Elliptic Partial Differential Operators

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Abstract

Elliptic partial differential operators have become an important class of operators in modern differential geometry, due in part to the Atiyah-Singer index theorem, which states that the index of an elliptic operator (defined in terms of the analytic properties of the operator) is equal to its topological index (defined purely in terms of topological data). In this paper, we give an introduction to the theory of elliptic partial differential operators on manifolds, with the main focus being to prove a generalised version of the Hodge-Decomposition theorem for elliptic partial differential operators.

Contents

1. Introduction

Elliptic partial differential operators have become an important class of operators in modern differential geometry, due in part to the Atiyah-Singer index theorem, which states that the index of an elliptic operator (defined in terms of the analytic properties of the operator) is equal to its topological index (defined purely in terms of topological data). In this paper, we give an introduction to the theory of elliptic partial differential operators on manifolds, with the main focus being to prove a generalised version of the Hodge-Decomposition theorem for elliptic partial differential operators.

This paper is divided into five parts. We begin with a summary of the theory that we need from differential geometry and functional analysis in Section 2. Then in Section 3 we introduce elliptic partial differential operators on vector bundles. We define the symbol of a partial differential operator, we define what it means for an operator to be elliptic, and we prove the existence of the formal adjoint of a partial differential operator. In Section 4 we digress to pursue essentials of the theory of pseudodifferential operators on complex-valued functions of several real variables. The main result from this section is the existence of a parametrix for an elliptic partial differential operator, which we use to prove various properties of elliptic partial differential operators. In Section 5, we globalise the results from the previous section to vector bundles, and then prove a decomposition theorem for elliptic partial differential operators, which states that the any section of a bundle can be written uniquely as a sum of two sections, one in the image of the operator, and the other in the kernel of its adjoint. From this we derive the Hodge-Decomposition theorem as a corollary.

2. Background

Vector Bundles

Let M be a smooth manifold, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. For an integer $p \geq 0$, a $\mathbb{K}^p$*-vector bundle over* M consists of a smooth manifold E and a surjective smooth submersion $\pi\colon E \longrightarrow M$ such that for each $x \in M$, the *fibre* $E_x := \pi^{-1}(x)$ has the structure of a p-dimensional vector space over $\mathbb{K}$, and such that for each $x \in E$, there is a neighbourhood U of x as well as a diffeomorphism $E|_U := \pi^{-1}(U) \to M \times \mathbb{K}^p$ which preserves fibres and is a linear isomorphism on each fibre. Such a diffeomorphism is called a *local trivialisation*; p is called the *rank* of the vector bundle. Note that for each p, the manifold $\underline{\mathbb{K}}^p_M := M \times \mathbb{K}^p$ is a $\mathbb{K}^p$-vector bundle over M called the *trivial* $\mathbb{K}^p$*-vector bundle over* M. A *section* of E is a smooth map $u\colon M \longrightarrow E$ such that $u_x := u(x) \in E_x$ for each $x \in M$.

Given two vector bundles E and F over M, we can form their *direct sum* $E \oplus F$ and *tensor product* $E \otimes F$, which are again vector bundles over M having fibres $(E \oplus F)_x = E_x \oplus F_x$ and $(E \otimes F)_x = E_x \otimes F_x$, respectively. For each nonnegative integer k, we can form the k^{th} *symmetric power* $\Sigma^k(E)$ of E as well as its k^{th} *exterior power* $\Lambda^k(E)$. These are again vector bundles over E, and they have fibres $(\Sigma^k(E))_x = \Sigma^k(E_x)$ and $(\Lambda^k(E))_x = \Lambda^k(E_x)$, respectively. We also denote the conjugate vector bundle of E by $\bar{E}$.

A *fibre metric* on E is a section of h of $E^* \otimes \bar{E}^*$ such that for each $x \in M$, the induced map $h_x\colon E_x \times E_x \longrightarrow \mathbb{K}$ is an inner product. When $\mathbb{K} = \mathbb{R}$, we can view h as a section of $\Sigma^2(E)$.

Riemannian Manifolds

Let M be a compact oriented manifold of dimension n endowed with a *Riemannian metric* g (a fibre metric on the tangent bundle TM).

The metric g provides us with canonical bundle isomorphisms $\flat\colon TM \to T^*M$ and $\sharp := \flat^{-1}\colon T^*M \to TM$ called the *musical isomorphisms* defined by $\xi^\flat = g_x(\cdot, \xi)$ for $\xi \in T_x^*M$. By declaring $\flat$ to be an isometry on each fibre, this induces a fibre metric on T^*M, which we also denote by g. In turn, this induces a fibre metric on $\Lambda^k(T^*M)$, again denoted by g, given on irreducibles by

$$g_x(\xi_1 \wedge \cdots \wedge \xi_k, \zeta_1 \wedge \cdots \wedge \zeta_k) = \det[g_x(\xi_i, \zeta_j)]_{ij}$$

The metric g allows us to define a canonical top form vol_g on M, called the *Riemannian volume form*, given in any positively-oriented local coordinate chart x by $\sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n$. If $f \in C^\infty(M)$, we define the integral of f over M by

$$\int_M f = \int_M f \,\mathrm{vol}_g \,.$$

Banach Spaces

A Banach space is a vector space X together with a norm $\|\cdot\|$ whose induced norm topology is complete. Given Banach spaces X and Y, a linear operator $T\colon X \to Y$ is *bounded* if there is a positive constant C such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. Recall that T is bounded if and only it is continuous, and that if T is bounded, its *operator norm* is

$$\begin{aligned} \|T\| &= \inf\{C > 0 : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\ &= \sup\{\|Tx\| : \|x\| = 1\}. \end{aligned}$$

We denote the space of bounded linear operators from X to Y by $\mathscr{B}(X, Y)$, which we make it into a Banach space by equipping it with the operator norm. The *dual* of X is the space $X^* := \mathscr{B}(X, \mathbb{K})$ consisting of all bounded linear functionals on X.

Among the many kinds of operators studied in functional analysis, there are two that will be especially important in our discussion in this paper; these are the compact operators and the Fredholm operators. An operator $T \in \mathscr{B}(X, Y)$ is *compact* if T maps bounded subsets of X to precompact subsets of Y. Equivalently, T takes bounded sequences in X to sequences with convergent subsequences in Y. In contrast, T is *Fredholm* if it is invertible modulo compact operators, which is to say there are operators $S \in \mathscr{B}(X, Y)$, $K \in \mathscr{B}(X, X)$ and $K' \in \mathscr{B}(Y, Y)$ with K

and K' compact such that

$$ST = \mathrm{id}_X - K \quad \text{and} \quad TS = \mathrm{id}_Y - K'.$$

Equivalently, T has closed image, finite-dimensional kernel and finite-dimensional cokernel:

Proposition 2.1. *A bounded linear operator $T\colon X \to Y$ is Fredholm if and only if* $\ker T$ *and* $\operatorname{coker} T := Y/\operatorname{im} T$ *are finite-dimensional.*

Proof. The proof of this fact requires some care. See for instance Theorem 5.1 in [GGK90]. □

Finally, it will be useful to know the (Banach space) dual of a bounded linear operator, as well as how the image and kernel of the operator are related to the image and kernel of the dual. Given $T \in \mathscr{B}(X, Y)$, the *dual* of T is the operator $T^* \in \mathscr{B}(Y^*, X^*)$ defined by $T^\dagger \alpha = \alpha \circ T$. Given a subset S of X, the *annihilator* of S is the subspace S° of X^* defined by

$$S^\circ = \{\alpha \in X^* : \alpha(x) = 0 \text{ for all } x \in S\},$$

and given a subset S of X^*, the *preannihiliator* of S is the subspace $S_\circ$ of X defined by

$$S_\circ = \{x \in X : \alpha(x) = 0 \text{ for all } \alpha \in S\}.$$

We have the following:

Proposition 2.2. *Let X and Y be Banach spaces.*

(a) If A is a linear subspace of X, then $(A^\circ)_\circ = \bar{A}$.

(b) If $T \in \mathscr{B}(X, Y)$, then $\ker T^\dagger = (\operatorname{im} T)^\circ$, and hence $(\ker T^\dagger)_\circ = \overline{\operatorname{im} T}$.

Proof. For the forward inclusion in (a), if $x \notin \bar{A}$ then by the Hahn-Banach Theorem, there is a linear functional $\alpha \in X^*$ with $\alpha = 0$ on $\bar{A}$ and $\alpha(x) \neq 0$, which is to say $x \notin (A^\circ)_\circ$. For the reverse inclusion, assume $x \in \bar{A}$, and choose a net (x_λ) in A converging to x. Then for any $\alpha \in A^\circ$, $\alpha(x) = \lim_\lambda \alpha(x_\lambda) = 0$, so $x \in (A^\circ)_\circ$.

Next, for the first part of (b), observe that $\alpha \in \ker T^\dagger \Leftrightarrow \alpha \circ T = 0 \Leftrightarrow \alpha = 0$ on $\operatorname{im} T \Leftrightarrow \alpha \in (\operatorname{im} T)^\circ$. The second part of (b) now follows immediately from (a). □

Hilbert Spaces

A Hilbert space is an inner product space $(H, \langle\cdot,\cdot\rangle)$ whose induced norm topology is complete. Every Hilbert space is a Banach space with its induced norm, so all results in the previous section apply to Hilbert spaces as well. We need the following two additional results specific to Hilbert spaces.

Proposition 2.3 (The Riesz Representation Theorem). *Let H be a Hilbert space. The map $H \to H^*$ sending $y \mapsto \langle \cdot, y\rangle$ is a linear isomorphism.*

Proof. That this map is linear follows immediately from linearity of $\langle \cdot, \cdot \rangle$ in its first argument. Injectivity follows easily: if we are given $y, z \in H$ satisfying $\langle \cdot, y\rangle = \langle \cdot, z\rangle$, then $0 = \langle \cdot, y - z\rangle$, so $0 = \langle y - z, y - z\rangle = \|y - z\|^2$, and therefore $y = z$.

Next we prove surjectivity. Let $\alpha \in H^*$. Since α is continuous, $\ker\alpha$ is closed, so we may write $H = \ker\alpha \oplus (\ker\alpha)^\perp$. If $\alpha = 0$, then $\alpha = \langle \cdot, 0\rangle$. Otherwise $\alpha \neq 0$, so we can find $y \in (\ker\alpha)^\perp$ nonzero. By normalising if necessary, we may assume that $\|y\| = 1$. Put $z = \overline{\alpha(y)}y$. Then $\|z\| = |\alpha(y)|$, and so $\alpha(z) = |\alpha(y)|^2 = \|z\|^2$. For any $x \in H$,

$$\langle x, z\rangle = \left\langle x - \frac{\alpha(x)}{\|z\|^2} z, z \right\rangle + \left\langle \frac{\alpha(x)}{\|z\|^2} z, z \right\rangle = \left\langle x - \frac{\alpha(x)}{\|z\|^2} z, z \right\rangle + \alpha(x).$$

Since $z \in (\ker\alpha)^\perp$, and since $\alpha(x - \alpha(x)z/\|z\|^2) = \alpha(x) - \alpha(x)\alpha(z)/\|z\|^2 = 0$, it follows that $\langle x - \alpha(x)z/\|z\|^2, z\rangle = 0$, and hence $\langle x, z\rangle = \alpha(x)$ for all $x \in H$. □

Corollary 2.4. *Let $T\colon H \to K$ be a linear map of Hilbert spaces. There is a unique linear map $T^*\colon K \to H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in H$ and $y \in K$. The map T^* is called the* adjoint *of T.*

Proof. Uniqueness follows by essentially the same argument as in the previous proof. We now prove existence. First we do this for the case where $K = H$. Given $y \in H$, $\langle T(\cdot), y\rangle \in H^*$, so by the previous proposition, $\langle T(\cdot), y\rangle = \langle \cdot, z_y\rangle$ for a unique $z_y \in H$. Define T^* by $T^*y = z_y$ for each y. Then T^* satisfies the desired property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. That T^* is linear follows from uniqueness of each z_y and linearity of $\langle \cdot, \cdot\rangle$.

Now we prove the general case. Form the Hilbert space $H \oplus K$ equipped with inner product $\langle x \oplus y, x' \oplus y'\rangle = \langle x, x'\rangle + \langle y, y'\rangle$, and consider the linear map $T'\colon H \oplus K \to H \oplus K$ defined by $S(x \oplus y) = 0 \oplus Tx$. By our argument in the preceding paragraph, there is a unique linear map $S^*\colon H \oplus K \to H \oplus K$ with the property that

$$\langle S(x \oplus y), x' \oplus y'\rangle = \langle x \oplus y, S^*(x' \oplus y)\rangle$$

for all $x, x' \in H$ and $y, y' \in K$. Note that

$$\langle S(x \oplus y), x' \oplus y'\rangle = \langle 0 \oplus Tx, x' \oplus y'\rangle = \langle Tx, y'\rangle.$$

If $\pi_H : H \oplus K \to H$ and $\pi_K : H \oplus K \to K$ are the natural projection maps, then

$$\langle x \oplus y, S^*(x' \oplus y')\rangle = \langle x, \pi_H(S^*(x' \oplus y')\rangle + \langle y, \pi_K(S^*(x' \oplus y')\rangle.$$

If we take $x' = 0$ and $y = 0$, then

$$\langle Tx, y'\rangle = \langle x, \pi_H(S^*(0 \oplus y')\rangle,$$

so $y' \mapsto \pi_H(S^*(0 \oplus y'))$ is the adjoint of T. □

Multi-Indices

It is customary to index differential operators by so-called multi-indices, as they provide a concise way to write down these operators. In the context of $\mathbb{R}^n$, or more generally an n-dimensional manifold, a *multi-index* is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers. We define the *order* of a multi-index α to be the sum $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and we define its *factorial* to be the product $\alpha! = \alpha_1! \cdots \alpha_n!$. We also define the *binomial coefficients*

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

for multi-indices α and β, as well as the *multinomial coefficients*

$$\binom{\alpha}{\beta_1, \ldots, \beta_m} = \frac{\alpha!}{\beta_1! \cdots \beta_m!}$$

for multi-indices $\alpha, \beta_1, \ldots, \beta_m$. We impose a partial order on multi-indices by declaring $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for each $i = 1, \ldots, n$. Given $x \in \mathbb{R}^n$, we put $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Analogously we define

$$\partial^\alpha = \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Note that

$$\partial_x^\beta x^\alpha = \begin{cases} \frac{\alpha!}{\beta!} x^{\alpha - \beta} & \text{if } \beta \leq \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The Leibniz rule for the derivative of the product of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ can be stated as

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha - \beta} g.$$

More generally, if we have functions $f_1, \ldots, f_m$, then

$$\partial^\alpha (f_1 \cdots f_m) = \sum_{\beta_1 + \cdots + \beta_m = \alpha} \binom{\alpha}{\beta_1, \ldots, \beta_m} \partial^{\beta_1} f_1 \cdots \partial^{\beta_m} f_m.$$

We also have Taylor's Theorem: if $f \colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then

$$f(x+y) = \sum_{|\alpha|<k} \frac{1}{\alpha!} \partial^\alpha f(x) y^\alpha + k \int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha \, dt$$

Finally, we introduce the following bit of non-standard notation: for a multi-index α, we put

$$\alpha^1, \ldots, \alpha^{|\alpha|} = \underbrace{1, \ldots, 1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{n, \ldots, n}_{\alpha_n \text{ times}},$$

so that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial}{\partial x^{\alpha^1}} \cdots \frac{\partial}{\partial x^{\alpha^{|\alpha|}}}.$$

3. Elliptic Partial Differential Operators

Partial Differential Operators

Let M be a smooth manifold of dimension n, and let E and F be $\mathbb{K}$-vector bundles over M of ranks p and q, respectively.

Definition. A linear map $P \colon \Gamma(E) \to \Gamma(F)$ is a *partial differential operator of order m*, where m is a nonnegative integer, if each point of M has a neighbourhood U over which is defined

(a) a local coordinate system $x \colon U \to \mathbb{R}^n$;

(b) local trivialisations $E|_U \to U \times \mathbb{K}^p$ and $F|_U \to U \times \mathbb{K}^q$ for E and F with induced local frames $e_1, \ldots, e_p$ and $f_1, \ldots, f_q$, respectively; and

(c) smooth matrix valued functions $A^\alpha \colon U \to \mathbb{K}^{q \times p}$ for each $|\alpha| \leq m$, with $A^\alpha \neq 0$ for some α with $|\alpha| = m$,

and such that with respect to this coordinate system and these trivialisations, if $u \in \Gamma(E)$ is given on U by $u = \sum_i u_i f_i$, then $Pu = \sum_i v_i f_i$ on U where

$$(v_1(x), \ldots, v_q(x)) = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} (u_1(x), \ldots, u_p(x)). \tag{3.1}$$

Essentially, a partial differential operator is a linear map taking sections of E to sections of F that acts like an ordinary differential operator in coordinates. We usually abuse notation by writing $P = \sum_{|\alpha|\leq m} A^\alpha \partial^{|\alpha|}/\partial x^\alpha$ rather than writing out all of (3.1).

One might ask whether there is a nice coordinate-free definition of a partial differential operator; for example, since the expressions $A^\alpha \otimes \partial^{|\alpha|}/\partial x^\alpha$ are local sections of the bundle $\mathrm{Hom}(E, F) \otimes \Sigma^{|\alpha|}(TM)$, one could ask whether P can be viewed as a global section of a direct sum of these bundles. The answer to this question in general is no because the coefficients A^α do not transform in the correct way under changing local coordinates and local trivialisations. However, if we look only at the terms with $|\alpha| = m$, we do actually get a well-defined section of the bundle $\mathrm{Hom}(E, F) \otimes \Sigma^m(TM)$.

Proposition 3.1. *The local expressions*

$$\sum_{|\alpha|=m} A^\alpha \otimes \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{3.2}$$

determine a well-defined global section $\sigma(P)$ *of the bundle* $\mathrm{Hom}(E, F)\otimes\Sigma^m(TM)$. $\sigma(P)$ *is called the* principal symbol *of the operator* P.

Proof. We must show that the expressions in (3.2) are unchanged when we change local coordinates and local trivialisations. First change local trivialisations via transition functions $g\colon U \to \mathrm{GL}(p, \mathbb{K})$ and $h\colon U \to \mathrm{GL}(q, \mathbb{K})$ for E and F, respectively. Then, by the Leibniz rule, P takes the form

$$P = h\left(\sum_{|\alpha|\leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}\right)g^{-1} = \sum_{|\alpha|\leq m} B^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where each B^α is a $q \times p$ matrix of smooth functions depending on the A^α, h and the derivatives of g^{-1} up to order m. Notice the only order m derivatives that show up in the above come from $\partial^{|\alpha|}/\partial x^\alpha g^{-1}$ terms where $|\alpha| = m$ and $\partial^{|\alpha|}/\partial x^\alpha$ does not differentiate g^{-1} after applying the Leibniz rule. Therefore

$$B^\alpha = hA^\alpha g^{-1} \quad \text{for } |\alpha| = m,$$

which is the correct transformation law to make (3.2) well-defined.

Next, we show the local expressions are unchanged when we change local coordinates. Let $y = y(x)$ be a change of local coordinates. For good measure, we include here a derivation of the transformation law for going between the two local frames $\{\partial^m/\partial x^\alpha\}_{|\alpha|=m}$ and $\{\partial^m/\partial y^\alpha\}_{|\alpha|=m}$ for $\Sigma^m(TM)$. By the transformation law

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}, \tag{3.3}$$

we have

$$\begin{aligned}\frac{\partial^m}{\partial x^\alpha} = \frac{\partial}{\partial x_{\alpha^1}} \cdots \frac{\partial}{\partial x_{\alpha^m}} &= \left(\sum_{i_1} \frac{\partial y_{i_1}}{\partial x_{\alpha^1}} \frac{\partial}{\partial y_{i_1}} \right) \cdots \left(\sum_{i_m} \frac{\partial y_{i_m}}{\partial x_{\alpha^m}} \frac{\partial}{\partial y_{i_m}} \right) \\ &= \sum_{i_1,\dots,i_m} \frac{\partial y_{i_1}}{\partial x_{\alpha^1}} \cdots \frac{\partial y_{i_1}}{\partial x_{\alpha^1}} \frac{\partial^m}{\partial y_{i_1} \cdots \partial y_{i_m}}\end{aligned}$$

as a section of $\Sigma^m(TM)$. What we want to do is to write the above sum in the form $\sum_{|\beta|=m} c_{\alpha\beta} \partial^m / \partial y^\beta$. Notice that for a given multi-index β with $|\beta| = m$, $\partial^m / \partial y^\beta := \partial^m / \partial y_{\beta^1} \cdots \partial y_{\beta^m} = \partial^m / \partial y_{i_1} \cdots \partial y_{i_m}$ if and only if $i_1, \dots, i_m$ is a permutation of $\beta^1, \dots, \beta^m$. Therefore

$$c_{\alpha\beta} = \sum_{\sigma \in S_m} \frac{\partial y_{\sigma(\beta^1)}}{\partial x_{\alpha^1}} \cdots \frac{\partial y_{\sigma(\beta^m)}}{\partial x_{\alpha^m}}. \tag{3.4}$$

Now let us see how the local representation for P changes. By (3.3), P takes the form

$$P = \sum_{|\alpha| \le m} A^\alpha \left(\sum_{i_1} \frac{\partial y_{i_1}}{\partial x_{\alpha^1}} \frac{\partial}{\partial y_{i_1}} \right) \cdots \left(\sum_{i_{|\alpha|}} \frac{\partial y_{i_{|\alpha|}}}{\partial x_{\alpha^{|\alpha|}}} \frac{\partial}{\partial y_{i_{|\alpha|}}} \right) = \sum_{|\alpha| \le m} C^\alpha \frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

where each C^α is a smooth function $U \to \mathbb{K}^{q \times p}$ depending on the A^α and the x_i-derivatives of the y_j up to order m. By the Leibniz rule, the order m terms in the above are exactly

$$\sum_{|\alpha|=m} \sum_{i_1,\dots,i_m} A^\alpha \frac{\partial y_{i_1}}{\partial x_{\alpha^1}} \cdots \frac{\partial y_{i_m}}{\partial x_{\alpha^m}} \frac{\partial^m}{\partial y_{i_1} \cdots \partial y_{i_m}} = \sum_{|\alpha|=m} \sum_{|\beta|=m} c_{\alpha\beta} \frac{\partial^m}{\partial y^\beta},$$

where the equality follows from essentially the same argument we used to prove (3.4). Hence

$$C^\beta = \sum_{|\alpha|=m} c_{\alpha\beta} A^\alpha \quad \text{for } |\alpha| = m.$$

By the above and (3.4),

$$\sum_{|\beta|=m} C^\beta \otimes \frac{\partial^m}{\partial y^\beta} = \sum_{|\beta|=m} \sum_{|\alpha|=m} c_{\alpha\beta} A^\alpha \otimes \frac{\partial^m}{\partial y^\beta} = \sum_{|\alpha|=m} A^\alpha \otimes \frac{\partial^m}{\partial x^\alpha},$$

which proves (3.2) is well-defined. □

Remark. Although the bundles $\mathrm{Hom}(E, F) \otimes \Sigma^m(TM)$ do not lend themselves to a coordinate-free definition of partial differential operators, there is a way to define partial differential operators in a coordinate free way if the vector bundles E is equipped with a connection ∇. Specifically, one can define a partial differential operator of order m to be a map $P\colon \Gamma(E) \to \Gamma(F)$ of the form

$$P = \sum_{j=1}^{m} A^j \cdot \nabla^j$$

where $A^j \in \Gamma(\mathrm{Hom}(E, F) \otimes (TM)^{\otimes j})$ for each j, $\nabla^j\colon \Gamma(E) \to \Gamma((T^*M)^{\otimes j} \otimes E)$ is the induced connection for each j, and $\cdot$ is natural pairing of $\mathrm{Hom}(E, F) \otimes (TM)^{\otimes j}$ with $(T^*M)^{\otimes j} \otimes E$ (the $\mathrm{Hom}(E, F)$ component pairs with the E component, and the $(TM)^{\otimes j}$ component pairs with the $(T^*M)^{\otimes j}$ component). Viewed in this way, it is easy to define the symbol of P: it is just $\sigma(P) = A^m$. Of course, now one has to check that the symbol does not depend on the choice of connection, so this will not circumvent the calculation in Proposition 3.1!

For our purposes, this alternative definition will just get in the way, and we will not mention it again. Keep in mind, however, that partial differential operators often come up in this alternative form in the literature.

Given vector spaces V and W as well as a covector $\alpha \in V^*$, the map

$$\begin{aligned} V \times V^m &\to W \\ (w, v_1, \ldots, v_m) &\mapsto w\alpha(v_1)\ldots\alpha(v_m) \end{aligned}$$

is multilinear in $w, v_1, \ldots, v_m$ and symmetric in $v_1, \ldots, v_m$, so it descends to a map $W \times \Sigma^m(V) \to W$ that sends $w \otimes v_1 \cdots v_m$ to $w\sigma(v_1) \cdots \alpha(v_m)$. Consequently, given a covector $\xi \in T_x^*M$, we can evaluate $\sigma(P)$ at ξ to get a linear map $\sigma_\xi(P)\colon E_x \to F_x$. Explicitly, with respect to local coordinates and trivialisations as in (3.1), if $\xi = \sum_i \xi_i dx_i|_x$, then

$$\sigma_\xi(P) = \sum_{|\alpha|=m} A^\alpha(x)\xi^\alpha$$

where $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Definition. A differential operator $P\colon \Gamma(E) \to \Gamma(F)$ is *elliptic* if for each $x \in M$ and each nonzero $\xi \in T_x^*M$, the map $\sigma_\xi(P)\colon E_x \to F_x$ is a linear isomorphism.

Note that for P to be elliptic, E and F must have the same rank. Another way to look at the symbol is as follows. Via the projection $\pi\colon T^*M \to M$, the bundles

E and F on M pull back to bundles π^*E and π^*F on T^*M, so we may view the symbol of P as a bundle map $\sigma(P)\colon \pi^*E \to \pi^*F$. Thus, the operator P is elliptic if and only if $\sigma(P)$ is an isomorphism away from the zero section. The principal symbol behaves nicely when viewed as a map from partial differential operators to bundle maps:

Proposition 3.2. *Let $P, P'\colon \Gamma(E) \to \Gamma(F)$ and $Q\colon \Gamma(F) \to \Gamma(L)$ be partial differential operators, suppose P and P' have the same order, and let $t, t' \in \mathbb{K}$. Then*

$$\sigma(tP + t'P') = t\sigma(P) + t'\sigma(P'),$$

and

$$\sigma(Q \circ P) = \sigma(Q) \circ \sigma(P).$$

Proof. Let $\xi \in T^*M$, let m be the order of P and P', and let m' be the order of Q. With respect to local trivialisation and local coordinates, write

$$P = \sum_{|\alpha|\le m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad P' = \sum_{|\alpha|\le m} B^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \quad \text{and} \quad Q = \sum_{|\beta|\le m'} C^\beta \frac{\partial^{|\beta|}}{\partial x^\beta}.$$

The first identity follows easily, since

$$tP + t'P' = \sum_{|\alpha|\le m} (tA^\alpha + t'B^\alpha) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

implies

$$\sigma_\xi(tP + t'P') = \sum_{|\alpha|=m} (tA^\alpha + t'B^\alpha)\xi^\alpha = t\sigma_\xi(P) + t'\sigma_\xi(P').$$

For the second identity, observe that by the Leibniz rule,

$$Q \circ P = \sum_{|\beta|\le m'} C^\beta \frac{\partial^{|\beta|}}{\partial x^\beta}\left(\sum_{|\alpha|\le m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}\right) = \sum_{|\beta|\le m'} \sum_{|\alpha|\le m} \sum_{\gamma\le\beta} \binom{\beta}{\gamma} C^\beta \frac{\partial^{|\gamma|} A^\alpha}{\partial x^\gamma} \frac{\partial^{|\alpha+\beta-\gamma|}}{\partial x^{\alpha+\beta-\gamma}}.$$

The highest order terms in this sum correspond to those multi-indices α, β and γ with $|\beta| = m'$, $|\alpha| = m$ and $\gamma = \beta$, so the highest order terms are

$$\sum_{|\beta|=m'} \sum_{|\alpha|=m} C^\beta A^\alpha \frac{\partial^{|\alpha+\beta|}}{\partial x^{\alpha+\beta}}.$$

Hence

$$\sigma_\xi(Q \circ P) = \sum_{|\beta|=n} \sum_{|\alpha|=m} C^\beta C^\alpha \xi^{\alpha+\beta} = \sigma_\xi(Q) \circ \sigma_\xi(P).$$

$\square$

Example 3.3. Consider the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ on k-forms. If $\eta \in \Omega^k(M)$ is given in a coordinate chart $x\colon U \to \mathbb{R}^n$ by

$$\eta = \sum_{i_1<\cdots<i_k} \eta_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

then

$$\begin{aligned} d\eta &= \sum_{i_1<\cdots<i_k} \sum_{j=1}^{k+1} \frac{\partial \eta_{i_1\ldots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_k} \\ &= \sum_{i_1<\cdots<i_{k+1}} \sum_{j=1}^{k+1} (-1)^{j-1} \frac{\partial \eta_{i_1\cdots\widehat{i_j}\cdots i_{k+1}}}{\partial x_{i_j}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \end{aligned}$$

on U. Thus, if for each $j = 0, \ldots, n$, we let $e_{i_1\cdots i_j}$ $(1 \leq i_1 < \cdots < i_j \leq n)$ be the standard basis vectors for $\Lambda^j(\mathbb{R}) = \mathbb{R}^{\binom{n}{j}}$, then

$$d = \sum_{i_1<\cdots<i_k} e_{i_1\cdots i_{k+1}} \sum_{j=1}^{n} (-1)^{j-1} e^{\mathsf{T}}_{i_1\cdots\widehat{i_j}\cdots i_{k+1}} \frac{\partial}{\partial x_{i_j}}$$

on U, so d is a partial differential operator of first order (note each of the terms $e_{i_1\cdots i_{k+1}} e^{\mathsf{T}}_{i_1\cdots\widehat{i_j}\cdots i_{k+1}}$ is an $\binom{n}{k+1} \times \binom{n}{k}$ matrix). Moreover, for any $\xi \in T_x^*M$ and any $\xi \in \Lambda^k(T_x^*M)$, if $= \sum_i \xi_i dx_i|_x$ and

$$\zeta = \sum_{i_1<\cdots<i_k} \zeta_{i_1\cdots i_k} dx_{i_1}|_x \wedge \cdots \wedge dx_{i_k}|_x,$$

then

$$
\begin{aligned}
\sigma_\xi(d)(\zeta) &= \sum_{i_1<\cdots<i_{k+1}} e_{i_1\cdots i_{k+1}} \sum_{j=1}^{k+1}(-1)^{j-1}\xi_{i_j}\zeta_{i_1\cdots\widehat{i_j}\cdots i_k} \\
&= \sum_{i_1<\cdots i_{k+1}} \sum_{j=1}^{k+1}(-1)^{j-1}\xi_{i_j}\zeta_{i_1\cdots\widehat{i_j}\cdots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_{k+1}} \\
&= \sum_{i_1<\ldots i_k} \sum_{j=1}^{n} \xi_j \zeta_{i_1\cdots i_k} dx_j \wedge dx_{i_1}\wedge\cdots dx_{i_k} \\
&= \xi\wedge\zeta.
\end{aligned}
$$

Thus d is highly nonelliptic: given *any* $\xi \in T_x^*M$ nonzero, we can choose a basis $\xi_1, \ldots, \xi_n$ for T^*M with $\xi_1 = \xi$, so that $\sigma_\xi(d)(\zeta) = 0$ where $\zeta := \xi_1 \wedge \cdots \wedge \xi_n \neq 0$.

L^2 Spaces and Formal Adjoints

Now assume the manifold M is compact, and equip the vector bundle E with a fibre metric $\langle\cdot,\cdot\rangle_E$. The fibre metric provides us with a natural inner product on $\Gamma(E)$ defined by

$$\langle u, v\rangle_{L^2(E)} = \int_M \langle u, v\rangle_E,$$

called the L^2-*inner product* on E. We denote its corresponding norm by $\|\cdot\|_{L^2(E)}$, and we denote the completion of $\Gamma(E)$ with respect to this norm by $L^2(E)$. Thus, $L^2(E)$ is a Hilbert space.

Proposition 3.4. *The equivalence class of the L^2-norm on a compact manifold is independent of the choice of fibre metric used to define it.*

Proof. Let $\langle\cdot,\cdot\rangle_E'$ be another fibre metric on E, and denote its corresponding L^2-norm by $\|\cdot\|_{L^2(E)}'$. We must show there are positive constants c and C such that

$$c\|u\|_{L^2(E)}' \leq \|u\|_{L^2(E)} \leq C\|u\|_{L^2(E)}'$$

for all $u \in \Gamma(E)$. Choose a finite cover $U_1, \ldots, U_N$ of M by open sets such that for each ℓ, $\bar{U}_\ell$ is compact and contained in the domain of a coordinate chart $x_\ell \colon V_\ell \to \mathbb{R}^n$ over which there is a local trivialisation $\varphi_\ell \colon E|_{V_\ell} \to V_\ell \times \mathbb{K}^p$. Also choose a partition of unity $\chi_1, \ldots, \chi_N$ subordinate to the cover $U_1, \ldots, U_N$.

Fix an index ℓ, and consider the map

$$\bar{U}_\ell \times S \to \mathbb{R}$$
$$(p,z) \mapsto \left|\varphi_\ell^{-1}(p,z)\right|_{E_p}$$

where $S := \{z \in \mathbb{K}^p : |z| = 1\}$ is the unit sphere in $\mathbb{K}^p$. Since this map is smooth, and since $\bar{U}_\ell \times S$ is compact, it attains a minimum c_ℓ and a maximum C_ℓ. Note that $C_\ell \geq c_\ell > 0$ since $|\varphi_\ell^{-1}(p,z)|_{E_p} = 0 \Rightarrow \varphi_\ell^{-1}(p,z) = 0 \Rightarrow z = 0$. Since $\varphi_\ell^{-1}(p,z) = |z|\varphi_\ell^{-1}(p,z/|z|)$ for $z \neq 0$, we have $c_\ell|z| \leq |\varphi_\ell^{-1}(p,z)|_{E_p} \leq C_\ell|z|$ for all $p \in U_\ell$ and $z \in \mathbb{K}^r$. By the same argument, there are constants $C'_\ell \geq c'_\ell > 0$ such that $c'_\ell|z| \leq |\varphi_\ell^{-1}(p,z)|'_{E_p} \leq C'_\ell|z|$ for all $p \in U_\ell$ and $z \in \mathbb{K}^p$. Hence for any $u \in \Gamma(E)$,

$$\frac{c_\ell}{C'_\ell}|u|'_E \leq |u|_E \leq \frac{C_\ell}{c'_\ell}|u|'_E \quad \text{on } E|_{U_\ell}.$$

Now by definition

$$\|u\|^2_{L^2(E)} = \sum_\ell \int_{U_\ell} \chi_\ell |u|^2_E \operatorname{vol}_{g_\ell} \quad \text{and} \quad \|u\|'^2_{L^2(E)} = \sum_\ell \int_{U_\ell} \chi_\ell |u|'^2_E \operatorname{vol}_{g_\ell},$$

so by our work above

$$\sum_\ell \frac{c_\ell^2}{C'_\ell} \int_{U_\ell} \chi_\ell |u|'^2_E \operatorname{vol}_{g_\ell} \leq \|u\|^2_{L^2(E)} \leq \sum_\ell \frac{C_\ell^2}{c'^2_\ell} \int_{U_\ell} \chi_\ell |u|'^2_E \operatorname{vol}_{g_\ell}.$$

Thus, for $c = \min\{c_\ell/C'_\ell : \ell = 1,\dots,N\}$ and $C = \max\{C_\ell/c'_\ell : \ell = 1,\dots,N\}$, we have $c\|u\|'_{L^2(E)} \leq \|u\|_{L^2(E)} \leq C\|u\|'_{L^2(E)}$. □

Now equip F with a fibre metric $\langle\cdot,\cdot\rangle_F$, so that $\Gamma(F)$ is equipped with an L^2-inner product $\langle\cdot,\cdot\rangle_{L^2(F)}$.

Proposition 3.5. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be a partial differential operator of order m. There is a unique linear operator $P^*\colon \Gamma(F) \to \Gamma(E)$ such that*

$$\langle Pu, v\rangle_{L^2(F)} = \langle u, P^*v\rangle_{L^2(E)} \tag{3.5}$$

for all $u \in \Gamma(E)$ and $v \in \Gamma(F)$. Moreover, P^* is an m^{th} order partial differential operator with symbol

$$\sigma(P^*) = (-1)^m \sigma(P)^*.$$

Consequently, P is elliptic if and only if P^* is elliptic.

Proof. Uniqueness follows easily: if $Q, Q'\colon \Gamma(F) \to \Gamma(E)$ are linear operators such that $\langle u, Qv\rangle_{L^2(E)} = \langle Pu, v\rangle_{L^2(F)} = \langle u, Q'v\rangle_{L^2(E)}$ for all $u \in \Gamma(E)$ and $v \in$

$\Gamma(F)$, then $0 = \langle u, (Q - Q')v\rangle_{L^2(E)}$ for all u and v, so by taking $u = (Q - Q')v$, we get $Q = Q'$.

Next we prove existence. Choose a cover of M by local coordinate charts $x^\ell \colon U_\ell \xrightarrow{\sim} \mathbb{R}^n$ for $\ell = 1, \dots, N$ over each of which are defined orthonormal local frames $e_1^\ell, \dots, e_p^\ell$ and $f_1^\ell, \dots, f_q^\ell$ for E and F, respectively. Let $u \in \Gamma(E)$ and $v \in \Gamma(F)$ be arbitrary. On each U_ℓ, write $P = \sum_{|\alpha| \le m} A_\ell^\alpha \partial^{|\alpha|}/\partial x^\alpha$, $u = \sum_i u_i^\ell e_i^\ell$ and $v = \sum_j v_j^\ell f_j^\ell$. Also set $u^\ell = (u_1^\ell, \dots, u_p^\ell)$ and $v^\ell = (v_1^\ell, \dots, v_q^\ell)$. Then

$$\begin{aligned}
\langle Pu, v\rangle_{L^2(F)} &= \sum_\ell \langle Pu, \chi_\ell v\rangle_{L^2(F)} \\
&= \sum_\ell \int_{U_\ell} \sum_{|\alpha| \le m} \left(A_\ell^\alpha \frac{\partial^{|\alpha|} u^\ell}{\partial x^\alpha}\right)^{\mathsf{T}} \chi_\ell \bar{v}^\ell \operatorname{vol}_{g_\ell} \\
&= \sum_\ell \int_{U_\ell} (u^\ell)^{\mathsf{T}} \frac{1}{\sqrt{\det g_\ell}} \sum_{|\alpha| \le m} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \left(\sqrt{\det g_\ell} (A_\ell^\alpha)^{\mathsf{T}} \chi_\ell \bar{v}^\ell\right) \operatorname{vol}_{g_\ell} \\
&= \sum_\ell \int_{U_\ell} (u^\ell)^{\mathsf{T}} \overline{P_\ell^* v} \operatorname{vol}_{g_\ell},
\end{aligned}$$

where the third line follows by integration by parts (there are no boundary terms because $x_\ell(U) = \mathbb{R}^n$), and in the final line we have defined $P_\ell^* \colon \Gamma(F) \to \Gamma(E)$ by

$$P_\ell^* v = \frac{1}{\sqrt{\det g_\ell}} \sum_{|\alpha| \le m} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \left(\sqrt{\det g_\ell} (\bar{A}_\ell^\alpha)^{\mathsf{T}} \chi_\ell v^\ell\right). \tag{3.6}$$

Note that the above makes sense as a global operator because the right-hand side is supported in U_ℓ. If we put $P^* = \sum_\ell P_\ell^*$, then

$$\langle Pu, v\rangle_{L^2(F)} = \sum_\ell \int_{U_\ell} (u^\ell)^{\mathsf{T}} \overline{P_\ell^* v^\ell} \operatorname{vol}_{g_\ell} = \sum_\ell \langle u, P_\ell^* v\rangle_{L^2(E)} = \langle u, P^* v\rangle_{L^2(E)},$$

which proves existence. Also, P^* is clearly an m^{th} order partial differential operator.

Finally, we compute $\sigma(P^*)$. By the Leibniz rule, the terms in (3.6) taking m^{th} order derivatives of v are exactly

$$\frac{1}{\sqrt{\det g_\ell}} \sum_{|\alpha| = m} (-1)^m \sqrt{\det g_\ell} (\bar{A}_\ell^\alpha)^{\mathsf{T}} \chi_\ell \frac{\partial^{|\alpha|} v^\ell}{\partial x^\alpha} = \chi_\ell (-1)^m \sum_{|\alpha| = m} (\bar{A}_\ell^\alpha)^{\mathsf{T}} \frac{\partial^{|\alpha|} v^\ell}{\partial x^\alpha}.$$

Hence

$$\sigma(P^*) = \sum_\ell \sigma(P_\ell^*) = \sum_\ell \chi_\ell (-1)^m \sigma(P)^* = (-1)^m \sigma(P)^*.$$

This completes the proof. $\square$

The operator P^* in the preceding proposition is called the *formal adjoint* of P. The word "formal" is here to emphasise that P^* as an operator $\Gamma(F) \to \Gamma(E)$ is not an actual adjoint of P, since $\Gamma(E)$ and $\Gamma(F)$ are not Hilbert spaces in general. Of course, P and P^* naturally extend to operators $P\colon L^2(E) \to L^2(F)$ and $P^*\colon L^2(F) \to L^2(E)$ where P^* really is the Hilbert space adjoint of P. Also, note that the identity (3.5) extends to an identity

$$\langle Pu, v\rangle_{L^2(F)} = \langle u, P^*v\rangle_{L^2(E)}$$

valid for all $u \in L^2(E)$ and $v \in L^2(F)$.

Example 3.6. Endow M with a Riemannian metric g. Recall that g induces a fibre metric on $\Lambda(T^*M)$, so we have an L^2-inner product on $\Omega(M)$ defined by

$$\langle \omega, \eta\rangle = \int_M g(\omega, \eta).$$

Also recall, from Example 3.3, that d is a first order partial differential operator. In this example, we shall compute the formal adjoint of d with respect to this L^2-inner product.

There is a unique linear operator $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$, called the *Hodge star operator*, with the property that $\omega \wedge *\eta = g(\omega, \eta)\,\mathrm{vol}_g$ for all $\omega, \eta \in \Omega^k(M)$. With respect to any local orthonormal frame $e_1, \dots, e_n$ for T^*M, $*$ is given by

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$$

where $i_1 < \cdots < i_k$, $j_1 < \dots < j_{n-k}$, $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, the plus sign is chosen if $e_{i_1}, \dots, e_{i_k}, e_{j_1}, \dots, e_{j_{n-k}}$ is positively oriented, and the minus sign is chosen otherwise. It is straightforward to show that $** = (-1)^{n(n-k)}$.

The main utility of the Hodge star operator, for us at least, is that it allows us to rewrite the L^2-inner product as follows: for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$,

$$\langle \omega, \eta\rangle_{L^2(\Lambda(T^*M))} = \begin{cases} \int_M \omega \wedge *\eta & \text{if } k = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for $\omega \in \Omega^{k-1}(M)$ and $\eta \in \Omega^k(M)$,

$$\begin{aligned} d(\omega \wedge *\eta) &= d\omega \wedge *\eta + (-1)^{k-1}\omega \wedge d*\eta \\ &= d\omega \wedge *\eta + (-1)^{k-1}\omega \wedge (-1)^{(k-1)(n-k+1)}***d*\eta \\ &= d\omega \wedge *\eta - \omega \wedge *d^*\eta \end{aligned}$$

where we have defined $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$d^* = (-1)^{(k-1)(n-k+1)+k} = (-1)^{n(k+1)+1} * d *. \tag{3.7}$$

By the Stokes Theorem,

$$\langle d\omega, \eta\rangle_{L^2(\Lambda(T^*M))} = \int_M d\omega \wedge *\eta = \int_M \omega \wedge *d^*\eta = \langle \omega, d^*\eta\rangle_{L^2(\Lambda(T^*M))}.$$

Now if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ where $\ell \neq k+1$, then

$$\langle d\omega, \eta\rangle_{L^2(\Lambda(T^*M))} = \langle \omega, d^*\eta\rangle_{L^2(\Lambda(T^*M))} = 0.$$

Hence d^* is the formal adjoint of d on $\Omega(M)$; it is called the *coexterior derivative*.

4. The Local Analysis

In this section, we develop the essentials of the theory of pseudodifferential operators. First working only with complex-valued functions of several real variables, we introduce the Sobolev spaces and the uniform spaces, and prove a theorem by Sobolev that says when a given Sobolev space can be embedded inside a uniform space. Then we introduce the pseudodifferential operators, which are natural extensions of partial differential operators, and derive their basic algebraic properties. In particular, we show that pseudodifferential operators naturally extend to bounded linear operators on certain Sobolev spaces. We introduce a notion of ellipticity for pseudodifferential operators, and show that elliptic pseudodifferential have inverses, called parametrices, up to infinitely smoothing operators.

Now equipped with a theory of pseudodifferential operators, and working over vector bundles over a manifold, we choose a good system of trivialisations for our vector bundles and coordinate charts for our manifold so that we may extend our theory to vector bundles by pasting together the local results using a partition of unity. We prove that elliptic partial differential operators on vector bundles are elliptic in the sense of pseudodifferential operators, and use this to construct parametrices for them. This is the main result from this section. Finally, we use the existence of a parametrix to prove various properties of elliptic partial differential operators.

The Fourier Transform

We being by recalling the classical Fourier transform in the sense of Schwarz. We will be working with smooth functions defined on $\mathbb{R}^n$ and taking values in $\mathbb{C}^p$ for some p. We denote by $C^\infty(\mathbb{C}^p)$ the set of all smooth functions $\mathbb{R}^n \to \mathbb{C}^p$, and we define the *Schwartz space* $\mathcal{S}(\mathbb{C}^p)$ to consist of all functions $u \in C^\infty(\mathbb{C}^p)$ such that

$$\sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha u(x) \right| < \infty$$

for all multi-indices α and β. Intuitively, functions belonging to $\mathcal{S}$ tend to zero rapidly near infinity. The *Fourier transform* of $u \in \mathcal{S}(\mathbb{C}^p)$ is the function $\hat{u} \colon \mathbb{R}^n \to \mathbb{C}^p$ defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

The following proposition is a list of the basic properties of the Fourier transform that we will need. Note that it will be useful to define the derivative operators $D^\alpha = i^{-|\alpha|}\partial^\alpha$ as well as the convolution of functions: the *convolution* of $\varphi \in \mathcal{S}(\mathbb{C})$ and $u \in \mathcal{S}(\mathbb{C}^p)$ is the function $\varphi * u \in \mathcal{S}(\mathbb{C}^p)$ defined by

$$\varphi * u(x) = \int_{\mathbb{R}^n} \varphi(y) u(x-y)\, dy.$$

We also write $\langle u, v\rangle = \int_{\mathbb{R}^n} uv^*$ for the L^2-inner product on $\mathcal{S}$.

Proposition 4.1.

(a) *The Fourier transform defines a bijection $\mathcal{S}(\mathbb{C}^p) \to \mathcal{S}(\mathbb{C}^p)$ with inverse*

$$\check{u}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\, d\xi.$$

(b) $\widehat{D_x^\alpha u} = \xi^\alpha \hat{u}$ *and* $D_\xi^\alpha \hat{u} = \widehat{(-x)^\alpha u}$ *for all* $u \in \mathcal{S}(\mathbb{C}^p)$.

(c) $\widehat{\varphi * u} = \hat{\varphi}\hat{u}$ *and* $\widehat{\varphi u} = (2\pi)^{-n} \hat{\varphi} * \hat{u}$ *for all* $\varphi \in \mathcal{S}(\mathbb{C})$ *and* $u \in \mathcal{S}(\mathbb{C}^p)$.

(d) $\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle$ *for all* $u, v \in \mathcal{S}(\mathbb{C}^p)$ *(Plancherel's Formula).*

Proof. See Lemma 7.1.3, Theorem 7.1.3 and Theorem 7.1.6 in [Hör03]. Alternatively, see Theorem 2 of Section 4.3 in [Eva10].

□

Sobolev Spaces and Uniform Spaces

For $s \in \mathbb{R}$, we define the *Sobolev s-norm* on $\mathcal{S}(\mathbb{C}^p)$ by

$$\|u\|_s = \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2\, d\xi \right)^{1/2}. \tag{4.1}$$

We denote by $L^2_s(\mathbb{C}^p)$ the completion of $\mathcal{S}(\mathbb{C}^p)$ in this norm; thus L^2_s is a Banach space. The main reason to introduce these spaces is the following:

Proposition 4.2. *For an integer $k \geq 0$, the Sobolev k-norm is equivalent to the norm $u \mapsto (\sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} |\partial^\alpha u|^2)^{1/2}$. Thus, a function $u \in C^\infty(\mathbb{C}^p)$ has integrable derivatives of orders up to k if and only if $\|u\|_k < \infty$.*

Proof. Proposition 4.1(b) implies $\int_{\mathbb{R}^n} |\partial^\alpha u|^2 = \int_{\mathbb{R}^n} |\xi^\alpha|^2 |u|^2$, so the result follows by noting that $c_k(1 + |\xi|)^{2k} \leq \sum_{|\alpha|\leq k} |\xi^\alpha|^2 \leq C_k(1 + |\xi|)^{2k}$ for some constants $c_k, C_k > 0$. □

Proposition 4.3. *For any real numbers s and s', if $s \geq s'$, then the natural inclusion $L^2_s(\mathbb{C}^p) \hookrightarrow L^2_{s'}(\mathbb{C}^p)$ is an embedding.*

Proof. If $s \geq s'$, then $(1 + |\xi|)^{2s'} \leq (1 + |\xi|)^{2s}$, and so (4.1) implies $\|u\|_s \leq \|u\|_{s'}$. □

For an integer $k \geq 0$, we define $C^k_b(\mathbb{C}^p)$ the set of all functions $u\colon \mathbb{R}^n \to \mathbb{C}^p$ having bounded continuous derivatives up to order k. We turn it into a Banach space by equipping it with the norm

$$\|u\|_{C^k_b} = \sum_{|\alpha|\leq k} \sup|\partial^\alpha u|. \tag{4.2}$$

We have the following theorem by Sobolev, which says that with a function u with bounded Sobolev s-norm for large s must have some degree of degree of smoothness.

Proposition 4.4 (Sobolev Embedding Theorem). *For each integer $k \geq 0$ and each real number $s > k + n/2$, there is a constant $C_{k,s} > 0$ such that $\|u\|_{C^k_b} \leq C_{k,s}\|u\|_s$ for all $u \in \mathcal{S}(\mathbb{C}^p)$. Consequently, the natural inclusion $L^2_s(\mathbb{C}^p) \hookrightarrow C^k_b(\mathbb{C}^p)$ is an embedding for each such s and k.*

Proof. The following proof is from Theorem 2.5 in [LM89]. Let $u \in \mathcal{S}(\mathbb{C}^p)$, and let α be any multi-index. By Proposition 4.1(a), $u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$, so $D^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi)\, d\xi$.

$$\begin{aligned}
|\partial^\alpha u(x)| &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|)^{|\alpha|} |\hat{u}(\xi)|\, d\xi \\
&\leq (2\pi)^{-n} \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2|\alpha| - 2s}\, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2\, d\xi \right)^{1/2} \\
&= C_{\alpha,s} \|u\|_s
\end{aligned}$$

where the second line follows by Hölder's inequality, and where in the last line we have defined $C_{\alpha,s} = (\int_{\mathbb{R}^n}(1 + |\xi|)^{2|\alpha|-2s}\, d\xi)^{1/2}$. Note that $C_{\alpha,s} < \infty$ because $2s - 2|\alpha| \geq 2s - 2k > n$. It follows that $\|u\|_{C_b^k} \leq (\sum_{|\alpha|\leq k} C_{\alpha,s}^2)^{1/2}\|u\|_s$. □

Proposition 4.5 (Rellich Lemma). *Let $\{u_\ell\}_{\ell=1}^\infty$ be a bounded sequence of compactly supported functions in $L_s^2(\mathbb{C}^p)$. Then for any $s' < s$, $\{u_\ell\}$ has a subsequence that is Cauchy in the norm $\|s\|$ and therefore converges in $L_s^2(\mathbb{C}^p)$.*

Proof. See Theorem 2.6 in [LM89]. □

Pseudodifferential Operators

Pseudodifferential operators are a natural generalisation of partial differential operators that arise when trying to find an approximate inverse to a partial differential operator. For the purposes of this section, a partial differential operator of order m will be a linear operator $P(x, D)\colon \mathcal{S}(\mathbb{C}^p) \to \mathcal{S}(\mathbb{C}^q)$ of the form

$$P(x, D)u(x) = \sum_{|\alpha|\leq m} A^\alpha(x)D^\alpha u(x)$$

where A^α is a smooth function $\mathbb{R}^n \to \mathbb{C}^{q\times p}$ for each α, and $A^\alpha \neq 0$ for some α with $|\alpha| = m$. Given $u \in \mathcal{S}(\mathbb{C}^p)$, using Proposition 4.1, we may write $u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi$, so that by interchanging the order of differentiation and integration,

$$\begin{aligned} P(x, D)u(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\sum_{|\alpha|\leq m} A^\alpha(x)D_x^\alpha(e^{ix\cdot\xi}\hat{u}(\xi))\, d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\, d\xi \qquad (4.3) \end{aligned}$$

where $P(x, \xi) := \sum_{|\alpha|\leq m} A^\alpha(x)\xi^\alpha$. Switching the order of differentiation and integration here is allowed because $\hat{u} \in \mathcal{S}(\mathbb{C}^p)$. Replacing $P(x, \xi)$ in (4.3) by a more general function of (x, ξ) defines a pseudodifferential operator.

For $m \in \mathbb{R}$, a smooth function $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^{q\times p}$ is called a *symbol of order m* if for any two multi-indices α and β, there is a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|} \quad \text{for all } x, \xi \in \mathbb{R}^n. \qquad (4.4)$$

We denote the set of all symbols of order m by $S^m(\mathbb{C}^{q\times p})$, and we set $S^{-\infty}(\mathbb{C}^{q\times p}) = \cap_{m\in\mathbb{R}} S^m(\mathbb{C}^{q\times p})$ and $S^\infty(\mathbb{C}^{q\times p}) = \cup_{m\in\mathbb{R}} S^m(\mathbb{C}^{q\times p})$. Note that if $a \in S^m(\mathbb{C}^{q\times p})$ and $b \in S^{m'}(\mathbb{C}^{r\times q})$, then $ab \in S^{m+\ell}(\mathbb{C}^{r\times p})$ and $\partial_\xi^\alpha\partial_x^\beta a \in S^{m-|\alpha|}(\mathbb{C}^{q\times p})$. Also, the function p in (4.3) is a symbol of order m provided each A^α is a bounded function of x.

For the remainder of this section, we proceed largely as Hörmander in [Hör07], and the proofs are much the same. The only difference is that we are working with functions taking values in $\mathbb{C}^{q\times p}$, while Hörmander works with functions taking values in $\mathbb{C}$.

Proposition 4.6. *If $a \in S^m(\mathbb{C}^{q\times p})$, then the formula*

$$a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, d\xi. \tag{4.5}$$

defines a bounded linear map $a(x,D)\colon \mathcal{S}(\mathbb{C}^p) \to \mathcal{S}(\mathbb{C}^q)$.

The operator $a(x,D)$ in this proposition is called a *pseudodifferential operator of order m*. We write $\operatorname{Op} S^m(\mathbb{C}^{q\times p}) = \{a(x,D) : a \in S^m(\mathbb{C}^{q\times p})\}$ for the space of pseudodifferential operators of order m.

Proof. By the Leibniz rule,

$$\begin{aligned} D_x^\alpha a(x,D)u(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \xi^{\alpha-\beta} a(x,\xi) D_x^\beta a(x,\xi) \right) \hat{u}(\xi)\, d\xi \\ &= (2\pi)^{-n} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (D_x^\beta a(x,\xi)) \xi^{\alpha-\beta} \hat{u}(\xi)\, d\xi. \end{aligned}$$

The integrands in the above are absolutely integrable because we can bound $|D_x^\beta a(x,\xi)| \le C_\alpha (1+|\xi|)^m$ and $|\xi^{\alpha-\beta}\hat{u}(\xi)| \le C'_\alpha(1+|\xi|)^{-m-n-1}$ for some positive constants C_α and C'_α. This justifies interchanging the order of integration and differentiation and shows that $a(x,D)u \in \mathcal{S}(\mathbb{C}^q)$. □

The following proposition provides a way to write a given symbol as a certain kind of asymptotic sum.

Proposition 4.7. *For each non-negative integer j, let $m_j \in \mathbb{R}$ and $a_j \in S^{m_j}(\mathbb{C}^{q\times p})$. Assume $m_j \to -\infty$ as $j \to \infty$, and set $m'_k = \max_{j\ge k} m_j$. Then there exists $a \in S^{m'_0}(\mathbb{C}^{q\times p})$ such that $\operatorname{supp} a \subseteq \cup_{j=0}^\infty \operatorname{supp} a_j$ and such that for every k*

$$a - \sum_{j<k} a_j \in S^{m'_k}(\mathbb{C}^{q\times p}).$$

The function a is unique modulo $S^{-\infty}(\mathbb{C}^{q\times p})$, and it has the same property relative to any rearrangement of the formal series $\sum_j a_j$. We write $a \sim \sum_j a_j$.

Proof. Choose a cutoff function $\chi\colon \mathbb{R}^n \to [0,1]$ in $\mathcal{S}(\mathbb{C})$ equal to 0 in a neighbourhood of 1. Then the function $(x,\xi) \mapsto \chi(\xi)$ lies in $S^0(\mathbb{C})$. We claim that for every multi-index α, there is a constant $C_\alpha > 0$ such that

$$\left|\partial_\xi^\alpha(1-\chi(\varepsilon\xi))\right| \leq C_\alpha \varepsilon(1+|\xi|)^{|\alpha|-1}.$$

For $\alpha = 0$, this is trivial. If $\alpha \neq 0$, then

$$\begin{aligned}\left|\partial_\xi^\alpha(1-\chi(\varepsilon\xi))\right| &= \varepsilon^{|\alpha|}\left|(\partial^\alpha\chi)(\varepsilon\xi)\right| \\ &\leq C_\alpha\varepsilon^{|\alpha|}(1+|\varepsilon\xi|)^{1-|\alpha|} \\ &= C_\alpha\varepsilon(1+|\xi|)^{1-|\alpha|}\left(\frac{1+|\xi|}{1/\varepsilon+|\xi|}\right)^{|\alpha|-1} \\ &\leq C_\alpha\varepsilon(1+|\xi|)^{1-|\alpha|}\end{aligned}$$

where the second lines follows because $\chi(\xi)$ viewed as a function of (x,ξ) is an element of $S^0(\mathbb{C}) \subseteq S^1(\mathbb{C})$. It follows that we can find a sequence $\varepsilon_j \to 0$ such that

$$\left|\partial_\xi^\alpha\partial_x^\beta((1-\chi(\varepsilon_j|\xi|)a_j(x,\xi))\right| < 2^{-j}(1+|\xi|)^{m_j+1-|\alpha|}$$

for all multi-indices α and β with $|\alpha|+|\beta| \leq j$. Define $A_j(x,\xi) = (1-\chi(\varepsilon_j|\xi|))a_j(x,\xi)$. Since χ has compact support, the sum $a = \sum_j A_j$ is locally finite, so $a \in C^\infty(\mathbb{C}^{q\times p})$. Given multi-indices α and β as well as a non-negative integer k, we can find an integer N large enough that $N \geq |\alpha|+|\beta|$ and that $m'_N + 1 \leq m'_k$. Observe that

$$\left|\partial_\xi^\alpha\partial_x^\beta\Big(a(x,\xi) - \sum_{j<N} A_j(x,\xi)\Big)\right| \leq \sum_{j\geq N} 2^{-j}(1+|\xi|)^{m_j+1-|\alpha|} \leq (1+|\xi|)^{m'_k-|\alpha|},$$

whence $a - \sum_{j<N} A_j \in S^{m'_k}(\mathbb{C}^{q\times p})$. Hence

$$a - \sum_{j<k} a_j = a - \sum_{j<N} A_j + \sum_{k\leq j<N} A_j + \sum_{j<k}(A_j - a_j) \in S^{m'_k}(\mathbb{C}^{q\times p})$$

because $\sum_{k\leq j<N} A_j \in S^{m'_k}(\mathbb{C}^{q\times p})$ and $\sum_{j<k}(A_j - a_j) \in S^{-\infty}(\mathbb{C}^{q\times p})$. □

We are interested in how pseudodifferential operators behave under taking formal adjoints and composition. Before we proceed, it will be instructive to do this for partial differential operators first.

Example 4.8. Let $P(x,D)\colon \mathcal{S}(\mathbb{C}^p) \to \mathcal{S}(\mathbb{C}^q)$ be a partial differential operator of order m given by $P(x,D) = \sum_{|\alpha|\leq m} A^\alpha(x)D^\alpha$. By integration by parts,

$$\begin{aligned}
\langle P(x,D)u, v\rangle &= \int_{\mathbb{R}^n} \sum_{|\alpha|\leq m} A^\alpha(x)D^\alpha u(x)v(x)^*\,dx \\
&= \int_{\mathbb{R}^n} u(x) \sum_{|\alpha|\leq m} D^\alpha(A^\alpha(x)^* v(x))^* \\
&= \langle u, P^*(x,D)v\rangle
\end{aligned}$$

where, by the Leibniz rule, $P^*(x,D)$ is given by

$$P^*(x,D) = \sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta} D^\beta A^\alpha(x)^* D^{\alpha-\beta}.$$

Thus

$$P^*(x,\xi) = \sum_{|\alpha|\leq m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta} D^\beta A^\alpha(x)^* \xi^{\alpha-\beta}. \tag{4.6}$$

Now let $Q(x,D)\colon \mathcal{S}(\mathbb{C}^q) \to \mathcal{S}(\mathbb{C}^r)$ be a partial differential operator of order m' given by $Q(x,D) = \sum_{|\beta|\leq m'} B^\beta(x)D^\beta$. Using the Leibniz rule, we compute

$$\begin{aligned}
QP(x,D) &= \sum_{|\beta|\leq m'} B^\beta(x)D^\beta\left(\sum_{|\alpha|\leq m} A^\alpha(x)D^\alpha\right) \\
&= \sum_{|\beta|\leq m'}\sum_{|\alpha|\leq m}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma} B^\beta(x)(D^\gamma A^\alpha(x))D^{\alpha+\beta-\gamma}.
\end{aligned}$$

Thus

$$QP(x,\xi) = \sum_{|\beta|\leq m'}\sum_{|\alpha|\leq m}\sum_{\beta\leq\gamma}\binom{\beta}{\gamma} B^\beta(x)D_x^\gamma A^\alpha(x)\xi^{\alpha+\beta-\gamma}. \tag{4.7}$$

We want to write the sum (4.7) in terms of $P(x,\xi)$ and $Q(x,\xi)$. We can get the $D_x^\gamma A^\alpha(x)\xi^\alpha$ terms simply via

$$D_x^\gamma P(x,\xi) = \sum_{|\alpha|\leq m} D_x^\gamma A^\alpha(x)\xi^\alpha. \tag{4.8}$$

For the $B^\beta(x)\xi^{\beta-\gamma}$ terms, we use the identity (2.1) to get

$$\frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma Q(x,\xi) = \sum_{\substack{|\beta|\leq m' \\ \gamma\leq\beta}}\binom{\beta}{\gamma} B^\beta(x)\xi^{\beta-\gamma}. \tag{4.9}$$

Multiplying (4.8) with (4.9), summing over γ and using (4.7) gives

$$QP(x,\xi) = \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_x^\gamma P(x,\xi) D_\xi^\gamma Q(x,\xi). \tag{4.10}$$

Before we proceed with the case of pseudodifferential operators, we prove the following technical result. If A is an $n \times n$ matrix with complex entries, we write $A(\xi) = A\xi \cdot \xi = \sum_{i,j} A_{ij}\xi_i\xi_j$ for the quadratic form, and analogously we put $A(D) = AD \cdot D = \sum_{i,j} A_{ij} D_i D_j$.

Lemma 4.9. *If A is an $n \times n$ matrix of complex numbers all with non-positive real parts, then the formula*

$$e^{A(D)}u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{A\xi\cdot\xi} \hat{u}(\xi)\, d\xi \tag{4.11}$$

defines a linear map $e^{A(D)} \colon \mathcal{S}(\mathbb{C}) \to \mathcal{S}(\mathbb{C})$. Moreover, if $|A| \leq 1$ and if ℓ is an integer $> n/2$, then for every integer $k \geq \ell$, there is a constant $C_k > 0$ such that

$$\left| e^{A(D)}u(x) - \sum_{j<k} \frac{1}{j!} A(D)^j u(x) \right| \leq C_k \|u\|_{C_b^{\ell+2k}} \tag{4.12}$$

whenever $u \in \mathcal{S}(\mathbb{C})$ and $normu_{C_b^{\ell+2k}} < \infty$. Moreover, if $\operatorname{dist}(x, \operatorname{supp} u) > 1$, then the sum in the above vanishes, and we have the stronger bound

$$\left| e^{A(D)}u(x) \right| \leq C_k |A|^{k+\ell} \operatorname{dist}(x, \operatorname{supp} u)^{-k} \|u\|_{C_b^{\ell+2k}}. \tag{4.13}$$

Proof. See Theorem 7.6.5 in Chapter 7 of [Hör03]. ◻

Given a symbol $a \in S^m(\mathbb{C}^{q\times p})$, by using the Fourier transform, we can write

$$a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy d\xi.$$

If we introduce the auxiliary function

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(y-x)\cdot\xi} a(x,\xi)\, d\xi, \tag{4.14}$$

called the *kernel* of a, then $a(x,D)$ takes the form

$$a(x,D)u(x) = \int_{\mathbb{R}^n} K(x,y) u(y)\, dy. \tag{4.15}$$

Note that $K(x,y)$ is equal to $(2\pi)^{-n}\hat{a}(x, y-x)$ where $\hat{a}$ is the Fourier transform of a in the ξ variable, so using Fourier inversion (Proposition 4.1), we obtain

$$a(x,\xi) = \int_{\mathbb{R}^n} K(x, x-y) e^{-iy\cdot\xi}\, dy. \tag{4.16}$$

This calculation tells us that we can define a pseudodifferential operator either by a symbol or by a kernel.

Proposition 4.10. *Let $a \in S^m(\mathbb{C}^{q\times p})$, and set $b(x,\xi) = e^{iD_x\cdot D_\xi}a^*(x,\xi)$. Then $b(x,D)$ is the adjoint of $a(x,D)$ with respect to the L^2-inner product, $b \in S^m(\mathbb{C}^{p\times q})$, and b has the asymptotic expansion*

$$b(x,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!}(iD_x \cdot D_\xi)^j a^*(x,\xi) = \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha a^*(x,\xi) \tag{4.17}$$

Notice the similarity between the expansion (4.17) and the formula for the formal adjoint of a partial differential operator in (4.6).

Proof. This proof is adapted from Theorem 18.1.7 in [Hör07] to cover the case of $C^{q\times p}$-valued functions.

By viewing $a(x,D)$ as the integral operator (4.15), the equation

$$\langle a(x,D)u, v\rangle = \langle u, b(x,D)v\rangle$$

is satisfied when $b(x,D)$ has kernel $\overline{K(y,x)}^{\mathsf{T}}$. Using (4.16), we compute the symbol of $b(x,D)$:

$$\begin{aligned} b(x,\xi) &= \int_{\mathbb{R}^n} \overline{K(x-y,x)}^{\mathsf{T}} e^{-iy\cdot\xi}\, dy \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{iy\cdot(\eta-\xi)} \overline{a(x-y,\eta)}^{\mathsf{T}}\, dy d\eta \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iy\cdot\eta} \overline{a(x-y,\xi-\eta)}^{\mathsf{T}}\, dy d\eta. \end{aligned}$$

Notice that the above is the convolution of the functions $(x,\xi) \mapsto e^{-ix\cdot\xi}$ and $(x,\xi) \mapsto \overline{a(x,\xi)}^{\mathsf{T}}$. Since the Fourier transform of the function $(x,\xi) \mapsto (2\pi)^{-n}e^{ix\cdot\xi}$ is the function $(x,\xi) \mapsto e^{ix\cdot\xi}$, Fourier inversion gives

$$b(x,\xi) = e^{iD_x\cdot D_\xi}\overline{a(x,\xi)}^{\mathsf{T}}.$$

Now we prove the expansion (4.17). Choose a smooth function $\chi\colon \mathbb{R}^n \to [0,1]$ such that $\chi(\xi) = 1$ when $|\xi| < 1/2$ and $\chi(\xi) = 0$ when $|\xi| > 1$. For each integer $\nu \geq 0$, set

$$a_\nu(x,\xi) = \chi(\xi/2^\nu)a^*(x,\xi), \qquad b_\nu(x,\xi) = e^{iD_x\cdot D_\xi}a_\nu(x,\xi).$$

Then $\chi \in S^{m+\ell}(\mathbb{C})$ implies $\partial_\xi^\alpha \partial_x^\beta a_\nu \in S^m(\mathbb{C}^{q\times p})$, so for some constant $C_{\alpha\beta}$,

$$|\partial_\xi^\beta \partial_x^\beta a_\nu(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|} \leq C_{\alpha\beta}(1+2^\nu)^{|m|}$$

whenever $(x,\xi) \in \operatorname{supp} a_\nu$. Hence Lemma 4.9 implies

$$\left| b_\nu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j a_\nu(x,\xi) \right| < C_k 2^{|m|\nu}$$

for some constant C_k. Given ξ, denote by μ the smallest nonnegative integer such that $|\xi| \le 2^{\mu+2}$. Then either $\mu = 0$ and $|\xi| \le 4$ or $\mu > 0$ and $2^{\mu+1} < |\xi| \le 2^{\mu+2}$. In the latter case, observe that $\operatorname{dist}((x,\xi), \operatorname{supp} a_\nu) \ge |\xi|/2 \ge (1+|\xi|)/4$, so by (4.13),

$$\left| b_\nu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j a_\nu(x,\xi) \right| < C'_k (1+|\xi|)^{|m|-k} \tag{4.18}$$

for some constant C'_k. Set $A_\nu(x,\xi) = a_{\nu+1}(x,\xi) - a_\nu(x,\xi)$ and $B_\nu(x,\xi) = b_{\nu+1}(x,\xi) - b_\nu(x,\xi)$, so that

$$a^*(x,\xi) - a_\mu(x,\xi) = \sum_{\nu\ge\mu} A_\nu(x,\xi), \qquad b(x,\xi) - b_\mu(x,\xi) = \sum_{\nu\ge\mu} B_\mu(x,\xi). \tag{4.19}$$

Since $a_\nu \in S^m(\mathbb{C}^{q\times p})$, and since $2^{\nu-1} \le |\xi| \le 2^{\nu+1}$ in $\operatorname{supp} A_\nu$, it follows that

$$|\partial_\xi^\alpha \partial_x^\beta A_\nu(x,\xi)| \le C'_{\alpha\beta} 2^{\nu(m-|\alpha|)}$$

for some constant $C'_{\alpha\beta}$. Since

$$B_\nu(x, 2^\nu \xi) = e^{iD_x\cdot D_\xi/2^\nu} A_\nu(x,\xi),$$

Lemma 4.9 gives

$$\left| B_\nu(x,2^\nu\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j A_\nu(x,2^\nu\xi) \right| \le C''_k 2^{(m-k)\nu}$$

for some constant C''_k. Thus

$$\left| B_\nu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j A_\nu(x,\xi) \right| \le C''_k 2^{(m-k)\nu}.$$

If we take $k > m$ and sum over $\nu \ge \mu$, we obtain

$$\begin{aligned}\left| \sum_{\nu\ge\mu}\left(B_\nu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j A_\nu(x,\xi) \right) \right| &\le C''_k \sum_{\nu\ge\mu} 2^{(m-k)\nu} \\ &\le C''_k 2^{(m-k)\nu+1} \\ &\le C'''_k (1+|\xi|)^{m-k} \end{aligned} \tag{4.20}$$

for constants C_k'' and C_k'''. By the triangle inequality,

$$\begin{aligned}
&\left|b(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j a^*(x,\xi)\right| \\
&\qquad \le \left|b(x,\xi) - b_\mu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j (a^*(x,\xi) - a_\mu(x,\xi))\right| \\
&\qquad\qquad + \left|b_\mu(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_\xi \cdot D_\xi)^j a_\mu(x,\xi)\right|.
\end{aligned}$$

The middle term in the above is by (4.19) equal to the left-hand side of (4.20). Hence (4.18) and (4.20) together yield

$$\left|b(x,\xi) - \sum_{j<k} \frac{1}{j!}(iD_x \cdot D_\xi)^j a^*(x,\xi)\right| \le C_k(1+|\xi|)^{|m|-k}$$

for a new constant C_k. This proves the expansion is valid. □

Proposition 4.11. *If $a \in S^m$ and $b \in S^{m'}$, then*

$$a(x,D)b(x,D) = c(x,D)$$

where $c \in S^{m+m'}$ is given by

$$b(x,\xi) = e^{iD_y \cdot D_\eta} a(x,\eta) b(y,\xi)|_{\eta=\xi, y=x}$$

and has the asymptotic expansion

$$\begin{aligned}
b(x,\xi) &\sim \sum_{j=0}^{\infty} \frac{1}{j!}(iD_y \cdot D_\eta)^j a(x,\eta) b(y,\xi)|_{\eta=\xi, y=x} \\
&= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi). \qquad (4.21)
\end{aligned}$$

Again notice the striking similarity between the expansion (4.21) and the corresponding formula for partial differential operators (4.10).

Proof. The Fourier transform of $b(y,D)u(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi} \hat{u}(\xi)\, d\xi$ is

$$\eta \mapsto (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{iy\cdot(\xi-\eta)} b(y,\xi) \hat{u}(\xi)\, d\xi dy,$$

so

$$a(x,D)b(x,D)u(x) = (2\pi)^{-n} \iiint_{\mathbb{R}^{3n}} e^{ix\cdot\eta + iy\cdot(\xi-\eta)} a(x,\eta) b(y,\xi) \hat{u}(\xi)\, d\xi dy \eta.$$

Hence

$$\begin{aligned} c(x,\xi) &= \iint_{\mathbb{R}^{2n}} e^{-i(x-y)\cdot(\xi-\eta)} a(x,\eta) b(y,\xi)\, dy d\eta \\ &= e^{iD_x \cdot D_\xi} a(x,\eta) b(y,\xi)|_{\eta=\xi, y=x}. \end{aligned}$$

The argument in the preceding proposition applied to the function $(x,y,\xi,\eta) \mapsto a(x,\eta)b(y,\xi)$ shows the expansion (4.21) is valid. $\square$

Proposition 4.12. *Let $a \in S^m(\mathbb{C}^{p\times p})$. Suppose there are positive constants c and C such that for all $x, \xi \in \mathbb{R}^n$ with $|\xi| \geq c$, the matrix inverse of $a(x,\xi)$ exists and satisfies the bound*

$$|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}.$$

Then there exists a symbol $b \in S^{-m}(\mathbb{C}^{p\times p})$ unique up to $S^{-\infty}(\mathbb{C}^{p\times p})$ such that

$$a(x,D)b(x,D) - \mathrm{id} \in \mathrm{Op}\, S^{-\infty}(\mathbb{C}^{p\times p}), \tag{4.22}$$

$$b(x,D)a(x,D) - \mathrm{id} \in \mathrm{Op}\, S^{-\infty}(\mathbb{C}^{p\times p}). \tag{4.23}$$

Proof. Choose a smooth function $\chi\colon \mathbb{R}^n \to [0,1]$ such that $\chi(\xi) = 0$ for $|\xi| \leq c$ and $\chi(\xi) = 1$ for $|\xi| \geq c+1$, and set $b(x,\xi) = \chi(\xi)a(x,\xi)^{-1}$. It suffices to show that for every pair of multi-indices α and β, there is a constant $C_{\alpha\beta}$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)^{-1}\right| \leq C_{\alpha\beta}(1+|\xi|)^{-m-|\alpha|} \tag{4.24}$$

whenever $|\xi| \geq c$. First we prove this for $\alpha = 0$ by inducting on β. The base case $\beta = 0$ holds true by assumption. Now let β be any nonzero multi-index and assume the result holds for all multi-indices of order $< |\beta|$. By differentiating both sides of $aa^{-1} = \mathrm{id}$ with respect to x_j, we get $\partial a^{-1}/\partial x_j = -a^{-1}(\partial a/\partial x_j)a^{-1}$, so with the Leibniz rule, we obtain

$$\partial_x^\beta a^{-1} = \sum_{\substack{\beta'+\beta''+\beta'''=\beta \\ \beta''\neq 0}} c_{\beta'\beta''\beta'''} \left(\partial_x^{\beta'} a^{-1}\right)\left(\partial_x^{\beta''} a\right)\left(\partial_x^{\beta'''} a^{-1}\right).$$

By the induction hypothesis, there are positive constants $C_{\beta'}$ and $C_{\beta'''}$ such that $|\partial_x^{\beta'} a^{-1}| \leq C_{\beta'}(1+|\xi|)^{-m}$ and $|\partial_x^{\beta'''} a^{-1}| \leq C_{\beta'''}(1+|\xi|)^{-m}$ whenever $|\xi| \geq c$. In addition, because $a \in S^m(\mathbb{C}^{p\times p})$, there is a constant $C_{\beta''} > 0$ such that $|\partial_x^{\beta''} a| \leq C_{\beta''}(1+|\xi|)^m$. Hence

$$\begin{aligned} \left|\partial_x^\beta a^{-1}\right| &= \sum_{\substack{\beta'+\beta''+\beta'''=\beta \\ \beta''\neq 0}} c_{\beta'\beta''\beta'''} C_{\beta'} C_{\beta''} C_{\beta'''} (1+|\xi|)^{-m+m-m} \\ &= C_\beta (1+|\xi|)^{-m} \end{aligned}$$

for some constant C_β. Now we prove (4.24) in general by inducting on $|\alpha|$ with β fixed. The base case $\alpha = 0$ is what we just proved. Let α be any nonzero multi-index, and assume the result holds for multi-indices with order $< |\alpha|$. By differentiating both sides of $a(x,\xi)a(x,\xi)^{-1} = \mathrm{id}$ with respect to ξ_j, we obtain

$$\frac{\partial a^{-1}}{\partial \xi_j} = -a^{-1}\frac{\partial a}{\partial \xi_j}a^{-1},$$

so by the Leibniz rule, for any multi-indices α and β with $\alpha \neq 0$, we have

$$\partial_\xi^\alpha \partial_x^\beta a^{-1} = \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ \alpha''\neq 0}} c^{\alpha'\alpha''\alpha'''}_{\beta'\beta''\beta'''}\big(\partial_\xi^{\alpha'}\partial_x^{\beta'}a^{-1}\big)\big(\partial_\xi^{\alpha''}\partial_x^{\beta''}a\big)\big(\partial_\xi^{\alpha'''}\partial_x^{\beta'''}a^{-1}\big)$$

for integral constants $c^{\alpha'\alpha''\alpha'''}_{\beta'\beta''\beta'''} > 0$. By the induction hypothesis, there are positive constants $C_{\alpha'\beta'}$ and $C_{\alpha'''\beta'''}$ such that $|\partial_\xi^{\alpha'}\partial_x^{\beta'}a^{-1}| \leq C_{\alpha'\beta'}(1+|\xi|)^{-m-|\alpha'|}$ and $|\partial_x^{\alpha'''\beta'''}a^{-1}| \leq C_{\alpha'''\beta'''}(1+|\xi|)^{-m-|\alpha'''|}$ whenever $|\xi| \geq C$. Because $a \in S^m(\mathbb{C}^{p\times p})$, there is a constant $C_{\beta''} > 0$ such that $|\partial_x^{\beta''}a| \leq C_{\beta''}(1+|\xi|)^{m-|\alpha''|}$. Thus

$$\begin{aligned}
&\big|\partial_\xi^\alpha \partial_x^\beta a^{-1}\big| \\
&\qquad \leq \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ \alpha''\neq 0}} c^{\alpha'\alpha''\alpha'''}_{\beta'\beta''\beta'''} C_{\alpha'\beta'}C_{\alpha''\beta''}C_{\alpha'''\beta'''}(1+|\xi|)^{-m-|\alpha'|+m-|\alpha''|-m-|\alpha'''|} \\
&\qquad = C_{\alpha\beta}(1+|\xi|)^{-m-|\alpha|}
\end{aligned}$$

for some constant $C_{\alpha\beta}$. This proves $b \in S^{-m}(\mathbb{C}^{p\times p})$.

With that out of the way, it now follows from the expansion in Proposition 4.11 that

$$a(x,D)b(x,D) = \mathrm{id} - r(x,D)$$

for some $r \in S^{-1}(\mathbb{C}^{p\times p})$. We want to invert $\mathrm{id} - r(x,D)$ by a Neumann series, so we set

$$b(x,D)r(x,D)^k = b_k(x,D), \qquad b_k \in S^{-m-k}(\mathbb{C}^{p\times p})$$

in Proposition 4.11. Using Proposition 4.7, we can find $c \in S^{-m}(\mathbb{C}^{p\times p})$ with the expansion $c \sim \sum_{j=0}^\infty b_j$. Observe that for any integer $k > 0$,

$$\begin{aligned}
a(x,D)\sum_{j<k} b_j(x,D) = \sum_{j<k} a(x,D)b(x,D)r(x,D)^j &= \sum_{j<k}(\mathrm{id} - r(x,D))r(x,D)^j \\
&= \mathrm{id} - r(x,D)^k,
\end{aligned}$$

and hence

$$a(x,D)c(x,D) - \mathrm{id} = a(x,D)\left(c(x,D) - \sum_{j<k} b_j(x,D)\right) - r(x,D)^k$$

is an element of $\mathrm{Op}\, S^{-k}(\mathbb{C}^{p\times p})$. This proves $a(x,D)c(x,D) - \mathrm{id} \in \mathrm{Op}\, S^{-\infty}(\mathbb{C}^{p\times p})$.

Uniqueness follows because if b and c are elements of $S^{-m}(\mathbb{C}^{p\times p})$ such that $a(x,D)b(x,D) - \mathrm{id}$ and $a(x,D)c(x,D) - \mathrm{id}$ both lie in $S^{-\infty}(\mathbb{C}^{p\times p})$, then

$$\begin{aligned} &c(x,D) - b(x,D) \\ &\qquad = c(x,D)(\mathrm{id} - a(x,D)b(x,D)) + (c(x,D)a(x,D) - \mathrm{id})b(x,D) \end{aligned}$$

is an element of $S^{-\infty}(\mathbb{C}^{p\times p})$. □

Proposition 4.13. *If $a \in S^m(\mathbb{C}^{q\times p})$ has compact x-support, then $a(x,D)$ extends to a bounded linear operator $a(x,D)\colon L^2_s(\mathbb{C}^p) \to L^2_{s-m}(\mathbb{C}^q)$ for every $s \in \mathbb{R}$.*

Proof. See Proposition 3.2 in [LM89]. □

This concludes the basic theory of pseudodifferential operators. In the next section, we show how to extend this theory to vector bundles. In order to do so, we need one more technical result, which is the following proposition. For a subset A of $\mathbb{R}^n$ and a real number $\varepsilon > 0$, we define $A_\varepsilon = \{x \in \mathbb{R}^n : \mathrm{dist}(x,A) \leq \varepsilon\}$. We say a symbol $a \in S^\infty(\mathbb{C}^{q\times p})$ is *ε-local* if for any compactly supported function $u \in C^\infty(\mathbb{C}^p)$, we have

$$\operatorname{supp} a(x,D)u \subseteq (\operatorname{supp} u)_\varepsilon.$$

Note that the total symbol of a partial differential operator is always 0-local and that the composition of an ε-local symbol with an ε'-symbol in the sense of Proposition 4.11 is $(\varepsilon + \varepsilon')$-local.

Proposition 4.14. *Let $a \in S^m(\mathbb{C}^{q\times p})$. For any $\varepsilon > 0$, there is an ε-local symbol $a_\varepsilon \in S^m(\mathbb{C}^{q\times p})$ such that $a - a_\varepsilon \in S^{-\infty}(\mathbb{C}^{q\times p})$.*

Proof. Choose a smooth function $b_\varepsilon \in S^0(\mathbb{C})$ such that $b = 1$ on a neighbourhood of the diagonal $x = \xi$ and such that $b(x,\xi) = 0$ whenever $|x - \xi| \geq \varepsilon$. Then the symbol $a_\varepsilon \in S^m(\mathbb{C}^{q\times p})$ given by $a_\varepsilon(x,D) = b_\varepsilon(x,D)a(x,D)$ is ε-local. Moreover, by Proposition 4.11, both a and a_ε have the expansion (4.21), hence $a - a_\varepsilon \in S^{-\infty}(\mathbb{C}^{q\times p})$. □

5. The Global Analysis

We are now in a position to globalise our results from the previous section to vector bundles. Let (M, g) be a compact n-dimensional Riemannian manifold, and let E and F be vector bundles over M of ranks p and q, respectively. Choose a cover for M by closed coordinate balls $y_\ell\colon \bar{U}_\ell \to \bar{\mathbb{B}}^n$ for $\ell = 1, \ldots, N$ over each of which E and F have trivialisations $\varphi_\ell\colon E|_{\bar{U}_\ell} \to \bar{U}_\ell \times \mathbb{K}^p$ and $\psi_\ell\colon E|_{\bar{U}_\ell} \to \bar{U}_\ell \times \mathbb{K}^q$, respectively. Moreover, choose coordinate charts and trivialisations in such a way that the balls $B_\ell := \{x \in U_\ell : |y_\ell(x)| < 1/\sqrt{2}\}$ of radius $1/\sqrt{2}$ cover M, and such that $\{x \in U_\ell : |y_\ell(x)| < 2/\sqrt{5}\} \subseteq U_\ell$ for each ℓ. Choose a partition of unity $\chi_1, \ldots \chi_N$ subordinate to the cover $B_1, \ldots, B_N$. For each ℓ, put

$$x_\ell := \frac{y_\ell}{\sqrt{1 - |y_\ell|^2}}\colon U_\ell \to \mathbb{R}^n,$$

so that $x_\ell(U_\ell) = \mathbb{R}^n$ and $x_\ell(B_\ell) = \mathbb{B}^n$. In addition, assume that $x_\ell^{-1}(2\mathbb{B}^n) \subseteq U_\ell$. Then, given $u \in \Gamma(E)$, we may write $u = \sum_\ell \chi_\ell u$, and we can view each $\chi_\ell u$ as a smooth function $\mathbb{R}^n \to \mathbb{C}^r$ supported in $\mathbb{B}^n$. A cover chosen in this way we will call a *good presentation*.

For a nonnegative integer k, we define the *uniform C^k-norm* on $C^k(E)$ by

$$\|u\|_{C^k} = \sum_{\ell=1}^{N} \|u_\ell\|_{C^k_b},$$

where $\|u_\ell\|_{C^k_b}$ is defined by (4.2) (note that this makes sense because each u_ℓ has compact support). Equipped with this norm, $C^k(E)$ is a Banach space. For $s \in \mathbb{R}$, we define the Sobolev s-norm on $\Gamma(E)$ by

$$\|u\|_s = \sum_{\ell=1}^{N} \|u_\ell\|_s$$

where $\|u_\ell\|_s$ is defined by (4.1). We denote the completion of $\Gamma(E)$ with respect to this norm by $L^2_s(E)$.

Proposition 5.1. *The L^2-norm $\|\cdot\|$ and the Sobolev 0-norm $\|\cdot\|_0$ are equivalent.*

Proof. Given sections of u and v of E, define

$$\langle u, v\rangle_E' = \sum_\ell \langle u_\ell, v_\ell\rangle$$

where $\langle u_\ell, v_\ell\rangle$ is defined via the trivialisation $E|_{U_\ell} \cong U_\ell \times \mathbb{K}^p$. This defines a metric on E, so by Proposition 3.4, the induced L^2-norm

$$\|u\|' = \left(\sum_\ell \|u_\ell\|^2\right)^{1/2}$$

is equivalent to the L^2-norm $\|\cdot\|$ determined by $\langle\cdot,\cdot\rangle_E$. The above norm is clearly equivalent to the Sobolev 0-norm. □

Our first two global results are the following, which are immediate consequences of Propositions 4.3 and 4.4 given how we have defined the Sobolev norms.

Proposition 5.2. *For any real numbers s and s', if $s \geq s'$, then the natural inclusion $L^2_s(E) \hookrightarrow L^2_{s'}(E)$ is an embedding.*

Proposition 5.3 (Sobolev Embedding Theorem). *For any non-negative integer k and any real number $s > k + n/2$, the natural inclusion $L^2_s(E) \hookrightarrow C^k(E)$ is an embedding.*

We are now in a position to extend pseudodifferential operators to vector bundles. We begin with the following motivating construction. Consider any coordinate chart $x\colon U \to \mathbb{R}^n$ of M over which the bundles E and F trivialise, say $e_1, \dots, e_p$ and $f_1, \dots, f_q$ are local frames for E and F over U, respectively. Given a symbol $a \in S^m(\mathbb{K}^{q\times p})$, we can construct an operator $A\colon \Gamma(E) \to \Gamma(F|_U)$ as follows: if $u = \sum_i u_i e_i$ on U, then we put $Au = \sum_j v_j f_j$ where $v_1, \dots, v_q$ are defined by

$$(v_1(x), \dots, v_q(x)) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)(\hat{u}_1(\xi), \dots, \hat{u}_p(\xi))\, d\xi. \tag{5.1}$$

If the functions $v_1, \dots, v_q$ are supported in U for each $u \in \Gamma(E)$, then the above defines a global operator $\Gamma(E) \to \Gamma(F)$ simply by setting $Au = 0$ outside $E|_U$. For example, if $\chi \in C^\infty(M)$ is a smooth function supported in U, then for any $u \in \Gamma(E)$, χAu and $A(\chi u)$ are both supported in U, and hence define global operators χA and $A\chi$ from $\Gamma(E)$ to $\Gamma(F)$.

Definition. A linear operator $A\colon \Gamma(E) \to \Gamma(F)$ is a *pseudodifferential operator of order m* if it can be written as a finite sum $A = \sum_\alpha A_\alpha$ of linear operators $A_\alpha\colon \Gamma(E) \to \Gamma(F)$ in the form (5.1). If $m \leq 0$, then A is a *smoothing operator of order m*. If A is a smoothing operator of order m for every $m \leq 0$, then it is an *infinitely smoothing operator*.

Note that any partial differential operator $P\colon \Gamma(E) \to \Gamma(F)$ of order m is automatically a pseudodifferential operator of order m. Our first result about pseudodifferential operators is the following, which is a direct consequence of Proposition 4.13 and the above definition.

Proposition 5.4. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be a pseudodifferential operator of order m. For each $s \in \mathbb{R}$, P extends to a bounded linear map $L^2_s(E) \to L^2_{s-m}(F)$.*

We now come to our first main result: the existence of a parametrix for an elliptic partial differential operator between vector bundles.

Theorem 5.5. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator of order m. There is a pseudodifferential operator $A\colon \Gamma(F) \to \Gamma(E)$ of order $-m$, unique up to infinitely smoothing operators, such that*

$$PA = \mathrm{id} - S', \qquad AP = \mathrm{id} - S.$$

for some infinitely smoothing operators $S\colon \Gamma(E) \to \Gamma(E)$ and $S'\colon \Gamma(F) \to \Gamma(F)$.

Proof. With respect to our good presentation, write $P = \sum_{|\alpha|\le m} A^\alpha_\ell \partial^{|\alpha|}/\partial x^\alpha$ on U_ℓ for each ℓ. By (4.3), P is given on U_ℓ by $P(\sum_i u_i e^\ell_i) = \sum_j v_j f^\ell_j$ where

$$(v_1(x), \dots, v_q(x)) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p_\ell(x,\xi)(\hat{u}_1(\xi), \dots, \hat{u}_p(\xi))\, d\xi$$

and $p_\ell(x,\xi) = \sum_{|\alpha|\le m} A^\alpha_\ell(x)\xi^\alpha$. Because U_ℓ is precompact, p_ℓ is bounded in the x-variable, and therefore p_ℓ is actually a symbol of order m. We show each p_ℓ satisfies the conditions of Proposition 4.12. Since P is elliptic, the matrix $q_\ell(x,\xi) = \sum_{|\alpha|=m} A^\alpha_\ell(x)\xi^\alpha$ is invertible for every $\xi \ne 0$. If we define $r_\ell = p_\ell - q_\ell$, then $p_\ell = q_\ell(\mathrm{id} + q_\ell^{-1} r_\ell)$. We intend to invert $\mathrm{id} + q_\ell^{-1} r$ by the Neumann series. Notice that for all $x, \xi \in \mathbb{R}^n$ and $\lambda > 0$, we have $q_\ell(x, \lambda\xi) = \lambda^m q_\ell(x,\xi)$. Using this and the identity $q_\ell q_\ell^{-1} = \mathrm{id}$, we obtain $q_\ell(x,\lambda\xi)^{-1} = \lambda^{-m} q_\ell(x,\xi)^{-1}$. Thus for any $\xi \ne 0$,

$$|q_\ell(x,\xi)^{-1}| = |\xi|^{-m} q_\ell(x, \xi/|\xi|) \le C|\xi|^{-m}$$

where

$$C = \max_{(x,\xi)\in\mathbb{R}^n\times\mathbb{S}^{n-1}} |q_\ell(x,\xi)^{-1}| \le \max_{(x,\xi)\in\bar{U}_\ell\times\mathbb{S}^{n-1}} |q_\ell(x_\ell(x),\xi)| < \infty$$

because $\bar{U}_\ell \times \mathbb{S}^{n-1}$ is compact. On the other hand, obviously there is a constant $C' > 0$ such that $|r_\ell(x,\xi)| \le C'|\xi|^{m-1}$. Thus $|(p_\ell^{-1} r)(x,\xi)| \le CC'|\xi|^{-1} < 1$ if $|\xi| > 1/CC'$, so the Neumann series

$$(\mathrm{id} + q_\ell(x,\xi)^{-1} r_\ell(x,\xi))^{-1} = \sum_{j=0}^{\infty} (-q_\ell(x,\xi)^{-1} r_\ell(x,\xi))^j$$

converges for $|\xi| > 1/CC'$. It follows that

$$p_\ell(x,\xi)^{-1} = (\mathrm{id} + q_\ell(x,\xi)^{-1} r_\ell(x,\xi)) q_\ell(x,\xi)^{-1}$$

exists for $|\xi| \geq 1/CC'$. From an argument similar to the one we used to obtain the constant C', it follows that

$$|p_\ell(x,\xi)^{-1}| \leq \frac{1}{1-1/CC'} |q_\ell(x,\xi)^{-1}| \leq \frac{C''}{1-1/CC'} (1+|\xi|)^{-m}$$

for some constant $C'' > 0$. Thus by Proposition 4.12, we can find symbols $a_\ell \in S^{-m}(\mathbb{C}^{p\times p})$ and $s_\ell, s'_\ell \in S^{-\infty}(\mathbb{C}^{p\times p})$ such that

$$p_\ell(x,D) a_\ell(x,D) = \mathrm{id} - s'_\ell(x,D), \qquad a_\ell(x,D) p_\ell(x,D) = \mathrm{id} - s_\ell(x,D).$$

Since $p_\ell c$ and $c p_\ell$ are infinitely smoothing symbols whenever $c \in S^{-\infty}$ is an infinitely smoothing symbol, we may, using Proposition 4.14, replace q_ℓ by a 1-local symbol with the above two equations holding true but for potentially different infinitely smoothing symbols s_ℓ and s'_ℓ. The above two equations then imply that s_ℓ and s'_ℓ must also be 1-local. Let A_ℓ, S_ℓ and S'_ℓ be the operators corresponding to a_ℓ, s_ℓ and s'_ℓ as defined by (5.1). 1-locality guarantees that these operators map into $\Gamma(F)$. By construction, they satisfy the identities

$$PA_\ell = \mathrm{id} - S_\ell, \qquad A_\ell P = \mathrm{id} - S_\ell.$$

Set

$$A = \sum_\ell \chi_\ell A_\ell, \qquad A' = \sum_\ell A_\ell \chi_\ell$$
$$S = \sum_\ell \chi_\ell S_\ell, \qquad S' = \sum_\ell S'_\ell \chi_\ell.$$

Then

$$PA'u = \sum_\ell PA_\ell(\chi_\ell u) = \sum_\ell \chi_\ell u - \sum_\ell S'_\ell(\chi_\ell u) = u - S'u$$

and

$$APu = \sum_\ell \chi_\ell A_\ell P u = \sum_\ell \chi_\ell u - \sum_\ell \chi_\ell S'_\ell = u - S'u,$$

so $PA' = \mathrm{id} - S'$ and $AP = \mathrm{id} - S$. These two relations imply $APA' = A' - SA' = A - AS'$, and hence $A - A' = AS' - SA'$ is an infinitely smoothing operator. It follows that both A and A' have the desired property. □

Corollary 5.6. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic partial differential operator of order m. The following hold:*

(a) *For any $s \in \mathbb{R}$ and any $u \in L^2_s(E)$, if Pu is smooth, then u is smooth.*

(b) *For each $s \in \mathbb{R}$, P extends to a Fredholm map $P\colon L^2_s(E) \to L^2_{s-m}(E)$ with kernel* $\ker P = \ker P|_{\Gamma(E)}$.

Proof. This argument is adapted from the proof to Theorem 5.2 in [LM89]. Choose Q, S and S' for P as in Theorem 5.5. By Proposition 5.4, P, Q, S and S' extend to bounded linear operators

$$P\colon L^2_s(E) \to L^2_{s-m}(F), \qquad Q\colon L^2_{s-m}(F) \to L^2_s(E),$$
$$S\colon L^2_s(E) \to L^2_s(E), \qquad S'\colon L^2_{s-m}(F) \to L^2_{s-m}(F)$$

satisfying

$$PQ = \mathrm{id} - S, \qquad QP = \mathrm{id} - S'.$$

For (a), if $u \in L^2_s(E)$ and Pu is smooth, then

$$u = QPu + S'u.$$

Since S' is infinitely smoothing, $S'u$ is smooth, and since Pu is smooth, QPu is smooth. Thus u is smooth. For (b), it suffices to show the operators S and S' are compact. This follows by the Rellich Lemma (Proposition 4.5) because S and S' only operate on functions with compact support. □

Theorem 5.7 (The Elliptic Decomposition Theorem). *Let M be a compact oriented Riemannian manifold, let E and F be vector bundles over M, and let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic partial differential operator. There is an L^2-orthogonal direct sum decomposition:*

$$\Gamma(F) = \operatorname{im} P \oplus \ker P^*.$$

Proof. This proof borrows arguments from Theorem 5.5 in Chapter III of [LM89] and from Theorem 3.7 in [Mar02]. The operator P and its formal adjoint P^* naturally extend to Fredholm operators $P\colon L^2_m(E) \to L^2(F)$ and $P^*\colon L^2(F) \to L^2_{-m}(E)$. Hence, with respect to the L^2-inner product $\langle \cdot, \cdot \rangle_{L^2(F)}$, we have the following orthogonal direct sum decomposition:

$$L^2(F) = (\ker P^*)^\perp \oplus \ker P^*.$$

We show

$$(\ker P^*)^\perp = (\ker P^\dagger)_\circ = \overline{P(L^2_m(E))} = P(L^2_m(E)),$$

where $P^\dagger\colon L^2(F)^* \to L^2_m(E)^*$ is the Banach space dual of P. The second equality is just Proposition 2.2(a), and the third equality holds because P is Fredholm. Let us prove the first equality. Note that $u \in (\ker P^\dagger)_\circ$ if and only if $\alpha(u) = 0$ for all

$\alpha \in L^2(F)$ with $\alpha \circ P = 0$. By the Riesz Representation Theorem (Proposition 2.3), this is equivalent to $\langle u, v\rangle_{L^2(F)} = 0$ for all $v \in L^2(F)$ such that $0 = \langle P(\cdot), v\rangle_{L^2(F)} = \langle \cdot, P^*v\rangle_{L^2(E)}$. Thus the equality amounts to showing $\langle \cdot, P^*v\rangle_{L^2(E)} = 0$ if and only if $P^*v = 0$. The reverse direction is obvious. For the forward direction, observe that $\langle \cdot, P^*v\rangle_{L^2(E)} = 0$ implies $\|P^*\|_{L^2(E)} = 0$, and, since $L^2_{-m}(E) \hookrightarrow L^2_0(E) \cong L^2(E)$ (Propositions 5.1 and 5.2), it follows that $P^*v = 0$.

We have shown

$$L^2(F) = P(L^2_m(E)) \oplus \ker P^*.$$

After intersecting both sides of the above with $\Gamma(F)$, we get

$$\Gamma(F) = (P(L^2_m(E)) \cap \Gamma(F)) \oplus (\ker P^* \cap \Gamma(F)) = P(\Gamma(E)) \oplus \ker P^*,$$

where we have used the fact $P(L^2_m(E)) \cap \Gamma(F) = P(\Gamma(E))$, which follows from Corollary 5.6(a), and the fact that $\ker P^* \subseteq \Gamma(F)$ from Corollary 5.6(b). This completes the proof. □

Corollary 5.8. *Let M be a compact manifold, let E, F and L be vector bundles over M, and let $P\colon \Gamma(E) \to \Gamma(F)$ and $Q\colon \Gamma(F) \to \Gamma(L)$ be partial differential operators. If for each $x \in M$ and each nonzero $\xi \in T^*_xM$, the sequence*

$$E_x \xrightarrow{\sigma_\xi(P)} F_x \xrightarrow{\sigma_\xi(Q)} L_x$$

is exact, then $PP^*+Q^*Q\colon \Gamma(E) \to \Gamma(E)$ is an elliptic partial differential operator, and we have the following L^2-orthogonal decomposition:

$$\Gamma(E) = (\ker P \cap \ker Q^*) \oplus \operatorname{im}(PP^* + Q^*Q).$$

Proof. That the operator $PP^* + Q^*Q$ is elliptic follows from the identity

$$\sigma(PP^* + Q^*Q) = \sigma(P)\sigma(P)^* + \sigma(Q)^*\sigma(Q)$$

as well as the following elementary linear algebra result, which we prove:

> *Let U, V and W be finite-dimensional inner product spaces, and let $T\colon U \to V$ and $S\colon V \to W$ be linear maps with adjoints T^* and S^*. If the sequence*
>
> $$U \xrightarrow{T} V \xrightarrow{S} W$$
>
> *is exact, then $TT^* + S^*S\colon V \to V$ is invertible.*

Assume the sequence above is exact. It suffices to show $R := TT^*+S^*S$ is injective, so let $v \in V$ be arbitrary, and assume $Rv = 0$. Then $0 = \langle Rv, v\rangle = |T^*v|^2 + |Sv|^2$, which implies $T^*v = 0$ and $Sv = 0$. By exactness, $v = Tu$ for some $u \in U$, and thus $|v|^2 = \langle Tu, v\rangle = \langle u, T^*v\rangle = 0$, so $v = 0$. $\square$

In Examples 3.3 and 3.6, we saw that the exterior derivative $d\colon \Omega(M)$ is a first order partial differential operator, and we computed its formal adjoint d^* with respect to L^2-norm on $\Lambda(T^*M)$ induced by the metric g. The operator $\Delta := dd^* + d^*d$ is called the *Hodge-Laplacian*; it sends k-forms to k-forms for each k. When $M = \mathbb{R}^n$ and $k = 0$, then $\Omega^0(M) = C^\infty(\mathbb{R}^n)$ and $\Delta\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is (the negative of) the ordinary Laplacian operator on functions of several real variables: $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$. A form $\omega \in \Omega(M)$ is *harmonic* if $\Delta\omega = 0$; we define

$$\mathscr{H}^k(M) = \ker(\Delta\colon \Omega^k(M) \to \Omega^k(M))$$

and

$$\mathscr{H}(M) = \ker(\Delta\colon \Omega(M) \to \Omega(M))$$

to be the spaces of harmonic k-forms and harmonic forms, respectively.

Corollary 5.9 (The Hodge Decomposition Theorem). *Let M be a compact manifold. There is the following L^2-orthogonal direct sum decomposition:*

$$\Omega(M) = \Delta(\Omega(M)) \oplus \mathscr{H}(M).$$

Proof. In Corollary 5.8, take $E = F = \Lambda(T^*M)$, $P = d$ and $Q = d^*$. Then $\Delta = PP^* + Q^*Q$, so all we need to show is show the sequence

$$\Lambda(T_x^*M) \xrightarrow{\sigma_\xi(d)} \Lambda(T_x^*M) \xrightarrow{\sigma_\xi(d)} \Lambda(T_x^*M) \tag{5.2}$$

is exact for each nonzero $\xi \in T_x^*M$ and $x \in M$.

Let $\xi \in T_x^*M$ be nonzero. If $\zeta \in \Lambda(T_x^*M)$, then

$$\sigma_\xi(d)(\sigma_\xi(d)(\zeta)) = \xi \wedge \xi \wedge \zeta = 0.$$

Conversely, let $\zeta \in \Lambda^{k-1}(T_x^*M)$, and assume $\sigma_\xi(d)(\zeta) = \xi \wedge \zeta = 0$. Choose a basis $\xi_1, \ldots, \xi_n$ for T_p^*M with $\xi_1 = \xi$, and write

$$\zeta = \sum_{1\le i_1<\cdots<i_k\le n} \zeta_{i_1\cdots i_k}\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$$

for real numbers $\zeta_{i_1\cdots i_k}$. Because $\xi \wedge \xi_1 = 0$,

$$\xi \wedge \zeta = \sum_{1<i_1<\cdots<i_k\le n} \zeta_{i_1\cdots i_k}\xi \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_k},$$

so $\zeta_{i_1\cdots i_k} = 0$ whenever $1 < i_1 < \cdots < i_k$. It follows that

$$\zeta = \xi \wedge \left(\sum_{1<i_2<\cdots<i_k\leq n} \zeta_{1 i_2\cdots i_k} \xi_{i_2} \cdots \wedge \xi_{i_k} \right),$$

and hence the sequence is exact. □

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