The Slice Theorem

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Abstract

This paper serves to give a comprehensive review of the slice theorem. Given a proper group action of a Lie group on a smooth manifold, the slice theorem provides a decomposition of a tubular neighbourhood of an orbit in terms of the group, and the normal space to the orbit at a point. In the first section of this paper, we spend time constructing and proving the slice theorem for a finite dimensional Lie group acting on a finite dimensional smooth manifold. In general, the slice theorem does not extend to infinite dimensional manifolds, but there are instances in which slice theorems have been constructed. Namely, the action of the group of diffeomorphisms on the space of metrics over a compact manifold, and the action of the gauge group on the space of connections over a compact Riemannian manifold both admit a slice theorem, provided we first extend to Sobolev spaces of high enough regularity. Consequently, we first spend time developing the necessary framework to understand these specific slice theorems, which is the focus of the rest of the paper.

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Chapter 1

The Slice Theorem in Finite Dimensions

The goal of this paper is to give a comprehensive review of the Slice Theorem. To put it loosely, the Slice Theorem says that if we have a Lie group G acting smoothly and properly on a smooth manifold M, then for any $x \in M$, there exists a G-invariant tubular neighbourhood of the orbit of x, which can be traced by translations of a "slice" at x, which is a submanifold containing x that is invariant under action by the isotropy group of x. The aim of this first chapter is to give a full treatment of the Slice Theorem in finite dimensions. To understand the actual statement of the theorem, as well as the proof, we must first equip ourselves with the prerequisite knowledge.

1.1 Proper maps

In this section we talk about all the relevant properties of proper maps that lend themselves to our study of proper group actions in Section 1.3.

Definition 1.1.1. Let X, Y be topological spaces. We say that a map $F : X \to Y$ is *proper* if for any compact set $K \subseteq Y$, $F^{-1}(K)$ is compact in X.

We now give a couple of important properties of proper maps which will be useful later.

Theorem 1.1.2. *Let X be a topological space, and let Y be a locally compact Hausdorff space. If a continuous map $F : X \to Y$ is a proper map, then F is closed.*

Proof. Let K be a closed subset of X. Let y be a limit point of $F(K)$ and let $U \ni y$ be precompact (here we use the fact that Y is locally compact). Then y is a limit point of $F(K) \cap \overline{U}$. Since F is proper, $F^{-1}(\overline{U})$ is compact, and thus $K \cap F^{-1}(\overline{U})$ is compact. Then $F(K \cap F^{-1}(\overline{U})) = F(K) \cap \overline{U}$ must be compact as well, as F is continuous. But Y is Hausdorff, so that $F(K) \cap \overline{U}$ must in fact be closed, hence contain all of its limit points. Thus $y \in F(K) \cap \overline{U}$ so $y \in F(K)$. Since y was an arbitrary limit point of $F(K)$, we conclude that $F(K)$ is closed. □

Lemma 1.1.3. *Let M, N be smooth manifolds and let $F : M \to N$ be an injective smooth immersion. If F is proper, then it is a smooth embedding.*

Proof. Since F is proper, it is closed by Theorem 1.1.2, and closed injective maps are homeomorphisms onto their images. □

Lemma 1.1.4. *Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map. Then the following are sufficient conditions for properness of F.*

1. *X is compact, and Y is Hausdorff.*
2. *F is closed with compact fibers.*
3. *F is a topological embedding with closed image.*

Proof. 1. Let X be compact, and Y Hausdorff. Let $K \subseteq Y$ be compact. Since Y is Hausdorff, K must be closed. Then by continuity of F, $F^{-1}(K)$ is closed in X, hence compact.

2. Suppose that F is closed with compact fibers. Let K be compact in Y. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $F^{-1}(K)$. By assumption, the fibre $F^{-1}(p) \subseteq X$ is compact, so there exists a finite subset $\Lambda_p \subseteq \Lambda$ such that $\{U_\alpha\}_{\alpha \in \Lambda_p}$ is an open cover of $F^{-1}(p)$. Now, since $X - \left(\bigcup_{\alpha \in \Lambda_p} U_\alpha\right)$ is closed in X, we obtain that $F\Big(X - \left(\bigcup_{\alpha \in \Lambda_p} U_\alpha\right)\Big)$ is closed in Y, hence $V_p = Y - F\Big(X - \left(\bigcup_{\alpha \in \Lambda_p} U_\alpha\right)\Big)$ is some open set in Y that contains $p \in K$. Then $\{V_p\}_{p \in K}$ is an open cover for K, and by compactness there exists a finite $K_0 \subseteq K$ such that $\bigcup_{p \in K_0} V_p$ covers K.

But observe that

$$\begin{aligned}
F^{-1}(V_p) &= F^{-1}\Big(Y \setminus F\Big(X \setminus (\bigcup_{\alpha\in\Lambda_p} U_\alpha)\Big)\Big) \\
&= X \setminus F^{-1}\Big(F\Big(X \setminus (\bigcup_{\alpha\in\Lambda_p} U_\alpha)\Big)\Big) \\
&\subseteq X \setminus \Big(X \setminus (\bigcup_{\alpha\in\Lambda_p} U_\alpha)\Big) \\
&= \bigcup_{\alpha\in\Lambda_p} U_\alpha,
\end{aligned}$$

so in fact we have

$$F^{-1}(K) \subseteq F^{-1}(\bigcup_{p\in K_0} V_p) = \bigcup_{p\in K_0}(\bigcup_{\alpha\in\Lambda_p} U_\alpha).$$

But since K_0 is finite, and each Λ_p is finite, we have a finite subset of Λ given by $\Lambda_0 = \bigcup_{p\in K_0} \Lambda_p$ so that $\{U_\alpha\}_{\alpha\in\Lambda_0}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Hence $F^{-1}(K)$ is compact.

3. Suppose that F is a topological embedding with closed image. Let K be compact in Y. Since $F(X)$ is closed in Y, we know that $K \cap F(X)$ is closed in K (with respect to the subspace topology), hence it is compact in K. But since compactness is an intrinsic property of topological spaces, we then know that $K \cap F(X)$ is compact in Y hence in $F(X)$ with respect to the subspace topology there as well. Finally, because F is a homeomorphism onto its image, $F^{-1}(K) = F^{-1}(F(X) \cap K)$ is compact. □

Lemma 1.1.5. *The restriction of a proper map to a closed subset is a proper map.*

Proof. Suppose $F : X \to Y$ is a proper map between topological spaces, and let $A \subseteq X$ be closed. Then for any compact set $K \subseteq Y$, we know that $F|_A(K) = A \cap F^{-1}(K)$, which is closed in $F^{-1}(K)$ with respect to the subspace topology. But then $A \cap F^{-1}(K)$ is compact as a closed subset of a compact set. □

1.2 Overview of Group actions

We assume that the reader has a basic understanding of Lie groups and general group actions. Note that much of this section comes from [14].

Definition 1.2.1. Let G be a Lie group, M a smooth manifold, and $\Phi : G \times M \to M$ be an action on M, that is, $\Phi(g, m) = g \cdot m$. The action is said to be a *smooth action* if the action map Φ is smooth.

Lemma 1.2.2. *If $\Phi : G \times M \to M$ is a smooth action of a Lie group G on a smooth manifold M, then $\Phi_g : M \to M$ defined by $\Phi_g(p) = g \cdot p$ is a diffeomorphism on M.*

Proof. Observe that the map defined by action of an element $g \in G$ on M is invertible:

$$g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = x$$

and right action by g^{-1} is by definition a smooth map as well. Hence action by g is a smooth map with smooth inverse, and thus a diffeomorphism. $\square$

The following is very useful, and helps us transfer some nice topological properties of M to the orbit space M/G in later sections.

Proposition 1.2.3. *The quotient map $\pi : M \to M/G$ is an open map.*

Proof. Let $U \subseteq M$ be open. Then $\pi(U)$ is the equivalence class $[u]_\sim$ for $u \in U$, where $x \sim u$ if and only if $x = g \cdot u$ for some $g \in G$. Thus $\pi(U)$ is the equivalence class comprised of the orbits of all elements in U, that is, GU. Then $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$, which is open by Lemma 1.2.2. $\square$

Definition 1.2.4. Let G be a Lie group that acts smoothly on the smooth manifolds M and N. A map $F : M \to N$ is said to be *equivariant* if for all $g \in G$ and $p \in M$,

$$F(g \cdot p) = g \cdot F(p).$$

The following theorem provides a very strong statement regarding maps that satisfy this property for certain group actions.

Theorem 1.2.5 (Equivariant Rank Theorem)**.** *Let M and N be smooth manifolds, and let G be a Lie group. Suppose that $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G action on M and* any *smooth G action on N. Then F has constant rank. Thus, if F is surjective then it is a smooth submersion. If F is injective, it is a smooth immersion. And if F is bijective, it is a diffeomorphism.*

Proof. Denote the group actions $\Phi : G \times M \to M$ and $\Psi : G \times N \to N$, and let Φ_g be as in Lemma 1.2.2, and Ψ_g be defined analogously. Fix any $p \in M$ and consider the following commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{dF_p} & T_{F(p)}N \\ {\scriptstyle d(\Phi_g)_p}\downarrow & & \downarrow{\scriptstyle (d\Psi_g)_{F(p)}} \\ T_{\Phi_g(p)}M & \xrightarrow[dF_{\Phi_g(p)}]{} & T_{\Psi_g(F(p))}N \end{array}$$

Since Φ_g and Ψ_g are diffeomorphisms, it must be that dF_p is of the same rank as $dF_{\Phi_g(p)}$. And since $\Phi : G \times M \to M$ is transitive, then by defintion, for any $q \in M$ there exists $g \in G$ so that $\Phi_g(p) = q$. Hence dF_p has the same rank as dF_q for any $q \in M$. Thus F must be of constant rank. The remaining statements now follow immediately from [14, Theorem 4.14]. $\square$

The Equivariant Rank Theorem helps us prove that the orbits are in fact submanifolds of M, but first we need the following fact about maps of constant rank.

Lemma 1.2.6 (Constant Rank Level Set Theorem)**.** *Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map with constant rank r. Each level set of F is a properly embedded submanifold of codimension r in M That is, it is an embedded submanifold $S \subseteq M$ and the inclusion map $\iota : S \hookrightarrow M$ is a proper map.*

Proof. Let $m = \dim M$, and $r = \text{rank} F$. Take some arbitrary $q \in N$ and consider the set $F^{-1}(q)$. By [14, Theorem 4.12] there exist charts (U, φ) centered at $p \in M$ and (V, ψ) centered at $q = F(p) \in N$ such that in these charts, (denoting $\widehat{F}$ as the coordinate representation of F).

$$\widehat{F}(x^1, ..., x^r, x^{r+1} ..., x^m) = (x^1, ..., x^r, 0, ..., 0).$$

Thus we can use this to compute

$$\begin{aligned} S \cap U &= \{(x^1, ..., x^r, x^{r+1}, ..., x^m) \in U | \widehat{F}(x^1, ..., x^r, x^{r+1}, ..., x^m) = (0, ..., 0)\} \\ &= \{(x^1, ..., x^r, x^{r+1}, ..., x^m) \in U | (x^1, ..., x^r, 0, .., 0) = (0, ..., 0)\} \\ &= \{(x^1, ..., x^r, x^{r+1}, ..., x^m) \in U | x^1 = \cdots = x^r = 0\} \end{aligned}$$

and thus S is an embedded manifold of dimension $m - r$. Furthermore, as $\{q\}$ is closed in N, $S = F^{-1}(q)$ must be closed in M by continuity of F. Then by Lemma 1.1.4 part 3, we conclude that F is proper. $\square$

Proposition 1.2.7. *Let $\Phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. For each $p \in M$, the orbit map $\Phi^{(p)} : G \to M$ defined by $\Phi^{(p)}(g) = g \cdot p$ is smooth and has constant rank, so the isotropy group $G_p = (\Phi^{(p)})^{-1}(p)$ is a properly embedded Lie subgroup of G. If $G_p = \{e\}$, then $\Phi^{(p)}$ is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M.*

Proof. The orbit map is smooth because it is a composition of smooth maps

$$G \cong G \times \{p\} \hookrightarrow G \times M \xrightarrow{\Phi} M.$$

Since Φ is a group action, it satisfies the property that for any $g, h \in G$, $(hg)\cdot p = h\cdot(g\cdot p)$, and thus we can write

$$\Phi^{(p)}(hg) = (hg) \cdot p = h \cdot (g \cdot p) = h \cdot \Phi^{(p)}(g),$$

so that $\Phi^{(p)} : G \to M$ is equivariant with respect to the left action of G on itself and the action on M.

Since groups act transitively on themselves, Theorem 1.2.5 tells us that $\Phi^{(p)}$ is a map of constant rank, and thus Lemma 1.2.6 tells us that the level set $(\Phi^{(p)})^{-1}(p)$ is an embedded submanifold of G, hence a Lie subgroup of G.

Finally, suppose that $G_p = \{e\}$. If $\Phi^{(p)}(g') = \Phi^{(p)}(g)$, then $g' \cdot p = g \cdot p$ so that $g'g^{-1} \cdot p = p$ and thus $g'g^{-1} = e$, hence $g = g'$, so that $\Phi^{(p)}$ is injective. Then again invoking Theorem 1.2.5, we get that $\Phi^{(p)}$ is an immersion, and thus $\Phi^p(G) = G(p)$ is an immersed submanifold of M. □

We conclude with a short Lemma that is of importance to us in the next section.

Lemma 1.2.8. *Let $\Phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. Then M admits a G-invariant open cover. That is, there exists an open cover $\{U_i\}_{i\in I}$ of M such that $g \cdot U_i = U_i$ for all $g \in G$.*

Proof. Let $\{V_i\}_{i\in I}$ be any open over of M, and set $U_i = G \cdot V_i$. Since any $g \cdot V_i$ is open by virtue of the action being a diffeomorphism, we can see that U_i is open, and $\{U_i\}_{i\in I}$ clearly covers M. □

1.3 Proper Group Actions on Manifolds

The orbits of a group acting on a manifold are very important when it comes to understanding the action. The orbits tell us which points are related by some element of the group. Given a point p on M, the orbit of p is essentially those points that can be "reached" by p under the action of G. Since G is a group, one can easily see that the orbits provide an equivalence relation on M. That is, $p \sim q$ if there exists some $g \in G$ such that $g \cdot p = q$. Because of the geometric structure added by these orbits, we often like to look at the "orbit space" M/G of the group action. The problem is, for some arbitrary group action the orbit space M/G need not have nice properties. However, under the additional assumption that the group action is *proper* we obtain rich geometric structure on the orbit space M/G. In this section we study this structure. We begin by talking about what it means for an action to be proper, then delve into all of the nice properties of proper actions, and conclude with showing that proper actions give rise to G-invariant Riemannian metrics, which are vital for proving the Slice Theorem in section 1.5.

Definition 1.3.1. A continuous left action by a Lie group G on a topological space X is said to be *proper* if the map

$$\begin{aligned} G \times M &\to M \times M \\ (g, p) &\mapsto (g \cdot p, p) \end{aligned}$$

is a proper map.

We denote this extended action map by $\Phi_{\text{ext}} : G \times M \to M \times M$.

A natural question when given the above definition is why we require $\Phi_{\text{ext}} : G \times M \to M \times M$ to be proper, and not just $\Phi : G \times M \to M$. It turns out asking that Φ_{ext} be proper is a weaker requirement than asking that Φ be proper. This should make sense intuitively, as Φ_{ext} contains Φ in the first component of its image, but also remembers the 'pre-action' element of M in the second component, which restricts the possible elements in the pre-image of a compact set in $M \times M$.

To highlight this discrepancy, consider the group action of $\mathbb{R}$ on $\mathbb{R}$ by addition. The map $\Phi_{\text{ext}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $(g, x) \mapsto (g + x, x)$ is proper: take any compact set $K \subseteq \mathbb{R} \times \mathbb{R}$. Since K is closed and bounded, then $\pi_1(K) \subseteq \mathbb{R}$ is a closed bounded set of $\mathbb{R}$, which must be contained in a set of the form $[a, b]$, and similarly, $\pi_2(K)$ is a closed set contained in some $[c, d]$. Then the pre-image of K is a set of elements $(g, x) \in \mathbb{R} \times \mathbb{R}$ such that $g + x \in [a, b]$ *and* $x \in [c, d]$. This is the important thing to note, since we require that $x \in [c, d]$, then for $g + x$ to be in $[a, b]$, we must have $g \geq c - a$ and $g \leq d - b$, hence our first factor is bounded also, and the pre-image of K is closed and bounded, thus compact.

The important thing to notice is that this same group action viewed as the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is *not* proper. Indeed, consider any point $x \in \mathbb{R}$. Then the pre-image of x is actually isomorphic to all of $\mathbb{R}$, since the pre-image of any point x is an equivalence class $[(g, y)]_\sim \in \mathbb{R} \times \mathbb{R}$, where $(g, y) \sim (g', y') \Leftrightarrow g + y = g' + y' = x$. So, we can see that this map being proper is a much stronger requirement.

Another question one might have is why we even want a proper map in the first place. Well, a group action being proper allows us to say a lot more about the action than otherwise. To see how bad things can get if the action is not proper, consider the action of $\mathrm{GL}_n(\mathbb{R})$ acting on $\mathbb{R}^n$ by left multiplication. To see why this action is not proper, observe that

$$\Phi_{\text{ext}}^{-1}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \supseteq \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \middle| t \in \mathbb{R} \right\},$$

which is clearly not compact. Now, let us take a look at how bad the orbit space M/G of this action is. We know that for any two nonzero $v, w \in \mathbb{R}^n$, there exists some $A \in \mathrm{GL}_n(\mathbb{R})$ such that $Av = w$. Thus, all nonzero elements are contained in a single orbit. Since $A\vec{0} = \vec{0}$ for all $A \in \mathrm{GL}_n(\mathbb{R})$, we see that $\vec{0}$ is in its own orbit. Note that any neighbourhood of the origin contains some non-zero vector. Now consider $\pi(\vec{0}) \in M/G$ (letting $\pi : M \to M/G$ be our quotient map) and let $U \ni \pi(\vec{0})$ be any open set. Then $\pi^{-1}(U)$ is open, and thus there exists some nonzero $v \in \mathbb{R}^n$ such that $v \in \pi^{-1}(U)$. But then $\pi(v) \in U$, and thus $U = M/G$ (since there are only the two orbits). So, we conclude that the orbit space M/G is just $\{0, 1\}$ with the Seirpinski topology: $\{\varnothing, \{1\}, \{0, 1\}\}$. So if the action is not proper, the orbit spaces do not even have to be Hausdorff.

Now, we denote the set $G(A|B) = \{g \in G | A \cap gB \neq \varnothing\}$.

Example 1.3.2. Let $\Phi : G \times G \to G$ be defined by $\Phi(g, h) = gh$. That is, Φ is group multiplication on G. Then recall that $\Phi_{\text{ext}} : G \times G \to G \times G$ is given by $\Phi_{\text{ext}}(g, h) = (gh, h)$. Then notice that if we define $\Psi(g, h) \coloneqq (gh^{-1}, h)$, then this is a smooth map as well. Now, observe

$$\begin{aligned} \Psi \circ \Phi_{\text{ext}}(g, h) &= \Psi(gh, h) \\ &= (ghh^{-1}, h) \\ &= (g, h), \end{aligned}$$

and

$$\begin{aligned}\Phi_{\text{ext}} \circ \Psi(g,h) &= \Phi_{\text{ext}}(gh^{-1},h)\\ &= ((gh^{-1})h,h)\\ &= (g,h).\end{aligned}$$

Thus $\Psi = \Phi_{\text{ext}}^{-1}$. Then since Φ_{ext}^{-1} is a smooth map, we obtain that for any compact $K \in G \times G$, $\Phi_{\text{ext}}^{-1}(K)$ must be compact, by continuity of Ψ. Hence Φ_{ext} is a proper map, and the action is proper.

So, we showed that groups act properly on themselves by left or right multiplication.

Lemma 1.3.3. *Let $\Phi : G \times M \to M$ be a continuous action of a topological group G on a Hausdorff topological space M. Then Φ is a proper action if and only if for all $A, B \subseteq M$ precompact (resp. compact), $G(A|B)$ is precompact (resp. compact).*

Proof. $(\Rightarrow)$: Suppose that Φ is a proper action. Let A and B be precompact. Clearly $G(\overline{A},\overline{B}) \supseteq G(A|B)$ so to prove both cases, it suffices to show that $G(\overline{A},\overline{B})$ is compact. Indeed, for the compact set

$$\overline{A} \times \overline{B} \subseteq M \times M,$$

we know that

$$\Phi_{\text{ext}}^{-1}(\overline{A} \times \overline{B}) = \{(g,m)|m \in \overline{B}, g \cdot m \in \overline{A}\}$$

which by properness of Φ_{ext} is compact. Then applying π_1 to both sides here we see that

$$\pi_1(\Phi_{\text{ext}}^{-1}(\overline{A} \times \overline{B})) = G(\overline{A}|\overline{B}),$$

which by continuity of π_1 is compact.

$(\Leftarrow)$: Suppose now that for all $A, B \subseteq M$ precompact (resp. compact), $G(A|B)$ is precompact (resp. compact). Let $K \subseteq M \times M$ be compact. Then $\pi_1(K), \pi_2(K) \subseteq M$ are compact. Since $K \subseteq \pi_1(K) \times \pi_2(K)$, and K is closed, it suffices to show that $\Phi_{\text{ext}}^{-1}(\pi_1(K) \times \pi_2(K))$ is compact. Indeed,

$$\begin{aligned}\Phi_{\text{ext}}^{-1}(\pi_1(K) \times \pi_2(K)) &= \{(g,m) \in G \times M | m \in \pi_2(K), g \cdot m \in \pi_1(K)\}\\ &= G(\pi_1(K)|\pi_2(K)) \times \pi_2(K)\end{aligned}$$

which is compact as the product of compact sets. $\square$

The following is easy to see if we recall that every set in a compact Hausdorff space is precompact.

Corollary 1.3.4. *Every continuous action by a compact Lie group on a Hausdorff topological space is proper.*

Proposition 1.3.5. *Let $\Phi : G \times M \to M$ be a proper action. Then $\forall x \in M$, G_x is compact.*

Proof. Let $x \in M$ be arbitrary, then

$$G_x = \{g \in G | g \cdot x = x\} = \Phi_{\text{ext}}^{-1}((x, x))$$

which is compact, since Φ_{ext} is proper. □

If we have the added property that the action is free, we in fact obtain a very well-known result which says that M/G is in fact a smooth manifold. We provide the following theorem without proof, which the interested reader can find in [14, Chapter 21].

Theorem 1.3.6 (Quotient Manifold Theorem)**.** *Let G be a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a smooth manifold of dimension $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Corollary 1.3.7. *Let G be a Lie group acting properly on M, and let $x \in M$. Then $\pi : G \to G/G_x$ is a principal G_x-bundle.*

Proof. Since G_x is compact by Proposition 1.3.5 it is closed, and thus by Example 1.3.2 and Lemma 1.1.5 G_x acts properly on G. Also, groups act freely on themselves, and thus by Theorem 1.3.6, we obtain that G/G_x is a smooth manifold, and we have a smooth submersion $\pi : G \to G/G_x$. Then around any point $gG_x \in G/G_x$, we know that π admits a smooth section σ. That is, we have a neighbourhood $U \ni gG_x$ and a smooth map $\sigma : U \to G$ such that $\pi \circ \sigma = \text{Id}_U$.

Now, we claim that the map

$$\begin{aligned} \varphi : U \times G_x &\to \pi^{-1}(U) \\ (gG_x, h) &\mapsto \sigma(gG_x)h \end{aligned}$$

is a diffeomorphism. Clearly φ is smooth. Take any $s \in \pi^{-1}(U)$. That is $s \in G$ such that $\pi(s) = sG_x \in U$. Then since $\pi \circ \sigma(sG_x) = \text{Id}_U(sG_x) = sG_x$, we see that $\sigma(sG_x)G_x = sG_x$, and thus there exists $h_s \in G_x$ so that $\sigma(sG_x)h_s = s$. But then

$$\varphi(sG_x, h_s) = \sigma(sG_x)h_s = s$$

thus φ is a surjection, and noting that $h_s = \sigma(sG_x)^{-1}s \in G_x$, we can define the inverse

$$\begin{aligned}\psi : \pi^{-1}(U) &\to G_x \times U \\ \psi(s) &= (sG_x, \sigma(sG_x)^{-1}s)\end{aligned}$$

and clearly this is smooth (as a product of smooth maps). Hence φ is a diffeomorphism.

Now observe that for any $h \in G_x$, we have

$$\begin{aligned}\psi(s)h &= (sG_x, \sigma(sG_x)^{-1}s) \cdot h \\ &= (sG_x, \sigma(sG_x)^{-1}sh) \\ &\overbrace{=}^{h \in G_x} \Big(shG_x, \sigma(shG_x)^{-1}sh\Big) \\ &= \psi(sh)\end{aligned}$$

hence ψ is G_x-equivariant, and thus provides a local trivialization of $\pi^{-1}(U) \cong U \times G_x$.

Now, suppose that we have two local trivializations (U, ψ_U) and (V, ψ_V) such that $U \cap V \neq \varnothing$. That is, we have open sets $U, V \subseteq G/G_x$ and local sections $\sigma_U : U \to G$ and $\sigma_V : V \to G$ such that $\pi \circ \sigma_U = \mathrm{Id}_U$ and $\pi \circ \sigma_V = \mathrm{Id}_V$.

Then observe that for the map $\psi_U \circ \psi_V^{-1} : (U \cap V) \times G_x \to (U \cap V) \times G_x$ we get

$$\begin{aligned}\psi_U \circ \psi_V^{-1}(sG_x, h) &= \psi_U(\sigma_V(sG_x)h) \\ &= \Big(\sigma_V(sG_x)hG_x, \sigma_U(\sigma_V(sG_x)hG_x)^{-1}\sigma_V(sG_x)h\Big)\end{aligned}$$

and although this last line looks messy, once we remember that $h \in G_x$, and that $\sigma_V(sG_x)G_x = sG_x$, so $\sigma_V(sG_x)hG_x = sG_x$, we obtain

$$\psi_U \circ \psi_V^{-1}(sG_x, h) = \Big(sG_x, \sigma_U^{-1}(sG_x)\sigma_V(sG_x)h\Big)$$

and so the transition functions are exactly $\tau_{VU}(sG_x) = \sigma_U^{-1}(sG_x)\sigma_V(sG_x)$. But then, since $\pi(\sigma_V(sG_x)) = sG_x = \pi(\sigma_U(sG_x))$, we obtain that $\sigma_U^{-1}(sG_x)\sigma_V(sG_x) \in G_x$. Then $\tau_{VU} : U \cap V \to G_x$, and $\pi : G \to G/G_x$ is a G_x-principal bundle. □

One nice property of compact Lie groups is that when acting on a manifold, the orbits are compact embedded submanifolds. For an arbitrary Lie group acting properly, although we do not necessarily get that the orbits are compact, we do still get something 'almost as nice'.

Most of the remaining results and their proofs can be found in [21, Theorem 4.2.4].

Theorem 1.3.8. *Let $\Phi : G \times M \to M$ be a proper smooth action of a Lie group G on a smooth manifold M. Then the map $\Phi_x : G/G_x \to M$ defined by $gG_x \mapsto g \cdot x$ is an embedding.*

Proof. Let $x \in M$ be arbitrary. We know from Corollary 1.3.7 that G/G_x is a smooth manifold.
<u>Claim:</u> Φ_x is a proper map.
Proof of Claim: Let $K \subseteq M$ be compact. Observe that

$$\begin{aligned}\Phi_x^{-1}(K) &= \{gG_x \in G/G_x | g \cdot x \in K\}\\ &= G/G_x\Big(K\Big|x\Big),\end{aligned}$$

which is compact by Lemma 1.3.3. ■

Since Φ_x is proper, by Lemma 1.1.3 it suffices to show that Φ_x is an injective immersion.

First suppose that $\Phi_x(g) = \Phi_x(h)$. Then $gx = hx$ and thus $h^{-1}gx = x$ so that $h^{-1}g \in G_x$, and $gG_x = hG_x$. Hence Φ_x is injective.

Since the tangent space of G/G_x at any point is canonically isomorphic to $T_e(G/G_x)$, and Φ_x is G-equivariant it suffices to show that $[(\Phi_x)_*]_{eG_x}$ is injective. Let $v \in T_{eG_x}(G/G_x)$ be such that $[(\Phi_x)_*]_{eG_x}(v) = 0$. Since $\pi : G \to G/G_x$ is a surjective submersion, we know that there exists some $\eta \in T_eG$ such that $[\pi_*]_e(\eta) = v$. Then, noticing that $\Phi(-, x) = \Phi_x \circ \pi$, we compute

$$\begin{aligned}[(\Phi(-, x))_*]_e(\eta) &= [(\Phi_x)_*]_{eG_x} \circ [\pi_*]_e(\eta)\\ &= [(\Phi_x)_*]_{eG_x}(v)\\ &= 0.\end{aligned}$$

Then consider the curve $\exp(t\eta)$:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(\exp(t\eta), x)\Big|_{t=s} &= \frac{\partial}{\partial t}\Phi(\exp((t+s)\eta), x)\Big|_{t=0}\\ &= \frac{\partial}{\partial t}\Phi(\exp(s)\eta, \exp(t\eta)x)\Big|_{t=0}\\ &= [(\Phi(\exp(s), -))_*]_x \circ [(\Phi(-, x))_*]_e(\eta)\\ &= 0,\end{aligned}$$

which is true for any $s \in \mathbb{R}$, hence $\exp(t\eta) \cdot x = x$. That is, $\pi(\exp(t\eta)) = eG_x$, for all $t \in \mathbb{R}$, so that $v = [\pi_*]_e(\eta) = 0$. Hence Φ_x is an immersion. □

Corollary 1.3.9. *For a smooth proper action of a Lie group G on a smooth manifold M, the orbits are closed submanifolds of M.*

We saw in Proposition 1.2.3 that $\pi : M \to M/G$ is always an open map. It turns out that if the action is proper, this map is also closed:

Proposition 1.3.10. *Let $\Phi : G \times M \to M$ be a smooth proper action of a Lie group G on a manifold M. Then $\pi : M \to G/M$ is closed, and M/G is locally compact Hausdorff and second countable.*

Proof. Let $A \subseteq M$ be closed. By Theorem 1.1.2, $\Phi_{\text{ext}}(A) = GA \times A$ is closed in $M \times M$, hence GA is closed since π_1 is closed. Since $\pi^{-1}(\pi(A)) = GA$, we get that $\pi(A)$ is closed by definition of the quotient topology.

Now take $x, y \in M$ such that $Gy \neq Gx$. We use a slight abuse of notation and let Gx, Gy denote the equivalence classes of their respective orbits in M/G. Since Gx and Gy are both closed, and M is normal, we know that there exist open sets $U \supseteq Gx$ and $V \supseteq Gy$ such that $\overline{U} \cap \overline{V} = \varnothing$. By Proposition 1.2.3, $\pi(U)$ is an open set containing Gx. And by the fact that π is closed, we know that $(M/G)\backslash\pi(\overline{U})$ is an open set containing Gy. So, M/G is Hausdorff.

Finally, second countability of M/G is a direct consequence of Proposition 1.2.3, as any countable basis of M descends to a countable basis of M/G. □

The following then follows from the fact that a second countable, locally compact Hausdorff space is paracompact, the proof of which can be found in [15][Theorem 4.77].

Corollary 1.3.11. *For a Lie group G acting properly on a manifold M, the orbit space M/G is paracompact.*

1.4 A G-invariant Riemannian metric

We now introduce the final (and arguably the most important) piece of machinery, which allows us to prove the Slice Theorem.

Theorem 1.4.1. *Let G be a Lie group acting properly on a smooth manifold M. There exists a G-invariant Riemannian metric on M.*

The proof of this is not trivial, and it uses the following two important topological properties that properness gives us.

Lemma 1.4.2. *Let G be a Lie group acting properly on a smooth manifold M. Then*

1. *Any cover of M by G-invariant open sets admits a subordinate partition of unity by G-invariant smooth functions.*

2. *The algebra of G-invariant smooth functions, $C^\infty(M)^G$, separates orbits of M.*

Proof of Lemma. Let $\{U_i\}_{i\in I}$ be a G-invariant open cover of M, which exists by Lemma 1.2.8.

Now, we know that by Proposition 1.2.3, $\pi : M \to M/G$ is an open map, and thus $\{\pi(U_i)\}_{i\in I}$ is an open cover of M/G. Since the action is proper, we have by Corollary 1.3.11 that M/G is paracompact, and thus there exists a locally finite refinement $\{W_\alpha\}_{\alpha\in\Lambda}$ of $\{\pi(U_i)\}_{i\in i}$. Now if there are multiple W_α's contained in a single $\pi(U_i)$, replacing them with

$$W_i = \bigcup_{\alpha\in\Lambda, W_\alpha\subseteq U_i} W_\alpha$$

preserves local finiteness, so that $\{W_i\}_{i\in I}$ is a locally finite cover of M/G. It is not difficult to see that $\{\pi^{-1}W_i\}_{i\in I}$ is a locally finite cover of M. Indeed, taking any p in M we know that there exists an open neighbourhood $B_\varepsilon(\pi(p))$ in M/G that intersects only finitely many W_i, and taking pre-images we see that $\{\pi^{-1}(W_i)\}_{i\in I}$ is a locally finite cover of M.

Now we know that there exists a locally finite partition of unity of M along with a mapping $\iota : \mathbb{N} \to I$ such that $\operatorname{supp}\psi_j \subseteq \pi^{-1}(W_{\iota(j)})$, and such that $\operatorname{supp}\psi_j$ is compact for each j. Now, choosing a right invariant Haar measure μ on G, we define the function

$$\psi_j^G(x) = \int_G \psi_j(g\cdot x)d\mu(g).$$

We claim that this is smooth: By smoothness of the group action and ψ_j, the integrand is smooth in x. Furthermore, as ψ_j is compactly supported, applying the Dominated Convergence Theorem allows us to differentiate under the integral sign.

Next we claim that $\operatorname{supp}\varphi_j^G \subseteq \pi^{-1}(W_\iota(j))$. Suppose towards a contradiction that $x \notin \pi^{-1}(W_{\iota(j)})$, but $x \in \operatorname{supp}(\psi_j^G)$. Since $x \notin \pi^{-1}(W_{\iota(j)})$, we know that $\psi_j(x) = 0$. Then there must be some $h \in G$ such that $\psi_j(h \cdot x) \neq 0$. But then $h \cdot x \in \pi^{-1}(W_{\iota(j)})$, so $[x] \in W_{\iota(j)}$, a contradiction. Hence $\operatorname{supp}(\psi_j^G) \subseteq \pi^{-1}(W_{\iota(j)})$.

Now, observe that the cover $\{\operatorname{supp}\psi_j^G\}$ need not be a locally finite; just because $x \in \operatorname{supp}\psi_j$ for finitely many j, there is no reason for us to believe that $(G\cdot x)\cap\operatorname{supp}\psi_j \neq \varnothing$ for only finitely many j. Since $C^\infty(M)$ is a Fréchet space, there exists a sequence of seminorms $\|\cdot\|_j$ on $C^\infty(M)$ defining the Fréchet topology. Since we can always define new seminorms $p_j(\cdot) = \sum_{k=1}^{j}\|\cdot\|_j$, we can assume that $\|\cdot\|_j$ is an increasing family of seminorms: $\|\cdot\|_j \leq \|\cdot\|_{j+1}$ for all j. Now, define

$$\tilde{\varphi}_i = \sum_{j\in\mathbb{N},\iota(j)=i} \frac{1}{2^j\|\psi_j^G\|_j}\psi_j^G,$$

and we can see that $\tilde{\varphi}_i$ is G-invariant and smooth. Furthermore, $\operatorname{supp}(\tilde{\varphi}_i)\subseteq\pi^{-1}(W_i)$, and thus $\{\operatorname{supp}(\tilde{\varphi}_i)\}_{i\in I}$ is a locally finite covering of M. Thus $\sum_{i\in I}\tilde{\varphi}_i < \infty$ everywhere and we can set

$$\varphi_i = \frac{\tilde{\varphi}_i}{\sum_{i\in I}\tilde{\varphi}_i}$$

so that φ_i is G-invariant and $\{\varphi_i\}$ is a partition of unity on M, subordinate to the cover $\{\pi^{-1}(W_i)\}_{i\in I}$ and thus subordinate to $\{U_i\}_{i\in I}$, as desired. This concludes the proof of the first statement.

Now, let $x, y \in M$ such that $\pi(x)\neq\pi(y)$. Since M/G is paracompact and Hausdorff, it is normal, and there exist disjoint neighbourhoods $W_x \ni \pi(x)$ and $W_y \ni \pi(y)$. Then $\{\pi^{-1}(W_x), \pi^{-1}(W_y), M\backslash((G\cdot x)\cup(G\cdot y))\}$ is a G-invariant open cover of M, and there exists a subordinate partition of unity $\{\varphi_x, \varphi_y, \varphi_0\}$, which separates $G\cdot x$ and $G\cdot y$. $\square$

We can now prove the existence of a G-invariant Riemannian metric on M.

Proof of Theorem 1.4.1. Let η be an arbitrary Riemannian metric on M, and let $(K_j)_{j\in\mathbb{N}}$ be a compact exhaustion of M. Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a set of cutoff functions $\chi_j : M \to [0,1]$ such that $\operatorname{supp}(\chi_j)\subseteq \operatorname{Int}(K_{j+1})$ and $\chi_j(x) = 1, \forall x \in K_j$. Then letting μ be a right invariant Haar measure on G (as in Lemma 1.4.2), we can define a symmetric 2-form on M with support in K_j by

$$\eta_j(x)(v,w) = \int_G \chi_j(g\cdot x)\eta(g\cdot x)([(\Phi_g)_*]_x(v), [(\Phi_g)_*]_x(w))d\mu(g)$$

where $v, w \in T_xM$. By construction η_j is G-invariant, and for $x \in G\cdot K_j$, it is in fact positive definite. Now since $\{G\cdot\operatorname{Int}(K_j)\}_{j\in\mathbb{N}}$ is a G-invariant open cover of M, by Lemma 1.4.2

there exists a subordinate G-invariant partition of unity $\{\varphi_j\}$. Then we can construct a Riemannian metric γ on all of M by setting

$$\gamma(x) = \sum_{j\in\mathbb{N}} \varphi_j(x)\eta_j(x)$$

for all $x \in M$. For all $g \in G$ and $v, w \in TM$ we see that

$$\begin{aligned} g \cdot \gamma(x)(v,w) &\coloneqq ((\Phi_g)^*)_x(\gamma(x)(v,w)) \\ &= \sum_{j\in\mathbb{N}}((\Phi_g)^*)_x(\varphi_j(x)\eta_j(x)(v,w)) \\ &= \sum_{j\in\mathbb{N}}(\varphi_j(g\cdot x))((\Phi_g)^*)_x\eta_j(x)(v,w)) \\ &= \sum_{j\in\mathbb{N}} \varphi_j(x)\eta_j(x)(v,w) \\ &= \gamma(x)(v,w) \end{aligned}$$

hence γ is a G-invariant Riemannian metric on M. $\square$

1.5 The Slice Theorem

We saw in Section 1.3 that for a proper action of G on M, the orbits were properly embedded submanifolds of M. Then if we equip M with a Riemannian metric, we obtain that for any $x \in M$, the Tubular Neighbourhood Theorem [13, Theorem 5.25] tells us that there exists a tubular neighbourhood of $G \cdot x$. But we can do much better than this. We showed in Theorem 1.4.1 that there exists a G-invariant Riemannian metric on M. It turns out that if the metric on M is a G-invariant metric, then we obtain a tubular neighbourhood that decomposes into G_x-invariant cross-sections, and that the cross section containing x 'traces out' the tubular neighbourhood, under the action of G. That is, action by G 'slides' this cross section, or rather normal submanifold at x, along the orbit so that it remains normal to the orbit as it traces out the entire tubular neighbourhood.

In this section we will formalize the idea of this 'cross-section' and then prove the existence of this 'fancy' tubular neighbourhood, which is precisely the statement of the Slice Theorem.

Before we state the slice theorem, we first introduce the notion of a 'slice'.

Definition 1.5.1. Let G be a Lie group acting on a smooth manifold M, and equip M with a G-invariant Riemannian metric γ as in Theorem 1.4.1. Consider for some $x \in M$, the normal space $V_x = T_xM/T_x(Gx)$ to the orbit Gx. For any $g \in G_x$, the differential of the action, $(\Phi_g)_*$, sends a vector in T_xM back into T_xM, and thus, since it is an isometry, induces an automorphism on V_x. This gives rise to a representation $\rho : G_x \to \mathrm{GL}(V_x)$, which is defined by $\rho(g)([v]) = q \circ (\Phi_g)_*(v)$ where $q : T_xM \to T_xM/T_x(Gx)$ is the quotient map. Now, since $G \to G/G_x$ is a principal G_x bundle by Corollary 1.3.7, we obtain the associated bundle $G \times_\rho V_x$. This is referred to as the *slice bundle*. For small enough ε, we know that the exponential is defined on $V_x^\varepsilon = \{v \in V_x | \|v\|_\gamma < \varepsilon\}$. We call V_x^ε a *slice* at x.

Theorem 1.5.2 (The Slice Theorem)**.** *Let $\Phi : G \times M \to M$ be a smooth proper action of a Lie group G on a smooth manifold M. Then for any $x \in M$, there exists a G-equivariant diffeomorphism from a G-invariant neighbourhood of the zero section of $G \times_\rho V_x$ onto a G-invariant neighbourhood of Gx such that the zero section is mapped to the orbit Gx.*

In order to prove the Slice Theorem, the following lemma from [11, Chapter 1.3] is useful:

Lemma 1.5.3. *Let $\Phi : G \times M \to M$ be a proper action. Then for all $x \in M$, and every open set $W \supseteq G_x$ in G, there exists a neighbourhood U of x such that $G(U|U) \subseteq W$.*

Proof of Lemma. Let V be a neighbourhood of x such that $G(V|V)$ is precompact (the existence of such a set is guaranteed by Lemma 1.3.3). Let $A = G(V|V) \backslash W \subseteq W^C$. If A is empty, we are done (as V would then be our desired neighbourhood), so we can assume it is non-empty. Since W^C is closed, $\overline{A} \subseteq W^C$, and thus $\overline{A} \cap G_x = \varnothing$. Hence for all $a \in \overline{A}$, there exist neighbourhoods C_a of a and V_a of x such that $C_aV_a \cap V_a = \varnothing$.

Then $\bigcup_{a \in \overline{A}} C_a$ is an open cover of $\overline{A}$. Note that $\overline{A}$ is compact, as $A \subseteq G(V|V)$, and $G(V|V)$ is precompact by choice of V. Then there exists a finite subset $F \subseteq \overline{A}$, so that $\bigcup_{a \in F} C_a$ is an open cover of $\overline{A}$. Then $U = V \cap \bigcap_{a \in F} V_a$ is open, and

$$G(U|U) \subseteq \cap_{a \in F} G(V_a|V_a) \subseteq G(V|V), \text{ and}$$
$$G(U|U) \cap A \subseteq (\bigcap_{a \in F} G(V_a|V_a)) \cap (\bigcup_{a \in F} C_a) = \varnothing$$

Hence it must be that $G(U|U) \subseteq W$. □

Proof of Slice Theorem. We proceed as in [11]. Let γ be a G-invariant metric on M, which exists by Theorem 1.4.1. Let x be any point in M. Take $\varepsilon > 0$ such that the exponential map exp is defined on the neighbourhood $V_x^\varepsilon = \{v \in V_x | \|v\|_\gamma < \varepsilon\}$. By G-invariance of γ, we know that for all $v \in V_x^\varepsilon$, and $g \in G$, we have the following:

$$\begin{aligned} g \cdot \exp(v) &= \exp(\rho(g)v), \\ d(x, \exp(v)) &= \|v\|_\gamma. \end{aligned}$$

Now, as per Definition 1.5.1, we know that $(G \times V_x^\varepsilon, \pi, G \times_{G_x} V_x^\varepsilon)$ is a G_x-principal bundle. Then we define

$$\begin{aligned} \phi : G \times V_x^\varepsilon &\to M, \\ (g, v) &\mapsto g \cdot \exp(v), \end{aligned}$$

which is smooth. And so we are left with the following commutative diagram:

$$\begin{tikzcd} G \times V_x^\varepsilon \arrow[r, "\phi"] \arrow[d, "\pi"'] & M \\ G \times_{G_x} V_x^\varepsilon \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Since ϕ is smooth, and π is surjective, we know that φ is smooth. And since ϕ is of maximal rank at $(e, 0)$, so too is φ at $\pi(e, 0)$.

Recall that $G \times_{G_x} V_x^\varepsilon$ is an associated bundle for the principal G_x-bundle $G \to G/G_x$, and $\dim(G \times_{G_x} V_x^\varepsilon) = \dim G/G_x + \dim V_x^\varepsilon = \dim M$. Hence by the inverse function theorem, there exists a neighbourhood of $\pi(e, 0)$ in $G \times_{G_x} V_x^\varepsilon$ over which φ is a diffeomorphism. Then there is a neighbourhood W of G_x, so that after possibly shrinking ε we have $\varphi|_{\pi(W \times V_x^\varepsilon)}$ is a diffeomorphism onto an open set in M. Now, by Lemma 1.5.3, after possibly shrinking ε (again), we can assume that the open ball $U = \{y \in M | d(x, y) < 2\varepsilon\}$ satisfies $G(U|U) \subseteq W$.

Recall that for any $g \in G$, $g \cdot - : M \to M$ is a diffeomorphism, so that $g \cdot \varphi$ is a diffeomorphism on $\pi(W \times V_x^\varepsilon)$. But φ respects the G-action. That is, for any $g \in G$, and any $[h, v] \in \pi(G \times V_x^\varepsilon)$, we have

$$\varphi(g \cdot [h, v]) = \varphi([gh, v]) = (gh) \cdot \exp(v) = g \cdot (h \cdot \exp(v)) = g \cdot \varphi([h, v]).$$

Thus since $G\pi(W \times V_x^\varepsilon) = \pi(G \times V_x^\varepsilon)$, we see that φ is of maximal rank on $\pi(G \times V_x^\varepsilon)$. Then to show that φ is a diffeomorphism, it remains only to show that it is injective. To obtain injectivity, it suffices to show that if $\phi(g, v) = \phi(g', v')$, then $[g', v'] \sim [g, v]$.

Indeed, suppose that $\phi(g,v) = \phi(g',v')$. Then by definition $g \cdot \exp(v) = g' \cdot \exp(v')$, so that $\exp(v) = g^{-1}g' \cdot \exp(v')$. Then

$$\begin{aligned} d(x, g^{-1}g' \cdot x) &\leq d(x, \exp(v)) + d(\exp(v), g^{-1}g'x) \\ &= \|v\| + \underbrace{d(g^{-1}g' \exp(v'), g^{-1}g'x)}_{=d(\exp(v'),x),\text{ by } G\text{-invariance}} \\ &= \|v\| + \|v'\| < 2\varepsilon \end{aligned}$$

Hence $g^{-1}g'x \in U$ so that $g^{-1}g' \in W$, and thus by injectivity on $\pi(W \times V_x^\varepsilon)$, $[e,v] = [g^{-1}g', v']$, so that $[g,v] = [g',v']$. $\square$

Chapter 2

Sobolev Spaces of Sections

Unfortunately, the Slice Theorem does not translate into infinite dimensions as nicely as we would like. This is largely due to the lack of an infinite dimensional inverse function theorem, which in the previous section we used to prove the Slice Theorem. However, given certain infinite dimensional manifolds, special cases of the Slice Theorem have been developed. In this chapter we consider two cases of an infinite dimensional analogue for the Slice Theorem. First we study the space of Riemannian metrics on a compact manifold M under the action by the group of diffeomorphisms on M. Second, we study the space of connections on a bundle E with unitary structure group G over a compact, oriented Riemannian manifold M being acted on by the gauge group.

2.1 Differential Operators on Vector Bundles

Since we cannot use the same techniques to establish these special infinite dimensional Slice Theorems as we did in the finite dimensional case, we need to develop a lot of new machinery. That is to say that switching to infinite dimensional manifolds and Lie groups requires a lot more analysis, for which this section provides a brief overview of the important results we need. We assume the reader has taken an introductory course in functional analysis.

Let E and F be vector bundles of rank r, l respectively, over a smooth manifold M and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of M such that E and F trivialize over each U_α, and we have coordinates $(x^1, ..., x^n)$ over each U_α.

Let $\Gamma(E)$ denote the space of smooth sections of the vector bundle E, and $\Gamma_c(E)$ denote the space of compactly supported smooth sections of the vector bundle E.

Let $\{s_1, ..., s_r\}$ be a local frame for E over U_α. Then we can write $\xi = \xi^i s_i$.

Definition 2.1.1. A *linear, differential operator of order k* is a linear map $P : \Gamma(E) \to \Gamma(F)$ such that in coordinates, and with respect to our local frames, we can write

$$\Big(P\xi(x)\Big)^i = \sum_{j=1}^{r} \sum_{0\leq|\lambda|\leq k} P_{ij}^{\lambda}(x) \frac{\partial^{|\lambda|}\xi^j}{(\partial x^1)^{\lambda_1}\cdots(\partial x^n)^{\lambda_n}}$$

where the λ are multi-indices $(\lambda_1, ..., \lambda_n)$, with $|\lambda| = \lambda_1 + \cdots + \lambda_n$, and for each i, j, $P_{ij} \in C^\infty(U_\alpha)$. Also it must be that there exists at least one non-zero P_{ij}^{λ} for $|\lambda| = k$ (otherwise the operator would be of order $q < k$ where q highest is the highest integer such that there is $P_{ij}^{\lambda} \neq 0$ for $|\lambda| = q$).

A familiar example of a differential operator is the gradient operator $\nabla : C^\infty(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$ defined in Cartesian coordinates by $\nabla = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}$. We can see from the expression of the gradient that it has order 1. Also, a very common differential operator that shows up in physics is the Laplacian $\Delta : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$, which is defined as the composition of the gradient with the divergence. That is, $\Delta = \operatorname{div}(\nabla f)$, and in Euclidean coordinates this looks like $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$, where we can see that clearly this is of order 2.

The exterior derivative $d : \Omega^m(M) \to \Omega^{m+1}(M)$ (where Ω^m denotes the space of m-forms on M) is another commonly seen differential operator, and has order 1.

The above differential operators are very well understood, and admit nice properties that do not necessarily hold true for any differential operator. However, we can characterize any differential operators by its *symbol*, which can tell us more about how the operator behaves.

Definition 2.1.2. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator of order k. Let $x \in M$, and take some $t \in T_x^*M$. Take f differentiable in a neighbourhood of x such that $f(x) = 0$ and $df_x = t$ and then define

$$\begin{aligned} \sigma_t(P) : E_x &\to F_x \\ \xi &\mapsto \frac{1}{k!}P(f^k\xi)(x). \end{aligned}$$

It is easy to see that $\sigma_t(P)$ is a linear map (as P is). Furthermore, by requiring that $f(x) = 0$, we obtain the property that the symbol depends only on the highest order derivatives of P. That is, the symbol of a differential operator P of order k depends only on the terms with multi-index λ such that $|\lambda| = k$.

We say that P has ***injective symbol*** at $x \in M$ if for all nonzero $t \in T_x^*M$ the map $\sigma_t(P) : E_x \to F_x$ is injective. If moreover $\sigma_t(P) : E_x \to F_x$ is *bijective*, then we say that P is ***elliptic*** at x.

If we assume that M is orientable, and equip M with a Riemannian metric g, we obtain a corresponding Riemannian volume measure V_g on M. Now, taking some inner product $(\cdot,\cdot)$ on E defined pointwise, for $\xi, \zeta \in \Gamma_c(E)$, $(\xi,\zeta)_E$ is compactly supported, and thus we define the L^2*-inner product* on $\Gamma_c(E)$ by

$$\langle \xi, \zeta \rangle_{L^2(E)} \coloneqq \int_M (\xi,\zeta) dV_g.$$

We denote the completion of $\Gamma_c(E)$ with respect to this induced norm of this inner product as $L^2(E)$, so $L^2(E)$ is a Hilbert space.

Similarly if we have a fibre metric $(\cdot,\cdot)_F$ on F defined pointwise, we can repeat the above construction to obtain the Hilbert space $L^2(F)$.

Definition 2.1.3. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator of order k. We say that the operator $P^* : \Gamma(F) \to \Gamma(E)$ is a *formal adjoint* of P if for all $\xi \in \Gamma(E)$ and $\zeta \in \Gamma_c(F)$ we have

$$\langle P\xi, \zeta \rangle_{L^2(F)} = \langle \xi, P^*\zeta \rangle_{L^2(E)}.$$

It turns out that for any given differential operator, it has a unique formal adjoint operator.

Proposition 2.1.4. *Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator of order k. Then there exists a unique map $P^* : \Gamma(F) \to \Gamma(E)$ as defined in Definition 2.1.3. Furthermore, for any $x \in M$ and $t \in T_x^*M$, we have $\sigma_t(P^*) = (-1)^k \sigma_t(P)^*$.*

Proof. See [20, Section 4] □

Corollary 2.1.5. *If $P : \Gamma(E) \to \Gamma(F)$ is a differential operator of order k with injective symbol, then P^*P is an elliptic differential operator.*

Proof. Indeed, P^*P has symbol $(-1)^k\sigma_t(P)^* \circ \sigma_t(P)$, and since $\sigma_t(P)$ is injective, $\sigma_t(P)^*$ must be surjective, hence $(-1)^k\sigma_t(P)^* \circ \sigma_t(P)$ is a bijection. □

The following example will be useful to us in section 3.6.

Example 2.1.6. Let γ be a Riemannian metric on M, and let $\mathrm{Sym}_2(M)$ denote the space of smooth symmetric 2-forms over M. Then

$$\begin{aligned} \alpha_\gamma : \mathfrak{X}(M) &\to \mathrm{Sym}_2(M) \\ X &\mapsto \mathcal{L}_X\gamma, \end{aligned}$$

where $\mathcal{L}_X\gamma$ is the Lie derivative of γ in the direction X, is a first order linear operator with injective symbol.

Proof. Let $X \in \mathfrak{X}(M)$, and $\omega \in T^*M \otimes T^*M$. [14, Example 12.35] shows us that we can write out the Lie derivative in coordinates as

$$\mathcal{L}_X\omega = \big(X\omega_{ij} + \omega_{kj}\frac{\partial X^k}{\partial x^i} + \omega_{ik}\frac{\partial X^k}{\partial x^j}\big)dx^i \otimes dx^j,$$

where it becomes evident that it is indeed a first order differential operator.

Now take $t \in T_x^*M$, and take $f \in C^\infty(M)$ such that $df_x = t$, and $f(x) = 0$. Then

$$\sigma_t(\alpha_\gamma) : T_xM \to (\mathrm{Sym}_2(M))_x \subseteq T^*M \otimes T^*M$$

is defined on some $v \in T_xM$ and if we extend it to some vector field $X \in \mathfrak{X}(M)$ (so that it makes sense to apply the Lie derivative),

$$\sigma_t(\alpha_\gamma)(v) = (L_{fX}\gamma)_x = t \otimes \iota_{X_x}\gamma + \iota_{X_x}\gamma \otimes t$$

where the second equality can be worked out by evaluating $(L_{fX}\gamma)(Y_1, Y_2)$ for vector fields Y_1, Y_2 and using various properties of the Lie derivative, and the fact that $f(x) = 0$. From this we can see that $\sigma_t(\alpha_\gamma)(v) \in (\mathrm{Sym}_2(M))_x$ is independent of the extension X of v.

It remains to show that $v \mapsto t \otimes \iota_v\gamma + \iota_v\gamma \otimes t$ is injective. Suppose $t \otimes \iota_v\gamma + \iota_v\gamma \otimes t = 0$. Then let $e^1 = t$, and extend to a basis $\{e^1, ..., e^n\}$ of T_x^*M. Then we write $\iota_v\gamma = \gamma_{ij}e^i(v)e^j$, so that

$$\gamma_{ij}e^i(v)e^j \otimes e^1 + \gamma_{ij}e^i(v)e^1 \otimes e^j = 0$$

and thus for any vectors $w = w^ie_i, u = u^ie_i \in T_xM$ we obtain

$$\gamma_{ij}v^iw^ju^1 = -\gamma_{ij}v^iu^jw^1. \tag{2.1}$$

Suppose $v^1 \neq 0$. Then letting $w = u = v$ gives us that

$$\gamma_{ij}v^iv^jv^1 = -\gamma_{ij}v^iv^jv^1,$$

and thus $\gamma_{ij}v^iv^j = 0$, so that $v = 0$ by positive definiteness of γ.

If instead $v^1 = 0$, then let $u = v$ and $w = e_1$, and substitute into (2.1) to give us

$$\gamma_{ij}v^iv^j = 0,$$

and again by positive definiteness, $v = 0$.

So, in either case, we must have $v = 0$, and thus $\sigma_t : T_xM \to (\mathrm{Sym}_2(M))_x$ is injective. □

Example 2.1.7. Let A denote a connection $\nabla^A : \Gamma(E) \to \Gamma(T^*M \otimes E)$. The induced connection on the endomorphism bundle $d_A : \Gamma(\mathrm{End}(E)) \to \Gamma(T^*M \otimes \mathrm{End}(E))$ is a first order linear operator with injective symbol.

Proof. For $\theta \in \Gamma(\mathrm{End}(E))$, we know that in local coordinates and with respect to some frame, we can write

$$d_A = d + [A, \cdot]$$

where A is the connection matrix for ∇^A with respect to the local frame. From this, it is clear that d_A is a first order linear differential operator.

Now, let $x \in M$ and let $t \in T_x^*M$ and $f \in C^\infty(M)$ such that $df_x = t$. Then for any $\beta \in \mathrm{End}(E)_x$

$$\begin{aligned}\sigma_t(d_A)(\beta) &= d_A(f\beta)(x)\\ &= \big(df \otimes \beta\big)(x) + \big(f \otimes d_A(\beta)\big)(x)\\ &= (t \otimes \beta)(x)\end{aligned}$$

Hence $\sigma_t(d_A) : \mathrm{End}(E)_x \to T_x^*M \otimes \mathrm{End}(E)_x$ is defined by $\beta \mapsto t \otimes \beta$, which is clearly injective. □

2.2 Sobolev Spaces and Embeddings

We would like to eventually work with the space of smooth sections of a vector bundle. However we run into the problem that this space is not a very 'nice' space. For instance, this space is not even complete. However, we can embed this space into much nicer spaces; Hilbert spaces even. In this section we walk through various equivalent constructions of these aforementioned Hilbert spaces. We then give important topological properties of these spaces.

For the remainder of this chapter, let E be a rank r vector bundle over a compact n-dimensional oriented manifold M, and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of M by domains of coordinate charts such that E is trivial over U_α, and let $\{\rho_\alpha\}_{\alpha\in\Lambda}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$. Since M is compact, we can assume that Λ is finite. Let $\Gamma(E)$ denote the space of smooth sections on E.

2.2.1 Sobolev spaces on M via coordinates

Since U_α is diffeomorphic to a subset of $\mathbb{R}^n$ we can use the standard Euclidean volume measure dV to define the standard L^2-norm $\|\cdot\|_{L^2(U_\alpha)}$ for smooth functions $f \in C^\infty(U_\alpha)$:

$$\|f\|_{L^2(U_\alpha)} = \Big(\int_{U_\alpha} |f|^2 dV\Big)^{\frac{1}{2}}.$$

Now, let λ be a multi-index, and let $(x^1, ..., x^n)$ denote the local coordinates on U_α. Then we denote $D_\lambda = \frac{\partial^{|\lambda|}}{(\partial x^1)^{\lambda_1}, ..., (\partial x^n)^{\lambda_n}}$.

Now, recall that for a section $\xi \in \Gamma(U_\alpha)$, we can use a local frame $\{s_1^\alpha, ..., s_r^\alpha\}$ on U_α to write $\xi = \xi^j s_j^\alpha$, where each $\xi^i \in C^\infty(U_\alpha)$.

Then for each ξ^j, $\xi \in \Gamma(U_\alpha)$, and $D_\lambda(\rho_\alpha \xi^j) \in C_0^\infty(U_\alpha)$. Then we can define the ***Sobolev k-norm*** of $\xi \in \Gamma(E|_{U_\alpha})$ on U_α by

$$\|\xi\|_{H^k(U_\alpha)} = \sum_{j=1}^{r} \sum_{0\leq|\lambda|\leq k} \|D_\lambda(\rho_\alpha \xi^j)\|_{L^2(U_\alpha)}. \tag{2.2}$$

Similarly, if $\xi \in \Gamma(E)$, we can express $\xi|_{U_\alpha} = \xi_\alpha$, and we know that ξ is well-defined on overlaps. So, we can write $D_\lambda \xi = \sum_{\alpha\in\Lambda}(D_\lambda \rho_\alpha \xi_\alpha^i)$. Since Λ is finite, this is well-defined. And thus we can write the ***Sobolev k-norm*** of $\xi \in \Gamma(E)$ as

$$\|\xi\|_{H^k(E)} = \sum_{\alpha\in\Lambda} \sum_{j=1}^{r} \sum_{0\leq|\lambda|\leq k} \|(D_\lambda(\rho_\alpha \xi^j))\|_{L^2(U_\alpha)}. \tag{2.3}$$

The reader is encouraged to check that (2.3) is in fact a norm on $\Gamma(E)$. Note that the norm (2.3) depends on the choice of coordinates and partition of unity, however the norms induced by different choices are all equivalent. We then define $H^k(E)$ as the completion of $\Gamma(E)$ with respect to this norm. Note that as this norm is induced via the L^2-inner product of the form

$$\langle \xi, \zeta \rangle_{H^k(E)} = \sum_{\alpha \in \Lambda} \sum_{j=1}^{r} \sum_{0 \leq |\lambda| \leq k} \langle D_\lambda(\rho_\alpha \xi^j), D_\lambda(\rho_\alpha \zeta^j) \rangle_{L^2(U_\alpha)} \tag{2.4}$$

we obtain that $H^k(E)$ is a Hilbert space.

2.2.2 Sobolev spaces via the Fourier Transform

We now provide a separate construction of the Sobolev space H^k, using the Fourier transform, which helps us to prove the simple version of the Sobolev Embedding theorem, as well as understand the importance of higher order terms in our differential operators. Something of importance to note is that defining the Sobolev norm with the Fourier transform on a manifold M must be done in coordinates, and thus extending it globally here is making use of the fact that M is compact.

On $C^\infty(\mathbb{R}^n)$, define the seminorms $q_{j,k}(u) = \sup_{x \in \mathbb{R}^n} \{(1 + |x|^2)^{\frac{j}{2}} |D_\lambda u(x)|, |\lambda| \leq k\}$. Then one can check that the set of all $u \in C^\infty(\mathbb{R}^n)$ such that every $q_{j,k}(u) < \infty$ with countable family of seminorms $\{q_{j,k}\}_{j,k \in \mathbb{N}}$ is a Fréchet space, called the *Schwartz space*, denoted $\mathscr{S}(\mathbb{R}^n)$. Intuitively, the functions in this space and their derivatives "decay fast enough". For example, $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$.

Let $u \in C_c^\infty(\mathbb{R}^n)$ and define the ***Fourier transform*** of u by

$$\hat{u}(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\zeta} dx$$

for $x, \zeta \in \mathbb{R}^n$ and $x \cdot \zeta = x^1\zeta^1 + \cdots + x^n\zeta^n$, the standard Euclidean dot product. Consider now the map $\mathscr{F}^* : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ defined by $(\mathscr{F}^* u)(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int u(x) e^{ix\cdot\zeta} dx$. The reader is encouraged to check that this is in fact an adjoint of $\mathscr{F}$ with respect to the L^2-inner product for any $u, f \in \mathscr{S}$. Using the Dominated Convergence theorem and Fubini's theorem, one can show that $\mathscr{F}^*\mathscr{F} = \mathscr{F}\mathscr{F}^* = I$ (see [24, Section 1.2]), and thus $\mathscr{F}^*$ acts as an inverse on $\mathscr{S}(\mathbb{R}^n)$. Then with respect to the L^2 inner product $(u, f) = \int u(x)\overline{f(x)}dx$, we have $(\mathscr{F}u, \mathscr{F}f)_{L^2} = (\mathscr{F}^*\mathscr{F}u, f)_{L^2} = (u, f)_{L^2}$, so that $\|\mathscr{F}u\|_{L^2} = \|u\|_{L^2}$, making $\mathscr{F}$ an isometry on $\mathscr{S}(\mathbb{R}^n)$. Also, since compactly supported functions are Schwartz functions, it follows that with respect to the L^2 inner product, we have $\overline{\mathscr{S}(\mathbb{R}^n)} = L^2(\mathbb{R}^n)$, and thus we recover Plancherel's theorem, that $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ for $u \in L^2(\mathbb{R}^n)$.

For any $u \in \mathscr{S}(\mathbb{R}^n)$, we compute

$$\begin{aligned}(D_\lambda)_x u(x) &= (D_\lambda)_x \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (D_\lambda)_x \big(\hat{u}(\zeta) e^{ix\cdot\zeta}\big) d\zeta \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (i^{|\lambda|}\zeta^\lambda) \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta,\end{aligned} \tag{2.5}$$

where $\zeta^\lambda = \zeta_1^{\lambda_1} \cdots \zeta_n^{\lambda_n}$. Also, the subscript for $(D_\lambda)_x$ is just there for clarity, and in the second line we used the Dominated Convergence theorem to move the operator $(D_\lambda)_x$ inside the integral. Now we again take the Fourier transform

$$\begin{aligned}\Big(\widehat{(D_\lambda)_x u(x)}\Big)(\zeta) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} i^{|\lambda|}\zeta^\lambda \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta e^{-ix\cdot\zeta} dx \\ &= \mathscr{F}\mathscr{F}^*(i^{|\lambda|}\zeta^\lambda \hat{u}(\zeta)) = i^{|\lambda|}\zeta^\lambda \hat{u}(\zeta)\end{aligned}$$

and thus

$$\begin{aligned}\|u\|_{H^k} &= \sum_{|\lambda|\le k} \|D_\lambda u(x)\|_{L^2} \\ &= \sum_{|\lambda|\le k} \|\widehat{D_\lambda u(x)}\|_{L^2} \\ &= \sum_{|\lambda|\le k} \|i^{|\lambda|}\zeta^\lambda \hat{u}(\zeta)\|_{L^2} \\ &= \sum_{|\lambda|\le k} \|\zeta^\lambda \hat{u}(\zeta)\|_{L^2} \\ &= \sum_{|\lambda|\le k} \int_{\mathbb{R}^n} |\zeta^\lambda|^2 |\hat{u}(\zeta)|^2 d\zeta.\end{aligned}$$

We now show that there exist constants C_1, C_2 such that $C_1(1+|\zeta|^2)^k \le \sum_{|\lambda|\le k} |\zeta^\lambda|^2 \le C_2(1+|\zeta|^2)^k$. The following clever proof for this inequality was relayed in [10]. We instead claim that there exists constants C_1 and C_2 such that for all $t \in \mathbb{R}$,

$$C_1(t^2 + |\zeta|^2)^k \le \sum_{|\lambda|\le k} t^{2k-2|\lambda|}\zeta^{2\lambda} \le C_2(t^2 + |\zeta|^2)^k.$$

Since $\zeta \in \mathbb{R}^n$, we have $\nu = (t, \zeta) \in \mathbb{R}^{n+1}$, and then we can write $\|\nu\|^{2k} = (\|\nu\|^2)^k = (t^2 + |\zeta|^2)^k$, and also $\sum_{|\lambda| \leq k} t^{2k-2|\lambda|}\zeta^{2\lambda} = \sum_{|\lambda|=k} \nu^{2\lambda}$. Observe that $f(\nu) \coloneqq \sum_{|\lambda|=k} \nu^{2\lambda}$ is a homogeneous polynomial of degree $2k$. That is, $f(\nu) = \|\nu\|^{2k} f(\frac{\nu}{\|\nu\|})$. Hence it suffices to prove the inequality on the unit sphere $\mathbb{S}^n$. Since f is positive and continuous, and $\mathbb{S}^n$ compact, we can take some upper bound C_2 for f on $\mathbb{S}^n$. That is $f(\nu) \leq C_2$ on $\mathbb{S}^n$. Furthermore, f obtains a minimum on $\mathbb{S}^n$, say at $(x_1, ..., x_{n+1})$. Then at least one x_i is non-zero, and

$$f(\nu) \geq x_i^{2k},$$

making $C_1 = x_i^{2k}$ an appropriate lower bound here. Thus we have proven the inequality on $\mathbb{S}^n$, as desired. Then setting $t = 1$, we obtain our desired inequality.

This tells us that for functions the k-Sobolev space $H^k(\mathbb{R}^n)$, (the functions in L^2 for which $D_\lambda u \in L^2$ for $|\lambda| \leq k$) we can equivalently define the Sobolev k-norm by

$$\|u\|_{H^k} \sim \int_{\mathbb{R}^n} (1 + |\zeta|^2)^k |\hat{u}(\zeta)|^2 d\zeta. \tag{2.6}$$

That is, we can define Sobolev space $H^k(\mathbb{R}^n)$ as the set of $u \in \mathscr{S}'$ such that $\hat{u}$ is locally square integrable and $(1 + |\xi|^2)^s \hat{u} \in L^2(\mathbb{R}^n)$, where we are defining $\mathscr{S}'$ to be the dual of the Fréchet space $\mathscr{S}$, which is the closure of $\mathscr{S}$ under the mapping $f(u) = \langle u, f\rangle = \int_{\mathbb{R}^n} u(x)f(x)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in the weighted L^2-space $(L^2(\mathbb{R}^n), (1 + |\zeta|^2)^k)$ (the proof for which looks the same as it does in the standard $L^2(\mathbb{R}^n)$), so too is $\mathscr{S}(\mathbb{R}^n)$. Then since $\mathscr{F} : H^k(\mathbb{R}^n) \to (L^2(\mathbb{R}^n), (1 + |\zeta|^2)^k)$ is an isometric isomorphism, we obtain that $\mathscr{F}^{-1}(\mathscr{S}(\mathbb{R}^n))$ is dense in $H^k(\mathbb{R}^n)$.

Proposition 2.2.1. *Let $u \in C^\infty(\mathbb{R})$. The norm defined by*

$$\|u\|_k \coloneqq \|u\|_{L^2} + \sum_{|\lambda|=k} \|D_\lambda u\|_{L^2} \tag{2.7}$$

is equivalent to the Sobolev k-norm on $\mathbb{R}^n$.

Proof. This is in fact true for all Sobolev spaces (not just the L^2-based ones as we are considering here), however the proof for a general Sobolev space is rather lengthy, and we refer the interested reader to [1, Theorem 5.2], which also proves it on more general domains than just $\mathbb{R}^n$.

Before we begin, let us recall the notation $\zeta^\lambda \coloneqq \zeta_1^{\lambda_1} \cdots \zeta_n^{\lambda_n}$ for a multi-index λ, and note that this is distinct from $|\zeta|^k = \left(\sqrt{\zeta_1^2 + \cdots + \zeta_n^2}\right)^k$, where $k \in \mathbb{N}$.

First observe that if $|\zeta| \leq 1$, then $(1+|\zeta|^2)^k \leq 2^k$, and if $|\zeta| \geq 1$, then

$$(1+|\zeta|^2)^k = \sum_{j=1}^{k} \binom{k}{j} |\zeta|^{2j} \leq \sum_{j=1}^{k} \binom{k}{j} |\zeta|^{2k} = 2^k |\zeta|^{2k}.$$

Then $(1+|\zeta|^2)^k \leq 2^k(1+|\zeta|^{2k})$, so $1+|\zeta|^{2k} \leq (1+|\zeta|^2)^k \leq 2^k(1+|\zeta|^{2k})$.

Also, letting $\binom{k}{\lambda} = \frac{k!}{\lambda_1! \cdots \lambda_n!}$,

$$\begin{aligned} |\zeta|^{2k} &= (|\zeta_1|^2 + \cdots + |\zeta_n|^2)^k \\ &= \sum_{|\lambda|=k} \binom{k}{\lambda} \zeta^{2\lambda} \end{aligned}$$

and thus since $\binom{k}{\lambda} \geq k$, we obtain $\sum_{|\lambda|=k} \zeta^{2\lambda} \leq |\zeta|^{2k} \leq \max_{|\lambda|=k} \binom{k}{\lambda} \sum_{|\lambda|=k} \zeta^{2\lambda}$, so in fact the norm in 2.6 is equivalent to $\int (1 + \sum_{|\lambda|=k} |\zeta^{2\lambda}|) |\hat{u}(\zeta)|^2 d\zeta$, and thus

$$\begin{aligned} \|u\|_{H^k} &\sim \int_{\mathbb{R}^n} (1 + \sum_{|\lambda|=k} \zeta^{2\lambda}) |\hat{u}(\zeta)|^2 d\zeta \\ &= \int_{\mathbb{R}^n} |\hat{u}(\zeta)|^2 d\zeta + \int_{\mathbb{R}^n} \sum_{|\lambda|=k} |\zeta^\lambda|^2 |\hat{u}(\zeta)|^2 d\zeta \\ &= \|\hat{u}(\zeta)\|_{L^2} + \sum_{|\lambda|=k} \|\zeta^\lambda \hat{u}(\zeta)\|_{L^2} \\ &= \|u(x)\|_{L^2} + \sum_{|\lambda|=k} \|D_\lambda u(x)\|_{L^2} \end{aligned} \tag{2.8}$$

which concludes the proof. □

Note that by our definition of the Sobolev space for sections of a vector bundle, the above proposition extends to $H^k(E)$ as well.

2.2.3 Sobolev Spaces via Connections

Let ∇^E be a connection on E, so $\nabla^E : \Gamma(E) \to \Gamma(T^*M \otimes E)$, and let ∇^{LC} denote the Levi-Civita connection on TM for our Riemannian metric g on M, and let ∇^{LC*} denote the induced connection on T^*M. Then $\nabla^{LC*} \otimes \nabla$ is the induced connection on $T^*M \otimes E$. That is $\nabla^{LC*} \otimes \nabla^E : \Gamma(T^*M \otimes E) \to \Gamma(T^*M \otimes T^*M \otimes E)$ and thus we can again use the Levi-Civita connection to give us an induced connection on $T^*M \otimes T^*M \otimes E$. Continuing in this manner, using the induced connection on the bundle $\bigotimes^k T^*M \otimes E$ (namely $\bigotimes^k \nabla^{LC*} \otimes \nabla^E$) we define the map

$$\begin{aligned} \nabla^k : \Gamma(E) &\to \Gamma(\bigotimes^k T^*M \otimes E) \\ \xi &\mapsto \underbrace{\nabla(\nabla(...\nabla}_{k\text{-times}}(\xi)))). \end{aligned}$$

It turns out that this is a differential operator of order k, and not only that, but we can use it in place of D_λ for our definition of the Sobolev norm. Indeed, given a Hermitian inner product h defined on the fibers of E, we obtain a pointwise defined inner product on $\bigotimes^k T^*M \otimes E$ defined as follows: Let U be an open set over which we have local coordinates $(x^1, ..., x^n)$ and a frame $\{s_1, ..., s_r\}$ for $E|_U$. Then we can write any $\xi \in \Gamma(\bigotimes^k T^*M \otimes E)$ locally as $\xi^\eta_{i_1...i_k} dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes s_\eta$. We then define the pointwise inner product $\langle \xi, \zeta \rangle = g^{i_1 j_1} ... g^{i_k j_k} h_{\eta\nu} \xi^\eta_{i_1...i_k} \zeta^\nu_{j_1...j_k}$.

Using compactness of M, and arguing by induction, one can prove the following.

Proposition 2.2.2. *The following norm on $\Gamma(E)$ is equivalent to the Sobolev k-norm:*

$$\|\xi\|^2_{H^k(M)} = \sum_{j=0}^{k} \int |\nabla^j \xi|^2_E dV_g. \tag{2.9}$$

2.2.4 Embeddings

On E, we denote by C^k the space of k-times continuously differentiable sections. Now, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of M over which we have coordinates and local trivializations of E. Then consider the shrunken open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ such that $\overline{V_\alpha} \subseteq U_\alpha$ (such a cover always exists). Again we write $\xi|_{U_\alpha}$ as ξ_α. We define the **uniform C^k norm** by

$$\|\xi\|_{C^k(E)} = \sum_{\alpha\in\Lambda} \sum_{j=1}^{r} \sum_{0\le|\lambda|\le k} \sup_{x\in V_\alpha} |(D_\lambda \xi^j_\alpha)|. \tag{2.10}$$

Note that $C^k(E)$ is a Banach space.

Remark 2.2.3. Given a partition of unity ρ_α corresponding to the cover $\{U_\alpha\}_{\alpha\in\Lambda}$ it turns out that the following definition for the C^k-norm is equivalent to ours

$$\|\xi\|_{C^k(E)} = \sum_{\alpha\in\Lambda}\sum_{j=1}^{r}\sum_{0\leq|\lambda|\leq k}\sup_{x\in U_\alpha}|(D_\lambda(\rho_\alpha\xi_\alpha^j)|. \tag{2.11}$$

This second definition looks much more similar to (2.3), and will thus be useful for us to transfer the following results on $\mathbb{R}^n$ to E.

It is easy to see from our local coordinate definition of Sobolev norm that if $u \in H^s$, then $(\sum_{|\lambda|=k} D_\lambda)u \in H^{s-k}$. We now present the most important result of this section.

Theorem 2.2.4 (Sobolev Embedding Theorem)**.** *If $s > \frac{n}{2} + k$, then $H^s(E) \subset C^k(E)$ and the embedding is continuous.*

Proof. We first prove this on $\mathbb{R}^n$, and then extend to sections of E.

Let $u \in H^s(\mathbb{R}^n)$. Then by (2.5), we have that

$$\begin{aligned}
|D_\lambda u(x)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|\zeta^\lambda||\hat{u}(\zeta)|d\zeta \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(1+|\zeta|^2)^{\frac{-s}{2}}(1+|\zeta|^2)^{\frac{s}{2}}|\zeta^\lambda||\hat{u}(\zeta)|d\zeta \\
&\underbrace{\leq}_{\text{Hölders inequality}} \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\int_{\mathbb{R}^n}\frac{|\zeta^\lambda|^2}{(1+|\zeta|^2)^s}d\zeta\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}(1+|\zeta|^2)^s|\hat{u}(\zeta)|^2d\zeta\right)^{\frac{1}{2}}
\end{aligned}$$

and thus squaring this gives us

$$|D_\lambda u(x)|^2 \leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{|\zeta^\lambda|^2}{(1+|\zeta|^2)^s}d\zeta\int_{\mathbb{R}^n}(1+|\zeta|^2)^s|\hat{u}(\zeta)|^2d\zeta.$$

Now we sum over all $|\lambda| \leq k$ to give us

$$\begin{aligned}
\|u(x)\|_{C^k(\mathbb{R}^n)} &= \sup_{x \in \mathbb{R}^n} \sum_{0 \leq |\lambda| \leq k} |D_\lambda u(x)|^2 \\
&\leq \sum_{0 \leq |\lambda| \leq k} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{|\zeta^\lambda|^2}{(1+|\zeta|^2)^s} d\zeta \int_{\mathbb{R}^n} (1+|\zeta|^2)^s |\hat{u}(\zeta)|^2 d\zeta \\
&= \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{\sum_{1 \leq \lambda \leq k} |\zeta^\lambda|^2}{(1+|\zeta|^2)^s} d\zeta \right) \left(\int_{\mathbb{R}^n} (1+|\zeta|^2)^s |\hat{u}(\zeta)|^2 d\zeta \right) \\
&\leq \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{C}{(1+|\zeta|^2)^{s-k}} d\zeta \right) \left(\int_{\mathbb{R}^n} (1+|\zeta|^2)^s |\hat{u}(\zeta)|^2 d\zeta \right)
\end{aligned}$$

where to obtain the last line we used the inequality $\sum_{1 \leq \lambda \leq k} |\zeta^\lambda|^2 \leq C(1+|\zeta|^2)^k$, proven in section 2.2.2. Then since for $s - k > \frac{n}{2}$, $(1+|\zeta|^2)^{s-k} \in L^1(\mathbb{R})$, we obtain the constant $C_1 = \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{C}{(1+|\zeta|^2)^{s-k}} d\zeta \right)$ such that

$$\|u(x)\|_{C^k(\mathbb{R}^n)} \leq C_1 \|u(x)\|_{H^s(\mathbb{R}^n)},$$

and thus $H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$.

To see why this extends to sections in $H^s(E)$, we consider the form of the $C^k(E)$ norm as given in (2.11). Since each $\rho_\alpha \xi_\alpha^i$ is a smooth function on $\mathbb{R}^n$, the result for $\mathbb{R}^n$ applies. □

Note that the following corollary is also true under the same regularity assumptions as the Sobolev Embedding Theorem ($s > \frac{n}{2}$), however since for our purposes in the later sections lower regularity provides no added benefits, we will prove it under a slightly weaker assumption.

Corollary 2.2.5. *For $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$, the space $H^s(\mathbb{R}^n)$ is a Banach algebra under the operation of pointwise multiplication.*

Proof. Suppose $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$ and take $u, v \in H^s(\mathbb{R}^n)$. Then, using the generalized Leibniz rule for multi-indices, we can write each $D_\lambda(uv) = \sum_{\alpha \leq \lambda} \binom{\lambda}{\alpha} D_\alpha u D_{\lambda-\alpha} v$, where $\alpha \leq \lambda$ means $\alpha^i \leq \lambda^i$ for all $1 \leq i \leq n$. Also, by Theorem 2.2.4 and our assumption on s, we know that

$u, v \in C^0(\mathbb{R})$, and thus we can write

$$\begin{aligned}\|uv\|_{H^s(\mathbb{R}^n)} &= \|uv\|_{L^2(\mathbb{R}^n)} + \sum_{|\lambda|=s} \|D_\lambda(uv)\|_{L^2(\mathbb{R}^n)} \\ &\leq \sup_{x\in\mathbb{R}^n} |v(x)| \|u\|_{L^2(\mathbb{R}^n)} + \sum_{|\lambda|=s} \sum_{\alpha\leq\lambda} \binom{\lambda}{\alpha} \|(D_\alpha u)(D_{\lambda-\alpha} v)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

But since $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$, for every $|\lambda| = s$, either $s - |\alpha| > \frac{n}{2}$ or $s - |\lambda - \alpha| > \frac{n}{2}$, and thus by Theorem 2.2.4 either $(D_\alpha u) \in C^0(\mathbb{R}^n)$ or $D_{\lambda-\alpha} v \in C^0(\mathbb{R}^n)$. As a result, we obtain

$$\begin{aligned}\|uv\|_{H^s(\mathbb{R}^n)} \leq \sup_{x\in\mathbb{R}^n} |v(x)| \|u\|_{L^2(\mathbb{R}^n)} + \sum_{|\lambda|=s} \Bigg(&\sum_{\alpha\leq\lambda; |\alpha|\leq\lfloor\frac{s}{2}\rfloor} \sup_{x\in\mathbb{R}^n} |D_\alpha u(x)| \|D_{\lambda-\alpha} v\|_{L^2(\mathbb{R}^n)} \\ &+ \sum_{\alpha\leq\lambda; |\alpha|\geq\lfloor\frac{s}{2}\rfloor} \sup_{x\in\mathbb{R}^n} |D_{\lambda-\alpha} v(x)| \|D_\alpha u\|_{L^2(\mathbb{R}^n)} \Bigg).\end{aligned} \tag{2.12}$$

Now, observe that

$$\begin{aligned}&\sum_{|\lambda|=s} \sum_{\alpha\leq\lambda; |\alpha|\leq\lfloor\frac{s}{2}\rfloor} \sup_{x\in\mathbb{R}^n} |D_\alpha u(x)| \|D_{\lambda-\alpha} v\|_{L^2(\mathbb{R}^n)} \\ &\leq \Big(\sum_{|\lambda|=s} \sum_{\alpha\leq\lambda; |\alpha|\leq\lfloor\frac{s}{2}\rfloor} \sup_{x\in\mathbb{R}} |D_\alpha u(x)| \Big) \Big(\sum_{|\lambda|=s} \sum_{\alpha\leq\lambda; |\alpha|\leq\lfloor\frac{s}{2}\rfloor} \|D_{\lambda-\alpha} v\|_{L^2(\mathbb{R})} \Big) \\ &\leq \|v\|_{C^{\lfloor\frac{s}{2}\rfloor}(\mathbb{R})} \|u\|_{H^{\lfloor\frac{s}{2}\rfloor}(\mathbb{R})} \\ &\leq C_0 \|v\|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})},\end{aligned}$$

for some $C_0 \in \mathbb{R}$, where in the last line we used Sobolev Embedding Theorem, and the easy to see fact that for $k < s$, $\|u\|_{H^k(\mathbb{R})} \leq \|u\|_{H^s(\mathbb{R})}$. We can use the exact same argument for the other summand from (2.12), to obtain

$$|uv\|_{H^s(\mathbb{R}^n)} \leq \sup_{x\in\mathbb{R}} |v(x)| \|u\|_{L^2(\mathbb{R}^n)} + C_1 \|v\|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})}.$$

Finally, again using the aforementioned fact, and a similarly easy to see fact that if $k < s$, $\|v\|_{C^k(\mathbb{R})} \leq \|v\|_{C^s(\mathbb{R})}$, we see that $\|v\|_{C^0(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} \leq \|v\|_{C^{\frac{s}{2}}} \|u\|_{H^s(\mathbb{R})}$. So again invoking the Sobolev Embedding Theorem,

$$|uv\|_{H^s(\mathbb{R}^n)} \leq C_2 \|v\|_{H^s(\mathbb{R})} \|u\|_{H^s(\mathbb{R})},$$

hence $H^s(\mathbb{R})$ is a Banach space under pointwise multiplication. $\square$

Corollary 2.2.6. *For a vector bundle E over a compact manifold M, $H^s(E)$ is a Banach algebra under the action of pointwise multiplication.*

As mentioned in the previous proof, it is not hard to see that for $k < s$, $\|\cdot\|_{H^s(E)}$ is stronger than the norm $\|\cdot\|_{H^k(E)}$. That is, $\|\cdot\|_{H^k(E)} \leq \|u\|_{H^s(E)}$, so clearly $H^k(E) \subseteq H^s(E)$. It turns out that we can say something even stronger than this.

Theorem 2.2.7 (Rellich Lemma)**.** *When $s > k$, the embedding $H^s(E) \subseteq H^k(E)$ is compact.*

Proof. See [24][Chapter 1, Section 5]. □

To understand another result of the Sobolev Embedding Theorem, we first introduce the notion of the Fréchet derivative.

Definition 2.2.8. Let $f : E \to F$ be a map between Banach Spaces. Suppose that f is differentiable at $\xi \in E$. Then there exists a continuous linear map $A_\xi : E \to F$ such that for some $\delta > 0$ and $\|\zeta\| < \delta$ we can write $f(\xi + \zeta) = f(\xi) + A(\zeta) + \varepsilon(\zeta)\|\zeta\|_E$, where $\lim_{\zeta \to 0} \varepsilon(\zeta) = 0$. Then supposing that f is differentiable on some open set $U \subseteq E$, we define the *Fréchet derivative* of f, denoted $Df : U \to L(E, F)$, by $D_\xi f = A_\xi$ as defined above.

Since $D_\xi f \in L(E, F)$, assuming that Df is differentiable in some open set $U \subseteq E$ we can again take the Fréchet derivative to give us $D(Df)) : U \to L(E, L(E, F))$.

We can continue in this way to define $D^k f : U \to L(E, L(E \cdots (L(E, F)) \cdots))$.

Since $L(E, L(E \cdots (L(E, F)) \cdots)) \cong \mathcal{L}^k(E, , E, \cdots , E, F)$ (where $\mathcal{L}(E, , E, \cdots , E, F)$ is the space of multilinear maps from $\underbrace{E \times \cdots \times E}_{k-\text{times}}$ to F) it is often easier to think about the k-th Fréchet derivative of f as a map $D^k f : U \to \mathcal{L}^k(E \times E \cdots \times E, F)$, writing its evaluation as $D^k f(\xi_1, ..., \xi_k) \in F$.

This allows us now to state the following Theorems, which we will need later on.

Theorem 2.2.9 (Banach Space Inverse Function Theorem)**.** *Let X and Y be Banach spaces, and let $U \subseteq X$ be an open neighbourhood of 0. Suppose $F : U \to Y$ is a map of class C^k, such that*

$$D_0 F : X \to Y$$

is a bounded linear isomorphism of X onto Y. Then there exists an open neighbourhood $V \subseteq U$ containing 0 such that $F\big|_V : V \to F(V)$ is a C^k-diffeomorphism.

Theorem 2.2.10. *Let E and F be vector bundles over M and let $f : E \to F$ be a smooth fiber preserving map. Then if $s > \frac{n}{2}$, the map $f \circ (-) : H^s(E) \to H^s(F)$ defined by $\alpha \mapsto f \circ \alpha$ is smooth and its derivatives satisfy $D^k_p(f \circ \alpha)(\xi_1, ..., \xi_k) = D_{\alpha(p)} f(\xi_1, ..., \xi_k)$*

Proof. See [19, pp. 41-42] □

2.3 Elliptic and Operators with Injective Symbol

In the previous section we saw that for the Sobolev norm, we can define it either using derivatives of all orders up to k, or only the derivatives of order k and zero. In this section we extend this idea, by looking at elliptic operators, which capture all the highest order differential information.

Definition 2.3.1. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator. Fix some $\xi, \eta \in \Gamma(E)$, and suppose that for all $\zeta \in \Gamma(F)$

$$\langle \eta, \zeta \rangle_{L^2(F)} = \langle \xi, P^* \zeta \rangle_{L^2(E)}.$$

Then we say that η is a weak solution to $P\xi = \eta$.

By continuity, a differential operator $P : \Gamma(E) \to \Gamma(F)$ of order k extends uniquely to a bounded linear map $P_s : H^s(E) \to H^{s-k}(F)$ (here $s > k$). If we recall that the adjoint $P^* : \Gamma(F) \to \Gamma(E)$ was defined by $\langle P\xi, \zeta \rangle = \langle \xi, P^* \zeta \rangle$, and by continuity this defining property also extends to $\xi, \zeta \in H^s$ giving us $P^*_s : H^s(F) \to H^{s-k}(F)$.

Theorem 2.3.2 (Elliptic Regularity Theorem)**.** *Let M be a compact manifold, and let E and F be vector bundles over M. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order k, and suppose $\eta \in L^1(E)$. Suppose that $\xi \in L^1(E)$ is a weak solution to $P\xi = \eta$. If $\eta \in H^s(E)$, then $\xi \in H^{s+k}(E)$ and there exists some constant $C > 0$ independent of ξ such that*

$$\|\xi\|_{H^{s+k}(F)} \leq C\big(\|P\xi\|_{H^s(F)} + \|\xi\|_{L^1(E)}\big). \tag{2.13}$$

Corollary 2.3.3. *Let $\xi \in \Gamma(E)$. And let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic differential operator of order s. The norm defined by*

$$\|\xi\|_{s_P} := \|\xi\|_{L^2} + \|P\xi\|_{L^2}$$

is equivalent to the $H^s(E)$ norm.

Proof. It suffices to find constants C_1 and C_2 such that

$$C_1\Big(\|P\xi\|_{L^2(E)} + \|\xi\|_{L^2(E)}\Big) \le \|\xi\|_{H^s(E)} \le C_2\Big(\|P\xi\|_{L^2(E)} + \|\xi\|_{L^2(E)}\Big).$$

From definition 2.1.2, we can see that if we take a constant $C_0 = \max_{1\le\|\lambda\|\le k, x\in M}\{\|P^\lambda_{ij}(x)\|_{L(E)}\}$ (where $\|\cdot\|_{L(E)}$ is the operator norm induced by the inner product on E, and P^λ_{ij} is the matrix coefficient for D_λ) then

$$\frac{1}{C_0}\|P\xi\|_{L^2(E)} \le \sum_{0\le|\lambda|\le k}\|D_\lambda\xi\|_{L^2(E)}$$

and thus

$$\begin{aligned}
\frac{1}{C_0}\|P\xi\|_{L^2(E)} + \|\xi\|_{L^2} &\le \sum_{0\le|\lambda|\le k}\|D_\lambda\xi\|_{L^2(E)} + \|\xi\|_{L^2(E)} \\
&= \sum_{1\le|\lambda|\le k}\|D_\lambda\xi\|_{L^2(E)} + 2\|\xi\|_{L^2(E)} \\
&\le 2\Big(\sum_{0\le|\lambda|\le k}\|D_\lambda\xi\|_{L^2(E)}\Big).
\end{aligned}$$

Hence letting $C_1 = \min\{\frac{1}{2C_0}, \frac{1}{2}\}$, we obtain

$$C_1\Big(\|P\xi\|_{L^2(E)} + \|\xi\|_{L^2}\Big) \le \sum_{0\le|\lambda|\le k}\|D_\lambda\xi\|_{L^2(E)}.$$

By Theorem 2.3.2, and the fact that $L^2 \subseteq L^1$ over compact domains, we obtain C_2 such that

$$\|\xi\|^s_H \le C_2\Big(\|P\xi\|_{L^2(E)} + \|\xi\|_{L^2}\Big).$$

Hence, we conclude that $\|\cdot\|_{H^s(E)} \sim \|\cdot\|_{s_P}$. $\square$

Theorem 2.3.4. *Let $P : \Gamma(E) \to \Gamma(F)$ be a smooth, linear, elliptic differential operator of order k, and let $s \ge k$. Then*

1. *The kernel of the extension $P : H^s(E) \to H^{s-k}(E)$ is a finite dimensional, closed subspace of $\Gamma(E)$. Note that $P^*_{s-k} : H^{s-k}(E) \to H^{s-2k}(E)$ must be elliptic as well, so that $\ker P^*_{s-k}$ also consists only of smooth sections.*

2. We obtain an L^2-orthogonal direct-sum decomposition

$$H^{s-k}(F) = P_s(H^s(E)) \oplus \ker P^*_{s-k}. \tag{2.14}$$

Proof. See [17, Section 3.1] $\square$

It turns out that in order to obtain the direct sum decomposition above, it suffices for P to only have injective symbol.

Theorem 2.3.5 (Berger-Ebin decomposition for operator with injective symbol)**.** *Let $P : \Gamma(E) \to \Gamma(F)$ be a smooth, linear differential operator of order k with injective symbol. Then for $s \geq k$, there is an L^2-orthogonal direct sum decomposition*

$$H^{s-k}(F) = P_s(H^s(E)) \oplus \ker P^*_{s-k}.$$

First we need some important facts.

Lemma 2.3.6. *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear map. Assume that W is an algebraic linear complement to $T(X)$ in Y, which we will denote by $Y = T(X) + W$. Then if W is closed, $T(X)$ is closed and $Y = T(X) \oplus W$ is a topological direct sum.*

Proof. This proof follows [20][Chapter VIII, Theorem 1]. Let $\pi : X \to X/\ker T$ denote the quotient map. Then $T(x) = \hat{T} \circ \pi(x)$, where $\hat{T} : X/\ker T \to Y$ is injective and bounded. Thus since we only need to show that $T(X) = \hat{T} : X/\ker T \to Y$ is closed in Y, it suffices to show it for T injective.

Let W be the closed algebraic complement of $T(X)$ in Y. Hence W is itself a Banach space. Consider the Banach space $X \oplus W$ with the direct product topology. Define $\tilde{T} : X \oplus W \to Y$ by $\tilde{T} = T(x) + z$. Then $\tilde{T}$ is a continuous bijection of Banach spaces, hence a Banach space isomorphism. Since X is closed in $X \oplus W$, it must be that $T(X)$ is closed in Y. $\square$

Lemma 2.3.7. *We have* $\ker P^*P = \ker P$ *and* $\ker PP^* = \ker P^*$.

Proof. These are both immediate from $\langle \xi, P^*P\xi\rangle_{L^2(E)} = \langle P\xi, P\xi\rangle_{L^2(E)}$ and $\langle \xi, PP^*\xi\rangle_{L^2(E)} = \langle P^*\xi, P^*\xi\rangle_{L^2(E)}$. $\square$

Lemma 2.3.8. *Let $P : \Gamma(E) \to \Gamma(F)$ be a continuous, linear differential operator with injective symbol. Then the following hold:*

1. $P_s(H^s(E)) = P_s P^*_{s+k}(H^{s+k}(F))$.
2. $P^*_{s-k}(H^{s-k}(F)) = P^*_{s-k}P_s(H^s(E))$
3. $\ker P^*_{s-k} \cap P_s(H^s(E)) = \{0\}$.

Proof. This proof is based on claims made in [23].

1. The $\supseteq$ direction is clear. For the reverse inclusion, consider $\xi \in P_s(H^s(E))$, then $\xi = P_s(\eta)$ for some $\eta \in H^s(E)$. Then By Corollary 2.1.5, P^*P is elliptic, so by Theorem 2.3.4, we have the orthogonal decomposition

$$H^{s-2k}(E) = P^*_{s-k}P_s(H^s(E)) \oplus \ker(P^*_{s-3k}P_{s-2k}) \underbrace{=}_{\text{Lemma 2.3.7}} P^*_{s-k}P_s(H^s(E)) \oplus \ker P_{s-2k}.$$

Hence $\eta = P^*_{s+k}P_{s+2k}(\nu) + w$ for some $\nu \in H^{s+2k}(E)$ and $w \in \ker P$. Then

$$\xi = P_s(\eta) = P_s(P^*_{s+k}P_{s+2k}(\nu)+w) = P_sP^*_{s+k}P_{s+2k}(\nu)+P_s(w) = P_sP^*_{s+k}(P_{s+2k}(v)) \in P_sP^*_{s+k}(H^{s+k}(F)).$$

2. The $\supseteq$ direction is clear. For the reverse direction, consider $\xi = P^*_{s-k}(\eta) \in P^*_{s-k}(H^{s-k}(E))$. By the orthogonal decomposition of H^{s-2k}, to show that $\xi \in P^*_{s-k}P_s(H^s(E))$ it suffices only to show that $\xi \perp \ker P_{s-2k}$. Indeed, for any $v \in \ker P_{s-2k}$ we have

$$\langle \xi, v\rangle = \langle P^*_{s-k}(\eta), v\rangle = \langle \eta, P_{s-2k}v\rangle = \langle \eta, 0\rangle = 0.$$

3. Take $\xi \in \ker P^*_{s-k} \cap P_s(H^s(E))$. Then $\xi = P_s(\eta)$, and $\langle P_s(\eta), P_s(\eta)\rangle = \langle \eta, P^*_{s-k}P_s(\eta)\rangle = 0$, so $\xi = P(\eta) = 0$. □

Proof of Theorem 2.3.5. Since $\ker P^*_{s-k}$ is closed in $H^{s-k}(F)$, to get the direct sum decomposition, by Lemma 2.3.6 it suffices to show that there exists an algebraic linear decomposition $H^{s-k}(F) = P_s(H^s(E)) + \ker P^*_{s-k}$.

Take some $\xi \in H^{s-k}(F)$. Then $P^*_{s-k}(\xi) \in H^{s-2k}(E)$, and then by part 2 of Lemma 2.3.8, $P^*_{s-k}(\xi) = P^*_{s-k}P_s(\eta)$ for some $\eta \in H^s(E)$. Then $\xi = P_s(\eta) + \nu$ for $\nu \in \ker P^*_{s-k}$. By Lemma 2.3.8 part 3, this η and $P(\nu)$ are unique. Hence we obtained the desired algebraic decomposition.

Moreover, for any $\eta \in H^s(E)$, and $\nu \in \ker P^*_{s-k}$ we have $(P_s\eta, \nu) = (\eta, P^*_{s-k}\nu) = 0$, so that they are orthogonal. Hence

$$H^{s-k}(F) = P_s(H^s(E)) \oplus \ker P^*_{s-k}$$

is an orthogonal decomposition.

□

Chapter 3

A Slice Theorem for the Moduli Space of Metrics

Let M^n be a compact manifold, and assume $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$, so that by Corollary 2.2.5 H^s is a Banach algebra. Let $\mathscr{D}$ be the group of diffeomorphisms on M, where the binary operation is composition. Let $\mathscr{M}$ denote the set of smooth Riemannian metrics on M. In this section we will see that for $\mathscr{D}$ acting on $\mathscr{M}$ by pullback, we obtain a 'slice theorem' on $\mathscr{M}$. That is, just as in the finite dimensional case for the proper action of a Lie group on a smooth manifold, we will show that for any $\gamma \in \mathscr{M}$, the orbit of γ looks a lot like a quotient of $\mathscr{D}$ by the isotropy group of γ, and this orbit admits a neighbourhood with cross-sections that are invariant under action by the isotropy group of γ.

This specific slice theorem was proved by David G. Ebin, and we will work through its construction following the ideas and arguments originally presented in [3] and [4].

Let us first discuss the action of $\mathscr{D}$ on $\mathscr{M}$.

Definition 3.0.1. We define the action of the diffeomorphism group $\mathscr{D}$ on the set of metrics $\mathscr{M}$ over M by $\Phi(\varphi, \gamma) = \varphi^*\gamma$, where $\varphi^*\gamma$ is the pullback of γ under the diffeomorphism φ, characterized by $(\varphi^*\gamma)_x(v, w) = \gamma_{\varphi(x)}(\varphi_* v, \varphi_* w)$.

Remark 3.0.2. For $\gamma \in \mathscr{M}$ and $\varphi, \psi \in \mathscr{D}$, we have $\mathrm{Id}^*\gamma = \gamma$, and

$$\Phi(\gamma, \psi \circ \varphi) = (\psi \circ \varphi)^*\gamma = \varphi^*(\psi^*\gamma) \ = \Phi(\psi^*\gamma, \varphi) \ = \Phi(\Phi(\gamma, \psi), \varphi).$$

Thus, the map $\Phi : \mathscr{M} \times \mathscr{D} \to \mathscr{M}$ defined by $\Phi(\gamma, \varphi) = \varphi^*\gamma$ is a right action of $\mathscr{D}$ on $\mathscr{M}$.

Now, we wish to study this action intensively and develop a slice theorem for it. However, neither $\mathscr{D}$ nor $\mathscr{M}$ are very nice spaces, in fact neither space is even complete. So, we will make use of the theory of Sobolev spaces we developed in the last chapter, embedding these into their own respective Sobolev spaces of sections of a bundle over M. We can then extend the group action to these much nicer spaces, whose richer geometric structure lend themselves to developing a slice theorem for this extended action. So, we shall spend the next couple sections developing the geometry of these spaces.

3.1 The Group $\mathscr{D}^{s+1}$

In this section we will look at the Sobolev space $\mathscr{D}^{s+1}$, and show that this is both a group and has infinite dimensional manifold structure.

Let N be some arbitrary manifold. We denote by $H^s(M,N)$ the space of s-regular maps from M to N, which is defined using our standard definitions of Sobolev spaces of sections as $H^s(M \times N)$, where $M \times N \to M$ is a trivial bundle over M.

The method that Ebin employs to show that $\mathscr{D}^{s+1}$ is a smooth manifold is to show that $\mathscr{D}^{s+1}$ is an open subset of the smooth manifold $H^s(M,M)$. So, first we must show that $H^s(M,N)$ is a smooth Hilbert manifold, modelled on the Hilbert space $H^s(f^*TN)$. We begin the process by constructing a chart around every point $f \in H^s(M,N)$.

Impose a Riemannian metric γ on N and take some $f \in H^s(M,N)$. By continuity of f, and compactness of M we know that $f(M)$ is compact in N and thus there is a uniform neighbourhood W of the zero section in $TN|_{f(M)}$ such that $\exp|_W$ is a diffeomorphism onto its image.

Denote

$$H^{s,\varepsilon}(f^*TN) \coloneqq \{\xi \in H^s(f^*TN) : \max_{p \in M} \|\xi(p)\|_\gamma < \varepsilon\},$$

so that $\xi(x) \in W$, for all $x \in M$. Note that by Theorem 2.2.4, $H^s(f^*TN) \hookrightarrow C^0(f^*TN)$, hence $H^{s,\varepsilon}(f^*TN)$ is an open neighbourhood of zero in $H^s(f^*TN)$.

We can also see that $x \mapsto \exp_{f(x)}(\xi(x)) \in H^s(M,N)$, since $\xi \in H^s(M,W)$, and $\exp \in C^\infty(W,N)$. Furthermore, as exp is a diffeomorphism on W, we obtain that the map $\Psi_f : H^{s,\varepsilon}(f^*TN) \to H^s(M,N)$ defined by $\xi \mapsto (x \mapsto \exp_{f(x)}(\xi(x)))$ is injective.

We use this map to define a set of local charts on $H^s(M,N)$ given by

$$\bigcup_{f \in H^s(M,N)} \left(H^{s,\varepsilon}(f^*TN), \Psi_f \right). \tag{3.1}$$

The proof that these in fact define smooth charts on $H^s(M,N)$ can be found in [3, pp. 47-50].

We can construct the manifold $C^1(M,N)$ analogously, modelled on the bundle $C^1(f^*TN)$.

We simplify to the case where $N = M$, and consider the set of maps from $M \to M$ that are continuous, invertible, and both the map and its inverse have continuous first derivatives.

Definition 3.1.1. Define $C^1(\mathscr{D}) \subseteq C^1(M,M)$ by $\{f \in C^1(M,M) : f \text{ is invertible and } f^{-1} \in C^1(M,M)\}$.

Proposition 3.1.2. *The set $C^1(\mathscr{D})$ is open in $C^1(M,M)$.*

Proof. See [8, Chapter 2, Theorem 1.6] □

Note that by our choice of s we have $s+1 > \frac{n}{2}+1$, and thus by the Sobolev Embedding Theorem 2.2.4 we have an embedding $\iota : H^{s+1}(M,M) \hookrightarrow C^1(M,M)$. We now use this fact to help us develop a notion of diffeomorphisms of regularity s.

Definition 3.1.3. Define $\mathscr{D}^{s+1} := H^{s+1}(M,M) \cap C^1(\mathscr{D})$.

Proposition 3.1.4. *The set $\mathscr{D}^{s+1}$ is a smooth manifold.*

Proof. Since $C^1(\mathscr{D}) \subseteq C^1(M,M)$ is open by Proposition 3.1.2, and $\iota : H^{s+1}(M,M) \hookrightarrow C^1(M,M)$ is a continuous embedding, we know that $\mathscr{D}^{s+1} = \iota^{-1}(C^1(\mathscr{D}))$ is open in $H^{s+1}(M,M)$. Hence since $H^{s+1}(M,M)$ is itself a smooth manifold, we obtain a smooth structure on $\mathscr{D}^{s+1}$. □

We need the following statement.

Lemma 3.1.5 (Theorem 1.2 from [7])**.** *Let M^n be a closed oriented manifold, N a smooth manifold. Suppose that $s > \frac{n}{2}+1$. Then for any $r \geq 0$,*

- $\mu : H^{s+r}(M,N) \times \mathscr{D}^s \to H^s(M,N)$, *defined by* $(f,\varphi) \mapsto f \circ \varphi$, *and*
- $\mathrm{inv} : \mathscr{D}^{s+r} \to \mathscr{D}^s$, *defined by* $\varphi \mapsto \varphi^{-1}$

are both C^r-maps.

Lemma 3.1.6. *For $\varphi \in \mathscr{D}^{s+1}$, $H^{s+1}(TM) \cong H^{s+1}(\varphi^*TM)$ are homeomorphic as Banach spaces.*

Proof. We define a map

$$\begin{aligned}\Theta : H^{s+1}(TM) &\to H^{s+1}(\varphi^* TM)\\ X &\mapsto \varphi^* X.\end{aligned}$$

Since regularity is preserved under composition, we can write its inverse map as

$$\begin{aligned}\Theta^{-1} : H^{s+1}(\varphi^* TM) &\to H^{s+1}(TM)\\ Y &\mapsto (\varphi^{-1})^* Y.\end{aligned}$$

So Θ is a bundle isomorphism. We wish to show that it is a homeomorphism of Sobolev spaces. By the Open Mapping Theorem, it suffices to show that the map $X \mapsto X \circ \varphi$ is a bounded linear map.

It is clear that this is a linear, bijective map, as $(Y + X) \circ \varphi = Y \circ \varphi + X \circ \varphi$.

To see that it is bounded, we let $N = TM$, and let $r = 0$, and then applying Lemma 3.1.5, we obtain that this is a C^0 map. Thus Θ is a bounded linear bijection between Banach spaces, and thus a homeomorphism by the Open Mapping Theorem. $\square$

Proposition 3.1.7. *$T_\varphi \mathscr{D}^{s+1} \cong H^{s+1}(TM)$ for any $\varphi \in \mathscr{D}^{s+1}$.*

Proof. Take some $\varphi \in \mathscr{D}^{s+1}$. By our construction of the charts (3.1), and by openness of $\mathscr{D}^{s+1}$ in $H^{s+1}(M, M)$, we know that there exists $\varepsilon > 0$ such that $\left(H^{s+1,\varepsilon}(\varphi^* TM), \Psi_\varphi\right)$ is a coordinate chart around φ in $\mathscr{D}^{s+1}$. But since $H^{s+1}(\varphi^* TM)$ is a Hilbert space, the tangent space at any point in $H^{s+1,\varepsilon}(\varphi^* TM)$ can be identified with $H^{s+1}(\varphi^* TM)$. Then by the Lemma, we obtain $T_\varphi \mathscr{D}^{s+1} \cong H^{s+1}(TM)$, the space of regularity $s+1$ vector fields on M. $\square$

Proposition 3.1.8. *The manifold $\mathscr{D}^{s+1}$ is a topological group.*

Proof. This is an immediate result of Lemma 3.1.5. $\square$

In particular, for diffeomorphisms $\varphi \in \mathscr{D} \subseteq \mathscr{D}^{s+1}$, we have that both post-composition and pre-composition are in fact smooth maps.

Proposition 3.1.9 (Proposition 3.4 in [4])**.** *Let $\varphi \in \mathscr{D}^{s+1}$ and let $R_\varphi, L_\varphi : \mathscr{D}^{s+1} \to \mathscr{D}^{s+1}$ denote right and left group multiplication (in this case, pre-composition and post-composition, respectively) by φ. Then R_φ is smooth, and if $\varphi \in \mathscr{D}$, then L_φ is smooth.*

Proof. Let $\psi \in \mathscr{D}^{s+1}$, and take some $X \in T_\psi \mathscr{D}^{s+1} = H^{s+1}(\psi^* TM) \cong H^{s+1}(TM)$. We recall that the charts on $\mathscr{D}^{s+1}$ are defined using a Riemannian metric γ on M, and the resulting exponential. We will show that R_φ is smooth in some open set $U \cong H^{s+1,\varepsilon}(\psi^* TM)$ containing ψ. Taking some $\eta \in U$, we have $\eta = \exp_{\psi(\cdot)}(X)$ for some $X \in H^{s+1,\varepsilon}(\psi^* TM)$, and so

$$\begin{aligned} R_\varphi(\eta) &= \exp_{\psi(\cdot)}(X) \circ \varphi \\ &= \exp_{\psi\circ\varphi(\cdot)}(X \circ \varphi), \end{aligned}$$

so we can see that R_φ acts on the model space $H^{s+1}(\psi^* TM)$ via pullback by φ. In Lemma **??** we showed that this was a bounded linear map on Banach spaces, hence it is smooth.

Now, we check the left action L_φ for smoothness. If we consider the trivial bundle $\pi : M \times M \to M$ with $\pi(x, y) = x$, then we can identify $H^s(M, M)$ with the space of H^s sections of $M \times M$, written $H^s(M \times M)$ using our notation, and where $\psi \in H^s(M, M)$ corresponds to the section $x \mapsto (x, \psi(x))$. Then for $\varphi \in \mathscr{D}^{s+1}$, we can view it as a smooth fiber preserving map $\varphi : M \times M \to M \times M$ defined by $(x, y) \mapsto (x, \varphi(y))$. Then by Theorem 2.2.10, we obtain that $\psi \mapsto \varphi \circ \psi$ is smooth. $\square$

3.2 The Manifold $\mathscr{M}^s$

We want a nice way to define Riemannian metrics of regularity s on M. Recall that by definition we define the space of smooth Riemannian metrics as $\mathscr{M} \subseteq \Gamma(\mathrm{Sym}_2(M))$, such that for all $\gamma \in \mathscr{M}$ and $x \in M$, $\gamma_x : T_x M \times T_x M \to \mathbb{R}$ is positive definite. We will use a similar method for defining the space of regularity s metrics over M, but in order to define such a set using a pointwise property as we do for smooth metrics, we will want these sections to be continuous.

Definition 3.2.1. Define the set $C^0(\mathscr{M}) \subseteq C^0(\mathrm{Sym}_2(M))$ by the property: $\gamma \in C^0(\mathscr{M})$ if and only if for every $x \in M$, $\gamma(x)$ is positive definite.

Proposition 3.2.2. *The set $C^0(\mathscr{M})$ is open in $C^0(\mathrm{Sym}_2(M))$.*

Proof. Consider some open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ such that there exist both coordinate charts and vector bundle trivializations over each U_α, and let $\{\rho_\alpha\}_{\alpha\in\Lambda}$ be a subordinate partition of unity. First note that the space of positive definite symmetric matrices is open in the set of symmetric matrices. Then consider some $\gamma \in C^0(\mathscr{M})$. Let us take an orthonormal frame for γ in each trivialization, so that the local representation γ^α is of the form $\gamma^\alpha = \mathbb{I}_n$. Let the pointwise fibre metric on $\mathrm{Sym}_2(M)$ be such that it induces the spectral norm $\|\cdot\|_2$. Then for $\varepsilon = 1$ if we take $\beta \in C^0(\mathrm{Sym}_2(M))$ such that

$$\|\beta - \gamma\|_{C^0(\mathrm{Sym}_2(M))} < 1,$$

we obtain by definition (2.10) that for all $\alpha \in \Lambda$,

$$\|\beta^\alpha - \gamma^\alpha\|_2 < 1.$$

Hence all eigenvalues of β are strictly positive, so that β is positive definite.

So for any $\gamma \in C^0(\mathscr{M})$ there exists an open ball in $C^0(\mathrm{Sym}_2(M))$ containing γ such that it is entirely contained within $C^0(\mathscr{M})$, so that $C^0(\mathscr{M})$ is open in $C^0(\mathrm{Sym}_2(M))$. $\square$

Since we have $s > \frac{n}{2}$, we know by the Sobolev Embedding Theorem 2.2.4 that $H^s(\mathrm{Sym}_2(M)) \subseteq C^0(\mathrm{Sym}_2(M))$, so the following definition makes sense.

Definition 3.2.3. Define $\mathscr{M}^s := H^s(\mathrm{Sym}_2(M)) \cap C^0(\mathscr{M})$.

Proposition 3.2.4. *The set $\mathscr{M}^s$ is a smooth manifold.*

Proof. The Sobolev Embedding Theorem 2.2.4 tells us that the embedding $\iota : H^s(\mathrm{Sym}_2(M)) \hookrightarrow C^0(\mathrm{Sym}_2(M))$ is continuous. Then since $C^0(\mathscr{M})$ is open in $C^0(\mathrm{Sym}_2(M))$, we obtain that $\mathscr{M}^s = \iota^{-1}(C^0(M))$ is open in $H^s(\mathrm{Sym}_2(M))$, which is a Hilbert space. Hence $\mathscr{M}^s$ is a smooth, infinite dimensional manifold. $\square$

Corollary 3.2.5. *$T_\gamma \mathscr{M}^s = H^s(\mathrm{Sym}_2(M))$, for all $\gamma \in \mathscr{M}^s$.*

Proof. This is immediate from the fact that $\mathscr{M}^s$ is open in the Hilbert space $H^s(\mathrm{Sym}_2(M))$. $\square$

Definition 3.2.6. Given $\gamma \in \mathscr{M}^s$, γ induces a pointwise inner product on T^*M via the musical isomorphism, and this extends to a pointwise inner product on $\bigotimes^k T^*M$ defined on simple tensors by $(\beta_1 \otimes \cdots \otimes \beta_k, \omega_1 \otimes \cdots \otimes \omega_k) = \prod_{i=1}^k (\beta_i, \omega_i)_\gamma$, and an inner product on $\mathrm{Sym}_2(T^*M)$, which here we will also denote $(\cdot,\cdot)_\gamma$. Then we define an inner product on $\Gamma(\mathrm{Sym}_2(T^*M))$ by

$$\langle \omega, \beta \rangle_\gamma := \int_M (\omega, \beta)_\gamma d\mu_\gamma, \tag{3.2}$$

where μ_γ is the volume form on M induced by γ.

Since by choice of s, $\gamma \in C^0(\mathscr{M})$, by compactness of M we obtain from the proof for [20, Chapter 9, Section 2, Theorem 1] that the above metric induces the standard H^0-topology (which is just the L^2-topology) on $H^0(\mathrm{Sym}_2(M))$.

Since this is a positive-definite inner product on $H^s(\mathrm{Sym}_2(M))$, we say that it is a weak Riemannian metric. However it is not a strong Riemannian metric, since the L^2-topology is strictly weaker than the H^s-topology on $H^s(\mathrm{Sym}_2(M))$.

However, we will see shortly that this still defines a connection and corresponding exponential map on $\mathscr{M}^s$, and thus will suffice for the purpose of developing the slice theorem.

First we will show that for fixed $\alpha, \beta \in H^s(\mathrm{Sym}_2(M))$, the map $\mathscr{M}^s \ni \gamma \mapsto \langle \alpha, \beta \rangle_\gamma \in \mathbb{R}$ is smooth.

We provide the following two results without proof. Their justifications can be found in [4, Section 4]

Proposition 3.2.7. *The weak Riemannian metric defined by* (3.2) *is smooth.*

Proposition 3.2.8. *The weak Riemannian metric* (3.2) *is invariant under action by the diffeomorphism group* $\mathscr{D}^{s+1}$.

In fact this weak metric can be used to induce a strong Riemannian metric on $\mathscr{M}^s$, and though it will be of less importance to us than the weak metric, it will still have the following property which we need in our proof of the slice theorem.

Proposition 3.2.9. *There exists a strong, continuous,* $\mathscr{D}^{s+1}$*-invariant Riemannian metric* $(\cdot,\cdot)^s$ *on* $\mathscr{M}^s$.

Proof. See [4, Section 4] □

3.3 The Quotient Group $\mathscr{D}^{s+1}/I_\gamma$

Definition 3.3.1. Fix $\gamma \in \mathscr{M}^s$ and denote by I_γ the isotropy group of γ in $\mathscr{D}^{s+1}$. That is, $I_\gamma = \{\varphi \in \mathscr{D}^{s+1} : \varphi^*\gamma = \gamma\}$.

Proposition 3.3.2. *For* $\gamma \in \mathscr{M} \subseteq \mathscr{M}^s$, $I_\gamma \subseteq \mathscr{D}^s$ *consists only of smooth diffeomorphisms. That is,* $I_\gamma \cap \mathscr{D} = I_\gamma$.

Proof. Take $\varphi \in I_\gamma$, so that for all $x \in M$ and all $v, w \in T_xM$, $(\varphi^*\gamma)_x(v,w) = \gamma_{\varphi(x)}(\varphi_* v, \varphi_* w) = \gamma_x(v,w)$. We define a differential operator $Q : \mathscr{D}^{s+1} \to \mathscr{M}^s$ by

$$Q(\varphi) = \varphi^*\gamma.$$

We notice a small problem in that Q is not linear, and so we cannot compute its symbol. However, we can surmount this obstacle by first linearizing Q.

Consider some curve passing through $\varphi \in \mathscr{D}^{s+1}$. Given our construction of charts on $\mathscr{D}^{s+1}$, we can consider the family of diffeomorphisms that arise from our charts for $\mathscr{D}^{s+1}$, given by $\varphi_t = \exp_{\varphi(\cdot)}(tX_{\varphi_t(\cdot)})$, for $X \in H^{s+1}(TM)$. Then $\frac{d}{dt}|_{t=0}\varphi_t = X$, and thus

$$D_\varphi Q(X) = \frac{d}{dt}|_{t=0}\varphi_t^*\gamma = \mathcal{L}_X\gamma,$$

and we know from Example 2.1.6 that this has injective symbol.

Now, let $J = D_\varphi Q : H^{s+1}(TM) \to H^s(\mathrm{Sym}_2(M))$ denote this linearization at φ. Then consider the operator $P = J^*Q : \mathscr{D}^{s+1} \to \mathscr{M}$, and observe that since J^* is a linear map, (since φ is fixed) we obtain at $\psi \in \mathscr{D}^{s+1}$ the linearization $D_\psi P = J^* D_\psi Q$, so setting $\psi = \varphi$, we obtain $D_\varphi P = J^*J$, and as J has injective symbol $D_\varphi P$ is elliptic by Corollary 2.1.5. Now, since P is a differential operator of order 2, and the linearization of P is elliptic at φ, we can apply elliptic regularity of P to φ (see [2, Lemma 17.16]). That is, if $P\varphi \in H^k$, there exists $C > 0$ such that

$$\|\varphi\|_{H^{k+2}(M,M)} \leq C\big(\|P\varphi\|_{H^k(M,M)} + \|\varphi\|_{L^2(M,M)}\big).$$

But then since $P\varphi = (D_\varphi Q)^*\gamma$, which is smooth, we obtain that for any $k \in \mathbb{N}$,

$$\|\varphi\|_{H^{k+2}(M,M)} < \infty,$$

hence $\varphi \in C^\infty(M,M)$. Hence $\varphi \in \mathscr{D}$. $\square$

We now wish to show that I_γ embeds into $\mathscr{D}^{s+1}$. We first note that by Myers-Steenrod, I_γ with the C^1-topology is a Lie group.

Definition 3.3.3. We let $\mathfrak{g} = T_{\mathrm{Id}} I_\gamma$ denote the Lie algebra of I_γ.

Proposition 3.3.4. *The inclusion $\iota : I_\gamma \subseteq \mathscr{D}^{s+1}$ is smooth.*

Proof. The full proof is very technical (see [3, pp. 94-100]), but we will give the main idea here.

Recall by Proposition 3.1.9, R_η is a smooth map, and at any $\eta \in I_\gamma$ we have $\iota = R_\eta \circ \iota \circ R_{\eta^{-1}}$. So it suffices to show that it is smooth at Id.

For a small enough neighbourhood U of Id in I_γ, the group exponential $\operatorname{Exp} : \mathfrak{g} \to I_\gamma$ is a diffeomorphism, and thus provides us with a chart. Recall that on $\mathscr{D}^{s+1}$ we have coordinates on some neighbourhood of the identity given by $(H^{s,\varepsilon}(TM), \Psi_{\mathrm{Id}})$ as in (3.1). Let exp be the Riemannian exponential on M which induces the aforementioned chart. Let U be small enough so that for all $X \in U$, $\iota(\operatorname{Exp}(X)) \in \Psi_{\mathrm{Id}}(H^{s+1,\varepsilon}(TM))$. Then note that since $\operatorname{Exp} : \mathfrak{g} \to I_\gamma$ is smooth, the isometry $\eta = \operatorname{Exp}(X) : M \to M$ is smoothly dependent on $X \in U$. In charts on $\mathscr{D}^{s+1}$, $\iota(\eta) = \eta$ is given by some vector field $Y_\eta \in H^{s+1}(TM)$ so that, as maps on M, $\eta(x) = (x \mapsto \exp_x((Y_\eta)_x)$. That is, we want to know that the vector field Y_η such that $\exp(Y_\eta) = \eta$ depends smoothly on η. Since the Riemannian exponential Exp is defined by a 2nd order ODE, this boils down to the smooth dependence of initial conditions on the solutions of ODEs. $\square$

We now state some definitions which we will be applying throughout the rest of this chapter.

Definition 3.3.5. Let M be a Banach manifold. Let N be a subset of M and assume that at every point $y \in N$ there exists some chart (U, ψ) centered at y, and some open sets U_1, U_2 such that $\psi : U \to U_1 \times U_2$ is a Banach Space isomorphism, and

$$\psi(Y \cap U) = U_1 \times \{0\}.$$

Then we say that N is a *submanifold* of M.

Definition 3.3.6. Let $f : M \to N$ be a smooth map between Banach manifolds. We say that

- f is an *immersion* if at every $x \in M$, $T_x f$ is an injective map, and $T_x f(T_x M)$ is closed with closed linear complement in $T_{f(x)}N$.
- f is a *submersion* if at every $x \in M$, $T_x f$ is a surjective map and $\ker T_x f$ is closed with closed linear complement in $T_x M$.
- f is an *embedding* if it is an immersion that is also a homeomorphism onto its image. In this case the image is a submanifold.

Proposition 3.3.7. *The inclusion $I_\gamma \subseteq \mathscr{D}^{s+1}$ is an embedding.*

Proof. See [4, Corollary 5.4] $\square$

Proposition 3.3.8. *The quotient space $\mathscr{D}^{s+1}/I_\gamma$ is a smooth manifold.*

Proof. See [4, p 24] $\square$

3.4 Properties of the Action

In order to understand the slice theorem for the space of Riemannian metrics, we must first gain a deeper understanding of the action map $\Phi : \mathscr{D}^{s+1} \times \mathscr{M}^s \to \mathscr{M}^s$.

Definition 3.4.1. We reuse our previous notation to denote also the extended action map $\Phi : \mathscr{D}^{s+1} \times \mathscr{M}^s \to \mathscr{M}^s$. This map acts in the exact same way, $\Phi(\varphi, \gamma) = \varphi^*\gamma$.

Proposition 3.4.2. *The extension of the action map $\Phi : \mathscr{D}^{s+1} \times \mathscr{M}^s \to \mathscr{M}^s$ defined by $\Phi(\varphi, \gamma) = \varphi^*\gamma$ is continuous in both arguments. Moreover for fixed $\gamma \in \mathscr{M}$, the map $\Phi(\cdot, \gamma) : \mathscr{D}^{s+1} \to \mathscr{M}^s$ is smooth.*

Proof. Recall that $\mathscr{M}^s$ is modelled on $H^s(\mathrm{Sym}_2(M))$. We know also that $\mathscr{D}^{s+1} \subseteq \mathscr{D}^s$. Then $\varphi \in \mathscr{D}^{s+1}$, and $d\varphi \in H^s(TM, TM)$. Then we write for any $v, w \in TM$,

$$\Phi(\varphi, \gamma)_x(v, w) = \gamma_{\varphi(x)}(d\varphi v, d\varphi w).$$

In local trivializations, the above takes the form of composition and matrix multiplication. The composition is continuous by Lemma 3.1.5, and the multiplication is continuous by Corollary 2.2.5.

Now fix some $\gamma \in \mathscr{M}$. Then for all $r > 0$, $\gamma \in H^{s+r}(\mathrm{Sym}_2(M))$. Thus by Lemma 3.1.5, the map $\Phi(\cdot, \gamma)$ is C^r for all $r > 0$, hence it is smooth. $\square$

3.5 The Orbits of the Action

Just as we did in section 1.3 for finite dimensional manifolds, we will now attempt to show that the orbits $\Phi(\mathscr{D}^{s+1}, \gamma)$ are in fact embedded submanifolds of $\mathscr{M}^s$.

For the remainder of this chapter, we assume $\gamma \in \mathscr{M}$.

Lemma 3.5.1. *$T_{\mathrm{Id}}\Phi_\gamma : T_{\mathrm{Id}}(\mathscr{D}^{s+1}) \to T_\gamma(\mathscr{M}^s)$ is the map $T_{\mathrm{Id}}\Phi_\gamma(X) = \mathcal{L}_X\gamma$.*

Proof. Take $X \in T_{\mathrm{Id}}\mathscr{D}^{s+1}$. Then by Proposition 3.1.7, we can think of X as an element of $H^{s+1}(TM)$. Recall that we have a chart near $\mathrm{Id} \in \mathscr{D}^{s+1}$ defined using the Riemannian exponential map and let $\varphi_t = [x \mapsto \exp_x(tX_x)]$ be a curve passing through Id. Then

$$T_{\mathrm{Id}}\Phi_\gamma(X) = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^*\gamma) = \mathcal{L}_X\gamma.$$

□

Corollary 3.5.2. *The subspace $T_{\mathrm{Id}}\,\Phi_\gamma(\mathscr{D}^{s+1})$ is closed in $T_\gamma\mathscr{M}^s$, and its complement is closed.*

Proof. By Example 2.1.6, we know that $T_{\mathrm{Id}}\Phi_\mu$ has injective symbol, and thus by Theorem 2.3.5, we obtain the decomposition

$$H^s(\mathrm{Sym}^2(T^*M)) = T_{\mathrm{Id}}\Phi_\gamma(H^{s+1}(TM)) \oplus \ker(T_{\mathrm{Id}}\Phi_\gamma)^*.$$

□

Now we show that the same thing applies at any point $\eta \in \mathscr{D}^{s+1}$ by writing $T_\eta\,\Phi_\gamma$ at any $\eta \in \mathscr{D}^{s+1}$ using $T_{\mathrm{Id}}\,\Phi_\gamma$ and conjugating.

Lemma 3.5.3. *We have $T_\eta\,\Phi_\gamma(T_\eta(\mathscr{D}^{s+1})) = \eta^* \circ T_{\mathrm{Id}}\,\Phi_\gamma \circ (T_\eta R_{\eta^{-1}})(T_\eta(\mathscr{D}^{s+1}))$*

Proof. Take $\xi \in \mathscr{D}^{s+1}$, then

$$\begin{aligned}\eta^* \circ \Phi_\gamma \circ R_{\eta^{-1}}(\xi) &= \eta^*(\xi \circ \eta^{-1})^*\gamma \\ &= \eta^*(\eta^{-1})^*\xi^*\gamma \\ &= \xi^*\gamma,\end{aligned}$$

so that $\eta^* \circ \Phi_\gamma \circ R_{\eta^{-1}} = \Phi_\gamma$. Then by Lemma 3.5.1, and the fact that the pullback is linear, we obtain

$$T_\eta\,\Phi_\gamma = \eta^* \circ T_{\mathrm{Id}}\,\Phi_\gamma \circ T_\eta R_{\eta^{-1}}.$$

□

Lemma 3.5.4. *The linear maps $\eta^* : H^s(\mathrm{Sym}_2(M)) \to H^s(\mathrm{Sym}_2(M))$ and $T_\eta R_{\eta^{-1}} : T_\eta\mathscr{D}^{s+1} \to T_{\mathrm{Id}}\mathscr{D}^{s+1}$ are Banach-space homeomorphisms.*

Proof. The linear map η^* is a Banach space homeomorphism on $H^s(\mathrm{Sym}_2(M))$ by Lemma 3.1.5.

Take $X \in H^{s+1}(\eta^* TM)$ and consider the curve $x \mapsto \exp_{\eta(x)}(tX_x)$ passing through η in the direction X. Then observe that

$$\begin{aligned} T_\eta R_{\eta^{-1}}(X) &= \left.\frac{d}{dt}\right|_{t=0} \exp_{\eta(\cdot)}(tX) \circ \eta^{-1} \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp_{\eta\circ\eta^{-1}(\cdot)}(tX \circ \eta^{-1}) \\ &= X \circ \eta^{-1}, \end{aligned}$$

and we already showed that this was a Banach space homeomorphism in Lemma 3.1.6. □

Corollary 3.5.5. *At any $\eta \in \mathscr{D}^{s+1}$, $T_\eta\, \Phi_\gamma(H^{s+1}(\eta^* TM))$ is closed in $H^s(\mathrm{Sym}_2(M))$ and has closed linear complement.*

Proof. By the preceding lemma we know that $T_\eta R_{\eta^{-1}} : T_\eta \mathscr{D}^{s+1} \xrightarrow{\cong} T_{\mathrm{Id}} \mathscr{D}^{s+1}$ and thus by Corollary 3.5.2, $H^s(\mathrm{Sym}_2(M)) = T_{\mathrm{Id}}\, \Phi_\gamma \circ T_\eta R_{\eta^{-1}}(T_\eta \mathscr{D}^{s+1}) \oplus \ker(T_{\mathrm{Id}}\Phi_\gamma)^*$. That is, $T_{\mathrm{Id}}\, \Phi_\gamma \circ T_\eta R_{\eta^{-1}}(T_\eta \mathscr{D}^{s+1})$ is closed in $H^s(\mathrm{Sym}_2(M))$ with closed linear complement. Then again by the preceding lemma, since η^* is a homeomorphism of Banach spaces, we obtain that

$$H^s(\mathrm{Sym}_2(M)) = \eta^* \circ T_{\mathrm{Id}}\, \Phi_\gamma \circ T_\eta R_{\eta^{-1}}(T_\eta \mathscr{D}^{s+1}) \oplus \eta^*(\ker(T_{\mathrm{Id}}\Phi_\gamma)^*).$$

Hence by Lemma 3.5.3, $T_\eta\, \Phi_\gamma(H^{s+1}(\eta^* TM))$ is closed in $H^s(\mathrm{Sym}_2(M))$. □

Definition 3.5.6. Define the map $\tilde{\Phi}_\gamma : \mathscr{D}^{s+1}/I_\gamma \to \mathscr{M}^s$ by $\tilde{\Phi}_\gamma(I_\gamma \eta) = \eta^*\gamma$. Note then that $\Phi_\gamma(\eta) = \tilde{\Phi}_\gamma \circ \pi(\eta)$, where $\pi : \mathscr{D}^{s+1} \to \mathscr{D}^{s+1}/I_\gamma$ is the quotient map.

Lemma 3.5.7. *The map $\pi : \mathscr{D}^{s+1} \to \mathscr{D}^{s+1}/I_\gamma$ is smooth.*

Proof. See [4, pp. 23-25] □

Proposition 3.5.8. *For any $\eta \in \mathscr{D}^{s+1}$, $T_{I_\gamma \eta}\, \tilde{\Phi}_\gamma$ is an immersion.*

Proof. Note that $T_\eta\, \Phi_\gamma = T_{I_\gamma \eta}\, \tilde{\Phi}_\gamma \circ T_\eta \pi$. Then since π is a submersion, we have

$$T_{I_\gamma \eta}\, \tilde{\Phi}_\gamma(T_{I_\gamma \eta}(\mathscr{D}^{s+1}/I_\gamma)) = T_\eta\, \Phi_\gamma(T_\eta \mathscr{D}^{s+1}),$$

and thus $T_{I_\gamma \eta}\, \tilde{\Phi}_\gamma(T_{I_\gamma \eta}(\mathscr{D}^{s+1}/I_\gamma))$ is closed in $T_\eta \mathscr{M}^s$ with closed linear complement by Corollary 3.5.5. It remains to show that $\tilde{\Phi}_\gamma$ is smooth and injective.

Recall first that by Proposition 3.4.2, Φ_γ is smooth. Note also that since π is smooth, and $\Phi_\gamma = \tilde{\Phi}_\gamma \circ \pi$, it must be that $\tilde{\Phi}_\gamma$ is smooth.

Let us now show that $\tilde{\Phi}_\gamma$ is injective. By our observation $T_\eta \Phi_\gamma = T_{I_\gamma \eta} \tilde{\Phi}_\gamma \circ T_\eta \pi$, it suffices to show that $\ker T_\eta \Phi_\gamma \subseteq \ker T_\eta \pi$. Let's first show that this is true at $\mathrm{Id} \in \mathscr{D}^{s+1}$. Take some $X \in \ker T_{\mathrm{Id}} \Phi_\gamma$, and let Θ be the flow of X. Since $X \in \ker T_{\mathrm{Id}} \Phi_\gamma$, and $T_{\mathrm{Id}} \Phi_\gamma = \alpha$, then $X \in \ker \alpha = \ker \alpha^* \alpha$ using Lemma 2.3.7. But $\alpha^*\alpha$ is an elliptic operator, so by part 2 of Theorem 2.3.4, X is a smooth vector field, and for all t we have $\Theta_t \in \mathscr{D} \subseteq \mathscr{D}^{s+1}$. Moreover, for all t_0, using Lemma 3.5.1 we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \Theta_t^* \gamma &= \left.\frac{d}{dt}\right|_{t=0} \Theta_{t+t_0}^* \gamma \\
&= \left.\frac{d}{dt}\right|_{t=0} (\Theta_t \circ \Theta_{t_0})^* \gamma \\
&= \left.\frac{d}{dt}\right|_{t=0} \Theta_{t_0}^* (\Theta_t^* \gamma) \\
&= \Theta_{t_0}^* \left(\left.\frac{d}{dt}\right|_{t=0} \Theta_t^* \gamma \right) \\
&= \Theta_{t_0}^* \Big(\mathcal{L}_X \gamma \Big) \\
&= \Theta_{t_0}^*(0) = 0,
\end{aligned}$$

and thus $\Theta_t^* \gamma = \Theta_0^* \gamma = \mathrm{Id}^* \gamma = \gamma$ and therefore $\Theta_t \in I_\gamma$, for all t. Hence $\pi(\Theta_t) = I_\gamma$ for all t, and thus $T_{\mathrm{Id}} \pi(X) = 0$.

Now let's extend this idea to some arbitrary $\eta \in \mathscr{D}^{s+1}$. Recall that Lemma 3.5.3 gives us $T_\eta \Phi_\gamma = \eta^* \circ T_{\mathrm{Id}} \Phi_\gamma \circ T_\eta R_{\eta^{-1}}$, and suppose for some $X \in H^{s+1}(\eta^* TM)$ we have $T_\eta \Phi_\gamma(X) = 0$. By virtue of Lemma 3.5.4, we have $T_\eta R_{\eta^{-1}}(X) = (\eta^{-1})^* X \in \ker(\eta^* \circ T_{\mathrm{Id}} \Phi_\gamma)$, and η^* is injective, so it must be that $(\eta^{-1})^* X \in \ker T_{\mathrm{Id}} \Phi_\gamma$. Then by the first part of this proof we have $(\eta^{-1})^* X \in \ker T_{\mathrm{Id}} \pi$. But π commutes with right multiplication, and thus $\pi = R_{I_\gamma \eta} \circ \pi \circ R_{\eta^{-1}}$, so that $T_\eta \pi = T_{I_\gamma} R_{I_\gamma \eta} \circ T_{\mathrm{Id}} \pi \circ T_\eta R_{\eta^{-1}}$, and thus

$$T_\eta \pi(X) = T_{I_\gamma} R_{I_\gamma \eta} \circ T_{\mathrm{Id}} \pi \circ T_\eta R_{\eta^{-1}}(X) = T_{I_\gamma} R_{I_\gamma \eta} \circ T_{\mathrm{Id}} \pi((\eta^{-1})^* X) = 0.$$

Hence $X \in \ker T_\eta \pi$. $\square$

Proposition 3.5.9. *$\tilde{\Phi}_\gamma : \mathscr{D}^{s+1}/I_\gamma \to \mathscr{M}^s$ is an embedding.*

Proof. See [3][Proposition 6.13]. □

Let us denote $\mathcal{O}_\gamma := \Phi_\gamma(\mathscr{D}^{s+1})$, the orbit of γ under action by $\mathscr{D}^{s+1}$.

Corollary 3.5.10. *The orbit $\mathscr{O}_\gamma$ is closed in $\mathscr{M}^s$.*

3.6 Ebin's Slice Theorem for the Moduli Space of Riemannian Metrics

Theorem 3.6.1. *Let $\gamma \in \mathscr{M}$. Then there exists a submanifold S_γ of $\mathscr{M}^s$ such that:*

1. *For all $\eta \in I_\gamma, \eta^*(S_\gamma) = S_\gamma$.*
2. *If $\eta \in \mathscr{D}^{s+1}$ is such that $\eta(S_\gamma) \cap S_\gamma \neq \varnothing$, then $\eta \in I_\gamma$.*
3. *If $\chi : U \to \mathscr{D}^{s+1}$ is a local cross section at the identity for $\pi : \mathscr{D}^{s+1} \to \mathscr{D}^{s+1}/I_\gamma$, and $\Psi : U \times S_\gamma \to \mathscr{M}^s$ is defined by $\Psi(u, s) = \chi(u)^* s$, then Ψ is a homeomorphism onto a neighbourhood of γ in $\mathscr{M}^s$.*

Proof. We follow the proof of Ebin in [4, Section 7, Theorem].

By Proposition 3.5.9, we know that the tangent bundle of the orbit $T(\mathscr{O}_\gamma)$ is a subbundle of $T\mathscr{M}^s|_{\mathscr{O}_\gamma}$. We would like to use the fact that at every $\eta^*\gamma$ in $\mathscr{O}_\gamma$, the tangent space splits as described in the previous section.

First we define the normal bundle, defined at any point $\eta^*(\gamma)$ by

$$\nu_{\eta^*(\gamma)} = \{\omega \in T_{\eta^*(\gamma)}\mathscr{M}^s|_{\mathscr{O}_\gamma} : \langle \omega, \beta \rangle_{\eta^*(\gamma)} = 0, \text{ for all } \beta \in T\mathscr{O}_\gamma\}.$$

Since the pointwise inner product $\langle \cdot, \cdot \rangle_{\eta^*\gamma}$ is only a weak Riemannian metric, we cannot immediately conclude that this is a smooth subbundle of $TM|_{\mathscr{O}_\gamma}$, there is lots of work we still must do to show this. It suffices to find a smooth surjective bundle map $P : T(\mathscr{O}_\gamma)|_{\mathscr{O}_\gamma} \to T\mathscr{O}_\gamma$ such that $\ker P = \nu(\mathscr{O}_\gamma)$.

Notation change: Recall that in Lemma 3.5.1, we showed that $T_{\mathrm{Id}}\,\Phi_\gamma$ was the same map as α defined in Example 2.1.6. So to save space, since we will be writing it a lot, we write α in place of $T_{\mathrm{Id}}\,\Phi_\gamma$ for the remainder of this proof.

We know from Corollary 3.5.2 that $\nu_\gamma(\mathscr{O}_\gamma) = \ker \alpha^*$, and thus by Proposition 3.2.8, we obtain that $\nu_{\eta^*(\gamma)} = \eta^*(\ker \alpha^*) = \eta^*(\nu_\gamma)$.

We will digress for a moment to talk about the operator $\alpha^*\alpha$. Recall that α is injective, so that $\alpha^*\alpha : H^{s+3}(TM) \to H^{s+1}(TM)$ is elliptic. Then by Theorem 2.3.4, we know that

$$H^{s+1}(TM) = \alpha^*\alpha(H^{s+3}(TM)) \oplus \ker(\alpha^*\alpha) \underbrace{=}_{\text{Lemma 2.3.7}} \alpha^*(H^{s+3}(TM)) \oplus \ker\alpha \tag{3.3}$$

and thus the restricted map $(\alpha^*\alpha)\big|_{\alpha^*\alpha(H^{s+3}(TM))}$ is an isomorphism, so it makes sense to take its inverse. We will denote this restriction by $(\alpha^*\alpha)\big|_1$ for short, since with our convention for writing the decomposition, this restriction is on the first factor. This ends the digression.

We will define P using a similar idea to the one used in Lemma 3.5.3, by conjugating with η^*. Define P at $\eta^*\gamma$ by

$$P_{\eta^*\gamma} : \eta^*\alpha\Big((\alpha^*\alpha)\big|_1\Big)^{-1}\alpha^*(\eta^{-1})^* : T_{\eta^*\gamma}\mathscr{M}^s|_{\mathscr{O}_\gamma} \to T_{\eta^*\gamma}\mathscr{O}_\gamma.$$

By Lemma 2.3.8, $\alpha^*(H^s(\mathrm{Sym}_2(M))) = \alpha^*\alpha(H^{s+1}(TM))$, so post-composing α^* with $\Big((\alpha^*\alpha)\big|_1\Big)^{-1}$ makes sense.

Let us do a quick sanity check and make sense of the domain and image spaces. We know from Lemma 3.5.4 that η^* is a linear homeomorphism of Banach spaces, so $(\eta^{-1})^* : T_{\eta^*\gamma}\mathscr{M}^s|_{\mathscr{O}_\gamma} \to T_{\mathrm{Id}}\mathscr{M}^s|_{\mathscr{O}_\gamma}$, which is the domain of α^*. Also, the image of α is $T_{\mathrm{Id}}\mathscr{O}_s$, and $\eta^*(T_{\mathrm{Id}}\mathscr{O}_\gamma) = T_{\eta^*\gamma}\mathscr{O}_\gamma$, by Lemma 3.5.3.

We claim that $\ker P_{\eta^*\gamma} = \eta^*(\ker\alpha^*)$. Indeed, suppose that $P_{\eta^*\gamma}(\omega) = 0$ for some $\omega \in H^s(\mathrm{Sym}_2(M))$. We know that η^* is a linear isomorphism, and $\Big((\alpha^*\alpha)\big|_1\Big)^{-1}$ is an isomorphism whose image is orthogonal to $\ker\alpha$:

$$\Big((\alpha^*\alpha)\big|_1\Big)^{-1}(\alpha^*(H^s(\mathrm{Sym}_2(M)))) = \alpha^*(H^{s+2}(\mathrm{Sym}_2(M))) \perp \ker\alpha^*\alpha \underbrace{=}_{\text{Lemma 2.3.7}} \ker\alpha.$$

Looking at the definition of P, every map in the composition is injective on the image of the preceding map, except α^*. Then if $P_{\eta^*\gamma}(\omega) = 0$, observe that

$$P_{\eta^*\gamma}(\omega) = \underbrace{\eta^*\alpha\Big((\alpha^*\alpha)\big|_1\Big)^{-1}}_{\text{injective on the image of }\alpha^*} \alpha^* \underbrace{(\eta^{-1})^*}_{\text{injective}}(\omega) = 0.$$

Thus it must be that $(\eta^{-1})^*\omega \in \ker \alpha^*$. Hence $\omega \in \eta^*(\ker \alpha^*)$, so that $\ker P = \ker \nu(\mathscr{O}_\gamma)$.

Let's show that P_η is surjective. By (3.3) and the discussion above, we know that

$$\Big((\alpha^*\alpha)\big|_1\Big)^{-1}(\alpha^*(H^s(\mathrm{Sym}_2(M)))) = \alpha^*(H^{s+2}(\mathrm{Sym}_2(M)))$$

is the orthogonal complement of $\ker \alpha$ in $H^{s+1}(TM)$, and thus

$$\alpha\Big((\alpha^*\alpha)\big|_1\Big)^{-1}(\alpha^*(H^s(\mathrm{Sym}_2(M)))) = \alpha(H^s(TM)) = T_\gamma\mathscr{O}_\gamma.$$

Then using that $\eta^* : T_\gamma\mathscr{O}_\gamma \to T_{\eta^*\gamma}\mathscr{O}_\gamma$ (as we showed above), we see that P_η is surjective.

We will take for granted that P_η depends smoothly on η. But the interested reader can find details in [4, pp. 32-33] and [23, Corollary 7.4 and 7.5]. Note that in the latter, α^* is denoted by δ.

So, at this point we have obtained that $\nu(\mathscr{O}_\gamma)$ is in fact a smooth vector bundle over $\mathscr{O}_\gamma$. Now, consider the exponential on $\mathscr{M}^s$ induced by the weak Riemannian metric. The restriction $\exp|_{\nu(\mathscr{O}_\gamma)}$ is a diffeomorphism on some neighbourhood of the zero section of ν. By Proposition 3.5.9, we can identify an open neighbourhood of the identity coset $U \ni I_\gamma$ in $\mathscr{D}^{s+1}/I_\gamma$ with an open neighbourhood of $\gamma \in \mathscr{O}_\gamma$. Let $\chi : U \to \mathscr{D}^{s+1}$ be a smooth local cross section at the identity (such a map exists by [4, Proposition 5.10]). Then since the weak metric is $\mathscr{D}^{s+1}$-invariant, we can take also a small neighbourhood V of the zero section in ν_γ such that the exponential is a diffeomorphism on $W = \{\eta^*\omega : \eta \in \chi(U), \omega \in V\} \subseteq \nu$.

Now we make use of the strong Riemannian metric from Proposition 3.2.9. We can thus take

$$V = \nu_\gamma^\varepsilon := \{\omega \in H^s(\mathrm{Sym}_2(M)) : (\omega,\omega)_\gamma^s < \varepsilon^2\},$$

for some $\varepsilon > 0$. Recall from Proposition 3.2.9 that $(\cdot,\cdot)^s$ is $\mathscr{D}^{s+1}$-invariant. Thus for any $\eta \in I_\gamma$, and $\beta, \omega \in H^s(\mathrm{Sym}_2(M))$, we have $(\eta^*\beta, \eta^*\omega)^s_{\eta^*\gamma} = (\beta,\omega)^s_\gamma$. Hence I_γ acts as by orthogonal transformations with respect to our strong metric on $H^s(\mathrm{Sym}_2(M))$, so our neighbourhood ν_γ^ε is invariant under action by I_γ. Adjust ε (hence V) and U so that $\exp|_W$ is still a diffeomorphism onto its image.

Since the strong Riemannian metric induces a distance function ρ on $\mathscr{M}^s$, then with respect to ρ we can take some $\delta > 0$ such that $B_{2\delta}(\gamma) \subseteq \exp(W)$. Then, we take some $\varepsilon_0 < \varepsilon$ giving us $\nu_\gamma^{\varepsilon_0}$, and take $U_0 \subseteq U$ such that $\exp(W_0) \subseteq B_\delta(\gamma)$. Then we define our 'slice' as

$$S_\gamma := \exp(\nu_\gamma^{\varepsilon_0}).$$

Now all that is left is to verify that S_γ does in fact satisfy properties 1-3.

1. Let $\eta \in I_\gamma$, and take some $s \in S_\gamma$. Then $s = \exp(\omega)$ for $\omega \in \nu_\gamma^{\varepsilon_0}$. Then since the weak Riemannian metric is $\mathscr{D}^{s+1}$-invariant, the exponential defined by the weak Riemannian metric is $\mathscr{D}^{s+1}$-equivariant. So we have

$$\eta^* \exp(\omega) = \exp(\eta^* \omega) \in S_\gamma,$$

since $\eta^*(\nu_\gamma) = \nu_{\gamma^*\eta} = \nu_\gamma$ and $(\eta\omega, \eta\omega)^s_\gamma = (\omega, \omega)$, as the strong metric is $\mathscr{D}^{s+1}$-invariant.

2. Take $g, h \in S_\gamma$ such that $\eta^* g = h$ for some $\eta \in \mathscr{D}^{s+1}$. Then

$$\begin{aligned}\rho(\gamma, \eta^*\gamma) &\leq \rho(\gamma, h) + \rho(h, \eta^*\gamma)\\ &= \rho(\gamma, h) + \rho(\eta^* g, \eta^*\gamma)\\ &\leq \delta + \delta,\end{aligned}$$

which tells us that $\eta \in \chi(U)$. Also, since $g, h \in S_\gamma \subseteq \exp(V)$, there exists v_g, $v_h \in V$ such that $g = \exp(v_g)$ and $h = \exp(v_h)$. Hence

$$\exp(v_h) = h = \eta^* g = \eta^* \exp(v_g) = \exp(\eta^* v_g).$$

But since $\eta \in \chi(U)$ and $v_g, v_h \in V \subseteq \nu_\gamma$, then $\eta^* v_g, v_h \in W$, hence by injectivity of exp on W, it must be that $\eta^* v_g = v_h$. Hence $\eta^*(\nu_\gamma) = \nu_\gamma$, so $\eta \in I_\gamma$.

3. Consider the set $W_2 = \{\eta^*\omega | \eta \in \chi(U), \omega \in \nu_\gamma^{\varepsilon_0}\}$. Then define

$$\begin{aligned}\Theta : U \times S_\gamma &\to \mathscr{M}^s,\\ (I_\gamma \eta, s) &\mapsto \eta^* s.\end{aligned}$$

Then as a composition of continuous maps, and by definition of W_2 and the fact that exp is a diffeomorphism on W_2, we obtain that Θ is a continuous bijection.

Now, consider some $g \in \exp(W_2)$. Then $g = \exp(\eta^* v_g)$ for some $\eta \in \chi(U)$ and $v_g \in \nu_\gamma^{\varepsilon_0}$, and we can see that $\eta^* v_g = \exp^{-1}(g)$. Hence letting $\pi_{\mathscr{O}_\gamma} : \nu \to \mathscr{O}_\gamma$ be the smooth bundle projection map, we can see that $I_\gamma \eta = \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g)$. Then $\eta = \chi \circ \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g)$, so $\eta^{-1} = (\chi \circ \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g))^{-1}$. Then since $g = \exp(\eta^* v_g)$, using the equivariance of the exponential, we see that $\exp(v_g) = (\eta^{-1})^* g = ((\chi \circ \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g))^{-1})^* g$, so putting it together we get

$$\Theta^{-1}(g) = \left(\chi \circ \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g), ((\chi \circ \pi_{\mathscr{O}_\gamma} \circ \exp^{-1}(g))^{-1})^* g\right),$$

which is continuous. Hence Θ is a homeomorphism onto its image. $\square$

Chapter 4

A Slice Theorem on the Moduli Space of Connections

As before, let (M^n, γ) be a compact, oriented Riemannian manifold. Assume $\lfloor \frac{s}{2} \rfloor > \frac{n}{2} + 1$, so that by Corollary 2.2.5 H^s is a Banach algebra. Let E be a bundle over M with compact structure group G. In this chapter we will construct a slice theorem for the space of connections $\mathscr{A}$, acted on by the gauge group $\mathscr{G}$. This slice theorem was originally proven by Simon Donaldson, and the construction here follows that in [22].

4.1 The group $\mathscr{G}^{s+1}$ and the space $\mathscr{A}^s$

Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a covering of M over which we have coordinates and local trivializations of our bundle E. We first find a nice way to define $H^s(\mathrm{Aut}(E))$ locally. Let $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$, and consider some open set U_α. Then consider $H^s(U, G)$, the set of H^s maps from U to the structure group G. Corollary 2.2.5 gives us that $H^s(U, G)$ has pointwise defined multiplication (hence composition of linear maps) and inversion. We say that our bundle E is an H^s-G-bundle if the transition functions can be given on intersections by maps in $H^s(U_\alpha \cap U_\beta, G)$.

Let P be the reduced frame bundle of E considered as an H^{s+1}-G-bundle and consider the corresponding gauge group $H^{s+1}(P \times_{\mathrm{Ad}} G)$. By Corollary 2.2.5, pointwise multiplication in $\mathscr{G}^{s+1}$ is a smooth map. In fact, inversion too is smooth on $\mathscr{G}^{s+1}$.

Proposition 4.1.1. *Multiplication $\mathscr{G}^{s+1} \times \mathscr{G}^{s+1} \to \mathscr{G}^{s+1}$ and inversion $\mathscr{G}^{s+1} \to \mathscr{G}^{s+1}$ are smooth maps.*

Proof. In Corollary 2.2.5 we showed for $\lfloor \frac{s}{2} \rfloor > \frac{n}{2}$ that pointwise multiplication was a bounded linear map on Sobolev spaces. Since bounded linear maps between Banach spaces are smooth, we obtain that pointwise multiplication is smooth.

As for inversion, note that since M is compact, and u is everywhere invertible over M, $|\det u|$ obtains a minimum. Then $u^{-1} = \frac{1}{\det u}\mathrm{adj}u$ is well-defined over all of M. Since adj has entries composed of polynomials in the entries of u, we obtain again from Corollary 2.2.5 that $u \mapsto u^{-1}$ is smooth. $\square$

Denote the Lie algebra bundle associated to the gauge group by $\mathfrak{g}_E \coloneqq P \times_{Ad} \mathfrak{g}$, where again P is the reduced frame bundle, and $\mathfrak{g} = \mathrm{Lie}(G)$.

We can write any smooth connection locally as $\nabla^A = d + A$, and we know that the connection matrix A transforms under change of frame $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ by the formula

$$A_\beta = g_{\alpha\beta}^{-1} d(g_\alpha \beta) + g_{\alpha\beta}^{-1} A g_{\alpha\beta},$$

hence transition functions of regularity $s+1$ give rise to connections of regularity s. In the remainder of this section, we will construct the notion of a regularity s connection, and then define an action of the Gauge group $\mathscr{G}^{s+1}$ on these connections.

We know that the difference between two smooth connections is always a smooth End$E$-valued 1-form. That is, picking some reference connection ∇^0, we can write any connection ∇ as

$$\nabla^A = \nabla^0 + A$$

where $A \in \Gamma(T^*M \otimes \mathfrak{g}_E)$.

We use this property to define the space of s-regular connections on E.

Definition 4.1.2. Fix some smooth reference section ∇^0, and define the space of s-regular connections as the affine space

$$\mathscr{A}^s = \{\nabla^0 + A : A \in H^s(T^*M \otimes \mathfrak{g}_E)\}.$$

We use the shorthand $\nabla^A = \nabla^0 + A$ throughout this chapter, and often times refer to this connection just by its defining 1-form A.

Note that since $\mathscr{A}^s$ is an affine space modelled on $H^s(\mathfrak{g}_E)$, we know that at any point $A \in \mathscr{A}^s$, $T_A\mathscr{A}^s = H^s(T^*M \otimes \mathfrak{g}_E)$. Let ν be a fibre metric on E. Then note that γ and ν induce an inner product on $\mathscr{A}$ as follows. For $\omega, \eta \in T_A\mathscr{A}^s \subseteq H^s(\mathrm{End}E)$, we use the induced pointwise inner product on sections of $a, b \in \Gamma(T^*M \otimes \text{End}E) \cong \Gamma(T^*M \otimes E^* \otimes E)$ to give us the L^2-inner product

$$\langle a, b \rangle = \int_M (\omega, \eta)_{(T^*M \otimes \mathfrak{g}_E)} dV_\gamma \tag{4.1}$$

where dV_γ is the Riemannian volume form induced by γ.

We denote $d_A : \Gamma(\text{End}(E)) \to \Gamma(T^*M \otimes \text{End}(E))$ as the induced connection on the endomorphism bundle. Note that here we are technically abusing notation by writing d_A in place of $(d_A)_s$, the unique extension of d_A to the Sobolev s-space.

Definition 4.1.3. We define a left action $\Phi : \mathscr{G}^{s+1} \times \mathscr{A}^s \to \mathscr{A}^s$ which we will just denote shorthand by $\Phi(u, \nabla^A) = u \cdot \nabla^A$ by its action on a section $\xi \in \Gamma(E)$:

$$(u \cdot \nabla^A)\xi = u\nabla^A(u^{-1}\xi),$$

we can see that this is equivalent to

$$\begin{aligned} u\nabla^A(u^{-1}\xi) &= ud_A(u^{-1})\xi + uu^{-1}\nabla^A(\xi) \\ &= (ud_A(u^{-1}) + \nabla^A)\xi. \end{aligned}$$

Then, noting that $d_A(\text{Id}_E) = 0$ to give us $ud_A(u^{-1}) = -d_A(u)u^{-1}$ we obtain

$$(ud_A(u^{-1}) + \nabla^A)\xi = (-d_A(u)u^{-1} + \nabla^A)\xi,$$

hence we can define the action without sections simply by

$$u \cdot \nabla^A = \nabla^A - d_A(u)u^{-1}. \tag{4.2}$$

Let us confirm that the action above really is a left action. For $u, v \in \mathscr{G}^s$, and $\nabla^A \in \mathscr{A}^s$, we have

$$\begin{aligned} (uv) \cdot \nabla^A &= \nabla^A - d_A(uv)(uv)^{-1} \\ &= \nabla^A - \big(d_A(u)v + ud_A(v)\big)v^{-1}u^{-1} \\ &= \nabla^A - d_A(u)u^{-1} - ud_A(v)v^{-1}u^{-1} \\ &= u \cdot \nabla^A - ud_A(v)v^{-1}u^{-1}. \end{aligned}$$

Now we consider some section $\xi \in \Gamma(E)$, which allows us to use the initial definition of $u \cdot \nabla^A$ to simplify

$$\begin{aligned}\big(u \cdot \nabla^A - ud_A(v)v^{-1}u^{-1}\big)\xi &= u\nabla^A(u^{-1}\xi) - ud_A(v)v^{-1}u^{-1}\xi \\ &= u\big(\nabla^A - d_A(v)v^{-1}\big)(u^{-1}\xi) \\ &= \Big(u \cdot \big(v \cdot \nabla^A\big)\Big)(\xi),\end{aligned}$$

confirming that this is indeed a left action.

Note that for $a \in H^s(T^*M \otimes \mathfrak{g}_E)$, for any section ξ we obtain

$$\begin{aligned}u \cdot (\nabla^A + a)\xi &= u(\nabla^A(u^{-1}\xi) + au^{-1}\xi) \\ &= u\nabla^A(u^{-1}\xi) + uau^{-1}\xi,\end{aligned}$$

and the first term we can see is just the standard action of u on ∇^A, so we obtain the general expression

$$u \cdot (\nabla^A + a) = u \cdot \nabla^A + \mathrm{Ad}_u(a). \tag{4.3}$$

Lemma 4.1.4. *For any $\omega \in H^s(\mathrm{End}(E))$, the covariant derivative satisfies the property $d_{u(A)}(u\omega u^{-1}) = ud_A(\omega)u^{-1}$.*

Proof. Let ξ be some section of E. Then using (4.2)

$$\begin{aligned}d_{u(A)}(uau^{-1})\xi &= (u \cdot \nabla^A)(u\omega u^{-1}\xi) - u\omega u^{-1}((u \cdot \nabla^A)\xi) \\ &= \nabla^A(u\omega u^{-1}\xi) - d_A(u)\omega u^{-1}\xi - u\omega u^{-1}\nabla^A\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= d_A(u)\omega u^{-1}\xi + u\nabla^A(\omega u^{-1}\xi) - u\nabla^A(\omega u^{-1}\xi) \\ &\quad - d_A(u)\omega u^{-1}\xi - u\omega u^{-1}\nabla^A\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= u\nabla^A(\omega u^{-1}\xi) - u\omega u^{-1}\nabla^A\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= ud_A(\omega)u^{-1}\xi + u\omega\nabla^A(u^{-1}\xi) - u\omega u^{-1}\nabla^A\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= ud_A(\omega)u^{-1}\xi + u\omega d_A(u^{-1})\xi + u\omega u^{-1}\nabla^A\xi - u\omega u^{-1}\nabla^A\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= ud_A(\omega)u^{-1}\xi + u\omega d_A(u^{-1})\xi + u\omega u^{-1}d_A(u)u^{-1}\xi,\end{aligned}$$

where in the second line we used equation (4.2). Recalling that $d_A(u^{-1})u = -u^{-1}d_A(u)$, so that $d_A(u^{-1}) = -u^{-1}d_A(u)u^{-1}$, we subsutute into the last line to obtain

$$\begin{aligned}d_{u(A)}(uau^{-1})\xi &= ud_A(\omega)u^{-1}\xi - u\omega u^{-1}d_A(u)u^{-1}\xi + u\omega u^{-1}d_A(u)u^{-1}\xi \\ &= ud_A(\omega)u^{-1}\xi.\end{aligned}$$

$\square$

Using this Lemma we can derive an analogous result that will help us to construct our 'slice' in a later section. But first we need the following handy fact.

Lemma 4.1.5. *There exists an* Ad*-invariant inner product on* $H^s(T^*M \otimes \mathfrak{g}_E)$.

Proof. Since G is compact, there exists an Ad-invariant inner product $(\cdot, \cdot)_{\mathfrak{g}}$ on $\mathfrak{g}$. Observe that the Riemannian metric γ and the inner product $(\cdot, \cdot)_{\mathfrak{g}}$ induce an inner product on $T^*M \otimes \mathfrak{g}_E$ defined on simple tensors by

$$(\alpha \otimes \omega, \beta \otimes \theta)_{T^*M \otimes \mathfrak{g}_E} = (\alpha, \beta)_\gamma \cdot (\omega, \theta)_{\mathfrak{g}},$$

and is thus also invariant under the adjoint action of G on $T^*M \otimes \mathfrak{g}$.

Then for the induced L^2-inner product on $H^s(T^*M \otimes \mathfrak{g}_E)$, we obtain on simple tensors $a = \alpha \otimes \omega, b = \beta \otimes \theta \in H^s(T^*M \otimes \mathfrak{g}_E)$ that

$$\langle a, b \rangle = \int_M (\alpha, \beta)_\gamma (\omega, \theta)_{\mathfrak{g}} dV_\gamma.$$

Then we observe that for $u \in \mathscr{G}^{s+1}$, since at every point $x \in M$, $u(x) \in G$, we obtain

$$\begin{aligned}
\langle \mathrm{Ad}_u(a), \mathrm{Ad}_u(b) \rangle &= \int_M (\alpha, \beta)_\gamma (\mathrm{Ad}_u(\omega), \mathrm{Ad}_u(\theta))_{\mathfrak{g}} dV_\gamma \\
&= \int_M (\alpha, \beta)_\gamma (\omega, \theta)_{\mathfrak{g}} dV_\gamma \\
&= \langle a, b \rangle.
\end{aligned}$$

Hence there exists an L^2-inner product on $H^s(T^*M \otimes \mathfrak{g}_E)$ that is invariant under the adjoint action by $\mathscr{G}^{s+1}$. □

Lemma 4.1.6. *For any* $a \in H^s(T^*M \otimes \mathrm{End}E)$*, the adjoint of the covariant derivative satisfies the property* $d^*_{u(A)}(u\omega u^{-1}) = u d^*_A(a) u^{-1}$*, where* $d_{u(A)}$ *denotes the covariant derivative induced by the connection* $u \cdot \nabla^A$*.*

Proof. We know that d^*_A is defined by the property that for any $k, s \geq 1$ and $\omega \in H^k(\mathrm{End}E)$ and $a \in H^s(T^*M \otimes \mathrm{End}E)$ we have $\langle a, d_A \omega \rangle_{L^2(M)} = \langle d^*_A a, \omega \rangle_{L^2(M)}$. Let $\omega \in H^s(\mathrm{End}E)$ be arbitrary, then using Lemma 4.1.4,

$$\begin{aligned}
\langle d^*_{u(A)}(uau^{-1}), \omega\rangle_{L^2(M)} &= \langle uau^{-1}, d_{u(A)}(\omega)\rangle_{L^2(M)} \\
&= \langle uau^{-1}, d_{u(A)}(uu^{-1}\omega uu^{-1})\rangle \\
&= \langle uau^{-1}, ud_A(u^{-1}\omega u)u^{-1}\rangle \\
&= \langle a, d_A(u^{-1}\omega u)\rangle \\
&= \langle d^*_A a, u^{-1}\omega u\rangle \\
&= \langle ud_A(a)u^{-1}, \omega\rangle,
\end{aligned}$$

where we invoked the fact that the induced inner product is invariant under gauge transformations. Since ω was arbitrary, we conclude that d^*_A too satisfies $d^*_{u(A)}(uau^{-1}) = ud_A(a)u^{-1}$. ☐

Proposition 4.1.7. *The group $\mathscr{G}^s$ is a smooth manifold.*

Proof. See [18, Proposition 2.11]. ☐

4.2 Properties of the Action

Proposition 4.2.1. *The action $\Phi : \mathscr{G}^{s+1} \times \mathscr{A}^s \to \mathscr{A}^s$ is a smooth map of Banach manifolds.*

Proof. See [18, Proposition 3.12]. ☐

4.3 The Orbits of the Action

As in previous chapters, we denote the orbit map of the action $\mathscr{G}^{s+1} \cdot A$ for some fixed connection A by $\Phi_{\mathrm{A}} : \mathscr{G}^{s+1} \to \mathscr{A}^s$, defined by $\Phi_{\mathrm{A}}(u) = u \cdot A$.

Lemma 4.3.1. *The tangent space $T_A\mathscr{O}_A$ is the image of $d_A(H^{s+1}(\mathfrak{g}_E))$.*

Proof. Let $\gamma(t) = \exp(tX)$ be a one-parameter subgroup of $\mathscr{G}^{s+1}$ in the direction $X \in H^{s+1}(\mathfrak{g}_E^s)$. Then for any section $\xi \in \Gamma(E)$,

$$\begin{aligned}\Big(\frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot\nabla^A\Big)\xi &= \frac{d}{dt}\Big|_{t=0} \exp(tX)\nabla^A\big(\exp(tX)^{-1}\xi\big)\\ &= \frac{d}{dt}\Big|_{t=0} \exp(tX)\Big(d_A\big(\exp(-tX)\big)(\xi) + \exp(-tX)\big(\nabla^A\xi\big)\Big)\\ &= X\Big(\big(d_A(\mathrm{Id}_E)\big)(\xi) + \mathrm{Id}_E\big(\nabla^A\xi\big)\Big) + \mathrm{Id}\Big(\big(d_A(-X)\big)(\xi) - X(\nabla^A\xi)\Big)\\ &= X\big(\nabla^A\xi\big) - \big(d_A(X)\big)(\xi) - X\big(\nabla^A\xi\big)\\ &= -\big(d_A(X)\big)(\xi),\end{aligned} \tag{4.4}$$

where in the second line we use the defining property of the covariant derivative d_A on the endomorphism bundle, and the third line is just the product rule. and thus the tangent space of $\mathscr{G}^s \cdot A$ at $\mathrm{Id}\cdot A = A$ is $-d_A\big((H^{s+1}(\mathfrak{g}_E)\big)$, which is the same as $d_A\big(H^{s+1}(\mathfrak{g}_E)\big)$ by linearity of d_A. □

Recall from example 2.1.7 that $d_A : \Gamma(\mathrm{End}(E)) \to \Gamma(T^*M \otimes \mathrm{End}(E))$ has injective symbol. Then the restriction $d_A\big|_{\mathfrak{g}_E} : \Gamma(\mathfrak{g}_E) \to \Gamma(T^*M \otimes \mathfrak{g}_E)$ too has injective symbol, and thus by Theorem 2.3.5, we obtain the L^2-orthogonal decomposition

$$H^s(T^*M \otimes \mathfrak{g}_E) = d_A(H^{s+1}(\mathfrak{g}_E)) \oplus \ker d_A^*.$$

We already noted that the left hand side above can be identified with $T_A\mathscr{A}^s$, and so we have a decomposition of $T_A\mathscr{A}^s$ that is orthogonal with respect to the inner product (4.1). We then define $V_A = \nabla^A + \ker d_A^*$, and

$$V_A^\varepsilon \coloneqq \{\nabla^A + a \in H^s(T^*M \otimes \mathfrak{g}_E) : a \in \ker d_A^* \text{ and } \|a\|_{H^s(T^*M\otimes\mathfrak{g}_E)} < \varepsilon\}.$$

Then the set V_A^ε is our candidate for the 'slice' at A in $\mathscr{O}_A$.

Before we can attempt to construct our slice theorem, we must first equip ourselves with the tools necessary to deal with non-trivial isotropy groups.

Again we denote by I_A the isotropy group of the connection $A \in \mathscr{A}^s$.

Since M is connected, the holonomy group Hol_A of a connection A is unique up to conjugacy. Hence it suffices to identify it with a representative at a single point. So, let Hol_A denote the holonomy group of ∇^A at some $x \in M$.

Lemma 4.3.2 (Lemma 4.2.8 in [22])**.** *For any connection A over a connected base M, I_A is isomorphic to the centralizer of* Hol_A *in G.*

Proof. Recall that $C_G(\mathrm{Hol}_A) = \{g \in G : gh = hg, \forall h \in \mathrm{Hol}_A\}$, and $I_A = \{u \in \mathscr{G}^s : u \cdot \nabla^A = \nabla^A\}$.

Let $u \in I_A$, we first show that at any point $x \in M$, $u_x h = h u_x$ for all $h \in \mathrm{Hol}_A$. We then show that this property is independent of the base point, and use this as the defining property for our isomorphism.

Take $h \in \mathrm{Hol}_A$, and let $\gamma : [0,1] \to M$ be a loop at x for which the parallel transport with respect to the connection ∇^A is represented by h. That is, for any $\xi \in E_x$, we have $h\xi = \Pi_\gamma \xi$.

Let $\alpha : [0,1] \to M$ be any piecewise smooth path with $\alpha(0) = x$. Consider the maps

$$\begin{aligned} \Pi_{\alpha(t)} \circ u_x :& [0,1] \times E_x \to E_{\alpha(t)} \\ u_{\alpha(t)} \circ \Pi_{\alpha(t)} :& [0,1] \times E_x \to E_{\alpha(t)}. \end{aligned}$$

We can see that $(\Pi_{\alpha(0)} \circ u_{\alpha(0)})(\xi) = u_x \xi$, and since

$$\begin{aligned} \nabla^A\big(u \circ \Pi_{\alpha(t)}\xi\big) &\overbrace{=}^{u \in I_A} u \nabla^A\Big(u^{-1} u \circ \Pi_{\alpha(t)}\xi\Big) \\ &= u\nabla^A\big(\Pi_{\alpha(t)}\xi\big) \\ &= 0, \end{aligned}$$

we obtain that $u_{\alpha(t)}\Pi_{\alpha(t)}\xi$ is the unique parallel transport of $u_x\xi$ along α. But by definition, so is $\Pi_{\alpha(t)} u_x \xi$. Hence $u_{\alpha(t)}\Pi_{\alpha(t)} = \Pi_{\alpha(t)} u_x$, and thus, in particular, for the loop γ at x, we have $u_x \Pi_\gamma = \Pi_\gamma u_x$.

Now, using the idea above, we define a map

$$\begin{aligned} \Psi : I_A &\to C_G(\mathrm{Hol}_A) \\ u &\mapsto u_x. \end{aligned}$$

It is clear that Ψ is well-defined. We show that Ψ is bijective by showing that it has an inverse.

Consider the map

$$\begin{aligned} \Theta : C_G(\mathrm{Hol}_A) &\to I_A \\ g &\mapsto \tilde{g}, \end{aligned}$$

where

$$\tilde{g}_y = \Pi_\alpha g \Pi_\alpha^{-1}$$

for some piecewise smooth path α with $\alpha(0) = x$, and $\alpha(1) = y$.

First note that $\tilde{g}_y$ is independent of path. Indeed, if α, and β are two such paths connecting x and y, then $\alpha \circ \beta^{-1}$ is a loop at x and so we consider $\Pi_\alpha^{-1}\Pi_\beta \in \mathrm{Hol}_A$

$$\Pi_\alpha^{-1}\Pi_\beta g \underbrace{=}_{g \in C_G(\mathrm{Hol}_A)} g\Pi_\alpha^{-1}\Pi_\beta,$$

therefore

$$\Pi_\alpha g \Pi_\alpha^{-1} = \Pi_\beta g \Pi_\beta^{-1}.$$

So, Θ is well-defined. And we showed earlier that $u_y\Pi_\alpha = \Pi_\alpha u_x$, so that $u_y = \Pi_\alpha u_x \Pi_\alpha^{-1}$, and thus if $g = u_x$, we obtain $\tilde{g}_y = u_y$ for all $y \in X$, so $\Theta = \Psi^{-1}$.

Hence $\Psi : I_A \xrightarrow{\cong} C_G(\mathrm{Hol}_A)$. □

Corollary 4.3.3. *For any $A \in \mathscr{A}^s$ the isotropy group $I_A \subseteq \mathscr{G}^{s+1}$ is a Lie group.*

Proof. Note that $C_G(\mathrm{Hol}_A)$ is closed in the Lie group G, hence itself a Lie group. Then since $I_A \cong C_G(\mathrm{Hol}_A)$, I_A is a Lie group. □

Lemma 4.3.4. *We have* $\mathrm{Lie}(I_A) = \ker\big(d_A : H^{s+1}(\mathfrak{g}_E) \to H^s(T^*M \otimes \mathfrak{g}_E)\big)$.

Proof. This just follows from the fact that $T_{\mathrm{Id}_E}\Phi_{\mathrm{A}} = -d_A$, as computed in Lemma 4.3.1. □

Lemma 4.3.5. *The tangent space at the identity $T_{\mathrm{Id}}\mathscr{G}^{s+1}$ splits into an L^2-orthogonal direct sum*

$$T_{\mathrm{Id}}\mathscr{G}^{s+1} = d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E)) \oplus \ker d_A.$$

Proof. Since d_A has injective symbol, $d_A^* d_A$ is elliptic, so by Theorem 2.3.4, we an L^2-orthogonal decomposition

$$H^{s+1}(g_E) = d_A^* d_A(H^{s+3}(\mathfrak{g}_E) \oplus \ker d_A^* d_A.$$

But we know from Lemma 2.3.7 that $\ker d_A^* d_A = \ker d_A$, and from Lemma 2.3.8, $d_A^* d_A(H^{s+3}(\mathfrak{g}_E)) = d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E))$. Hence, recalling that $T_{\mathrm{Id}}\mathscr{G}^{s+1} = H^{s+1}(\mathfrak{g}_E)$, we are done. □

Lemma 4.3.6. *The slice V_A is invariant under action by I_A.*

Proof. Take $u \in I_A$ and $a \in \ker d_A^*$. Now since $u \in I_A$, using equation (4.3) we get

$$u \cdot (\nabla^A + a) = \nabla^A + \mathrm{Ad}_u(a)$$

and thus the statement amounts to showing that $\mathrm{Ad}_u(a) \in \ker d_A^*$. Indeed,

$$d_A^*(uau^{-1}) \overbrace{=}^{u\in I_A} d_{u(A)}^*(uau^{-1}) \overbrace{=}^{\text{Lemma 4.1.6}} ud_A^*(a)u^{-1} \overbrace{=}^{a\in\ker d_A^*} u0u^{-1} = 0.$$

□

Lemma 4.3.7. *The subgroup I_A is closed in $\mathscr{G}^{s+1}$.*

Proof. This is an immediate consequence of Proposition 4.2.1. □

4.4 Donaldson's Slice Theorem for the Moduli Space of Connections

Since I_A is a subgroup of $\mathscr{G}^{s+1}$, it acts freely on $\mathscr{G}^{s+1}$ and thus $\mathscr{G}^{s+1} \to \mathscr{G}^{s+1}/I_A$ is a principal I_A bundle. Also, in Lemma 4.3.6 we showed that for $u \in I_A$, for any $\nabla^A + a \in V_A$ we have $\Phi(u, A+a) = u \cdot (\nabla^A + a) \in V_A$. Thus the associated bundle $\mathscr{G}^{s+1} \times_{I_A} \ker d_A^*$ is also a smooth Banach manifold. We are now ready to state the slice theorem.

Theorem 4.4.1. *For any $A \in \mathscr{A}^s$ there exists $\varepsilon > 0$ such that $\mathscr{G}^{s+1} \times_{I_A} V_A^\varepsilon$ is diffeomorphic to a neighbourhood of $\mathscr{O}_A$ in $\mathscr{A}^s$.*

Proof. We use the arguments and ideas given in [5] and [9].

We define a map

$$\begin{aligned} \Psi : \mathscr{G}^{s+1} \times_{I_A} V_A &\to \mathscr{A}^s, \\ [u, \nabla^A + a] &\mapsto u \cdot (\nabla^A + a), \end{aligned}$$

and wish to show that for small enough ε, $\Psi|_{\mathscr{G}^{s+1}\times_{I_A} V_A^\varepsilon} : \mathscr{G}^{s+1} \times_{I_A} V_A^\varepsilon \to \mathscr{A}^s$ is a diffeomorphism onto its image, which necessarily contains $\mathscr{O}_A$.

Recall that by Lemma 4.3.5 we have the decomposition

$$T_{\mathrm{Id}_E}\mathscr{G}^{s+1} = d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E) \oplus \ker d_A.$$

Also remember that $\mathrm{Lie}(I_A) = \ker d_A$, and denote $I_A^\perp := \exp(d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E))$.

We now show that there is a neighbourhood of $(\mathrm{Id}_E, 0) \in I_A^\perp \times V_A$ that is a diffeomorphic to a neighbourhood of $\nabla^A \in \mathscr{A}^s$. Define the map

$$\begin{aligned} \tilde{\Psi} &: I_A^\perp \times V_A \to \mathscr{A}^s, \\ &(g, \nabla^A + a) \mapsto g \cdot (\nabla^A + a). \end{aligned}$$

We compute the differential at $(\mathrm{Id}_E, \nabla^A)$ in the direction of $(X, a) \in T_{\mathrm{Id}_E} I_A^\perp \times T_A V_A$:

$$\begin{aligned} D_{(\mathrm{Id}_E,0)}\tilde{\Psi}(X,a) &= \frac{d}{dt}\bigg|_{t=0} \Big(\exp(tX) \cdot (\nabla^A + ta) \Big) \\ &= \frac{d}{dt}\bigg|_{t=0} \Big(\exp(tX) \cdot \nabla^A + \exp(tX) ta \exp(tX)^{-1} \Big) \\ &= -d_A(X) + a \end{aligned}$$

where in the last line we used equation (4.4). But we know that $T_A\mathscr{A}^s = d_A(H^{s+1}(\mathfrak{g}_E)) \oplus \ker d_A^*$, and by Lemma 4.3.5, we have

$$d_A(H^{s+1}(\mathfrak{g}_E)) = d_A(d_A^*(H^{s+1}(T^*M \otimes \mathfrak{g}_E)) = d_A(T_{\mathrm{Id}} I_A^\perp).$$

Hence $D_{(\mathrm{Id}_E,\nabla^A)}\tilde{\Psi}$ is a linear isomorphism, and thus again invoking Theorem 2.2.9, and denoting $I_A^{\perp,\delta} := \{u \in I_A^\perp : \|u - \mathrm{Id}_E\|_{H^{s+1}(P\times_{\mathrm{Ad}}G)} < \delta\}$, we obtain $\delta > 0$ and $\varepsilon > 0$ such that for $I_A^{\perp,\delta}$, and V_A^ε, $\tilde{\Psi}\big|_{I_A^{\perp,\delta}\times V_A^\varepsilon} : I_A^{\perp,\delta} \times V_A^\varepsilon \to U_A \subseteq \mathscr{A}^s$ is a diffeomorphism onto its image. Now we analyze $\mathscr{G}^{s+1}$ near the identity to equip us with the second key observation needed to show that Ψ is a diffeomorphism.

Let $\mu : I_A^\perp \times I_A$ be the multiplication map, $\mu(g,h) = gh$ in $\mathscr{G}^{s+1}$, which is smooth by Proposition 4.1.1. Then for $X \in d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E))$ and $Y \in \ker d_A$, we have

$$D_{(\mathrm{Id}_E,\mathrm{Id}_E)}\mu(X,Y) = X + Y,$$

hence $D_{(\mathrm{Id}_E,\mathrm{Id}_E)}\mu$ is just the identity map on $T_{\mathrm{Id}_E}\mathscr{G}^{s+1} = d_A^*(H^{s+2}(T^*M \otimes \mathfrak{g}_E)) \oplus \ker d_A$. Thus, by Theorem 2.2.9, there exists an open neighbourhood $U_1 \times U_2$ of $(\mathrm{Id}_E, \mathrm{Id}_E) \in I_A^\perp \times I_A$ such that $\mu|_{U_1\times U_2}$ is a diffeomorphism onto its image, which we can assume to be some ball $B_{\delta_0}(\mathrm{Id})$ in $\mathscr{G}^{s+1}$. Then every element $u \in B_{\delta_0}(\mathrm{Id}_E)$ decomposes uniquely as $u = gh$ for $g \in U_1 \subseteq I_A^\perp$ and $h \in U_2 \subseteq I_A$.

Now pick $\delta > 0$ such that $I_A^{\perp,\delta} \subseteq U_1$, and observe that that $B_{\delta_0}(\mathrm{Id}_E) \times_{I_A} V_A^\varepsilon$ is diffeomorphic to $I_A^{\perp,\delta} \times V_A^\varepsilon$. To see this, define

$$\begin{aligned}\Theta : B_{\delta_0}(\mathrm{Id}_E) \times_{I_A} V_A &\to I_A^{\perp,\delta} \times V_A \\ [gh, \nabla^A + a] &\mapsto (g, h^{-1} \cdot (\nabla^A + a)).\end{aligned}$$

This is well-defined by the aforementioned unique decomposition of $u \in B_{\delta_0}(\mathrm{Id}_E)$. It is easy to verify that for any $\varepsilon > 0$ this has inverse map

$$\begin{aligned}\vartheta : I_A^{\perp,\delta} \times V_A^\varepsilon &\to B_{\delta_0}(\mathrm{Id}_E) \times_{I_A} V_A^\varepsilon \\ (g, \nabla^A + a) &\mapsto [g, \nabla^A + a].\end{aligned}$$

Indeed

$$\Theta(\vartheta(g, \nabla^A + a)) = \Theta([g, \nabla^A + a]) = (g\nabla^A + a)$$

and

$$\vartheta(\Theta([g, \nabla^A + a])) = \vartheta(g, \nabla^A + a) = [g, \nabla^A + a].$$

Since ϑ is clearly smooth, Θ is a diffeomorphism.

Then, after possibly shrinking δ and choosing $\varepsilon > 0$ such that Ψ is a local diffeomorphism, we obtain

$$\tilde{\Psi} \circ \Theta : B_{\delta_0}(\mathrm{Id}_E) \times_{I_A} V_A^\varepsilon \to U_A \subseteq \mathscr{A}^s$$

is a diffeomorphism. However, this is not quite what we were after. We want to extend this to the entire orbit $\mathscr{O}_A$.

Observe first that since for any $v \in \mathscr{G}^{s+1}$, v acts on an element $[u, \nabla^A + a] \in \mathscr{G}^{s+1} \times_{I_A} V_A^\varepsilon$ via $v \cdot [u, \nabla^A + a] = [vu, \nabla^A + a]$, and since the left translation map on $\mathscr{G}^{s+1}$ is a diffeomorphism, we know that $B_{\delta_0}(v) \cong B_{\delta_0}(\mathrm{Id}_E)$. Also, observe that

$$v \cdot \Psi([u, \nabla^A] + a) = v \cdot (u \cdot (\nabla^A + a)) = vu \cdot (\nabla^A + a) = \Psi([vu, \nabla^A + a]),$$

so Ψ respects the G-action. Hence

$$\Psi|_{B_{\delta_0}(v) \times_{I_A} V_A^\varepsilon} = L_v \circ \tilde{\Psi} \circ \Theta \circ L_{v^{-1}} : B_{\delta_0}(v) \times_{I_A} V_A^\varepsilon \to U_{v(A)} \subseteq \mathscr{A}^s$$

is a diffeomorphism onto some neighbourhood $U_{v(A)}$ of $v \cdot \nabla^A$. But this only tells us that Ψ is a local diffeomorphism. We do not know yet that Ψ is injective over its entire domain.

To obtain this we use the following result, adapted to fit our context and notation.

Lemma 4.4.2 (Lemma 10.14 in [9])**.** *There exists* $\varepsilon_0 > 0$ *such that if* $\nabla^{B_1}, \nabla_2^B \in V_A^{\varepsilon_0}$ *are equivalent by any gauge transformation* $u \in \mathscr{G}^{s+1}$*, then* $u = u_0\gamma$*, where* $\gamma \in I_A$ *and* $u_0 \in B_{\delta_0}(\mathrm{Id}_E)$*.*

Now take ε_0 as in the lemma. Suppose for $(u, \nabla^B), (u', \nabla^{B'}) \in \mathscr{G}^{s+1} \times V_A^{\varepsilon_0}$ we have $\Psi([u, \nabla^B]) = \Psi([u', \nabla^{B'}])$. Then $u \cdot \nabla^B = u' \cdot \nabla^{B'}$ and thus $\nabla^B = (u^{-1}u') \cdot \nabla^{B'}$. Hence $u^{-1}u' = u_0\gamma$, for $\gamma \in I_A$ and $u_0 \in B_{\delta_0}(\mathrm{Id}_E)$. Then

$$\Psi([\mathrm{Id}_E, \nabla^B]) = \Psi([u_0\gamma, \nabla^{B'}]) = \Psi([u_0, \gamma^{-1} \cdot \nabla^{B'}])$$

but $[\mathrm{Id}_E, \nabla^B], [u_0, \gamma^{-1} \cdot \nabla^{B'}] \in B_{\delta_0} \times_{I_A} V_A^{\varepsilon_0}$, and thus since Ψ is a diffeomorphism on $B_{\delta_0} \times_{I_A} V_A^{\varepsilon_0}$, it must be that $\mathrm{Id}_E = u_0$, and $\nabla^B = \gamma^{-1} \cdot \nabla^{B'}$, so that Ψ is injective on all of $\mathscr{G}^{s+1} \times_{I_A} V_A^{\varepsilon_0}$.

Hence

$$\Psi : \mathscr{G}^{s+1} \times_A V_A^{\varepsilon_0} \to U_{\mathscr{O}_A}$$

is a diffeomorphism onto a tubular neighbourhood of $\mathscr{O}_A$.

$\square$

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