CALCULATIONS ON DIRAC AND SPINOR VALUED FORMS

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Abstract. In this paper we review some material on Dirac bundles and spin geometry, and do some calculations on Dirac bundle valued forms and spinor valued forms.

Contents

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I would like to thank my supervisor Spiro Karigiannis, who made it possible for me to keep studying mathematics.

Introduction

In this paper we do several calculations on Dirac bundle valued forms and spinor valued forms. In section 1, we review some background material on Clifford algebras and their modules. In section 2, we define Dirac bundles and Dirac operators. In section 3, we define a linear operator s, and use it to obtain a decomposition for Dirac bundle valued forms, in analogy to the Lefschetz decomposition of complex valued forms obtained using the Lefschetz operator. In section 4, we use s to obtain identities that are in analogy to the Kähler identities obtained using the Lefschetz operator, investigate the compatibility between the exterior covariant derivative and the decomposition, then obtain a version of the generalized

Bochner identity on Dirac bundle valued forms. In section 5, we review the definition of spin groups using Clifford algebras, the construction of the unique nontrivial double covers from spin groups to special orthogonal groups, and the spin Lie algebra structures. In section 6, we review the definition of spin structures, spin manifolds, spinor bundles, and the Dirac bundle structures on spinor bundles. In section 7, we state a theorem of Lichnerowicz, and then use its proof to calculate the curvature term in our version of the generalized Bochner identity on spinor valued forms.

Notation

Throughout we use the Einstein summation convention, which means that we sum over repeated indices. Since we are always working with Riemannian manifolds, we do not distinguish between vector fields and 1-forms. For convenience, whenever we work over a bundle locally around a point p, we pick an orthonormal frame e_i of the (co)tangent bundle which is parallel at p. That is, $(\nabla e_i)_p = 0$ for all i.

1. Clifford Algebras and Modules

In this section we recall some background material on Clifford algebras and Clifford modules, mostly taken from chapter 1 of [1].

Definition 1.1. Let V be a finite dimensional real vector space equipped with an inner product. The *Clifford algebra* associated with V is defined to be

$$\mathrm{Cl}(V) := T(V)/(v \otimes v + \langle v, v\rangle 1),$$

where $T(V) = \sum_{i=0}^{\infty} \otimes^i V$ is the tensor algebra of V.

We identify V with the image of the natural embedding $V \hookrightarrow T(V) \twoheadrightarrow \mathrm{Cl}(V)$. We have that for any $v, w \in V$, their Clifford product in $\mathrm{Cl}(V)$ satisfy the relation $vw+wv = -2\langle v, w\rangle$. Alternatively, if $e_1, \ldots, e_n$ is an orthonormal basis for V, then $\mathrm{Cl}(V)$ is the real unital algebra generated by $e_1, \ldots, e_n$ with relations $e_ie_j + e_je_i = -2\delta_{ij}$. We let Cl_n denote $\mathrm{Cl}(\mathbb{R}^n)$ where the inner product on $\mathbb{R}^n$ is given by the dot product.

There is a canonical vector space isomorphism (not algebra isomorphism) between the exterior algebra ΛV and $\mathrm{Cl}(V)$, which sends $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to $e_{i_1} \cdots e_{i_k}$, where $e_1, \ldots, e_n$ is an orthonormal basis for V. So, the dimension of $\mathrm{Cl}(V)$ is 2^n.

Example 1.2. $\mathrm{Cl}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, the complex numbers, since $\mathrm{Cl}(\mathbb{R})$ is the real algebra generated by a single element e, with relation $e^2 = -1$.

$\mathrm{Cl}(\mathbb{R}^2)$ is isomorphic to $\mathbb{H}$, the quaternions. To see the this, notice that $\mathrm{Cl}(\mathbb{R}^2)$ is 4-dimensional, and spanned by $\{1, e_1, e_2, e_1e_2\}$, and these satisfy the relations

$$(e_1)(e_2) = (e_1e_2), \quad e_2(e_1e_2) = e_1, \quad (e_1e_2)e_1 = e_2, \quad e_1^2 = e_2^2 = (e_1e_2)^2 = -1.$$

The standard basis $e_1, \dots, e_n \in \mathbb{R}^n$ in Cl_n generate a finite group F_n, which is called the *Clifford group*. Concretely, F_n has generators $e_1, \dots, e_n, -1$, and relations $e_i^2 = -1$ for all i, $e_i e_j = (-1) e_j e_i$. Cl_n is isomorphic to the real group algebra $\mathbb{R}F_n$, modulo the additional relation $(-1) + 1 = 0$.

Proposition 1.3. *Let W be a module over a Clifford algebra Cl_n. Then there exists an inner product $\langle \cdot, \cdot \rangle$ on W such that Clifford multiplication by unit vectors $e \in V$ is orthogonal. That is,*

$$\langle ew, ew' \rangle = \langle w, w' \rangle$$

for all $w, w' \in W$ and $e \in \mathbb{R}^n$ with $\|e\| = 1$.

Proof. Choose any inner product $\langle \cdot, \cdot \rangle'$ on W, and obtain $\langle \cdot, \cdot \rangle$ by averaging it over the Clifford group F_n. That is, set

$$\langle w, w' \rangle := \frac{1}{|F_n|} \sum_{e \in F_n} \langle ew, ew' \rangle'.$$

By construction, for any $w \in \mathbb{R}^n$, we have $\langle e_i w, e_i w \rangle = \langle w, w \rangle$, and that for $i \neq j$, $\langle e_i w, e_j w \rangle = \langle e_j e_i w, -w \rangle = \langle e_i e_j w, w \rangle = -\langle e_j w, e_i w \rangle = 0$. So for any $e = a_i e_i$ with $\sum_i a_i^2 = 1$, we have

$$\langle ew, ew \rangle = \sum_i a_i^2 \langle e_i w, e_i w \rangle + \sum_{i \neq j} a_i a_j \langle e_i w, e_j w \rangle = \langle w, w \rangle.$$

Since the norm completely determines the inner product that induced the norm, and the norm is invariant under multiplication by unit vectors, we have that the inner product is also invariant by multiplication by unit vectors. □

There is a $\mathbb{Z}$-filtration on $\mathrm{Cl}(V)$, induced by the natural filtration on the tensor algebra. Under the isomorphism between $\mathrm{Cl}(V)$ and ΛV, the filtrations are the same. Unlike the alternating algebra, this filtration on $\mathrm{Cl}(V)$ is not induced from a $\mathbb{Z}$-grading of $\mathrm{Cl}(V)$, since "degree 1" elements $v \in V$ would square to "degree 0".

However, there is a $\mathbb{Z}/2$-grading on $\mathrm{Cl}(V)$, induced by the natural $\mathbb{Z}/2$-grading on the tensor algebra, where the even $\mathbb{Z}$-degree elements are considered to have degree 0, and the odd $\mathbb{Z}$-degree elements are considered to have degree 1. Alternatively, the $\mathbb{Z}/2$-grading on $\mathrm{Cl}(V)$ can be obtained by the following: the involution $\alpha(v) = -v$ on V induces an involution α on $\mathrm{Cl}(V)$, giving a decomposition into the 1 and -1 eigenspaces, which we call $\mathrm{Cl}(V)^0$ and $\mathrm{Cl}(V)^1$.

2. Dirac Bundles and Dirac Operators

There is a canonical representation

$$cl(\rho_n) : SO(n) \to Aut(\mathrm{Cl}_n),$$

since each orthogonal transformation on $\mathbb{R}^n$ induces an orthogonal transformation of Cl_n (an orthogonal transformation A on $\mathbb{R}^n$ induces a transformation on the tensor algebra, and sends elements of the ideal $\langle v \otimes v + \langle v, v\rangle\rangle$ to elements of the form $Av \otimes Av + \langle v, v\rangle = Av \otimes Av + \langle Av, Av\rangle$, so it preserves the ideal).

Let E be an oriented Riemannian vector bundle on a manifold M, which gives an $SO(n)$-principal bundle on M.

Definition 2.1. The *Clifford bundle* $\mathrm{Cl}(E)$ is defined to be the associated bundle

$$\mathrm{Cl}(E) = P_{SO}(E) \times_{cl(\rho_n)} \mathrm{Cl}_n.$$

Let M be an n-dimensional Riemannian manifold, $\mathrm{Cl}(M)$ be its Clifford bundle on the tangent bundle. Let ∇ denote the canonical connection on $\mathrm{Cl}(M)$ (which is the connection on the bundle of forms induced from the Levi-Civita connection on the tangent bundle of M). We recall the definition of a Dirac bundle from chapter 5 of [1].

Definition 2.2. A vector bundle S over M with a fibre metric $\langle \cdot, \cdot\rangle$ and a connection ∇^S compatible with the fibre metric (for which we will often simply write ∇) is called a Dirac bundle if S is a bundle of modules over $\mathrm{Cl}(M)$, and for each $p \in M$, and each $\sigma, \tau \in S_p$ and unit vector $e \in T_p(M)$, we have

$$\langle e\sigma, \tau\rangle = -\langle \sigma, e\tau\rangle,$$

and that ∇^S is a derivation. That is, for each $\phi \in \Gamma(\mathrm{Cl}(M))$, $\sigma \in \Gamma(S)$, we have

$$\nabla^S(\phi\sigma) = (\nabla\phi)\sigma + \phi(\nabla^S\sigma).$$

Example 2.3. The Clifford bundle itself with left multiplication as its module structure, and the obvious metric and connection, is a Dirac bundle.

To see this, first note that under the canonical isomorphism $\Lambda\mathbb{R}^n \cong \mathrm{Cl}_n$, the representation $cl(\rho_n)$ is identified with $\Lambda\rho_n$. So the connection of $\mathrm{Cl}(E) \cong \Lambda E$ that we described before is the same as the connection induced onto the associated bundle $\mathrm{Cl}(E)$ from the connection on $P_{SO}(E)$. Since $cl(\rho_n)$ maps into the automorphisms of Cl_n, we have that the induced Lie algebra map $cl(\rho_n)_*$ goes from $\mathfrak{so}_n$ to $\mathrm{Der}(\mathrm{Cl}_n)$, which then implies that the connection is a derivation on sections.

Example 2.4. Some other examples of Dirac bundles include spinor bundles, which appear in section 6.

Let S be a Dirac bundle over a Riemannian manifold M. Write $\Lambda^k_S = \Lambda^k \otimes S$ for the bundle of S-valued k-forms. Λ^k_S has a natural fibre metric induced from the fibre metrics on Λ^k and S given by $\langle \alpha \otimes \tau, \beta \otimes \sigma\rangle = \langle \alpha, \beta\rangle\langle \tau, \sigma\rangle$. In addition, Λ_S also has a connection induced from the connections on Λ^k and S given by $\nabla(\alpha \otimes \tau) = \nabla\alpha \otimes \tau + \alpha \otimes \nabla\tau$, making Λ_S also a Dirac bundle.

Definition 2.5. Let S be a Dirac bundle over M. Define the *Dirac operator* $D \colon \Gamma(S) \to \Gamma(S)$ by the formula (using the Einstein summation convention)

$$D\sigma = e_i \nabla_i \sigma$$

at $p \in M$ (remember that $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of TM).

The expression $e_i\nabla_i\sigma$ is well-defined, because if we choose a different local orthonormal frame $\tilde{e}_i = A_{ij}e_j$,

$$\tilde{e}_i\nabla_{\tilde{e}_i}\tau = A_{ij}e_j\nabla_{A_{ik}e_k}\tau = A_{ij}A_{ik}e_k\nabla_j\tau = e_j\nabla_j\tau.$$

As a special case of the above definition, the Dirac operator on the Dirac bundle Λ_S is given by

$$D(\alpha\otimes\sigma) = \nabla_i\alpha\otimes e_i\sigma + \alpha\otimes e_i\nabla_i\sigma.$$

Note that the Dirac operator on Λ_S preserves degree.

Proposition 2.6. *The Dirac operator is formally self adjoint. That is,*

$$\langle D\sigma,\tau\rangle = \langle\sigma, D\tau\rangle$$

for all compactly supported $\sigma,\tau\in\Gamma(S)$ (where the inner product is the L^2 inner product).

Proof. As usual, assume that $e_1, ..., e_n$ is an orthonormal frame parallel at a point $p\in M$.

$$\begin{aligned}\langle D\sigma,\tau\rangle_p =&\langle e_j\nabla_j\sigma,\tau\rangle_p\\ =&-\langle\nabla_j\sigma, e_j\tau\rangle_p\end{aligned}$$

which by metric compatibility,

$$\begin{aligned}=&-e_j\langle\sigma, e_j\tau\rangle_p + \langle\sigma,\nabla_j(e_j\tau)\rangle_p\\ =&-e_j\langle\sigma, e_j\tau\rangle_p + \langle\sigma, e_j\nabla_j\tau\rangle_p.\end{aligned}$$

Let V be the vector field defined by the condition that for all vector fields W,

$$\langle V, W\rangle = -\langle\sigma, W\tau\rangle.$$

Then,

$$\begin{aligned}\mathrm{div}(V)_p =&\langle\nabla_j V, e_j\rangle_p\\ =&e_j\langle V, e_j\rangle_p - \langle V,\nabla_j e_j\rangle_p\\ =&e_j\langle V, e_j\rangle_p\\ =&-e_j\langle\sigma, e_j\tau\rangle_p.\end{aligned}$$

Then, we have that

$$\langle D\sigma,\tau\rangle = \mathrm{div}(V) + \langle\sigma, D\tau\rangle,$$

and the proposition follows. ∎

3. A Lefschetz Decomposition for Dirac Bundle Valued Forms

The content of this section have all appeared in [2], however for the main result, Proposition 3.2, we give a different proof here.

Throughout this section we let V be an n-dimensional real inner product vector space, $\{e_1,\dots,e_n\}$ an orthonormal basis of V, and S a module over $\mathrm{Cl}(V)$. We write Λ_S for $\Lambda V\otimes S$.

Definition 3.1. Define $s : \Lambda_S \to \Lambda_S$ by the formula:

$$s(\alpha \otimes \sigma) = (e_i \wedge \alpha) \otimes (e_i \sigma).$$

We sometimes write s_k for the restriction of s to the subspace $\Lambda_S^k = \Lambda^k V \otimes S$ (when we need to emphasize it), but more often than not we will not distinguish between s and s_k. Note that s raises degree by 1.

Note that s is well-defined, since if we let $e_i' = A_{ij} e_j$ be another orthonormal basis of V, where A is an orthogonal matrix, then

$$\begin{aligned} (e_i' \wedge \alpha) \otimes (e_i' \sigma) &= (A_{ij} e_j \wedge \alpha) \otimes (A_{ik} e_k \sigma) \\ &= A_{ij} A_{ik} (e_j \wedge \alpha) \otimes (e_k \sigma) \\ &= \delta_{jk} (e_j \wedge \alpha) \otimes (e_k \sigma) \\ &= (e_j \wedge \alpha) \otimes (e_j \sigma). \end{aligned}$$

Analogous to the Lefschetz decomposition of complex-valued differential forms using the Lefschetz operator, we give a decomposition of S-valued forms using s. We write s^* for the adjoint of s. Note that s^* lowers degree by 1.

Proposition 3.2. *Let $P_k = \ker s_k^*$. We have that*

$$\Lambda_S^k = P_k \oplus s P_{k-1} \oplus \cdots \oplus s^k P_0,$$

where direct summands are orthogonal to each other (on Λ_S, we put the inner product induced by wedge product and tensor product).

Proof. Note that we have $P_k = \ker s_k^* = (\operatorname{Im} s_{k-1})^\perp$.

We use induction on k. The proposition is clearly true for $k = 0$. To prove the induction step, we use a lemma:

Lemma 3.3. *Given any $x \in P_k$ and $y \in \operatorname{Im} s_{k-1}$, we have that $\langle s^j x, s^j y \rangle = 0$ for all $j \geq 0$. Equivalently, instead of for all $y \in \operatorname{Im} s_{k-1}$, the above holds for any y of the form $y = s(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \otimes \gamma), \gamma \in S$.*

Assuming this lemma, and that the proposition is true for $k - 1$, we have that

$$s P_{k-1}, \ldots, s^k P_0$$

are pairwise orthogonal subspaces of Λ_S^k. Since

$$P_k = (\operatorname{Im} s_{k-1})^\perp = (s P_{k-1} \oplus \cdots \oplus s^k P_0)^\perp,$$

we have that $P_k, \ldots, s^k P_0$ are pairwise orthogonal, and their direct sum is all of Λ_S^k. $\square$

It remains to prove the lemma:

Proof of lemma. We use induction on j. The case $j = 0$ is true by definition.

Assume that the claim is true for j (for all $x \in P_k, y = s(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \otimes \gamma), \gamma \in S$). We want to show that for any $A_{i_1 \cdots i_k i} e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes \sigma_i \in P_k$, and any element of Λ_S^{k-1} of the form $e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \otimes \gamma$, we get

$$\begin{aligned}\langle A_{i_1 \cdots i_k i} e_{l_1} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes e_{l_1} \cdots e_{l_j} \sigma_i, \\ e_{l'_1} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \otimes e_{l'_1} \cdots e_{l'_j} \gamma \rangle = 0.\end{aligned}$$

Note that the left hand side of the above (desired) equation can be rewritten as the following:

$$A_{i_1 \cdots i_k i} \langle e_{l_1} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{l'_1} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \rangle \langle e_{l_1} \cdots e_{l_j} \sigma_i, e_{l'_1} \cdots e_{l'_j} \gamma \rangle.$$

We examine the above sum by fixing l_1 to be its various values.

When $l_1 \neq l'_1, \ldots, l'_j, j_1, \ldots, j_{k-1}$, the corresponding terms vanish by definition of the inner product on forms.

When $l_1 = l'_1$, since

$$\begin{aligned}&\langle e_{l_1} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{l_1} \wedge e_{l_2} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \rangle \\ =&\langle e_{l_2} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{l_2} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \rangle\end{aligned}$$

and

$$\langle e_{l_1} \cdots e_{l_j} \sigma_i, e_{l_1} e_{l'_2} \cdots e_{l'_j} \gamma \rangle = \langle e_{l_2} \cdots e_{l_j} \sigma_i, e_{l'_2} \cdots e_{l'_j} \gamma \rangle.$$

We get these terms in total:

$$A_{i_1 \cdots i_k i} \langle e_{l_2} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{l'_2} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \rangle \langle e_{l_2} \cdots e_{l_j} \sigma_i, e_{l'_2} \cdots e_{l'_j} \gamma \rangle,$$

Which is 0 by the inductive hypothesis.

When $l_1 = l'_i$, we can bring the $e_{l'_i}$ to the front by possibly introducing a minus sign, so the sum of the terms also vanish.

When $l_1 = j_1$, we can bring e_{j_1} to the front and possibly introduce a sign. Now we fix l'_j to be a particular number, and we see that by bringing e_{l_1} in $\langle e_{l_1} \cdots e_{l_j} \sigma_i, e_{l'_1} \cdots e_{l'_j} \gamma \rangle$ to the end and possibly introduce a sign (so we get the terms $\langle e_{l_2} \cdots e_{l_j} \sigma_i, e_{l'_1} \cdots e_{l'_{j-1}} (e_{l'_j} e_{l_1} \gamma) \rangle$), we get a sum of these terms: (without summing over l'_j)

$$\begin{aligned}&A_{i_1 \cdots i_k i} \langle e_{l_2} \wedge \cdots \wedge e_{l_j} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{l'_1} \wedge \cdots \wedge e_{l'_j} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{k-1}} \rangle \\ &\langle e_{l_2} \cdots e_{l_j} \sigma_i, e_{l'_1} \cdots e_{l'_{j-1}} (e_{l'_j} e_{l_1} \gamma) \rangle,\end{aligned}$$

which is 0 by the inductive hypothesis. Now we sum over l'_j, and still have 0.

When $l_1 = j_i$ for $i \neq 1$, we do the same thing and obtain a sum of 0. □

Proposition 3.4. *The adjoint s^* of s is given by*

$$s^*(\alpha \otimes \sigma) = -e_i \lrcorner \alpha \otimes e_i \sigma.$$

Proof. We compute

$$\begin{aligned}\langle s(\alpha\otimes\sigma),\beta\otimes\tau\rangle &= \langle e_i\wedge\alpha\otimes e_i\sigma,\beta\otimes\tau\rangle\\ &= \langle e_i\wedge\alpha,\beta\rangle\langle e_i\sigma,\tau\rangle\\ &= \langle\alpha,e_i\lrcorner\beta\rangle\langle\sigma,-e_i\tau\rangle\\ &= \langle\alpha\otimes\sigma,-e_i\lrcorner\alpha\otimes e_i\sigma\rangle.\end{aligned}$$

□

Proposition 3.5. *Let x be an S-valued k-form. We have the following formula for the commutator:*

$$[s^*,s]x=(n-2k)x.$$

Proof. Let $\alpha\otimes\tau$ be an S-valued k-form.

$$\begin{aligned}s^*s(\alpha\otimes\tau) &= s^*(e_i\wedge\alpha\otimes e_i\tau)\\ &= -e_j\lrcorner(e_i\wedge\alpha)\otimes e_je_i\tau,\end{aligned}$$

$$\begin{aligned}ss^*(\alpha\otimes\tau) &= -s(e_j\lrcorner\alpha\otimes e_j\tau)\\ &= -e_i\wedge(e_j\lrcorner\alpha)\otimes e_ie_j\tau,\end{aligned}$$

so

$$[s^*,s](\alpha\otimes\tau)=e_i\wedge(e_j\lrcorner\alpha)\otimes e_ie_j\tau-e_j\lrcorner(e_i\wedge\alpha)\otimes e_je_i\tau$$

since interior product is an antiderivation of degree -1,

$$=e_i\wedge(e_j\lrcorner\alpha)\otimes e_ie_j\tau+e_i\wedge(e_j\lrcorner\alpha)\otimes e_je_i\tau-\delta_{ij}\alpha\otimes e_je_i\tau$$

since $e_ie_j+e_je_i=-2\delta_{ij}$, and $e_i\wedge(e_i\lrcorner\alpha)=k\alpha$ (this is most easily seen by writing α as a sum of terms of the form $e_{i_1}\wedge\cdots\wedge e_{i_k}$),

$$\begin{aligned}&= e_i\wedge(e_j\lrcorner\alpha)\otimes(-2\delta_{ij})\tau-\delta_{ij}\alpha\otimes e_je_i\tau\\ &= -2k(\alpha\otimes\tau)+n(\alpha\otimes\tau).\end{aligned}$$

□

Remark 3.6. The triple $(s,s^*,[s^*,s])$ forms an $\mathfrak{sl}_2$-representation.

Proposition 3.7. *Let x be an S-valued k-form. We have the following formula for the commutator:*

$$[s^*,s^m]x=m(n-2k-m+1)s^{m-1}x.$$

Proof. We use induction on m:

The case $m=1$ is the previous proposition.

Assume $[s^*,s^m]x=m(n-2k-m+1)s^{m-1}x$. Then,

$$s^*s^{m+1}x=s^*s^msx$$

which by the induction hypothesis (and remember that sx is a $k+1$-form),

$$=s^ms^*sx+m(n-2k-m-1)s^mx$$

which by the previous lemma,

$$\begin{aligned} &= s^m(n-2k)x + s^m s s^* x + m(n-2k-m-1)s^m x \\ &= s^{m+1}s^*x + (m+1)(n-2k-m)s^m x. \end{aligned}$$

□

The next proposition is in analogy to Proposition 1.2.30 in [4].

Proposition 3.8. *(1) If $2k > n$, then $P_k = 0$*
(2) For $2k \le n$, $s^{n-2k}\colon P_k \to \Lambda_S^{n-k}$ is injective
(3) $s^{n-2k}\colon \Lambda_S^k \to \Lambda_S^{n-k}$ is an isomorphism.
(4) For $2k \le n$, $P_k = \ker(s^{n-2k+1})$.

Proof. Suppose $0 \ne \alpha \in P_k$. Pick $m > 0$ minimal with $s^m\alpha = 0$. Then,

$$0 = [s^*, s^m]\alpha = m(n-2k-m+1)s^{m-1}\alpha.$$

Since m is minimal, $s^{m-1}\alpha \ne 0$. So, $n-2k = m-1$. If we assume $2k > n$, then we get a contradiction, since $m-1 \ge 0$, thus $\alpha = 0$, which is part (1). If we assume $2k \le n$, then since $m = n-2k+1$ is minimal for $s^m\alpha = 0$, we get that $s^{n-2k}\alpha \ne 0$, which is part (2).

For part (3), since by part (2), $s^{n-2(k-i)}$ is injective on P_{k-i}, we have that s^{n-2k} is injective on s^iP_{k-i}. So s^{n-2k} is injective on all direct summands of Λ_S^k, and since it maps different direct summands to different direct summands of Λ_S^{n-2k} (by the decomposition given by Proposition 3.2), we have that $s^{n-2k}\colon \Lambda_S^k \to \Lambda_S^{n-k}$ is injective. By counting dimensions, we see that it is an isomorphism.

For part (4), let $2k \le n$. From part (2), we know that $P_k \subseteq \ker(s^{n-2k+1})$. Now let $\alpha \in \Lambda_S^k \cap \ker(s^{n-2k+1})$. Then,

$$s^{n-2k+2}s^*\alpha = (s^{n-2k+2}s^* - s^*s^{n-2k+2})\alpha = -(n-2k+2)s^{n-2k+1}\alpha = 0.$$

From part (2) we also know that s^{n-2k+2} is injective on Λ_S^{k-1}, so $s^{n-2k+2}s^*\alpha = 0$ gives $s^*\alpha = 0$. So $\alpha \in P_k$. □

4. Some Calculations on Dirac Bundle Valued Forms

Recall that given a connection ∇ on a vector bundle E, the *exterior covariant derivative* $d^\nabla : \Lambda_E^k \to \Lambda_E^{k+1}$ is defined by

$$d^\nabla(\alpha \otimes \tau) = d\alpha \otimes \tau + (-1)^k \alpha \wedge \nabla\tau.$$

Using a local orthonormal frame, we have that

$$d^\nabla(\alpha \otimes \tau) = d\alpha \otimes \tau + e_i \wedge \alpha \otimes \nabla_i\tau.$$

Let S be a Dirac bundle of the tangent bundle of a Riemannian manifold. Let s act on the space of S-valued differential forms. We obtain the following analogue of the Kähler identities. Note that in the computations below, we always compute things locally at a point p, and we repeatedly use the fact that we choose e_i to be parallel at the point p, meaning $(\nabla e_i)_p = 0$ for all i. Also note that $(\nabla e_i)_p = 0$ does not imply $(\nabla\nabla_j e_i)_p = 0$.

The following is in analogy to the Kähler identities, which can be found in, for example, Proposition 3.1.12 in [4].

Proposition 4.1. *We have the following identities:*

$$\begin{aligned}
\{d^\nabla, s\} &= 0, \\
\{d^\nabla, s^*\} &= -D, \\
\{(d^\nabla)^*, s^*\} &= 0, \\
\{(d^\nabla)^*, s\} &= -D.
\end{aligned}$$

Proof. Let $\alpha \otimes \tau$ be an S-valued k-form. The first identity:

$$\begin{aligned}
d^\nabla s(\alpha \otimes \tau) &= d^\nabla(e_i \wedge \alpha \otimes e_i\tau) \\
&= d(e_i \wedge \alpha) \otimes e_i\tau + (-1)^{k+1} e_i \wedge \alpha \wedge \nabla(e_i\tau),
\end{aligned}$$

$$\begin{aligned}
sd^\nabla(\alpha \otimes \tau) &= s(d\alpha \otimes \tau + (-1)^k \alpha \wedge \nabla\tau) \\
&= e_i \wedge d\alpha \otimes e_i\tau + (-1)^k e_i \wedge \alpha \wedge e_i\nabla\tau,
\end{aligned}$$

since $d(e_i \wedge \alpha) = -e_i \wedge d\alpha$ and $\nabla(e_i\tau) = e_i\nabla\tau$, we see that $\{d^\nabla, s\} = 0$.

The second identity (repeatedly using the identity $d = e_i \wedge \nabla_i$):

$$\begin{aligned}
s^* d^\nabla(\alpha \otimes \tau) =& s^*(d\alpha \otimes \tau + (-1)^k \alpha \wedge \nabla\tau) \\
=& - e_i \lrcorner (e_l \wedge \nabla_l \alpha) \otimes e_i\tau - (-1)^k e_i \lrcorner (\alpha \wedge (e_i\nabla\tau))
\end{aligned}$$

using $e \lrcorner (\omega \wedge \omega') = (e \lrcorner \omega) \wedge \omega' + (-1)^{\deg(\omega)} \omega \wedge (e \lrcorner \omega')$

$$\begin{aligned}
=& - \delta_{il} \nabla_l \alpha \otimes e_i\tau + e_l \wedge (e_i \lrcorner \nabla_l \alpha) \otimes e_i\tau \\
& - (-1)^k (e_i \lrcorner \alpha) \wedge (e_i \nabla\tau) - (-1)^k (-1)^k \alpha \wedge e_i \lrcorner (e_i\nabla\tau),
\end{aligned}$$

since $e_i \lrcorner (e_i\nabla\tau) = e_i \lrcorner \nabla(e_i\tau) = \nabla_i(e_i\tau) = e_i\nabla_i\tau$

$$\begin{aligned}
=& - \delta_{il} \nabla_l \alpha \otimes e_i\tau + e_l \wedge (e_i \lrcorner \nabla_l \alpha) \otimes e_i\tau \\
& - (-1)^k (e_i \lrcorner \alpha) \wedge (e_i\nabla\tau) - \alpha \wedge (e_i\nabla_i\tau), \\
d^\nabla s^*(\alpha \otimes \tau) =& e_l \wedge \nabla_l(e_i \lrcorner \alpha) \otimes e_i\tau - (-1)^{k+1} (e_i \lrcorner \alpha) \wedge (e_i\nabla\tau).
\end{aligned}$$

When we add these two expressions, the first term and last term of $s^* d^\nabla(\alpha \otimes \tau)$ together give $-D(\alpha \otimes \tau)$, and the third term of $s^* d^\nabla(\alpha \otimes \tau)$ cancels out with the second term of $d^\nabla s^*(\alpha \otimes \tau)$. The second term of $s^* d^\nabla(\alpha \otimes \tau)$ together with the first term of $d^\nabla s^*(\alpha \otimes \tau)$ gives

$$e_l \wedge (e_i \lrcorner \nabla_l \alpha - \nabla_l(e_i \lrcorner \alpha)) \otimes e_i\tau = 0.$$

The third and fourth identities are obtained by taking the adjoints of the first and second identities, and using that the Dirac operator is formally self-adjoint. $\square$

The Lefschetz type decomposition obtained in Proposition 3.2 also gives a decomposition of the bundle Λ^k_S into subbundles $P_k \oplus sP_{k-1} \oplus \cdots \oplus s^{k-1}P_1 \oplus s^k P_0$.

The following is in analogy to the proof of part (ii) of Proposition 3.1.12 in [4].

Proposition 4.2. *Let $\alpha \in \Gamma(P_k)$. We have that*

$$d^\nabla \alpha = y_{k+1} + sy_k + s^2 y_{k-1},$$

where $y_i \in \Gamma(P_i)$.

Proof. We can assume that $2k \leq n$, since otherwise $\alpha = 0$.

Decompose $d^\nabla \alpha = y_{k+1} + sy_k + \cdots + s^{k+1}y_0$, where each $y_i \in P_i = \Lambda^i_S \cap \ker(s^{n-2i+1})$ (by Proposition 3.8).

Since for any $\beta \in \Gamma(P_i) = \Gamma(\Lambda^i_S \cap \ker(s^{n-2i+1}))$,

$$d^\nabla s^{n-2i+1}\beta = 0 = s^{n-2i+1} d^\nabla \beta = 0$$

by the first identity in Proposition 4.1, we have that the individual terms in the equation

$$s^{n-2k+1}(y_{k+1} + sy_k + \cdots + s^{k+1}y_0) = s^{n-2k+1} d^\nabla \alpha = 0$$

are zero (since the decomposition is a direct sum decomposition):

$$s^{(n-2k+1)+(i+1)} y_{k-i} = 0.$$

Since on P_{k-i} (therefore $\Gamma(P_{k-i})$), s^m is injective for all $m \leq n-2(k-i)$, we get that $y_{k-i} = 0$ for $i \geq 2$. □

Define the Laplacians $\Delta_{d^\nabla} := \{d^\nabla, (d^\nabla)^*\} = d^\nabla (d^\nabla)^* + (d^\nabla)^* d^\nabla$ and $\Delta_D = D^2$. Let R_{ij} be the curvature operator $R_{ij} := \nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[e_i, e_j]}$ (where ∇ denotes connections on different bundles, depending on what we put in $R_{ij}(\cdot)$), and define the differential operators

$$K(\alpha \otimes \tau) = e_i \wedge R_{ij}(\alpha) \otimes e_j \tau + e_i \wedge \alpha \otimes e_j R_{ij}(\tau),$$

$$\mathcal{R}(\phi) = \frac{1}{2} e_i e_j R_{ij}(\phi).$$

One thing to note is that we sometimes put S-valued forms in $R_{ij}()$, so explicitly

$$R_{ij}(\alpha \otimes \tau) = R_{ij}(\alpha) \otimes \tau + \alpha \otimes R_{ij}(\tau).$$

Proposition 4.3. *We have the following identities:*

1. $\{s, D\} = -2d^\nabla, \quad \{s^*, D\} = -2(d^\nabla)^*$,
2. $[d^\nabla, D] = K$,
3. $[K, s^*] = 2\mathcal{R} + 2e_i \wedge (e_j \lrcorner R_{ij})$,
4. $\Delta_D = \Delta_{d^\nabla} + \mathcal{R} + e_i \wedge (e_j \lrcorner R_{ij})$.

Proof. The first identity:

$$\begin{aligned}
sD(\alpha\otimes\tau) =& e_j\wedge\nabla_i\alpha\otimes e_je_i\tau + e_j\wedge\alpha\otimes e_je_i\nabla_i\tau\\
=& -e_j\wedge\nabla_i\alpha\otimes e_ie_j\tau - e_j\wedge\nabla_i\alpha\otimes 2\delta_{ij}\tau\\
& -e_j\wedge\alpha\otimes e_ie_j\nabla_i\tau - e_j\wedge\alpha\otimes 2\delta_{ij}\nabla_i\tau\\
Ds(\alpha\otimes\tau) =& \nabla_i(e_j\wedge\alpha)\otimes e_ie_j\tau + e_j\wedge\alpha\otimes e_i\nabla_i(e_j\tau).
\end{aligned}$$

When we add these two expressions, the third term of $sD(\alpha\otimes\tau)$ and the second term of $Ds(\alpha\otimes\tau)$ cancel out, and the first term of $sD(\alpha\otimes\tau)$ cancels out with the first term of $Ds(\alpha\otimes\tau)$. The two remaining terms give $-2d^\nabla$, which gives $\{s,D\}=-2d^\nabla$. Taking the adjoint gives $\{s^*,D\}=-2(d^\nabla)^*$.

The second identity:

$$\begin{aligned}
d^\nabla D(\alpha\otimes\tau) =& d\nabla_i\alpha\otimes e_i\tau + e_j\wedge\nabla_i\alpha\otimes\nabla_j(e_i\tau)\\
& + d\alpha\otimes e_i\nabla_i\tau + e_j\wedge\alpha\otimes\nabla_j(e_j\nabla_i\tau),\\
Dd^\nabla(\alpha\otimes\tau) =& \nabla_i d\alpha\otimes e_i\tau + d\alpha\otimes e_i\nabla_i\tau\\
& + \nabla_i(e_j\wedge\alpha)\otimes e_i\nabla_j\tau + e_j\wedge\alpha\otimes e_i\nabla_i\nabla_j\tau.
\end{aligned}$$

When we take $[d^\nabla,D]$, the second term of $d^\nabla D(\alpha\otimes\tau)$ cancels with the third term of $Dd^\nabla(\alpha\otimes\tau)$, and the third term of $d^\nabla D(\alpha\otimes\tau)$ cancels with the second term of $Dd^\nabla(\alpha\otimes\tau)$. Using $d=e_i\wedge\nabla_i$, we are left with

$$\begin{aligned}
[d^\nabla,D](\alpha\otimes\tau) =& e_j\wedge\nabla_j\nabla_i\alpha\otimes e_i\tau + e_j\wedge\alpha\otimes e_j\nabla_j\nabla_i\tau\\
& - e_j\wedge\nabla_i\nabla_j\alpha\otimes e_i\tau - e_j\wedge\alpha\otimes e_j\nabla_i\nabla_j\tau\\
=& K(\alpha\otimes\tau).
\end{aligned}$$

The third identity:

$$Ks^*(\alpha\otimes\tau) = -e_j\wedge R_{ji}(e_k\lrcorner\alpha)\otimes e_ie_k\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_iR_{ji}(e_k\tau)$$

since curvature operator R_{ij} is a derivation,

$$\begin{aligned}
=& -e_j\wedge\lrcorner e_kR_{ji}\alpha\otimes e_ie_k\tau - e_j\wedge R_{ji}(e_k)\lrcorner\alpha\otimes e_ie_k\tau\\
& - e_j\wedge(e_k\lrcorner\alpha)\otimes e_ie_kR_{ji}\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_iR_{ji}(e_k)\tau,
\end{aligned}$$

$$\begin{aligned}
s^*K(\alpha\otimes\tau) =& -e_k\lrcorner(e_j\wedge R_{ji}\alpha)\otimes e_ke_i\tau - e_k\lrcorner(e_j\wedge\alpha)\otimes e_ke_iR_{ji}\tau\\
=& -R_{ji}\alpha\otimes e_je_i\tau + e_j\wedge(e_k\lrcorner R_{ji}\alpha)\otimes e_ke_i\tau - e_k\lrcorner(e_j\wedge\alpha)\otimes e_ke_iR_{ji}\tau\\
=& -R_{ji}\alpha\otimes e_je_i\tau - e_j\wedge(e_k\lrcorner R_{ji}\alpha)\otimes e_ie_k\tau - 2\delta_{ik}e_j\wedge(e_k\lrcorner R_{ji}\alpha)\otimes\tau\\
& -\alpha\otimes e_je_iR_{ji}\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_ie_kR_{ji}\tau - 2\delta_{ik}e_j\wedge(e_k\lrcorner\alpha)\otimes R_{ji}\tau\\
=& -R_{ji}\alpha\otimes e_je_i\tau - e_j\wedge(e_k\lrcorner R_{ji}\alpha)\otimes e_ie_k\tau - 2e_j\wedge(e_i\lrcorner R_{ji}\alpha)\otimes\tau\\
& -\alpha\otimes e_je_iR_{ji}\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_ie_kR_{ji}\tau - 2e_j\wedge(e_i\lrcorner\alpha)\otimes R_{ji}\tau.
\end{aligned}$$

When taking $[K, s^*]$, the first term of $Ks^*(\alpha\otimes\tau)$ cancels with the second term of $s^*K(\alpha\otimes\tau)$, and the third term of $Ks^*(\alpha\otimes\tau)$ and fifth term of $s^*K(\alpha\otimes\tau)$ cancel, so we are left with

$$\begin{aligned}
[K, s^*](\alpha\otimes\tau) =& - e_j \wedge R_{ji}(e_k)\lrcorner\alpha \otimes e_i e_k\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_i R_{ji}(e_k)\tau \\
&+ R_{ji}\alpha\otimes e_j e_i\tau + 2e_j\wedge(e_i\lrcorner R_{ji}\alpha)\otimes\tau \\
&+ \alpha\otimes e_j e_i R_{ji}\tau + 2e_j\wedge(e_i\lrcorner\alpha)\otimes R_{ji}\tau \\
=& - e_j\wedge R_{jikl}(e_l\lrcorner\alpha)\otimes e_i e_k\tau - e_j\wedge(e_k\lrcorner\alpha)\otimes e_i R_{jikl}e_l\tau \\
&+ R_{ji}\alpha\otimes e_j e_i\tau + 2e_j\wedge(e_i\lrcorner R_{ji}\alpha)\otimes\tau \\
&+ \alpha\otimes e_j e_i R_{ji}\tau + 2e_j\wedge(e_i\lrcorner\alpha)\otimes R_{ji}\tau
\end{aligned}$$

since $R_{jikl} = -R_{jilk}$,

$$\begin{aligned}
=& R_{ji}\alpha\otimes e_j e_i\tau + 2e_j\wedge(e_i\lrcorner R_{ji}\alpha)\otimes\tau \\
&+ \alpha\otimes e_j e_i R_{ji}\tau + 2e_j\wedge(e_i\lrcorner\alpha)\otimes R_{ji}\tau \\
=& e_i e_j R_{ij}(\alpha\otimes\tau) + 2e_i\wedge(e_j\lrcorner R_{ij}(\alpha\otimes\tau)) \\
=& 2\mathcal{R}(\alpha\otimes\tau) + 2e_i\wedge(e_j\lrcorner R_{ij}(\alpha\otimes\tau)).
\end{aligned}$$

The fourth identity:

By expanding out the brackets, we get the following identity between formal variables a, b, c:

$$\{a, \{b, c\}\} = [[a, b], c] + \{b, \{a, c\}\}.$$

Applying this, and Proposition 4.1, we get

$$\begin{aligned}
\Delta_D =& \frac{1}{2}\{D, D\} \\
=& -\frac{1}{2}\{D, \{d^\nabla, s^*\}\} \\
=& -\frac{1}{2}([[D, d^\nabla], s^*] + \{d^\nabla, \{D, s^*\}\}) \\
=& \frac{1}{2}[K, s^*] + \{d^\nabla, (d^\nabla)^*\} \\
=& \mathcal{R} + e_i\wedge(e_j\lrcorner R_{ij}) + \Delta_{d^\nabla}.
\end{aligned}$$

□

Remark 4.4. In the case of an S-valued 0-form (which is just a section of S), $d^\nabla = \nabla$, $(d^\nabla)^* = 0$, and interior product is 0, so the middle term is also 0, the fourth identity above reads

$$D^2 = \nabla^*\nabla + \mathcal{R},$$

which appears in chapter 2, section 5 of [1] under the name "general Bochner identity".

Proposition 4.5. *We have the following identities:*

1. $[\Delta_D, s] = 2K$,
2. $[\Delta_{d^\nabla}, s] = -K$,

(3) $[s, K] = 4(d^\nabla)^2$.

Proof. In this proof we use Proposition 4.1 and Proposition 4.3 throughout. The first identity:

$$\begin{aligned} sD^2 =&(-Ds - 2d^\nabla)D \\ =&-DsD - 2d^\nabla D \\ =&D(Ds + 2d^\nabla) - 2d^\nabla D \\ =&D^2 s + 2Dd^\nabla - 2d^\nabla D \\ =&D^2 s - 2K. \end{aligned}$$

The second identity:

$$\begin{aligned} \Delta_{d^\nabla} s =&d^\nabla (d^\nabla)^* s + (d^\nabla)^* d^\nabla s \\ =&-d^\nabla (s(d^\nabla)^* + D) - (d^\nabla)^* s d^\nabla \\ =&s d^\nabla (d^\nabla)^* - d^\nabla D + (s(d^\nabla)^* + D)d^\nabla \\ =&s(d^\nabla (d^\nabla)^* + (d^\nabla)^* d^\nabla) - K. \end{aligned}$$

The third identity:

$$\begin{aligned} sK =&s d^\nabla D - sDd^\nabla \\ =&-d^\nabla s D + (Ds + 2d^\nabla)d^\nabla \\ =&d^\nabla (Ds + 2d^\nabla) - Dd^\nabla s + 2(d^\nabla)^2 \\ =&Ks + 4(d^\nabla)^2. \end{aligned}$$

□

Corollary 4.6. *The operator $\Delta_D + 2\Delta_{d^\nabla}$ commutes with s. So $(\Delta_D + 2\Delta_{d^\nabla})s^m\alpha \in \Gamma(s^m P_k)$ for any $m \geq 0$ and $\alpha \in \Gamma(P_k)$.*

Proof. Since $\Delta_D + 2\Delta_{d^\nabla}$ commutes with s, and $\Delta_D + 2\Delta_{d^\nabla}$ is self-adjoint, we have that $\Delta_D + 2\Delta_{d^\nabla}$ also commutes with s^*. So, if $\alpha \in \Gamma(P_k)$, which means $s^*\alpha = 0$, we have that $s^*(\Delta_D + 2\Delta_{d^\nabla})\alpha = 0$, so $(\Delta_D + 2\Delta_{d^\nabla})\alpha \in \Gamma(P_k)$. Since $\Delta_D + 2\Delta_{d^\nabla}$ commutes with s, we can apply s repeatedly to both sides of $(\Delta_D + 2\Delta_{d^\nabla})\alpha \in \Gamma(P_k)$, and get $(\Delta_D + 2\Delta_{d^\nabla})s^m\alpha \in \Gamma(s^m P_k)$. □

5. Spin Groups and Their Lie Algebras

In this section we construct the spin groups, and then construct (unique) nontrivial 2-1 coverings from spin groups to special orthogonal groups, following chapter 1, section 2 of [1].

The group of units $\mathrm{Cl}^\times(V)$ consists of all non-zero elements of V (along with other elements), since every non-zero vector in V squares to a non-zero real number.

Definition 5.1. $P(V)$ is defined to be the subgroup of $\mathrm{Cl}^\times(V)$ generated by all non-zero elements of V. The *pin group* $\mathrm{Pin}(V)$ is defined to be the subgroup of $P(V)$ generated by unit length vectors, and the *spin group* $\mathrm{Spin}(V)$ is defined by

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V).$$

Alternatively, $\mathrm{Spin}(V)$ is the subgroup of $P(V)$ generated by elements of the form vw, where $v, w \in V, \|v\| = \|w\| = 1$.

Recall that the involution $\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is induced from $\alpha(v) = -v$ for $v \in V$.

Definition 5.2. The *twisted adjoint representation*

$$\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V) \to GL(\mathrm{Cl}(V))$$

is defined by

$$\widetilde{\mathrm{Ad}}_\varphi(y) = \alpha(\varphi) y \varphi^{-1}.$$

Note that on $\mathrm{Cl}^0(V)$ (and therefore $\mathrm{Spin}(V)$), the twisted adjoint representation is equal to the "adjoint representation" given by $\mathrm{Ad}_\varphi(y) = \varphi y \varphi^{-1}$, since $\alpha = \mathrm{Id}$ on $\mathrm{Cl}^0(V)$.

The next proposition says that by the twisted adjoint representation, vectors act by reflection across the hyperplane they define.

Proposition 5.3. *Let $v, w \in V$ with $v \neq 0$. We have that*

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{\langle v, w\rangle}{\|v\|^2} v.$$

Proof. Since $v^{-1} = -v/\|v\|^2$ and $vw + wv = -2\langle v, w\rangle$, we have that

$$\|v\|^2 \widetilde{\mathrm{Ad}}_v(w) = -\|v\|^2 v w v^{-1} = vwv = (-wv - 2\langle v, w\rangle)v = \|v\|^2 w - 2\langle v, w\rangle v. \qquad \square$$

As a corollary, we have that $\widetilde{\mathrm{Ad}}_v(V) = V$, and since $P(V)$ is generated by the non-zero vectors, we have that $\widetilde{\mathrm{Ad}}_\varphi(V) = V$ for all $\varphi \in P(V)$. Define

$$\widetilde{P}(V) := \{\varphi \in \mathrm{Cl}^\times(V) : \widetilde{\mathrm{Ad}}_\varphi(V) = V\}.$$

On $\widetilde{P}(V)$, $\widetilde{\mathrm{Ad}}$ is therefore also a representation with vector space V instead of $\mathrm{Cl}(V)$.

Proposition 5.4. *The kernel of $\widetilde{\mathrm{Ad}} : \widetilde{P}(V) \to GL(V)$ is $\mathbb{R}^\times$, the group of non-zero real numbers.*

Proof. Pick an orthonormal basis $e_1, \ldots, e_n$ for V. Let $\varphi \in \mathrm{Cl}^\times(V)$ be in the kernel of $\widetilde{\mathrm{Ad}}$, that is, $\alpha(\varphi)v = v\varphi$ for all $v \in V$. Decompose $\varphi = \varphi_0 + \varphi_1$ by the $\mathbb{Z}/2$-grading of $\mathrm{Cl}(V)$. Since α acts by 1 on even elements and -1 on odd elements, we have that

$$v\varphi_0 = \varphi_0 v, -v\varphi_1 = \varphi_1 v. \tag{5.5}$$

φ_0 and φ_1 (as with any element in the Clifford algebra) can be written as polynomial expressions in $e_1, \ldots, e_n$. Repeatedly using $e_i e_j = -2\delta_{ij} - e_j e_i$ shows that φ_0 can be written as $\varphi_0 = a_0 + e_1 a_1$, where a_0 and a_1 are polynomials in $e_2, \dots, e_n$. Both a_0 and $e_1 a_1$ are even, so we have that a_0 is even and a_1 is odd. Setting $v = e_1$ in (5.5) shows that

$$e_1 a_0 + e_1^2 a_1 = a_0 e_1 + e_1 a_1 e_1 = e_1 a_0 - e_1^2 a_1.$$

So $a_1 = 0$. So φ_0 does not involve e_1 terms. Similarly, φ_0 does not involve any of the e_i terms. So φ_0 is a real number.

An analogous argument shows that φ_1 does not involve any of the e_i terms. Since φ_1 is odd, this means that $\varphi_1 = 0$.

So, φ is a non-zero real number. Conversely, any non-zero real number is clearly in the kernel of $\widetilde{\mathrm{Ad}}$. $\square$

Proposition 5.6. $\widetilde{\mathrm{Ad}}$ *gives homomorphisms* $\mathrm{Pin}(V) \to O(V)$, *and* $\mathrm{Spin}(V) \to SO(V)$.

Proof. Notice that since any element in $\mathrm{Pin}(V)$ is a product of unit length vectors, and unit length vectors act by reflections under the representation $\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V) \to GL(V)$, we have that the representation restricted to $\mathrm{Pin}(V)$ is an orthogonal representation on V.

Analogously, since any element in $\mathrm{Spin}(V)$ is a product of an even number of unit length vectors, which act by a composition of an even number of reflections under the representation $\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V) \to GL(V)$, we have that the representation restricted to $\mathrm{Spin}(V)$ is a special orthogonal representation. $\square$

Proposition 5.7. $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to O(V)$, $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to SO(V)$ *are double covers.*

Proof. By viewing $\mathrm{Cl}(V)$ as a left $\mathrm{Cl}(V)$ module by multiplication, Proposition 1.3 gives an inner product on $\mathrm{Cl}(V)$ that is invariant under multiplication by unit length vectors. Under such an inner product, since all elements of $\mathrm{Pin}(V)$ are products of several unit length vectors with the real number 1, and that $1 \in \mathrm{Pin}(V)$, we have that all elements of $\mathrm{Pin}(V)$ have the same norm as 1. So by Proposition 5.4, the kernel of $\widetilde{\mathrm{Ad}}$ as a homomorphism from $\mathrm{Pin}(V)$ is $\mathrm{Pin}(V) \cap \mathbb{R} = \{1, -1\}$.

Since $1, -1 \in \mathrm{Spin}(V)$, we have that the kernel of $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to SO(V)$, which is the restriction of the kernel of $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to O(V)$ to $\mathrm{Spin}(V)$, is also $\{1, -1\}$.

Since orthogonal groups are generated by reflections, and special orthogonal groups are generated by pairwise products of reflections, we have that these homomorphisms are surjective. Thus they are double covers. $\square$

Proposition 5.8. *The double covers* $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to O(V)$, $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to SO(V)$ *are nontrivial.*

Proof. We prove this by showing that there is a path connecting -1 to 1 in $\mathrm{Spin}(V)$. Pick $e_1, e_2 \in V$ orthogonal, then

$$\gamma(t) = (e_1 \cos t + e_2 \sin t)(e_2 \sin t - e_1 \cos t)$$

for $t \in [0, \frac{\pi}{2}]$ does the job. $\square$

We always take $V = \mathbb{R}^n$ in the above, and will write $\mathrm{Pin}(n), \mathrm{Spin}(n)$ for $\mathrm{Pin}(\mathbb{R}^n), \mathrm{Spin}(\mathbb{R}^n)$.

Next we discuss the Lie algebras $\mathfrak{spin}_n$. The Lie algebra of $\mathrm{Cl}_n^\times$ (the group of units in Cl_n) is simply Cl_n with the commutator bracket. Since $\mathrm{Spin}(n)$ is a compact subgroup of $\mathrm{Cl}_n^\times$, we have that $\mathfrak{spin}_n$ is a Lie subalgebra of Cl_n.

Proposition 5.9. *Under the canonical isomorphism $\mathrm{Cl}_n \cong \Lambda\mathbb{R}^n$, the Lie subalgebra $\mathfrak{spin}_n$ of Cl_n is identified with $\Lambda^2\mathbb{R}^n$.*

Proof. For each $i \leq j$, consider the curve

$$\begin{aligned}\gamma(t) =&(e_i \cos t + e_j \sin t)(-e_i \cos t + e_j \sin t)\\ =&(\cos^2 t - \sin^2 t) + 2e_ie_j \sin t \cos t\\ =&\cos(2t) + \sin(2t)e_ie_j.\end{aligned}$$

These curves lie in $\mathrm{Spin}(n)$, and they meet the identity. Their tangent vectors at the identity are $2e_ie_j$. Hence, $\mathfrak{spin}_n$ contains $\Lambda^2\mathbb{R}^n = \mathrm{span}\{e_ie_j\}$. Since the e_ie_j's are linearly independent, and $\dim(\mathfrak{spin}_n) = \dim(\mathfrak{so}_n) = n(n-1)/2$, we conclude that $\mathfrak{spin}_n = \Lambda^2\mathbb{R}^n$. □

$\mathfrak{so}_n$ is the Lie algebra of all real $n \times n$ skew-symmetric matrices. $\mathfrak{so}_n$ is generated by the elementary transformations $v \wedge w$, given by

$$(v \wedge w)(x) = \langle v, x\rangle w - \langle w, x\rangle v.$$

Note that if $i < j$, then $e_i \wedge e_j$ is the matrix with ijth entry 1, jith entry -1, and all other entries 0, and they form a basis for $\mathfrak{so}_n$.

We have that the adjoint representation $\mathrm{Ad} : \mathrm{Spin}(n) \to SO(n)$ is the nontrivial double cover (since on the even part of the Clifford algebra, $\mathrm{Ad} = \widetilde{\mathrm{Ad}}$). This induces a Lie algebra isomorphism $\mathrm{Ad}_* : \mathfrak{spin}_n \to \mathfrak{so}_n$.

Proposition 5.10. *The Lie algebra isomorphism above is given explicitly on basis elements $\{e_ie_j\}_{i<j}$ by*

$$\mathrm{Ad}_*(e_ie_j) = 2e_i \wedge e_j.$$

Consequently,

$$\mathrm{Ad}_*^{-1}(v \wedge w) = \frac{1}{4}[v, w].$$

Proof. The curve

$$\gamma(t) = \cos(t) + \sin(t)e_ie_j$$

has $\gamma(0) = 1, \gamma'(0) = e_ie_j$. Then

$$\mathrm{Ad}_*(e_ie_j) = \frac{d}{dt}\mathrm{Ad}(\gamma(t))|_{t=0}.$$

Let $x \in \mathbb{R}^n$. We have that

$$\mathrm{Ad}(\gamma(t))(x) = \gamma(t)x\gamma(t)^{-1},$$

and since $(\gamma^{-1})'(0) = -\gamma'(0) = -e_ie_j$, we have that

$$\begin{aligned}\mathrm{Ad}_*(e_ie_j)(x) =& e_ie_jx - xe_ie_j \\ =& e_ie_jx + (e_ix + 2\langle e_i, x\rangle)e_j \\ =& e_ie_jx - e_ie_jx - 2\langle e_j, x\rangle e_i + 2\langle e_i, x\rangle e_j \\ =& 2(e_i \wedge e_j)(x).\end{aligned}$$

For the formula for $\mathrm{Ad}_*^{-1}(v \wedge w)$, it is enough to notice that on basis elements $e_i \wedge e_j$, we have

$$\mathrm{Ad}_*^{-1}(e_i \wedge e_j) = \frac{1}{2}e_ie_j = \frac{1}{4}[e_i, e_j]. \qquad \square$$

Corollary 5.11. *Let $\Delta : \mathrm{Spin}(n) \to SO(W)$ be a representation obtained by restricting a representation of Cl_n. Let $\Delta_* : \mathfrak{so}_n \to \mathfrak{so}(W)$ be the associated Lie algebra representation. Then on elementary transformations $v \wedge w \in \mathfrak{so}_n$,*

$$\Delta_*(v \wedge w) = \frac{1}{4}[v, w].$$

6. Spin Manifolds and Spinor Bundles

In this section we review some material on spin manifolds, mostly taken from the beginning of chapter 2 of [1]. [5], [6], [7] are some other useful references.

Let $\pi : E \to M$ be a rank n real orientable Riemannian vector bundle over a manifold M, which means that there is a smoothly varying inner product (that is, a Riemannian metric) on the fibres, and there exists a smoothly defined orientation on the fibres. Any vector bundle admit Riemannian metrics, while not all vector bundles are orientable. A necessary and sufficient topological criterion for orientability of a vector bundle is that the first Stiefel-Whitney class of the vector bundle vanishes.

A Riemannian metric on a vector bundle is equivalently a reduction of the structure group of the vector bundle from $GL(n,\mathbb{R})$ to $O(n)$, and a vector bundle being orientable is equivalently the existence of a structure group reduction from $GL(n,\mathbb{R})$ to $GL_+(n,\mathbb{R})$ (the group of matrices with positive determinant). A real oriented Riemannian vector bundle is therefore a vector bundle with a structure group $SO(n)$, together with an orientation on a fibre. So, a real orientable Riemannian vector bundle E comes with a principal $SO(n)$-bundle which we denote by $P_{SO}(E)$.

Definition 6.1. A *spin structure* on E is a lifting of the $SO(n)$ structure group to a $\mathrm{Spin}(n)$ structure group. More precisely, a *spin structure* on the real orientable Riemannian vector bundle E is a principal $\mathrm{Spin}(n)$-bundle $P_{Spin}(E)$ over M, together with a 2-sheeted covering bundle map

$$\xi : P_{Spin}(E) \to P_{SO}(E)$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}(E)$ and $g \in \mathrm{Spin}(n)$ (here ξ_0 is the canonical 2-sheeted covering map from $\mathrm{Spin}(n)$ to $SO(n)$). An orientable vector bundle is called *spinnable* if it can be equipped with a spin structure, and is called *spin* if we fix such a spin structure.

Remark 6.2. A necessary and sufficient topological criterion of the spinnability of an orientable vector bundle is that the second Stiefel-Whitney class of the vector bundle vanishes. A proof of this can be found in chapter 2, section 2 of [1].

Definition 6.3. Given a spin vector bundle E and a module V over the Clifford algebra Cl_n, we can form the associated bundle of the principal bundle $P_{Spin}(E)$ and the representation V (where V is now viewed as a representation of $\mathrm{Spin}(n)$). Associated bundles constructed this way are called *spinor bundles* corresponding to the spin structure.

Spinor bundles are canonically bundles of modules over the Clifford bundle $\mathrm{Cl}(E)$: define the *adjoint representation*

$$\mathrm{Ad} : \mathrm{Spin}(n) \to Aut(\mathrm{Cl}_n)$$

given by $\mathrm{Ad}_g(\varphi) = g\varphi g^{-1}$ for $g \in Spin_n \subset \mathrm{Cl}_n$. Since $\mathrm{Ad}_{-1} = \mathrm{id}$, we have that Ad factors through the projection map $\mathrm{Spin}(n) \to SO(n)$ to give us a representation Ad' of $SO(n)$. In fact, Ad' is the same representation as the representation $cl(\rho_n)$ from section 2. This gives the Clifford bundle a spinor bundle structure:

$$\mathrm{Cl}(E) = P_{Spin}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_n.$$

Definition 6.4. On a spinor bundle S, the structure of a bundle of modules over the Clifford bundle is given in the following way (where μ denotes the representation V of $\mathrm{Spin}(n)$):

$$\mathrm{Cl}(E) \oplus S = P_{Spin}(E) \times_{\mathrm{Ad}\oplus\mu} (\mathrm{Cl}_n \oplus V) \to P_{Spin}(E) \times_\mu V,$$
$$[p, \varphi, v] \mapsto [p, \varphi v].$$

A *spin manifold* is a Riemannian manifold together with a spin structure on its tangent bundle. When we say "spinor bundle on a spin manifold", we would then mean a spinor bundle coming from the spin structure on the tangent bundle of the manifold. When we work over the tangent bundle, we will use $P_{SO}(M)$ and $P_{Spin}(M)$ for $P_{SO}(TM)$ and $P_{Spin}(TM)$ (for example). From now on we only work with spin structures and spinor bundles over the tangent bundle of a spin manifold.

Next we describe the Dirac bundle structures of spinor bundles. Let $S = \mathrm{Cl}(M) \times_\mu V$ be a spinor bundle. We have already shown S is a bundle of modules over $\mathrm{Cl}(M)$. Proposition 1.3 gives an inner product on V invariant under multiplication by unit vectors, and this inner product induces a metric on the associated bundle S invariant under multiplication by unit vectors, which is the property needed in the definition of a Dirac bundle. It remains to find a compatible connection.

Definition 6.5. On a Riemannian manifold M, we have the Levi-Civita connection, which induces a canonical connection on its $O(n)$-bundle, and in turn induces a canonical connection on its $SO(n)$-bundle, $P_{SO}(M)$, if M is orientable. If M is spin, then we can pullback the connection on $P_{SO}(M)$ to get a connection on $P_{Spin}(M)$, by the map ξ. This connection on $P_{Spin}(M)$ would then induce a connection ∇^S on its associated bundles, namely the spinor bundles.

Proposition 6.6. *The spin connection on S is a derivation with respect to the Clifford module structure. That is, for each $\phi \in \Gamma(\mathrm{Cl}(M))$, $\sigma \in \Gamma(S)$, we have*

$$\nabla^S(\phi\sigma) = (\nabla\phi)\sigma + \phi(\nabla^S\sigma).$$

Proof. The representations $cl(\rho_n) = \mathrm{Ad}$ and μ preserve the module multiplication. That is,

$$\mu(g)(\varphi\sigma) = (cl(\rho_n)(g)\varphi)(\mu(g)\sigma)$$

for all $g \in \mathrm{Spin}(n), \varphi \in \mathrm{Cl}_n, \sigma \in V$. Differentiating at the identity gives that for each $A \in \mathfrak{so}_n = \mathfrak{spin}_n$,

$$(\mu_* A)(\varphi\sigma) = ((cl(\rho_n)_* A)\varphi)\sigma + \varphi((\mu_* A)\sigma).$$

This then implies that the spin connection is a derivation on sections. □

So, spinor bundles are Dirac bundles. Next we describe the spin connection more concretely.

Let $f \in V$, and $e_1, ..., e_n$ be an orthonormal local frame of TM. These together determine two local sections of S by the following: $e_1, ..., e_n$ is equivalent to a local section of $P_{SO}(M)$, which we can lift to two local sections α, β of $P_{Spin}(M)$. Then $[\alpha, f]$ and $[\beta, f]$ give the two local sections of S (and $[\alpha, f] = -[\beta, f]$).

Proposition 6.7. *Let $e_1, ..., e_n$ be an orthonormal local frame of TM, where we can write the Levi-Civita connection as*

$$\nabla e_i = \omega_{ji} \otimes e_j$$

where ω is an $\mathfrak{so}_n$-valued 1-form. Let $f \in V$. Then the connection ∇^S on S on sections obtained by the procedure above, $\sigma = [\alpha, f], [\beta, f]$ is given by the formula

$$\nabla\sigma = \frac{1}{4}\omega_{ji} \otimes e_i e_j \sigma.$$

Remark 6.8. Note that any local section of S can be written as a $C^\infty(M)$-linear combination of sections obtained like this (if we take $f_1, ..., f_m$ to be an orthonormal basis of V, then $[\alpha, f_1], ..., [\alpha, f_m]$ is a local orthonormal frame of S).

Proof. We have that $\mathfrak{so}_n$ is generated by the elementary transformations $x \wedge y, x, y \in \mathbb{R}^n$, given by $(x \wedge y)(v) := \langle x, v\rangle y - \langle y, v\rangle x$.

Denote the induced map on Lie algebras by subscript $*$. By Corollary 5.11, we have that $\mu_*(x \wedge y) = \frac{1}{4}[x, y]$ (that is, left multiplication by $\frac{1}{4}[x, y] \in \mathrm{Cl}_n$). Also recall that from the discussion after Definition 6.3, we have that on Spin_n, $cl(\rho_n) = \mathrm{Ad}$. Hence, by Corollary 5.11,

$$cl(\rho_n)_*(x \wedge y) = \mathrm{Ad}_*(x \wedge y) = \mathrm{ad}_{\frac{1}{4}[x,y]}.$$

That is,

$$\mathrm{Ad}_*(x \wedge y)(\varphi) = \frac{1}{4}[[x, y], \varphi].$$

Let $i \neq j$. Since $e_i e_j = -e_j e_i$, we have that by taking $x = e_i, y = e_j$,

$$\mathrm{Ad}_*(e_i \wedge e_j)(\varphi) = \frac{1}{2}[e_i e_j, \varphi].$$

Thus by pulling back the $P_{SO}(M)$ connection 1-form $\tilde{\omega} = \frac{1}{2}\omega_{ji} e_i \wedge e_j$ to a connection 1-form on $P_{Spin}(M)$ via ξ, we get the $\frac{1}{4}\omega_{ji} e_i e_j$ (since on each fibre, the map between these two principal bundles is the adjoint representation, and since connection 1-forms are forms with values in the Lie algebra, we can pullback connection 1-forms via the associated Lie algebra homomorphism). We then view $P_{Spin}(M)$ as a subbundle of $P_{SO}(S)$, then we have that for sections of $P_{SO}(S)$ of the form $[\alpha, f], [\beta, f]$, the desired formula holds:

$$\nabla \sigma = \frac{1}{4}\omega_{ji} \otimes e_i e_j \sigma.$$

□

Corollary 6.9. *Let Ω be the curvature 2-form on $P_{SO}(M)$, and let S be any spinor bundle associated to M. Then the curvature R of S induced by the above connection is given locally by*

$$R\sigma = \frac{1}{4}\Omega_{ji} \otimes e_i e_j \sigma,$$

where $e_1, ..., e_n$ is a local orthonormal frame of TM. In particular, for any two tangent vectors V and W at p, the curvature transformation $R_{V,W} : S_p \to S_p$ is given by

$$R_{V,W}\sigma = \frac{1}{4}\langle R_{V,W} e_i, e_j \rangle e_i e_j \sigma.$$

Remark 6.10. The formulas in Corollary 6.9 work for any section $\sigma \in \Gamma(S)$, since the curvature operator is a tensor.

7. Some Calculations on Spinor Valued Forms

In the case of a spinor bundle, the curvature term in the general Bochner identity can be greatly simplified:

Theorem 7.1. *(Lichnerowicz, [3]). Let M be a spin manifold and S be a spinor bundle. Let κ denote the scalar curvature of M. Then,*

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa.$$

Proof. The general Bochner identity mentioned in Remark 4.4 states $D^2 = \nabla^* \nabla + \mathcal{R}$, where

$$\mathcal{R} = \frac{1}{2} e_i e_j R_{ij}$$

by the second formula in Corollary 6.9,

$$
\begin{aligned}
=&\frac{1}{8}R_{ijkl}e_ie_je_ke_l\\
=&\frac{1}{8}\sum_l\Big(\frac{1}{3}\sum_{\substack{i,j,k\\ \text{distinct}}}(R_{ijkl}+R_{kijl}+R_{jkil})e_ie_je_k+\sum_{ij}R_{ijil}e_ie_je_i+\sum_{ij}R_{ijjl}e_ie_je_l\Big)e_l.
\end{aligned}
$$

The first big term is zero by the Bianchi identity, and we can rearrange the last two terms and change some indices around to get

$$
=\frac{1}{4}R_{ijil}e_je_l=-\frac{1}{4}\mathrm{Ric}(e_j,e_l)e_je_l=\frac{1}{4}\kappa. \qquad \square
$$

We can attempt to obtain an analogous formula starting from the fourth identity of Proposition 4.3:

Proposition 7.2. *Let M be a spin manifold and S be a spinor bundle. Let κ denote the scalar curvature, and* Ric *denote the Ricci transformation extended to ΛTM as a derivation. Then,*

$$
\Delta_D(\alpha\otimes\sigma)=\Delta_{d^\nabla}(\alpha\otimes\sigma)+\frac{1}{4}\kappa(\alpha\otimes\sigma)-\mathrm{Ric}(\alpha)\otimes\sigma+\frac{3}{4}R_{ij}\alpha\otimes e_ie_j\sigma.
$$

Proof. The curvature term in the fourth identity of Proposition 4.3 is explicitly:

$$
\frac{1}{2}R_{ij}\alpha\otimes e_ie_j\sigma+\frac{1}{2}\alpha\otimes e_ie_jR_{ij}\sigma+e_i\wedge(e_j\lrcorner R_{ij}\alpha)\otimes\sigma+e_i\wedge(e_j\lrcorner\alpha)\otimes R_{ij}\sigma.
$$

The second term is $\frac{1}{4}\kappa(\alpha\otimes\sigma)$, by the calculation in the proof of Theorem 7.1.

The fourth term: by the second formula in Corollary 6.9,

$$
e_i\wedge(e_j\lrcorner\alpha)\otimes R_{ij}\sigma=\frac{1}{4}R_{ijkl}e_i\wedge(e_j\lrcorner\alpha)\otimes e_ke_l\sigma=\frac{1}{4}R_{kl}\alpha\otimes e_ke_l\sigma,
$$

which together with the first term, add up to

$$
\frac{3}{4}R_{ij}\alpha\otimes e_ie_j\sigma.
$$

The third term: since R_{ij} is a derivation, we just need to check that for 1-forms α, that

$$
-\mathrm{Ric}(\alpha)=e_i\wedge(e_j\lrcorner R_{ij}\alpha).
$$

Note that $e_j\lrcorner R_{ij}\alpha=\langle R_{ij}\alpha,e_j\rangle$. By the symmetry of the Riemann curvature tensor,

$$
\langle R_{ij}\alpha,e_j\rangle=\langle R_{e_j,\alpha}(e_j),e_i\rangle.
$$

So,

$$
e_i\wedge(e_j\lrcorner R_{ij}\alpha)=\langle R_{e_j,\alpha}(e_j),e_i\rangle e_i=R_{e_j,\alpha}(e_j)=-\mathrm{Ric}(\alpha). \qquad \square
$$

It is unknown to the author what applications this formula has.

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