^8) - p_1(C^8) \wedge p_1(C^8) + p_1(C^8) \wedge p_1(C^8) = 0 \bmod 7$$

from which it follows that

$$2q(C^8) - \sigma(C^8) = 0 \bmod 7. \tag{5.1}$$

Let I_1', I_2' be the quadratic forms defined above on $H^4(B_1^8, M^7), H^4(B_2^8, M^7)$ respectively and let I be the intersection form on C^8. We will show that I is the "direct sum" of I_1' and $-I_2'$ in the sense that $I([\alpha]) = I_1'([\alpha_1]) - I_2'([\alpha_2])$ for appropriate forms α_1, α_2.

We compute

$$
\begin{aligned}
I([\alpha]) &= \int_{C^8} \alpha \wedge \alpha \\
&= \int_{\iota(B_1^8)\cup\iota(\bar{B}_2^8)} \alpha \wedge \alpha \\
&= \int_{\iota(B_1^8)} \alpha \wedge \alpha - \int_{\iota(B_2^8)} \alpha \wedge \alpha & (5.2)\\
&= \int_{B_1^8} \iota^*(\alpha) \wedge \iota^*(\alpha) - \int_{B_2^8} \iota^*(\alpha) \wedge \iota^*(\alpha) & (5.3)\\
&= \int_{B_1^8} \iota^*(\alpha - \alpha|_{M^7}) \wedge \iota^*(\alpha - \alpha|_{M^7}) \\
&\quad - \int_{B_2^8} \iota^*(\alpha - \alpha|_{M^7}) \wedge \iota^*(\alpha - \alpha|_{M^7}) & (5.4)\\
&= \int_{B_1^8} \alpha_1 \wedge \alpha_1 - \int_{B_2^8} \alpha_2 \wedge \alpha_2 \\
&= I_1'([\alpha_1]) - I_2'([\alpha_2]) & (5.5)
\end{aligned}
$$

(5.2) follows from additivity of the integral, with the negative sign arising from the reversed orientation of $\bar{B}_2^8$, (5.3) follows from diffeomorphism invariance of the integral and naturality of pullback with respect to $\wedge$. (5.4) is gotten by subtracting the integrals $\int_{B_i^8} \alpha|_{M^7}$ which vanish because $\alpha|_{M^7}$ is supported on a set of measure zero in B_i^8. Finally, (5.5) follows because the forms α_1, α_2 defined by (5.3) are now representatives of cohomology classes in $H^4(B_1^8, M^7), H^4(B_2^8, M^7)$ respectively, by construction.

Thus, by additivity of signature, we have $\sigma(C^8) = \sigma'(B_1^8) - \sigma'(B_2^8)$, and $q(C^8) = q(B_1^8) - q(B_2^8)$ as the curvature form of C^8 restricted to B_i^8 is the curvature form on B_i^8. Combining these facts with (5.1) we find

$$2q(B_1^8) - \sigma'(B_1^8) = 2q(B_2^8) - \sigma'(B_2^8) \bmod 7$$

as desired.

□

This equivalence class will be called **Milnor's invariant**, denoted by $\lambda(M^7)$.

Corollary 5.2. *If* $\lambda(M^7) \neq 0 \bmod 7$*, then* M^7 *is not the boundary of any 8-manifold with fourth Betti number zero.*

In particular, M^7 is not the 7-sphere, for then it would bound the 8-ball which of course has all Betti numbers but the first equal to 0.

Proof. Suppose M^7 does bound such an 8-manifold. Then ι an isomorphism forces $H^4(B^8, M^7) = 0$, and thus the quadratic form defined above is identically zero. In particular, $\lambda(M^7) = 0$. □

All that remains is to find a suitable 8-manifold with Milnor's space as boundary and compute $\lambda(M^7)$. If it is non-vanishing, then Milnor's space cannot be diffeomorphic to the 7-sphere, by the corollary above.

5.2 Computing $\lambda(M^7_k)$

Proceeding as above, we consider the following [3]:

Theorem 5.3. *The invariant* $\lambda(M^7_k) = (k^2 - 1) \bmod 7$*, where* M^7_k *denotes Milnor's space (cf. chapter 3). In particular, for* $k^2 \neq 1 \bmod 7$*,* M^7_k *is homeomorphic but not diffeomorphic to the 7-sphere with its standard smooth structure.*

Proof. This is all rather involved, requiring several more facts about characteristic classes that are not readily deducible from what has already been introduced, as well as familiarity with some unfamiliar spaces. Therefore we just give a sketch of Milnor's original proof; what should be noted is that all the machinery is in place, and what we have left is "essentially" a computation, though a difficult one nevertheless.

Let B^8_k denote the total space of the 4-cell bundle gotten from ξ_{hj} by "filling-in" the fibres. Conceretely, the manifold is constructed the same as M^7_k, pasting together two copies of $\mathbb{R}^4 \times D^4$ instead of $\mathbb{R}^4 \times S^3$, and clearly has M^7_k as boundary. Since the fibres are contractible, $H^4(B^8_k) = H^4(S^4)$ and thus generated by the single element $\beta = \pi^*_k \alpha$, where π^*_k is the projection map associated with our 4-cell bundle and α is the standard generator for $H^4(S^4)$. Choosing appropriate orientations for M^7_k, B^8_k we must have $\int_{B^8_k} \iota^{-1}(\beta) \wedge \iota^{-1}(\beta) = 1$, by virtue of β being a generator. Thus the signature $\sigma'(B^8_k) = 1$.

One may further compute, via an appropriate "Whitney sum" decomposition of the tangent bundle of B^8_k, and using the fact that B^8_1 is the quaternionic projective plane with an 8-cell removed, that the Pontrjagin class $p_1(B^8_k) = \pm 2k\beta$. Thus we find that

$$q'(B^8_k) = \int_{B^8_k} \iota^{-1}(\pm 2k\beta) \wedge \iota^{-1}(\pm 2k\beta) = 4k^2 \int_{B^8_k} \iota^{-1}(\beta) \wedge \iota^{-1}(\beta) = 4k^2$$

and in particular,

$$2q'(B^8_k) - \sigma'(B^8_k) = \lambda(M^7_k) = 8k^2 - 1 = (k^2 - 1) \bmod 7$$

as promised. $\square$

6 Epilogue

We have come a long way. In considering smooth structures, we saw that while a given manifold admits uncountably many distinct smooth structures if it admits one at all, in dimensions $1, 2, 3$ every smooth manifold had essentially a unique smooth structure, up to diffeomorphism. In investigating Milnor's discovery of "exotic" manifolds, possessing several distinct smooth structures even up to diffeomorphism, we found a particularly simple way of determining whether a given manifold was a topological sphere, by means of Morse functions. Having determined the underlying topological manifold of Milnor's space M_k^7, we showed that for certain k, namely $k^2 \neq 1 \bmod 7$, the spaces could not be diffeomorphic to the standard 7-sphere, by means of an invariant $\lambda(M_k^7)$, defined in terms of the Pontrjagin classes of a particular manifold admitting M_k^7 as boundary.

The key step in our proof of the invariance of $\lambda(M_k^7)$ lay in the Hirzebruch signature theorem. Indeed, the construction of M_k^7 was rather elementary, as were the Morse-theoretical techniques used in showing that Milnor's space was a topological 7-sphere; all the difficulty lay in the Hirzebruch signature theorem, on which another paper could be written in and of itself.

There have been many further developments related to exotic spheres since Milnor's seminal 1956 paper. Shimada proved the existence of exotic 15-spheres a few years later [10], and Milnor followed up with work on exotic $(4k-1)$-spheres, for appropriate values of k [11]. Furthermore, Milnor and Kervaire elucidated more structure to Milnor's family of exotic 7-spheres by demonstrating that their diffeomorphism classes form an abelian group of order 28, with group operation given by connect sum and the standard 7-sphere serving as identity element [12]. Kervaire later went on to demonstrate that there are topological manifolds which admit no smooth structure [13]. As mentioned in the introduction, Taubes found uncountably many distinct smooth structures on $\mathbb{R}^4$, using techniques — due to Donaldson — rather different from the ones considered here. Whether there are exotic 4-spheres remains an open question; perhaps this is a problem on which you, dear reader, can make some progress.

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