Riemannian Immersions and Submersions

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Abstract

We discuss the Gauss, Codazzi, and Ricci equations for a Riemannian immersion. The O'Neill tensors for a Riemannian submersion are introduced and applied to the Fubini-Study metric on $\mathbb{CP}^n$. Moreover, we discuss the case of a vector bundle equipped with a connection and fibre metric over a Riemannian manifold and show how to equip the vector bundle with an induced Riemannian metric. This construction is analyzed as a Riemannian immersion and a Riemannian submersion, and the precise conditions under which the O'Neill tensors vanish is determined.

Contents

1 Introduction

The purpose of this project is to introduce Riemannian immersions and submersions. The reader will need to be familiar with the basics of smooth manifold theory and vector bundles at the level of [3]. It will be helpful for the reader if they have some familiarity with Riemannian geometry, including parallel transport, geodesics, and the exponential map. See for example, [1] or [4]. While we will review the basics of Riemannian geometry that we will use, this review is brief and most theorems are not proved.

In section 2, we briefly review immersions, submersions, and embeddings in the context of smooth manifold theory, and we discuss some examples of each. We then review the basic definitions and important theorems in Riemannian geometry, including the curvature tensor symmetries, the Levi-Civita connection of a Riemannian manifold, and the Ricci, scalar, and sectional curvatures. At the end of section 2, we consider a vector bundle $E \to M$, where (M, g) is a Riemannian manifold, and E is equipped with a connection ∇^E and fibre metric h. We show how to naturally construct a Riemannian metric $\widehat{g}$ on E.

In section 3, we define a *Riemannian immersion* and the second fundamental form. We use the second fundamental form to prove the Gauss, Codazzi, and Ricci equations, which relate the curvature of the ambient manifold to the curvature of the submanifold. We then specialize to the *hypersurface* case, where the difference in dimension between the ambient and submanifold is 1. Section 3 ends with an introduction of the Gaussian and mean curvature of Euclidean hypersurfaces and a proof of Gauss's Theorema Egregium.

Finally, we introduce *Riemannian submersions* in section 4. The Fubini-Study metric and the vector bundle construction from section 2 are discussed as examples. The O'Neill tensors are defined for a Riemannian submersion, and the fundamental equations for the O'Neill tensors are proved. We end by computing the O'Neill tensors for the vector bundle construction and computing the curvature of the Fubini-Study metric using the fundamental equations.

2 Preliminaries

We briefly review immersions, submersions, embeddings, connections, and Riemannian manifold theory in order to fix notation and conventions. Most results in this section are not proved.

2.1 Immersions, Submersions, and Embeddings

Let M, N be smooth manifolds and $F\colon M \to N$ be a smooth map. The *rank* of F at $p \in M$ is the rank of the linear map $[F_*]_p\colon T_pM \to T_{F(p)}N$. We say that F has *constant rank* if its rank is the same at all $p \in M$. The most important types of constant rank maps are those with maximum rank:

Definition 2.1 *Let $F\colon M \to N$ be a smooth map.*
a. $F\colon M \to N$ is called an immersion *if $[F_*]_p$ is injective for all $p \in M$.*

b. $F\colon M \to N$ is called an embedding *if F is an injective immersion and F is a homeomorphism onto its image. That is, $U \subseteq M$ is open in M if and only if $F(U)$ is open in $F(M) \subseteq N$, where $F(M)$ is given the subspace topology of N.*

c. $F\colon M \to N$ is called a submersion *if $[F_*]_p$ is surjective for all $p \in M$.*

Notice that $\dim M \leq \dim N$ whenever $F\colon M \to N$ is an immersion, and $\dim M \geq \dim N$ whenever $F\colon M \to N$ is a submersion.

Example 2.2 Suppose $M, M_1, \ldots, M_k$ are smooth manifolds.
a. If $p_i \in M_i$ for each i, then the map $\iota_j\colon M_j \to M_1 \times \cdots \times M_k$ given by

$$\iota_j(q) = (p_1, \ldots, p_{j-1}, q, p_{j+1}, \ldots, p_k)$$

is an embedding.

b. If $\gamma\colon J \to M$ is a smooth curve and M is a smooth manifold, then γ is an immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$. In particular, if V is a smooth vector field on M and p is a regular point, then the integral curve $\Theta^{(p)}$ passing through p at $t = 0$ is an immersion.

c. $f\colon M \to N$ is a local diffeomorphism if and only if f is an immersion and a submersion.

d. The natural map $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ of a nonzero $(n+1)$-tuple onto its equivalence class is a submersion.

e. We claim that the map $x\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$x(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$$

is an immersion of $\mathbb{R}^2$ into $\mathbb{R}^4$ whose image $x(\mathbb{R}^2)$ is a torus $\mathbb{T}^2$. We compute the matrix of the pushforward at (θ, φ).

$$[x_*]_p = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\partial(\cos(\theta))}{\partial\theta} & \frac{\partial(\cos(\theta))}{\partial\varphi} \\ \frac{\partial(\sin(\theta))}{\partial\theta} & \frac{\partial(\sin(\theta))}{\partial\varphi} \\ \frac{\partial(\cos(\varphi))}{\partial\theta} & \frac{\partial(\cos(\varphi))}{\partial\varphi} \\ \frac{\partial(\sin(\varphi))}{\partial\theta} & \frac{\partial(\sin(\varphi))}{\partial\varphi} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\varphi \\ 0 & \cos\varphi \end{pmatrix}$$

Clearly the columns of the above matrix are always linearly independent, so $[x_*]_p$ has full rank. Thus, $x\colon \mathbb{R}^2 \to \mathbb{R}^4$ is an immersion. Moreover, $\|x(\theta,\varphi)\| = 1$ for all θ and φ, so x is an immersion of $\mathbb{R}^2$ into S^3. Notice that the image of x is a torus $\mathbb{T}$.

f. If $E \to M$ is a vector bundle over M, then $\pi\colon E \to M$ is a submersion. In particular, the map $\pi : TM \to M$ given by $\pi(p, X_p) = p$ is a submersion.

g. Suppose $\pi\colon E \to M$ is a rank r vector bundle over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and define the *zero section* as

$$Z = \{\vartheta \in E \colon \vartheta \text{ is the zero element in the vector space } \pi^{-1}(\pi(\vartheta)).\},$$

and give Z the subspace topology of E. We claim that Z can be given a smooth structure such that Z is a smooth manifold and the inclusion $\iota\colon Z \to E$ is an embedding.

Let $\vartheta \in Z$, define $p = \pi(\vartheta)$, and let $\{s_1, \dots, s_k\}$ be an $\mathbb{R}$-local frame for E on a neighborhood U of p. Here $k = 2r$ if $\mathbb{K} = \mathbb{C}$ and $k = r$ if $\mathbb{K} = \mathbb{R}$. By further restricting U, we may assume that U is the domain of a local chart (U, φ) for M. Now define a map $\theta\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^k$ by

$$\theta(y^i s_i(p)) = (\varphi(p), y^1, \dots, y^k)$$

for any $y^i \in \mathbb{R}$. Using the inverse function theorem, it is not difficult to verify that $(\pi^{-1}(U), \theta)$ gives a smooth chart for E.

In terms of this local frame, we have $Z \cap \pi^{-1}(U)$ contains exactly points of the form $(x^1, \dots, x^n, 0, \dots, 0)$ where $(x^1, \dots, x^n)$ is a point in U. Thus, we can construct a local chart $(Z \cap \pi^{-1}(U), \varphi^Z)$ for Z from the chart (U, φ) on M in the obvious way. Doing so for every such chart on M endows Z with a smooth structure such that Z is naturally diffeomorphic to M.

In local coordinates, the inclusion map $\iota\colon Z \to E$ is given by $\iota(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$, which is clearly an immersion. Moreover, since Z was given the subspace topology, ι is a topological embedding. So ι is an embedding, as claimed. //

It is not always easy to check the constant rank condition explicitly. Thankfully, there is a simple local criterion for submersions.

Theorem 2.3 (From [4, Theorem 4.6]) *Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a smooth map. Then π is a submersion if and only if for each $p \in M$ there is an open neighborhood $U \subseteq N$ of $\pi(p)$ and a smooth map $\sigma : U \to M$ such that $\sigma(\pi(p)) = p$ and $\pi \circ \sigma = \mathrm{Id}\colon U \to U$.*

As constant rank maps, immersions and submersions enjoy particularly nice local representations.

Theorem 2.4 (From [1, Remark III6.4]) *Let M, N be smooth m,n-manifolds, respectively, and let $F\colon M \to N$ be a smooth map of constant rank k. If $p \in M$, then there exist charts (U, φ) for M and (V, ψ) for N such that $\varphi(p) = (0, \dots, 0)$, $\psi(F(p)) = (0, \dots, 0)$, and $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is given by*

$$\widehat{F}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0).$$

Moreover, we may assume that $\varphi(U) = C^m_\epsilon(0)$ and $\psi(V) = C^n_\epsilon(0)$, where $C^l_\epsilon(0) \subseteq \mathbb{R}^l$ denotes an open cube of breadth ϵ about 0.

Moreover, every immersion is locally an embedding.

Theorem 2.5 (From [1, Theorem III4.12]) *Let $F : M \to N$ be an immersion. Then each $p \in M$ has a neighborhood U such that $F|_U$ is an embedding of U in M.*

Corollary 2.6 *Every immersion is locally injective.*

We now want to consider the notion of an *embedded submanifold* and its relation to immersions and embeddings. If $k \leq n$ and U is an open subset of $\mathbb{R}^n$, then a *k-slice* of U is a set of the form

$$S = \{(x^1, \dots, x^n) \in U \colon x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

where $c^{k+1}, \dots, c^n \in \mathbb{R}$ are constants. Now suppose that M is a smooth manifold and (U, φ) is a chart on M. A subset $S \subseteq U$ is called a *k-slice of U* if $\varphi(S)$ is a k-slice of $\varphi(U)$.

Definition 2.7 *Let M be a smooth manifold and $N \subseteq M$. Then N is called an* embedded submanifold of dimension k *or an* embedded k-submanifold *if for every $p \in N$ there is a chart (U, φ) with $p \in U$ such that $N \cap U$ is a k-slice for U.*

The following theorem justifies the name *embedded submanifold*:

Theorem 2.8 (From [4, Thm. 8.2]) *Let M be a smooth n-manifold and let $N \subseteq M$ be an embedded k-dimensional submanifold of M. With respect to the subspace topology, N is a smooth manifold of dimension k and the inclusion map $\iota\colon N \to M$ is an embedding of N into M.*

Example 2.9

a. Let $U \subseteq \mathbb{R}^k$ be open and let $F\colon U \to \mathbb{R}^n$ be smooth. Then the graph $\Gamma(F)$ is an embedded k-submanifold of $\mathbb{R}^{k+n}$.

b. The n-sphere S^n is an embedded submanifold of $\mathbb{R}^{n+1}$.

These examples show that our definition of an embedded submanifold agrees with our intuition of what a "submanifold" should look like in Euclidean space. Moreover, there is a natural connection between embeddings and embedded submanifolds.

Theorem 2.10 (From [3, Proposition 5.2]) *Let M, N be smooth m,n-manifolds, respectively, and let $F\colon M \to N$ be an embedding. Then $F(M)$ is an embedded submanifold of N.*

Using slice coordinates, a vector field can be extended from an embedded submanifold to an open set of the ambient manifold.

Theorem 2.11 *Suppose $M \subseteq \widetilde{M}$ is an embedded submanifold and X is a vector field on M. Then there is a vector field $\widetilde{X}$ defined on an open subset of $\widetilde{M}$ containing M such that for all $p \in M$, $[\iota_*]_p X_p = \widetilde{X}_p$.*

Proof. First, note that since $[\iota_*]_p$ is injective, we can identify T_pM with $[\iota_*]_p T_pM$ for all $p \in M$. Let $\{(\varphi_\alpha, \widetilde{U}_\alpha)\}$ be a set of slice charts for $\widetilde{M}$ such that $\cup_{\alpha \in A}(\widetilde{U}_\alpha \cap M)$ covers M.

In the slice coordinates $(\varphi_\alpha, \widetilde{U}_\alpha)$, X is given by

$$X(\varphi_\alpha^{-1}(x^1, \ldots, x^k)) = \sum_{i=1}^{k} f_\alpha^i(\varphi_\alpha^{-1}(x^1, \ldots, x^k, 0, \ldots, 0))\partial_i,$$

where $f_\alpha^i \in C^\infty(\widetilde{U}_\alpha \cap M)$. We canonically extend each f_α^i to $\widetilde{f}_\alpha^i \in C^\infty(\widetilde{U}_\alpha)$ by defining $\widetilde{f}_\alpha^i(\varphi_\alpha^{-1}(x^1, \ldots, x^n)) = f_\alpha^i(\varphi_\alpha^{-1}(x^1, \ldots, x^k, 0, \ldots, 0))$. Define $\widetilde{X}_\alpha = \sum_{i=1}^k \widetilde{f}_\alpha^i \partial_i$. By construction, $\widetilde{X}_\alpha$ is a smooth extension of X to $\Gamma(\widetilde{U}_\alpha)$.

Now let $\{\rho_\alpha\}$ be a partition of unity subordinate to the $\widetilde{U}_\alpha$. Define

$$\widetilde{X} = \sum_\alpha \rho_\alpha \widetilde{X}_\alpha \in \Gamma(\cup_{\alpha \in A} \widetilde{U}_\alpha).$$

By construction, $\widetilde{X}$ is a smooth extension of X to a neighborhood of M in $\widetilde{M}$. ∎

Finally, we can use submersions to construct embedded submanifolds of the domain by taking level sets. This provides a very nice connection between submersions and embeddings. In fact, we do not even need the full strength of a submersion to do this. Instead, we only need to know the map's behavior at points in the level set. To make this precise, suppose $F\colon M \to N$ is a smooth map. Then p is called a *regular point* if $[F_*]_p$ is surjective at p. If $q \in F(M)$, the level set $F^{-1}(q)$ is called a *regular level set* if every $p \in F^{-1}(q)$ is a regular point. Notice in particular that every level set of a submersion is regular.

Theorem 2.12 (From [3, Corollary 5.14]) *Let M, N be smooth manifolds and let $F : M \to N$ be smooth map. Every regular level set of F is an embedded submanifold of M whose codimension is equal to* $\dim N$. *In particular, any level set of a submersion $F : M \to N$ is an embedded submanifold of M.*

2.2 Connections, Curvature, and Torsion

Remark 2.13 If E is a vector bundle over M, we denote by $\Gamma(E)$ the smooth sections of E, and we denote by E_p the fibre of E over $p \in M$. That is, $E_p = \pi^{-1}(p)$.

Definition 2.14 *If $E \to M$ is a vector bundle, then a* connection *on a E is a map*

$$\nabla \colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E) \colon (X, \sigma) \mapsto \nabla_X \sigma$$

such that

a. $\nabla_X \sigma$ is $C^\infty(M)$-linear in X

b. $\nabla_X \sigma$ is $\mathbb{R}$-linear in σ

c. $\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X \sigma$ for all $f \in C^\infty(M)$ and $X \in \Gamma(TM)$.

Example 2.15 Let $(x^1, \dots, x^n)$ be the standard global coordinates for $\mathbb{R}^n$. The Euclidean connection on $T\mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\overline{\nabla}$ and satisfies $\overline{\nabla}_{\partial_i}\partial_j = 0$ for all $1 \le i, j \le n$.

Since $\nabla_X \sigma$ is $C^\infty(M)$-linear in X but only $\mathbb{R}$-linear in σ, $(\nabla_X \sigma)(p)$ depends on σ in a neighborhood of p but only depends on X at the point p. Thus, we often write $\nabla_{X_p} Y$ instead of $(\nabla_X Y)(p)$. In fact, the following proposition shows that $\nabla_X Y$ depends only on the values of Y on a particular curve.

Proposition 2.16 (See [4, Proposition 4.26]) *Let M be a smooth manifold with a connection ∇ on E, and let $X_p \in T_pM$. Suppose $\sigma, \tau \in \Gamma(E)$ and $\gamma \colon I \to M$ is a curve such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. If σ and τ agree on the image of γ, then $\nabla_{X_p}\sigma = \nabla_{X_p}\tau$.*

Often we are interested in smooth sections of a vector bundle over a curve. Indeed, suppose $\gamma \colon I \to M$ is a curve in M and V is a smooth section of E over γ. That is, $V \colon I \to TM$ is a smooth map such that $V_t \in E_{\gamma(t)}$ for all $t \in I$. We denote the space of all such maps by $\Gamma(\gamma^* E)$.

Theorem 2.17 (See [4, Theorem 4.24]) *Let M be a manifold with connection ∇ on E, and $\gamma \colon I \to M$ be a smooth curve. Then ∇ induces a unique $\mathbb{R}$-linear map $D_t \colon \Gamma(\gamma^* E) \to \Gamma(\gamma^* E)$ such that* *a.* $D_t(fV) = f'V + fD_tV$ *for all* $f \in C^\infty(I)$

b. If $V \in \Gamma(\gamma^*E)$ *and for some* $t \in I$*,* V_t *can be extended to a smooth section of* E*, say* $\widetilde{V}$*, in a neighborhood of* $\gamma(t)$*, then*

$$D_tV = \nabla_{\gamma'(t)}\widetilde{V}.$$

A connection on a vector bundle $E \to M$ induces a *curvature tensor* on E. The geometric meaning of such a tensor will become clear in the context of Riemannian geometry.

Definition 2.18 *Suppose* $E \to M$ *is a vector bundle and* ∇ *is a connection on* E*. Then the* curvature *of* ∇ *is the map* $R^\nabla\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ *given by*

$$R^\nabla(X,Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma.$$

Notice that $R^\nabla(X,Y)\sigma$ is skew-symmetric in X and Y. Using Definition 2.14 and properties of the Lie bracket, it is easy to show that R^∇ is $C^\infty(M)$-linear in each of its three arguments.

From now on, we consider the case $E = TM$, and we say ∇ is a connection on M. In a set of local coordinates $(x^1, \ldots, x^n)$ on M, we define the local functions R^l_{ijk} for the curvature tensor R by $R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l$. In the case $E = TM$ there is another important tensor called the *torsion*.

Definition 2.19 *Let* ∇ *be a connection on a smooth manifold* M*. The* torsion *of* ∇ *is the map* $T^\nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ *given by*

$$T^\nabla(X,Y) = \nabla_XY - \nabla_YX - [X,Y].$$

∇ *is said to be* torsion free *if* $T^\nabla = 0$*.*

Notice that T^∇ is skew-symmetric in its arguments. Using Definition 2.14 and properties of the Lie bracket, it is easy to show that T^∇ is $C^\infty(M)$-linear in each argument and is thus a $(2,1)$-tensor.

2.3 Riemannian Manifolds

Definition 2.20 *A* Riemannian manifold *is a pair* (M, g) *where* M *is a smooth manifold, and* g *is a fibre metric on* TM*. That is,* $g \in \Gamma(T^*M \otimes T^*M)$ *is a smooth (2,0)-tensor which is symmetric and positive definite at every point.*

In local coordinates we can write $g = g_{ij} dx^i \otimes dx^j$, where (g_{ij}) is a symmetric, positive definite matrix of smooth functions defined on the domain of the coordinate chart. Since $g_{ij} = g_{ji}$, we write $g = g_{ij} dx^i dx^j$ where $dx^i dx^j = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$ is the symmetrized tensor product. If g is understood, we often denote $g(X, Y)$ by $\langle X, Y \rangle$.

Example 2.21 Let $(x^1, \ldots, x^n)$ be the standard global coordinates on $\mathbb{R}^n$. The Euclidean metric on $\mathbb{R}^n$ is defined by $\overline{g}\,(\partial_i, \partial_j) = \delta_{ij}$.

The notion of an isomorphism in the category of Riemannian manifolds brings us to the definition of an *isometry*.

Definition 2.22 *Suppose* (M, g)*,* (N, h) *are Riemannian manifolds. An* isometry *from* M *to* N *is a map* $F : (M, g) \to (N, h)$ *such that* F *is a diffeomorphism from* M *to* N *and* $F^* h = g$*.*

Since we are largely interested in local properties of Riemannian manifolds, we introduce a more restricted notion of local equivalence through a *local isometry*.

Definition 2.23 *Let* (M, g) *and* (N, h) *be Riemannian manifolds. A* local isometry *from* M *to* N *is a map* $F : (M, g) \to (N, g)$ *such that for each* $p \in M$ *there is an open set* $U \subseteq M$ *containing* p *such that* $F|_U \colon (M|_U, g|_U) \to (N_{F(U)}, h|_{F(U)})$ *is an isometry.*

An important property of immersions is their ability to pull back metrics to metrics.

Proposition 2.24 *Suppose* $i \colon M \to N$ *is an immersion and* g *is a metric on* N*. Then* $i^* g$ *is a metric on* M

Proof. Since g is symmetric and bilinear, so is $i^* g$. It remains to show positive-definiteness. Suppose that $X_p \in T_p M$ such that $(i^* g)_p (X_p, X_p) = 0$. That is, $g_{i(p)}([i_*]_p X_p, [i_*]_p X_p) = 0$. Since g is positive definite, this implies $[i_*]_p X_p = 0$. Since i is an immersion, $X_p = 0$. ■

Example 2.25 Consider Example 2.2e, and suppose $\mathbb{T}^2$ is given the induced metric from $\mathbb{R}^4$. Let us show that the curvature of $\mathbb{T}^2$ is zero. By restricting x to a suitably small neighborhood and restricting the range to the image of x on this neighborhood, we obtain a smooth map $y = x|_U \colon U \to x(U)$ where $x(U)$ is an embedded submanifold of $\mathbb{R}^4$. Since $[x_*]_{(\theta,\varphi)}$ is injective at every point, $[y_*]_{(\theta,\varphi)}$ is invertible on U. By further restricting U we may assume y is a diffeomorphism by the inverse function theorem. Locally, $x = \iota \circ y$, where $\iota : x(U) \to \mathbb{R}^4$ is the inclusion map. Locally, the metric on the torus is given by $\iota^* \overline{g} = (y^{-1})^* (x^* \overline{g})$. Since y is a diffeomorphism, to show that this metric induces zero sectional curvature it suffices to show that $x^* \overline{g}$ is a metric on $\mathbb{R}^2$ which induces zero Riemannian curvature on $\mathbb{R}^2$.

$$x^* \overline{g} = \frac{1}{\sqrt{2}} \left(d(\cos\theta)^2 + d(\sin\theta)^2 + d(\cos\varphi)^2 + d(\sin\varphi)^2 \right)$$

$$= \frac{1}{\sqrt{2}} \left(\sin^2(\theta) d\theta^2 + \cos^2(\theta) d\theta^2 + \sin^2(\varphi) d\varphi^2 + \cos^2(\theta) d\varphi^2 \right)$$
$$= \frac{1}{\sqrt{2}} \left(d\theta^2 + d\varphi^2 \right).$$

The metric is a positive scalar multiple of the Euclidean metric, and thus the Riemann curvature tensor is zero. //

Since a Riemannian manifold (M, g) is equipped with a metric, there are a special class of connections on M.

Definition 2.26 *Suppose (M, g) is a Riemannian manifold. A connection ∇ is said to be* metric compatible *if for all $X, Y, Z \in \Gamma(TM)$, we have*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

One may wonder about the existence of torsion free and metric compatible connections. This leads us to the following important theorem.

Theorem 2.27 (Fundamental Theorem of Riemannian Geometry)
Let (M, g) be a Riemannian manifold. There is a unique connection ∇ on M that is torsion free and metric compatible. We call this connection the Levi-Civita *connection of (M, g).*

From now on, we assume that the connection on a Riemannian manifold is its Levi-Civita connection unless otherwise stated. Moreover, whenever $X, Y \in \Gamma(TM)$, we denote $g(X, Y)$ by $\langle X, Y\rangle$. Likewise, the curvature tensor of a Riemannian manifold is with respect to the Levi-Civita connection. Using the metric and curvature tensor, we construct the Riemann curvature tensor.

Definition 2.28 *Let (M, g) be a Riemannian manifold with curvature R. Then the Riemann curvature tensor is a the smooth section $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$ given by $R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ for all $X, Y, Z, W \in \Gamma(TM)$.*

In local coordinates, we define $R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_l) = \langle R_{ijk}^m \partial_m, \partial_l\rangle = R_{ijk}^m g_{ml}$. The Riemann curvature tensor satisfies several symmetries, which we summarize.

Proposition 2.29 (From [4, Proposition 7.12]) *Let (M, g) be a Riemannian manifold, with Riemannian curvature R. Then*

a. $R(X, Y, Z, W) = -R(Y, X, Z, W)$

b. $R(X, Y, Z, W) = -R(X, Y, W, Z)$

c. $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

d. $R(X,Y,Z,W) = R(Z,W,X,Y)$

Using the Riemann curvature, one can define the sectional curvature. Later, we will see that the sectional curvature measures the curvature of a two-dimensional submanifold of M.

Definition 2.30 *Let (M,g) be a Riemannian manifold, let $p \in M$, and let L_p be a two-dimensional subspace of T_pM. Let $\{X_p, Y_p\}$ be a basis of L_p. Then the* sectional curvature *of L_p is given by*

$$sec_p(L_p) = \frac{R(X_p, Y_p, Y_p, X_p)}{|X_p \wedge Y_p|^2}$$

where we define $|X_p \wedge Y_p|^2 = \langle X_p, X_p \rangle \langle Y_p, Y_p \rangle - \langle X_p, Y_p \rangle^2$.

Note that it is not difficult to prove that $|X_p \wedge Y_p|$ defines a norm on the vector space $\bigwedge^2 T_pM$. We must show that the previous definition does not depend on the choice of basis.

Proposition 2.31 *The definition of sectional curvature does not depend on the choice of basis.*

Proof. Let $p \in M$ and let $\{X_p, Y_p\}$ and $\{Z_p, W_p\}$ be two bases of T_pM. We can write $Z_p = aX_p + bY_p$ and $W_p = cX_p + dY_p$ for some $a, b, c, d \in \mathbb{R}$. Using the symmetries of the Riemann curvature tensor, it is easy to show that

$$R(Z_p, W_p, W_p, Z_p) = (ad - bc)^2 R(X_p, Y_p, Y_p, X_p),$$
$$|Z_p \wedge W_p|^2 = (ad - bc)^2 |X_p \wedge Y_p|^2.$$

■

We say a manifold has *constant sectional curvature* if for all $p \in M$ and all two dimensional subspaces L_p, K_p of T_pM, we have $sec_p(L_p) = sec_p(K_p)$. In this case, the sectional curvature is simply a smooth function on M.

Proposition 2.32 (From [4, Proposition 8.36]) *Suppose (M,g) has constant sectional curvature $C \in C^\infty(M)$. Then the Riemann curvature is given by*

$$R(X,Y,Z,W) = C(\langle X,W \rangle \langle Y,Z \rangle - \langle X,Z \rangle \langle Y,W \rangle) \text{ for all } X,Y,Z,W \in \Gamma(TM)$$

Finally, we review the Ricci curvature. To do so, we first take a linear algebra digression. Suppose V is a vector space with inner product $\langle . \rangle$, and $A \colon V \to V$ is linear. Suppose

$\{e_1, \ldots, e_n\}$ is a basis of V. Let $g_{ij} = \langle e_i, e_j \rangle$ and let g^{ij} be the inverse matrix. Then

$$\langle Ae_i, e_l \rangle g^{il} = \langle A_i^j e_j, e_l \rangle g^{il} = A_i^j g_{jl} g^{il} = A_i^j \delta_j^i = \sum_{i=1}^{n} A_i^i = \operatorname{tr}(A).$$

We can now use this characterization of the trace to obtain a local coordinate expression for the Ricci curvature.

Definition 2.33 *Let (M, g) be a Riemannian manifold. Fix $p \in M$ and $X_p, Y_p \in T_pM$. Consider the linear operator A_p on T_pM defined by $Z_p \mapsto R(Z_p, X_p)Y_p$. Define $Ric_p(X_p, Y_p) = \operatorname{tr}(A_p)$.*

By the preceding discussion, in local coordinates we have $Ric(X, Y) = \langle R(\partial_i, X)Y, \partial_l \rangle g^{il}$, so Ric is indeed a smooth $(2, 0)$-tensor. In local coordinates, we write $R_{jk} = Ric(\partial_j, \partial_k) = R_{ijkl} g^{il}$.

Definition 2.34 *The* scalar curvature *of a Riemannian manifold (M, g) is the smooth function on M defined by $Sc(p) = tr_g(Ric_p)$.*

Notice that in local coordinates, $Sc = R_{jk} g^{jk}$, so Sc is indeed smooth.

Next, we review the *gradient* of a smooth function on a Riemannian manifold M. Define the *flat* operator $\flat \colon \Gamma(TM) \to \Gamma(T^*M)$ by

$$\flat(X)(Y) = \langle X, Y \rangle$$

for all $X, Y \in \Gamma(TM)$. The flat operator is clearly tensorial, so $\flat \in \Gamma(T^*M \otimes T^*M)$. For every $p \in M$, $\flat_p \colon T_pM \to T_p^*M$ is injective since the metric is positive definite. Thus, $\flat_p$ is an injective linear map between vector spaces of the same dimension, and is thus an isomorphism. Let $\sharp_p \colon T_p^*M \to T_pM$ denote the inverse map.

Now consider a local coordinate frame $\{\partial_1, \ldots, \partial_n\}$ of TM. Then we can write $\flat(\partial_i) = A_{ik} dx^k$ for local smooth functions A_{ik} on M. Then

$$g_{ij} = \langle \partial_i, \partial_j \rangle = \flat(\partial_i)(\partial_j) = A_{ik} dx^k(\partial_j) = A_{ij},$$

so $[g_{ij}]$ is the matrix of $\flat$ with respect to the local bases $\{\partial_1, \ldots, \partial_n\}$ of TM and $\{dx^1, \ldots, dx^n\}$ of T^*M. Thus, $[g^{ij}(p)] = [g_{ij}(p)]^{-1}$ is the matrix for $\sharp_p$. Since the g^{ij} are smooth functions and both $\{\partial_1, \ldots, \partial_n\}$ and $\{dx^1, \ldots, dx^n\}$ are smooth local frames for TM and T^*M, respectively, $\sharp_p$ extends naturally to a tensorial map $\sharp \colon \Gamma(T^*M) \to \Gamma(TM)$. Now if $f \in C^\infty(M)$, define the *gradient* of f to be

$$\boldsymbol{\nabla}(f) = \sharp(df).$$

In local coordinates, $\boldsymbol{\nabla}(f) = \sharp(\frac{\partial f}{\partial x^i} dx^i) = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$. Now let Y be any other smooth vector field on M, say $Y = Y^i \frac{\partial}{\partial x^i}$. Then

$$
\begin{aligned}
\langle \boldsymbol{\nabla} f, Y\rangle &= \left\langle g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}, Y^k\frac{\partial}{\partial x^k}\right\rangle \\
&= g^{ij}Y^k\frac{\partial f}{\partial x^i}\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle \\
&= g^{ij}g_{jk}Y^k\frac{\partial f}{\partial x^i} \\
&= Y^k\frac{\partial f}{\partial x^k} \\
&= Yf,
\end{aligned}
$$

which is exactly how one would expect the gradient to behave.

Finally, we define the Hessian and state its relation to the gradient.

Definition 2.35 *Let (M, g) be a Riemannian manifold, and let $f \in C^\infty(M)$. Then the Hessian of F,* $\mathrm{Hess}(f)$*, is defined for all vector fields $X, Y \in \Gamma(TM)$ by* $\mathrm{Hess}(f)(X, Y) = X(Yf) - (\nabla_X Y)(f)$.

Proposition 2.36 *Let (M, g) be a Riemannian manifold, let $f \in C^\infty(M)$. Then* $\mathrm{Hess}(f)$ *is symmetric and* $\mathrm{Hess}(f)(X, Y) = \langle \nabla_X(\boldsymbol{\nabla} f), Y\rangle$ *for all $X, Y \in \Gamma(TM)$.*

2.4 Induced Riemannian Metric on a Vector Bundle

Let (M, g) be a Riemannian manifold, suppose $\pi : E \to M$ is a rank r vector bundle over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and suppose h is a fibre metric on E and ∇^E is a connection on E. In this section we want to construct a Riemannian metric $\widehat{g}$ on E in terms of g, h, and ∇^E. We will analyze this construction as a *Riemannian immersion* (where M is the zero section of E) and as a *Riemannian submersion* in later sections.

Fix $p \in M$, and let $\{s_1, \ldots, s_k\}$ be an $\mathbb{R}$-local frame for E on a neighborhood U of p. Here $k = 2r$ if $\mathbb{K} = \mathbb{C}$ and $k = r$ if $\mathbb{K} = \mathbb{R}$. Fix a chart (U, φ) for M about p. As in Example 2.2g, we can define local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^k)$ for E, where $(x^1, \ldots, x^n, y^1, \ldots, y^k)$ corresponds to the point $y^i s_i(\varphi^{-1}(x^1, \ldots, x^n))$ of E. We refer to the (x^i) coordinates as the *base coordinates*, and we refer to the (y^j) coordinates as the *fibre coordinates*. Notice that the (y^j) coordinates are defined once a local frame is chosen. Given this local frame, we let Γ^k_{ij} be the Christoffel symbols for this local frame. That is, $\Gamma^l_{ij} s_l = \nabla^E_{\frac{\partial}{\partial x^i}} s_j$.

Definition 2.37 *Suppose $p\colon N \to L$ is any submersion of smooth manifolds, and let $q \in N$. Then the* vertical subspace *of T_qN is the subspace $V_q = \ker [p_*]_q$.*

Lemma 2.38 *Consider the submersion $\pi\colon E \to M$ discussed above, and fix $\vartheta \in E$. In terms of the local coordinates constructed above, the set $\{\frac{\partial}{\partial y^1}\big|_\vartheta, \dots, \frac{\partial}{\partial y^k}\big|_\vartheta\}$ is a basis for V_ϑ.*

Proof. This follows immediately from the fact that π is given in local coordinates by

$$\pi(x^1, \dots, x^n, y^1, \dots, y^k) = (x^1, \dots, x^n).$$

■

Let $\gamma\colon [0,1] \to M$ be a smooth curve with $\gamma(0) = p$. Given any $\vartheta \in E_p = \pi^{-1}(p)$, the connection allows us to define the parallel transport $\widehat{\gamma}_\vartheta\colon [0,1] \to E$ such that $\widehat{\gamma}_\vartheta(0) = \vartheta$ and $\widehat{\gamma}_\vartheta(t) \in E_{\gamma(t)}$ for all $t \in [0,1]$.

We call $\widehat{\gamma}_\vartheta$ the *horizontal lift* of γ with initial point ϑ. It depends on γ, ϑ, and the connection ∇ on E. It is called a *lift* because $\pi \circ \widehat{\gamma}_\vartheta = \gamma$. A smooth curve on E is called a horizontal curve if it is the horizontal lift of a smooth curve on M as described above. A tangent vector in $T_\vartheta E$ is called *horizontal* if it is the velocity vector at ϑ of a horizontal curve passing through ϑ.

Proposition 2.39 *Suppose $\gamma'(0) = W^i \frac{\partial}{\partial x^i}\big|_p$ and that in local coordinates $\vartheta = (\varphi(p), V^1, \dots, V^k)$. Then*

$$(\widehat{\gamma}_\vartheta)'(0) = W^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l},$$

where the vectors are evaluated at ϑ and the Christoffel symbols are evaluated at p.

Proof. In the local coordinates $(x^1, \dots, x^n, y^1, \dots, y^k)$ for E, write

$$\widehat{\gamma}_\vartheta(t) = (x^1(t), \dots, x^n(t), y^1(t), \dots, y^k(t)).$$

Since $\widehat{\gamma}_\vartheta(t) \in E_{\gamma(t)}$ for all $t \in [0,1]$, the local coordinate expression for γ is

$$\gamma(t) = (x^1(t), \dots, x^n(t)).$$

Thus, we have

$$W^i \frac{\partial}{\partial x^i}\bigg|_p = \gamma'(0) = \frac{\mathrm{d}x^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial}{\partial x^i}\bigg|_p,$$

so $W^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}\big|_{t=0}$ for each i.

Using the local frame $\{s_1, \dots, s_r\}$, we can write $\widehat{\gamma}_\vartheta = c^j s_j(\gamma(t)) = (\gamma(t), c^1(t), \dots, c^k(t))$, so in fact $y^j(t) = c^j(t)$ for each j. From the parallel transport equation we obtain

$$\frac{\mathrm{d}y^l}{\mathrm{d}t} + \Gamma^l_{ij}(\gamma(t))\frac{\mathrm{d}x^i}{\mathrm{d}t} y^j(t) = 0.$$

Evaluating at $t = 0$ gives

$$\left.\frac{\partial y^l}{\partial t}\right|_{t=0} = -\Gamma^l_{ij} W^i V^j,$$

where the Christoffel symbol is assumed to be evaluated at $\gamma(0) = p$. Putting it all together, we have

$$\begin{aligned}
(\widehat{\gamma}_\vartheta)'(0) &= \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial x^i}\right|_\vartheta + \left.\frac{\mathrm{d}y^l}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial y^l}\right|_\vartheta \\
&= W^i \left.\frac{\partial}{\partial x^i}\right|_\vartheta - \Gamma^l_{ij} W^i V^j \left.\frac{\partial}{\partial y^l}\right|_\vartheta
\end{aligned}$$

■

Proposition 2.40 *Let H_ϑ be the set of horizontal tangent vectors at $\vartheta \in E_p$. Then H_ϑ is a subspace of $T_\vartheta E$, and there is a canonical isomorphism $h_\vartheta : T_p M \to T_\vartheta E$ that takes $W_p \in T_p M$ to its horizontal lift $h_\vartheta(W_p) \in H_\vartheta$.*

Proof. By Proposition 2.39, all horizontal tangent vectors at $\vartheta \in E_p$ are of the form

$$W^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l}.$$

Take any two horizontal vectors at ϑ, say

$$\begin{aligned}
&W^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l}, \\
&Z^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} Z^i V^j \frac{\partial}{\partial y^l},
\end{aligned}$$

and let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
&a\left(W^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l}\right) + b\left(Z^i \frac{\partial}{\partial x^i} - \Gamma^l_{ij} Z^i V^j \frac{\partial}{\partial y^l}\right) \\
&= (aW^i + bZ^i)\frac{\partial}{\partial x^i} - \Gamma^l_{ij}(aW^i + bZ^i) V^j \frac{\partial}{\partial y^l}
\end{aligned}$$

is horizontal, so H_ϑ is a subspace. For any $W_p = W^i \frac{\partial}{\partial x^i}\big|_p \in T_pM$ let $h_\vartheta(W_p)$ be its horizontal lift. Namely,

$$h_\vartheta(W_p) = W^i \frac{\partial}{\partial x^i}\bigg|_p - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l}\bigg|_p.$$

This map is clearly linear and invertible and thus an isomorphism. ■

As claimed, this horizontal subspace is complementary to the vertical subspace of T_pM.

Proposition 2.41 $T_\vartheta E = V_\vartheta \oplus H_\vartheta$.

Proof. First, let $W_\vartheta \in T_\vartheta E$ be arbitrary, say $W_\vartheta = a^i \frac{\partial}{\partial x^i}\big|_\vartheta + b^i \frac{\partial}{\partial y^i}\big|_\vartheta$. Then

$$\begin{aligned}
W_\vartheta &= a^i \frac{\partial}{\partial x^i}\bigg|_\vartheta + b_i \frac{\partial}{\partial y^i}\bigg|_\vartheta \\
&= \left(a^i \frac{\partial}{\partial x^i}\bigg|_\vartheta - \Gamma^l_{ij} a^i V^j \frac{\partial}{\partial y^l}\bigg|_\vartheta \right) \\
&\qquad - \left(\Gamma^l_{ij} a^i V^j \frac{\partial}{\partial y^j}\bigg|_\vartheta + b^i \frac{\partial}{\partial y^i}\bigg|_\vartheta \right) \\
&\in H_\vartheta + V_\vartheta
\end{aligned}$$

So $T_\vartheta = V_\vartheta + H_\vartheta$. Now suppose $W_\vartheta \in V_\vartheta \cap H_\vartheta$. Since $W_\vartheta \in H_\vartheta$, W_ϑ has the form

$$W_\vartheta = W^i \frac{\partial}{\partial x^i}\bigg|_\vartheta - \Gamma^l_{ij} W^i V^j \frac{\partial}{\partial y^l}\bigg|_\vartheta.$$

But since $W_\vartheta \in V_\vartheta$, we must have $W_i = 0$ for all i. But then $W_\vartheta = 0$. Hence $V_\vartheta \cap H_\vartheta = \{0\}$, which completes the proof. ■

Let $A_\vartheta, B_\vartheta \in T_\vartheta E$ be two tangent vectors to E at the point $\vartheta \in E_p \subset E$. Then we can find smooth curves v and w on E with $v(0) = w(0) = \vartheta$ and $v'(0) = A_\vartheta$ and $w'(0) = B_\vartheta$. Notice that v and w project down to smooth curves on M. Explicitly, define

$$\begin{aligned}
\gamma(t) &= (\pi \circ v)(t), \\
\tau(s) &= (\pi \circ w)(s).
\end{aligned}$$

We can regard v and w as smooth sections of E *over* the curves γ and τ, respectively. Recall that $(D_t v)(t)$ and $(D_s w)(s)$ are the covariant derivatives *along* γ and η, respectively, so in particular $(D_t v)(t) \in T_{\gamma(t)}M$ and $(D_s w)(s) \in T_{\eta(s)}M$.

Define an inner product $\widehat{g}_\vartheta$ on $T_\vartheta E$ by

$$\widehat{g}_\vartheta(A_\vartheta, B_\vartheta) = g_p([\pi_*]_\vartheta A_\vartheta, [\pi_*]_\vartheta B_\vartheta) + h_p((D_t v)(0), (D_s v)(0)),$$

where $\pi : E \to M$ is the natural projection. It is clear that $\widehat{g}_\vartheta$ is a symmetric bilinear form on $T_\vartheta E$.

Proposition 2.42 *$\widehat{g}_\vartheta$ is an inner product on $T_\vartheta E$.*

Proof. It suffices to show that $\widehat{g}_\vartheta$ is positive definite. Suppose that $\widehat{g}_\vartheta(A_\vartheta, A_\vartheta) = 0$. We must show that $A_\vartheta = 0$. Let (U, φ) be a chart on M containing p, and let $(\pi^{-1}(U), \theta)$ be the induced chart on E. In local coordinates, we can write

$$v(t) = (x^1(t), \dots, x^n(t), y^1(t), \dots, y^r(t)).$$

Now we have

$$A_\vartheta = v'(0) = \left.\frac{\partial x^i}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial x^i}\right|_\vartheta + \left.\frac{\partial y^k}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial y^k}\right|_\vartheta .$$

In local coordinates, it is easy to show that the covariant derivative of v along γ at $t = 0$ is given by

$$(D_t v)(0) = \left(\left.\frac{\mathrm{d}y^k}{\mathrm{d}t}\right|_{t=0} + y^j(0) \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} \Gamma^k_{ij}(p) \right) s_k(p). \tag{1}$$

Now since $0 = \widehat{g}_\vartheta(A_\vartheta, A_\vartheta) = g_p([\pi_*]_\vartheta A_\vartheta, [\pi_*]_\vartheta A_\vartheta) + h_p((D_t v)(0), (D_t v)(0))$ and g_p and h_p are nonnegative on $T_p E$, we must have both terms equal to zero. Since g_p is positive definite on E_p, we must have $[\pi_*]_\vartheta A_\vartheta = 0$, which implies

$$\begin{aligned} 0 = [\pi_*]_\vartheta A_\vartheta &= [\pi_*]_\vartheta \left(\left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial x^i}\right|_\vartheta + \left.\frac{\mathrm{d}y^k}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial y^k}\right|_\vartheta \right) \\ &= \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial x^i}\right|_p , \end{aligned} \tag{2}$$

Thus, we conclude that $\left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_p = 0$ for all i. Again, since $h_p((D_t v)(0), (D_t v)(0)) = 0$ and h_p is positive definite, (1) and (2) give

$$\begin{aligned} 0 = (D_t v)(0) &= \left(\left.\frac{\mathrm{d}y^k}{\mathrm{d}t}\right|_{t=0} + y^j(0) \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{t=0} \Gamma^k_{ij}(p) \right) s_k(p) \\ &= \left.\frac{\mathrm{d}y^k}{\mathrm{d}t}\right|_{t=0} s_k(p) \end{aligned}$$

which implies $\left.\frac{\mathrm{d}y^k}{\mathrm{d}t}\right|_{t=0} = 0$ for all k. All together, $A_\vartheta = 0$, so $\widehat{g}_\vartheta$ is positive definite. ■

Thus, we have shown that $\widehat{g}_\vartheta$ is an inner product on $T_\vartheta E$ for each $\vartheta \in E$. Moreover, the next proposition shows that $\widehat{g}$ is *smoothly varying* over E.

Proposition 2.43 *In local coordinates, we have*

$$\widehat{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} + \Gamma^k_{ip}\Gamma^l_{jq}y^p y^q h_{kl},$$
$$\widehat{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \Gamma^k_{ip}y^p h_{kj},$$
$$\widehat{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = h_{ij},$$

so $\widehat{g}$ *is a* smooth *tensor, and thus a smooth Riemannian metric on* E*.*

Proof. Define a smooth curve on E in local coordinates by $v(t) = (x^1(t), \ldots, x^n(t), y^1(t), \ldots, y^r(t))$ where $v(0) = \vartheta \in E$, and the coordinate functions satisfy $\frac{\mathrm{d}x^k}{\mathrm{d}t} = \delta_{ik}$ and $\frac{\mathrm{d}y^j}{\mathrm{d}t} = 0$. Now let $A_\vartheta = v'(0) = \frac{\partial}{\partial x^i}\big|_\vartheta$. So (1) gives

$$\begin{aligned}(D_t v)(0) &= \left(\frac{\mathrm{d}y^k}{\mathrm{d}t}\bigg|_{t=0} + y^a(0)\frac{\mathrm{d}x^i}{\mathrm{d}t}\bigg|_{t=0}\Gamma^k_{ia}(p)\right)s_k(p) \\ &= y^a(0)\Gamma^k_{ia}(p)s_k(p). \end{aligned} \tag{3}$$

Moreover, (2) gives

$$\begin{aligned}[\pi_*]_\vartheta A_\vartheta &= [\pi_*]_\vartheta\left(\frac{\mathrm{d}x^l}{\mathrm{d}t}\bigg|_{t=0}\frac{\partial}{\partial x^l}\bigg|_\vartheta + \frac{\mathrm{d}y^k}{\mathrm{d}t}\bigg|_{t=0}\frac{\partial}{\partial y^k}\bigg|_\vartheta\right) \\ &= \frac{\partial}{\partial x^i}\bigg|_p. \end{aligned} \tag{4}$$

From (3) and (4) we obtain

$$\begin{aligned}\widehat{g}_v\left(\frac{\partial}{\partial x^i}\bigg|_\vartheta, \frac{\partial}{\partial x^j}\bigg|_\vartheta\right) &= g_p\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p\right) + h_p\left(y^a(0)\Gamma^k_{ia}(p)s_k(p), y^b(0)\Gamma^l_{jb}(p)s_l(p)\right) \\ &= g_{ij}(p) + \Gamma^k_{ia}(p)\Gamma^l_{jb}(p)y^a(0)y^b(0)h_{kl}(p).\end{aligned}$$

As a local function on E, we have

$$\widehat{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} + \Gamma^k_{ia}\Gamma^l_{jb}y^a y^b h_{kl},$$

as claimed.

Now $A_\vartheta = \frac{\partial}{\partial y^j}\big|_\vartheta$ corresponds to $\frac{\mathrm{d}x^i}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}y^k}{\mathrm{d}t} = \delta_{jk}$. Again, (1) gives

$$(D_t v)(0) = \left(\frac{\mathrm{d}y^k}{\mathrm{d}t}\bigg|_{t=0} + y^a(0)\frac{\mathrm{d}x^i}{\mathrm{d}t}\bigg|_{t=0}\Gamma^k_{ia}(p)\right)s_k(p) = s_j(p), \tag{5}$$

so

$$\widehat{g}_\vartheta\left(\left.\frac{\partial}{\partial x^i}\right|_\vartheta,\left.\frac{\partial}{\partial y^j}\right|_\vartheta\right)=g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p,0|_p\right)+h_p(y^a(0)\Gamma^k_{ia}(p)s_k(p),s_j(p))$$
$$=\Gamma^k_{ia}(p)y^a(0)h_{jk}(p).$$

As a local function on M, we have

$$\widehat{g}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^j}\right)=\Gamma^k_{ia}y^a h_{jk},$$

as claimed. Finally, we have

$$\widehat{g}_\vartheta\left(\left.\frac{\partial}{\partial y^i}\right|_\vartheta,\left.\frac{\partial}{\partial y^j}\right|_\vartheta\right)=g_p\left(0|_p,0|_p\right)+h_p(s_i(p),s_j(p))=h_{ij}(p).$$

As a local function on M, this gives $\widehat{g}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)=h_{ij}$, as claimed. Since $\widehat{g}$ is tensorial and bilinear, we have $\widehat{g}(X,Y)$ is smooth for all smooth vector fields $X,Y\in\Gamma(TE)$, which shows that $\widehat{g}$ is a smooth Riemannian metric on E. ∎

Thus, we have constructed a smooth Riemannian metric $\widehat{g}$ on E. This metric has the property that the complementary horizontal subspace of $T_\vartheta E$ we have defined is the $\widehat{g}$-perp space of V_ϑ.

Proposition 2.44 *With respect to the metric $\widetilde{g}$ on E, the* orthogonal complement *of V_ϑ is precisely H_ϑ.*

Proof. We previously saw that that H_ϑ was an n-dimensional subspace of $T_\vartheta E$ spanned by vectors of the form

$$h_k=\left.\frac{\partial}{\partial x^k}\right|_\vartheta-\Gamma^i_{kj}V^j\left.\frac{\partial}{\partial y^i}\right|_\vartheta.$$

Furthermore, we found that the vertical vectors in V_ϑ are spanned by vectors of the form $v_l=\left.\frac{\partial}{\partial y^l}\right|_\vartheta$. Writing $\vartheta=(\varphi(p),V^1,\dots,V^k)$ in local coordinates, we have

$$\widehat{g}(h_k,v_l)=\widehat{g}\left(\frac{\partial}{\partial x^k}-\Gamma^i_{kj}V^j\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^l}\right)$$
$$=\widehat{g}\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial y^l}\right)-\Gamma^i_{kj}V^j\widehat{g}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^l}\right)$$
$$=V^a\Gamma^m_{ka}h_{ml}-\Gamma^i_{kj}V^jh_{il}=0.$$

So H_ϑ is a subspace of $V_\vartheta^{\perp_{\widehat{g}}}$. But H_ϑ and $V_\vartheta^{\perp_{\widehat{g}}}$ both have dimension n, so they are equal. ∎

3 Riemannian Immersions

Assume $(\widetilde{M}, \widetilde{g})$ (dimension n) and (M, g) (dimension k) are Riemannian manifolds with Levi-Civita connections $\widetilde{\nabla}$ and ∇, respectively, and that $i\colon M \to \widetilde{M}$ is an injective immersion with $i^*\widetilde{g} = g$. Then i is called a *Riemannian immersion.*

Fix $p \in M$. By Theorem 2.5 and Theorem 2.10, there is an open neighborhood U of $p \in M$ such that $i|_U$ is an embedding and $i(U)$ as an embedded submanifold of $\widetilde{M}$. Since we will only study local properties of Riemannian immersions, we are thus permitted to assume that M is an embedded submanifold of $\widetilde{M}$. Now let $\iota\colon M \to \widetilde{M}$ be the inclusion map $\iota(p) = p$. We have $[\iota_*]_p T_pM \subseteq T_p\widetilde{M}$, and since $[\iota_*]_p$ is injective, we can identify T_pM with $[\iota_*]_p T_pM$ and thus consider $T_pM \subseteq T_p\widetilde{M}$ for any $p \in M$. With this identification, we first discuss *local adapted frames.*

Proposition 3.1 *For every $p \in M$, there exists a local orthonormal frame $\{E_1, \ldots, E_n\}$ for $T\widetilde{M}$ in a neighborhood $U \subseteq \widetilde{M}$ of p such that for every $q \in M \cap U$, $\{E_1|_q, \ldots, E_k|_q\}$ is a basis for T_qM.*

Proof. Since M is locally an embedded submanifold of $\widetilde{M}$, let $(x^1, \ldots, x^n)$ be slice coordinates for $\widetilde{M}$ on a neighborhood $U \subseteq \widetilde{M}$. Then $\{\partial_1, \ldots, \partial_n\}$ is a local frame for TM, and for each $q \in M \cap U$, $\{\partial_1|_q, \ldots, \partial_k|_q\}$ is a basis for T_qM. Performing Gram-Schmidt on this local frame gives a smooth local orthonormal frame $\{E_1, \ldots, E_n\}$ on U such that $\operatorname{span}\{E_1|_q, \ldots, E_k|_q\} = \operatorname{span}\{\partial_1|_q, \ldots, \partial_k|_q\} = T_qM$ for each $q \in M \cap U$. ∎

Fix $p \in M$, and define $N_pM = (T_pM)^\perp$. Then $T_p\widetilde{M} = T_pM \oplus N_pM$. This allows us to decompose, over the points of M, the tangent bundle to $\widetilde{M}$ into the part tangent to M and the part normal to M.

Example 3.2 Consider again the immersion in Example 2.2e. We claim that the vectors

$$\begin{aligned} e_1 &= (-\sin\theta, \cos\theta, 0, 0), \\ e_2 &= (0, 0, -\sin\varphi, \cos\varphi) \end{aligned}$$

form an orthonormal basis of the tangent space, and the vectors

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \\ n_2 &= \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi) \end{aligned}$$

form an orthonormal basis of the normal space. From the matrix computation for $[x_*]_p$ in Example 2.2e, it is clear that $e_1 = [x_*]_{(\theta,\varphi)}\partial_\theta$ and $e_2 = [x_*]_{(\theta,\varphi)}\partial_\varphi$, so e_1 and e_2 are in the tangent space. It remains to show that $\{e_1, e_2, n_1, n_2\}$ is an orthonormal set of vectors. Indeed, we have

$$\begin{aligned}
\langle e_1, e_1 \rangle &= \sin^2\theta + \cos^2\theta = 1,\\
\langle e_1, e_2 \rangle &= 0,\\
\langle e_1, n_1 \rangle &= \frac{1}{\sqrt{2}}\left(-\sin\theta\cos\theta + \cos\theta\sin\theta\right) = 0,\\
\langle e_1, n_2 \rangle &= \frac{1}{\sqrt{2}}\left(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi\right) = 0,\\
\langle e_2, e_2 \rangle &= \sin^2\varphi + \cos^2\varphi = 1,\\
\langle e_2, n_1 \rangle &= \frac{1}{\sqrt{2}}\left(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi\right) = 0,\\
\langle e_2, n_2 \rangle &= \frac{1}{\sqrt{2}}\left(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi\right) = 0,\\
\langle n_1, n_1 \rangle &= \frac{1}{2}\left(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = 1,\\
\langle n_1, n_2 \rangle &= \frac{1}{2}\left(-\cos^2\theta - \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = \frac{1}{2}(-1+1) = 0,\\
\langle n_2, n_2 \rangle &= \frac{1}{2}\left(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi\right) = 1,
\end{aligned}$$

which completes the proof. //

While we are able to decompose $T_p\widetilde{M}$ at each $p \in M$, we would like to see if such a decomposition can be done in a smoothly varying way. Define $NM = \bigsqcup_{p\in M} N_pM$. To give NM the structure of a vector bundle we use a construction lemma from [3].

Lemma 3.3 (From [3, Lemma 10.6]) *Suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k. Let $E = \bigsqcup_{p\in M} E_p$, and let $\pi \colon E \to M$ be the map that takes each element of E_p to p. Suppose furthermore that we are given the following data:*

a. an open cover $\{U_\alpha\}_{\alpha\in A}$ of M,

b. for each $\alpha \in A$, a bijective map $\Phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$,

c. for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Then E has a unique topology and smooth structure making it into a smooth manifold and a smooth rank k vector bundle over M with π as projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

We now apply the construction lemma to the restriction $T\widetilde{M}|_M$ of $T\widetilde{M}$ to M.

Proposition 3.4 *$T\widetilde{M}|_M$ and NM can be given vector bundle structures such that $T\widetilde{M}|_M = TM \oplus NM$.*

Proof. Let $\{U_\alpha\}$ be an open cover of M such that there is an adapted orthonormal frame $\{E_{\alpha,1}, \ldots, E_{\alpha,n}\}$ of $T\widetilde{M}$ in an $\widetilde{M}$-open neighborhood $\widetilde{U}_\alpha \supseteq U_\alpha$. If $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \varnothing$, write $E_{\alpha,i} = (A_{\alpha\beta})_i^j E_{\beta,j}$. By construction of adapted orthonormal frames, we have

$$(A_{\alpha\beta})|_M = \begin{pmatrix} B_{\alpha\beta} & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix}$$

where $B_{\alpha\beta}, C_{\alpha\beta}$ are matrices of smooth functions on $U_\alpha \cap U_\beta$ of sizes $k \times k$ and $(n-k)\times(n-k)$ respectively.

Let $\pi\colon T\widetilde{M} \to M$ be the natural projection. For NM, define bijective functions $\Phi_\alpha\colon NM \cap \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n-k}$ to be

$$\Phi_\alpha\left(\sum_{i=k+1}^{n} a_i E_{\alpha,i}(p)\right) = (p, (a_{k+1}, \ldots, a_n)^T).$$

Then

$$\begin{aligned}(\Phi_\beta \circ \Phi_\alpha^{-1})(p, (a_{k+1}, \ldots, a_n)^T) = \Phi_\beta\left(\sum_{i=k+1}^{n} a_i E_{\alpha,i}(p)\right) &= \Phi_\beta\left(\sum_{i,j=k+1}^{n} ((A_{\alpha\beta}(p))_i^j a_i) E_{\beta,j}(p)\right) \\ &= (p, C_{\alpha\beta}^T(p)(a_{k+1}, \ldots, a_n)^T).\end{aligned}$$

By the previous lemma, NM is a vector bundle such that $\pi|_{NM}$ is the projection and $\{(\Phi_\alpha, U_\alpha)\}$ are local trivializations.

Similarly, $\Psi_\alpha\colon TM \cap \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ given by $\Psi_\alpha\left(\sum_{i=1}^k a_i E_{\alpha,i}(p)\right) = (p, (a_1, \ldots, a_k)^T)$ are local trivializations for TM, with transition matrices $B_{\alpha\beta}^T$.

Finally, $T\widetilde{M}$ already has the structure of a vector bundle, and $T\widetilde{M}$ has local trivializations $\Theta_\alpha\colon \pi^{-1}(\widetilde{U}_\alpha) \to \widetilde{U}_\alpha \times \mathbb{R}^n$ given by

$$\Theta_\alpha\left(\sum_{i=1}^{n} a_i E_{\alpha,i}(p)\right) = (p, (a_1, \ldots, a_k)^T).$$

A similar calculation as done with NM shows that the smooth transition matrices are $A_{\alpha\beta}^T$. Replacing each $\widetilde{U}_\alpha$ with U_α in the definition of the trivialization maps and transition matrices and then applying the previous lemma, we see that $T\widetilde{M}|_M$ is a vector bundle. Moreover, the transition matrices for $T\widetilde{M}|_M$ are the $A_{\alpha\beta}^T|_M = B_{\alpha\beta}^T \oplus C_{\alpha\beta}^T$, which exactly says that $T\widetilde{M}|_M = TM \oplus NM$. ■

We can now define the tangential and normal projections.

Definition 3.5 *Define the natural projections $\pi^\top \colon T\widetilde{M}|_M \to TM$, $\pi^\perp \colon T\widetilde{M}|_M \to NM$ to be the unique functions satisfying*

$$\begin{aligned}\pi^\top(a_1E_1 + \cdots + a_nE_n) &= a_1E_1 + \cdots + a_kE_k,\\ \pi^\perp(a_1E_1 + \cdots + a_nE_n) &= a_{k+1}E_{k+1} + \cdots + a_nE_n\end{aligned}$$

for any adapted local orthonormal frame $\{E_1, \ldots, E_n\}$ and smooth functions $a_i \in C^\infty(M)$

Proposition 3.6 *$\pi^\top$ and $\pi^\perp$ are smooth maps.*

Proof. We show that $\pi^\top$ is smooth. The argument for $\pi^\perp$ is similar. Let $p \in M$, and fix a chart $\varphi_\alpha \colon V_\alpha \to M$ such that $V_\alpha \subseteq U_\alpha$ where U_α is as in Proposition 3.4. Then $\theta_\alpha = (\varphi_\alpha \times \mathrm{Id}) \circ \Theta_\alpha|_{\pi^{-1}(V_\alpha)}$ is a local chart for $T\widetilde{M}|_M$, and $\psi_\alpha = (\varphi_\alpha \times \mathrm{Id}) \circ \Psi_\alpha|_{\pi^{-1}(V_\alpha)}$ is a local chart for TM, and in these coordinates $\pi^\top$ is the map $(x^1, \ldots, x^k, y^1, \ldots, y^n) \mapsto (x^1, \ldots, x^k, y^1, \ldots, y^k)$, which is clearly smooth. ■

By their definition, $\pi^\top$ and $\pi^\perp$ clearly induce $C^\infty(M)$-linear maps $\pi^\top \colon \Gamma(T\widetilde{M}|_M) \to \Gamma(TM)$ and $\pi^\perp \colon \Gamma(T\widetilde{M}|_M) \to \Gamma(NM)$. This means that any smooth vector field $X \in \Gamma(T\widetilde{M}|_M)$ can be decomposed as

$$X = X^\top + X^\perp$$

where $X^\top \in \Gamma(TM)$ is a tangent vector field over M and $X^\perp \in \Gamma(NM)$ is a normal vector field over M. Using this decomposition, we can define the second fundamental form. First we prove a lemma.

Lemma 3.7 *Suppose $X, Y \in \Gamma(T\widetilde{M}|_M)$. Let $\widetilde{X}_1, \widetilde{X}_2$ be extensions of X and let $\widetilde{Y}_1, \widetilde{Y}_2$ be extensions of Y to an $\widetilde{M}$-open subset of M (this is possible by Theorem 2.11). Then for all $p \in M$,*

$$(\widetilde{\nabla}_{\widetilde{X}_1}\widetilde{Y}_1)(p) = (\widetilde{\nabla}_{\widetilde{X}_2}\widetilde{Y}_2)(p)$$

Proof. By Proposition 2.16, $(\widetilde{\nabla}_{\widetilde{X}_i}\widetilde{Y}_i)(p)$ depends only on $(\widetilde{X}_i)_p = (X_i)_p$ and the values of $\widetilde{Y}_i$ along the image of any curve $\gamma \colon I \to M$ such that $\gamma'(0) = X_p$. But $\widetilde{Y}_i = Y$ along the image of any such curve. ■

The previous lemma shows that for $X, Y \in \Gamma(T\widetilde{M}|_M)$, we can define

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})|_M \in \Gamma(T\widetilde{M}|_M),$$

where $\widetilde{X}, \widetilde{Y}$ are *any* extensions of X and Y. Using this fact, we are able to define the second fundamental form.

Definition 3.8 (Second Fundamental Form)
The Second Fundamental Form *is the map* $B\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$ *given by* $B(X,Y) = (\widetilde{\nabla}_X Y)^{\perp}$.

Although not immediately obvious, we will see that the second fundamental form is symmetric and depends on its arguments pointwise. To prove this, we need a lemma.

Lemma 3.9 *Suppose X, Y are vector fields on $\widetilde{M}$ such that $X_p, Y_p \in T_pM$ for all $p \in M$. Then $[X,Y]_p \in T_pM$ for all $p \in M$.*

Proof. Making explicit our identification of T_pM with $[\iota_*]_p T_pM$, there are $U, V \in \Gamma(TM)$ such that $[\iota_*]_p U_p = X_p$ and $[\iota_*]_p V_p = Y_p$. But then $[\iota_*]_p [U,V]_p = [X,Y]_p$, which says that $[X,Y]_p \in [\iota_*]_p T_pM$ for each $p \in M$. Thus, $[X,Y]$ is tangent to M at each point $p \in M$. ■

Proposition 3.10 *B is a smooth section of $T^*M \otimes T^*M \otimes NM$ and is symmetric.*

Proof. For $X, Y \in \Gamma(TM)$, $B(X,Y) = (\widetilde{\nabla}_X Y)^{\perp}$ depends pointwise on X, so it suffices to show that B is symmetric. Indeed,

$$B(X,Y) - B(Y,X) = (\nabla_X Y - \nabla_Y X)^{\perp} = [X,Y]^{\perp} = 0,$$

where the last equality follows from the previous lemma. ■

Before looking at the geometric interpretation of the second fundamental form, we see that it relates the Levi-Civita connection on M to the Levi-Civita connection on $\widetilde{M}$.

Proposition 3.11 *Let $X, Y \in \Gamma(TM)$. Then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y).$$

Proof. We must show that $(\widetilde{\nabla}_X Y)^{\top}$ is the Levi-Civita connection on M. It is easy to see that $(\widetilde{\nabla}_X Y)^{\top}$ is $\mathbb{R}$-linear in X and Y.

Let $f \in C^\infty(M)$. Suppose the chosen extensions for X and Y are $\widetilde{X}$ and $\widetilde{Y}$, respectively. Without loss of generality, we may assume that the extensions for fX and fY are $\tilde{f}\widetilde{X}$ and $\tilde{f}\widetilde{Y}$, where $\tilde{f}$ extends f. Then

$$(\widetilde{\nabla}_{fX}Y)^\top = ((\widetilde{\nabla}_{\tilde{f}\widetilde{X}}\widetilde{Y})|_M)^\top = (\tilde{f}|_M(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y})|_M)^\top = f(\widetilde{\nabla}_X Y)^\top.$$

Since X is a vector field tangent to M at all $p \in M$, $(\widetilde{X}\widetilde{f})|_M$ depends only on the values of $\widetilde{X}$ and $\widetilde{f}$ on M. That is, $(\widetilde{X}\widetilde{f})(p) = (Xf)(p)$ for all $p \in M$. Thus, we have

$$(\widetilde{\nabla}_X(fY))^\top = ((\widetilde{\nabla}_{\widetilde{X}}(\tilde{f}\widetilde{Y}))|_M)^\top = (((\widetilde{X}\tilde{f})\widetilde{Y} + \tilde{f}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y})|_M)^\top = (Xf)Y + f(\widetilde{\nabla}_X Y)^\top.$$

Therefore, this defines a connection. Fix $X, Y, Z \in \Gamma(TM)$. Using the fact that $\widetilde{\nabla}$ is metric compatible and all the vector fields are tangent to M, we have

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \widetilde{\nabla}_X Y, Z\rangle + \langle Y, \widetilde{\nabla}_X Z\rangle \\ &= \langle (\widetilde{\nabla}_X Y)^\top, Z\rangle + \langle Y, (\widetilde{\nabla}_X Z)^\top\rangle, \end{aligned}$$

so the connection is metric compatible. Finally, Lemma 3.9 gives

$$(\widetilde{\nabla}_X Y)^\top - (\widetilde{\nabla}_Y X)^\top = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\top = [X, Y]^\top = [X, Y],$$

so the connection is torsion free. ■

A similar formula can be given for vector fields along curves.

Proposition 3.12 *Let $\gamma \colon I \to M$ be a smooth curve and V a vector field on M along γ. Then*

$$\widetilde{D}_t V = D_t V + B(\gamma', V).$$

Proof. Let $\{E_1, \ldots, E_n\}$ be a local adapted orthonormal frame, so $\{E_1, \ldots, E_k\}$ is a local orthonormal frame for TM. Thus, we can write

$$V_t = V^j(t)E_j(t),$$

where j sums from 1 to k, and the E_j are viewed as vector fields over γ (as functions of t) as well as vector fields on M. Using the fact that the E_j as vector fields over γ are extendible, we have

$$\widetilde{D}_t V = (V^j)' E_j + V^j \widetilde{D}_t E_j = (V^j)' E_j + V^j \widetilde{\nabla}_{\gamma'} E_j.$$

A similar result holds for $D_t V$. Thus,

$$(\widetilde{D}_t V - D_t V)(t) = V^j(t)(\widetilde{\nabla}_{\gamma'(t)} E_j - \nabla_{\gamma'(t)} E_j) = V^j(t) B(\gamma'(t), E_j(t)) = B(\gamma'(t), V(t)).$$

■

We are now able to see the geometric interpretation of the second fundamental form. Fix $X_p \in T_pM$, and let $\gamma\colon I \to M$ be the M-geodesic with initial data X_p. Intuitively, γ is a "straight" curve in M. By Proposition 3.12, we see that $\widetilde{D}_t\gamma' = B(\gamma', \gamma')$. In particular, we have

$$(\widetilde{D}_t\gamma')(0) = B(\gamma'(0), \gamma'(0)) = B(X_p, X_p).$$

Thus, B measures the failure of "straight lines" in M to be "straight lines" in the ambient manifold $\widetilde{M}$. This means that B in some sense captures the curvature of M as "seen from" the perspective of the ambient manifold.

These considerations motivate us to consider the case where geodesics in M are geodesics in $\widetilde{M}$.

Definition 3.13 *M is called* totally geodesic *if every M-geodesic is an $\widetilde{M}$-geodesic.*

Proposition 3.14 *The following are equivalent:*

a. M is totally geodesic,

b. If $X_p \in T_pM$, then the $\widetilde{M}$-geodesic γ with initial data $\gamma'(0) = X_p$ stays in M on some neighborhood $(-\epsilon, \epsilon)$.

c. $B = 0$.

Proof. We prove $a \implies b$, $b \implies c$, and $c \implies a$. First, suppose M is totally geodesic, and let $X_p \in T_pM$. Let $\gamma_{X_p}\colon I \to M$ be the M-geodesic with initial data X_p. By assumption, γ_{X_p} coincides with the $\widetilde{M}$-geodesic with initial data X_p on its domain I. By uniqueness of geodesics in $\widetilde{M}$, the $\widetilde{M}$-geodesic with initial data X_p lies in M for all $t \in I$.

Now suppose b holds. We want to show $B = 0$. Fix any $X_p \in T_pM$, and let γ_{X_p} be the $\widetilde{M}$-geodesic with initial data X_p. By restricting the domain of γ_{X_p} to a sufficiently small neighborhood $(-\epsilon, \epsilon)$, we may assume that γ_{X_p} lies in M. Applying Proposition 3.12 at $t = 0$ with $V = \gamma'$, we obtain $0 = (\widetilde{D}_t\gamma')(0) = (D_t\gamma')(0) + B(X_p, X_p)$. Since $(D_t\gamma')(0)$ and $B(X_p, X_p)$ are orthogonal, this implies $B(X_p, X_p) = 0$. Since $X_p \in T_pM$ and $p \in M$ were arbitrary, $B(v, v) = 0$ for all $v \in TM$. Since B is symmetric, this implies $B = 0$.

Finally, $c \implies a$ is immediate from Proposition 3.12 with $V = \gamma'$. ■

We will see that given a product manifold $M_1 \times M_2$, any submanifold of the form $M_1 \times \{q\}$ is totally geodesic. To do this, we need to determine the Levi-Civita connection for $M_1 \times M_2$ in terms of the Levi-Civita connections of M_1 and M_2, which requires a fairly lengthy lemma.

Lemma 3.15 *Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 be the Riemannian connection of M_1 and let ∇^2 be the Riemannian connection of M_2. The Levi-Civita connection of $M_1 \times M_2$ is the unique connection satisfying*

$$\nabla_{X_1+X_2}(Y_1+Y_2) = \nabla^1_{X_1}Y_1 + \nabla^2_{X_2}Y_2 \tag{6}$$

whenever $X_1, Y_1 \in \Gamma(TM_1)$ and $X_2, Y_2 \in \Gamma(M_2)$.

Proof. We first show that there is at most one connection on $M_1 \times M_2$ satisfying (6). Let $(p,q) \in M_1 \times M_2$, let $(x^1, \ldots, x^k)$ be local coordinates on M_1 about p and let $(y^1, \ldots, y^n)$ be local coordinates on M_2 about q, so $(x^1, \ldots, x^k, y^1, \ldots, y^n)$ are local coordinates for $M_1 \times M_2$ about (p,q). Fix $X, Y \in \Gamma(T(M_1 \times M_2))$. Locally, we can write

$$X = a^i \frac{\partial}{\partial x^i} + b^j \frac{\partial}{\partial y^j}, \tag{7}$$

$$Y = c^k \frac{\partial}{\partial x^k} + d^r \frac{\partial}{\partial y^r}. \tag{8}$$

Thus, if ∇ is some connection on $M_1 \times M_2$ satisfying (6), we can use the product rule to decompose and then apply (6).

$$\begin{aligned}
\nabla_X Y &= a^i \nabla_{\frac{\partial}{\partial x^i}} Y + b^j \nabla_{\frac{\partial}{\partial y^j}} Y \\
&= a^i \left(\frac{\partial c^k}{\partial x^i}\frac{\partial}{\partial x^k} + c^k \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} + \frac{\partial d^r}{\partial x^i}\frac{\partial}{\partial y^r} + d^r \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^r} \right) \\
&\quad + b^j \left(\frac{\partial c^k}{\partial y^j}\frac{\partial}{\partial x^k} + c^k \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^k} + \frac{\partial d^r}{\partial y^j}\frac{\partial}{\partial y^r} + d^r \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^r} \right) \\
&= \left(a^i \frac{\partial c^k}{\partial x^i} + b^j \frac{\partial c^k}{\partial y^j} \right) \frac{\partial}{\partial x^k} + \left(a^i \frac{\partial d^r}{\partial x^i} + b^j \frac{\partial d^r}{\partial y^j} \right) \frac{\partial}{\partial y^r} + a^i c^k \nabla^1_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} + b^j d^r \nabla^2_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^r} \\
&= \left(a^i \frac{\partial c^l}{\partial x^i} + b^j \frac{\partial c^l}{\partial y^j} + a^i c^k \Gamma^l_{ik} \right) \frac{\partial}{\partial x^l} + \left(a^i \frac{\partial d^s}{\partial x^i} + b^j \frac{\partial d^s}{\partial y^j} + b^j d^r \Omega^s_{jr} \right) \frac{\partial}{\partial y^s},
\end{aligned} \tag{9}$$

where Γ^l_{ik} and Ω^s_{jr} are the Christoffel symbols of ∇^1, ∇^2, respectively. Thus, (9) shows that any connection satisfying (6) must satisfy (9), and thus there is at most one connection satisfying (7).

We want to define ∇ using (9). However, we must ensure that such a definition is independent of the choice of coordinates. Thus, with our fixed coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^n)$, let us define the the local operator

$$A(X,Y) = \left(a^i \frac{\partial c^l}{\partial x^i} + b^j \frac{\partial c^l}{\partial y^j} + a^i c^k \Gamma^l_{ik}\right) \frac{\partial}{\partial x^l} + \left(a^i \frac{\partial d^s}{\partial x^i} + b^j \frac{\partial d^s}{\partial y^j} + b^j d^r \Omega^s_{jr}\right) \frac{\partial}{\partial y^s},$$

whenever the smooth local vector fields X, Y are given by (7) and (8). Clearly $A(X,Y)$ is smooth on its domain of definition since the coefficient functions are all smooth. Moreover, suppose f is a smooth function on $M_1 \times M_2$, and X and Y are given by (7) and (8). Then

$$\begin{aligned}
A(fX,Y) &= \left((fa^i)\frac{\partial c^l}{\partial x^i} + (fb^j)\frac{\partial c^l}{\partial y^j} + (fa^i)c^k\Gamma^l_{ik}\right)\frac{\partial}{\partial x^l} \\
&\qquad + \left((fa^i)\frac{\partial d^s}{\partial x^i} + (fb^j)\frac{\partial d^s}{\partial y^j} + (fb^j)d^r\Omega^s_{jr}\right)\frac{\partial}{\partial y^s} \\
&= f\left[\left(a^i \frac{\partial c^l}{\partial x^i} + b^j \frac{\partial c^l}{\partial y^j} + a^i c^k \Gamma^l_{ik}\right) \frac{\partial}{\partial x^l} + \left(a^i \frac{\partial d^s}{\partial x^i} + b^j \frac{\partial d^s}{\partial y^j} + b^j d^r \Omega^s_{jr}\right) \frac{\partial}{\partial y^s}\right] \\
&= fA(X,Y)
\end{aligned}$$

It is also easy to see from the definition of A that if Z is another such local vector field, then $A(X+Z,Y) = A(X,Y)+A(Z,Y)$ and $A(X,Y+Z) = A(X,Y)+A(X,Z)$. Finally, we must verify the product rule for A. If X and Y are given by (7) and (8), then

$$\begin{aligned}
A(X,fY) &= \left(a^i \frac{\partial (fc^l)}{\partial x^i} + b^j \frac{\partial (fc^l)}{\partial y^j} + a^i fc^k \Gamma^l_{ik}\right) \frac{\partial}{\partial x^l} + \left(a^i \frac{\partial (fd^s)}{\partial x^i} + b^j \frac{\partial (fd^s)}{\partial y^j} + b^j fd^r \Omega^s_{jr}\right) \frac{\partial}{\partial y^s} \\
&= \left(a^i c^l \frac{\partial f}{\partial x^i} + b^j c^l \frac{\partial f}{\partial y^j}\right)\frac{\partial}{\partial x^l} + f\left(a^i \frac{\partial c^l}{\partial x^i} + b^j \frac{\partial c^l}{\partial y^j} + a^i c^k \Gamma^l_{ik}\right)\frac{\partial}{\partial x^l} \\
&\qquad + \left(a^i d^s \frac{\partial f}{\partial x^i} + b^j d^s \frac{\partial f}{\partial y^j}\right)\frac{\partial}{\partial y^s} + f\left(a^i \frac{\partial d^s}{\partial x^i} + b^j \frac{\partial d^s}{\partial y^j} + b^j d^r \Omega^s_{jr}\right)\frac{\partial}{\partial y^s} \\
&= \left[\left(a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial y^j}\right)f\right]\left(c^l\frac{\partial}{\partial x^l}\right) + \left[\left(a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial y^j}\right)f\right]\left(d^s\frac{\partial}{\partial y^s}\right) + fA(X,Y) \\
&= \left[\left(a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial y^j}\right)f\right]\left(c^l\frac{\partial}{\partial x^l} + d^s\frac{\partial}{\partial y^s}\right) + fA(X,Y) \\
&= (Xf)Y + fA(X,Y).
\end{aligned}$$

Thus, A is a local connection on $M_1 \times M_2$. Now suppose that $(\hat{x}^1, \ldots, \hat{x}^k, \hat{y}^1, \ldots, \hat{y}^n)$ are other local coordinates on $M_1 \times M_2$, and furthermore suppose that the chart domains of $(x^1, \ldots, x^k, y^1, \ldots, y^n)$ and $(\hat{x}^1, \ldots, \hat{x}^k, \hat{y}^1, \ldots, \hat{y}^n)$ overlap. We define a local connection $\hat{A}$ as we did for A, but in the new coordinate system. That is, if X, Y are local smooth vector fields of the form

$$X = \hat{a}^i \frac{\partial}{\partial \hat{x}^i} + \hat{b}^j \frac{\partial}{\partial \hat{y}^j},$$

$$Y = \hat{c}^k \frac{\partial}{\partial \hat{x}^k} + \hat{d}^r \frac{\partial}{\partial \hat{y}^r},$$

then $\hat{A}(X, Y)$ is defined as

$$\hat{A}(X,Y) = \left(\hat{a}^i \frac{\partial \hat{c}^l}{\partial \hat{x}^i} + \hat{b}^j \frac{\partial \hat{c}^l}{\partial \hat{y}^j} + \hat{a}^i \hat{c}^k \hat{\Gamma}^l_{ik} \right) \frac{\partial}{\partial \hat{x}^l} + \left(\hat{a}^i \frac{\partial \hat{d}^s}{\partial \hat{x}^i} + \hat{b}^j \frac{\partial \hat{d}^s}{\partial \hat{y}^j} + \hat{b}^j \hat{d}^r \hat{\Omega}^s_{jr} \right) \frac{\partial}{\partial \hat{y}^s}.$$

We must show that A and $\hat{A}$ agree on the intersection of their domain. Since both are tensorial in the first factor and satisfy the product rule, it suffices to show that

$$A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right) = \hat{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right) \tag{10}$$

$$A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = \hat{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) \tag{11}$$

$$A\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^r}\right) = \hat{A}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^r}\right) \tag{12}$$

We show (10) and (11), as (12) is similar to (11). For (10), note that $b^j = c^k = 0$ for all j and r, and d^s is constant, so from the definition of A, we have

$$A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right) = 0.$$

Similarly,

$$\hat{A}\left(\frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial \hat{y}^r}\right) = 0.$$

But then we have

$$\begin{aligned}
\hat{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right) &= \hat{A}\left(\frac{\partial \hat{x}^t}{\partial x^i}\frac{\partial}{\partial \hat{x}^t}, \frac{\partial \hat{y}^u}{\partial y^r}\frac{\partial}{\partial \hat{y}^u}\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\hat{A}\left(\frac{\partial}{\partial \hat{x}^t}, \frac{\partial \hat{y}^u}{\partial y^r}\frac{\partial}{\partial \hat{y}^u}\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\left(\frac{\partial^2 \hat{y}^u}{\partial \hat{x}^t \partial y^r}\frac{\partial}{\partial \hat{y}^u} + \frac{\partial \hat{y}^u}{\partial y^r}\hat{A}\left(\frac{\partial}{\partial \hat{x}^t}, \frac{\partial}{\partial \hat{y}^u}\right)\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\left(0 \cdot \frac{\partial}{\partial \hat{y}^u} + \frac{\partial \hat{y}^u}{\partial y^r} \cdot 0\right) \\
&= 0.
\end{aligned}$$

This proves (10). We now prove (11). First, note that from the definition of A, we have

$$A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = \Gamma^l_{ik}\frac{\partial}{\partial x^l}.$$

Similarly,
$$\hat{A}\left(\frac{\partial}{\partial \hat{x}^i}, \frac{\partial}{\partial \hat{x}^k}\right) = \hat{\Gamma}^l_{ik}\frac{\partial}{\partial \hat{x}^l}. \tag{13}$$
Performing a computation similar to as with $\hat{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^r}\right)$ and substituting (13), we have
$$\begin{aligned}
\hat{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) &= \hat{A}\left(\frac{\partial \hat{x}^t}{\partial x^i}\frac{\partial}{\partial \hat{x}^t}, \frac{\partial \hat{x}^u}{\partial x^k}\frac{\partial}{\partial \hat{x}^u}\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\left(\frac{\partial^2 \hat{x}^u}{\partial \hat{x}^t \partial x^k}\frac{\partial}{\partial \hat{x}^u} + \frac{\partial \hat{x}^u}{\partial x^k}\hat{A}\left(\frac{\partial}{\partial \hat{x}^t}, \frac{\partial}{\partial \hat{x}^u}\right)\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\left(\frac{\partial^2 \hat{x}^u}{\partial \hat{x}^t \partial x^k}\frac{\partial}{\partial \hat{x}^u} + \frac{\partial \hat{x}^u}{\partial x^k}\hat{\Gamma}^l_{tu}\frac{\partial}{\partial \hat{x}^l}\right) \\
&= \frac{\partial \hat{x}^t}{\partial x^i}\left(\frac{\partial^2 \hat{x}^u}{\partial \hat{x}^t \partial x^k}\frac{\partial x^v}{\partial \hat{x}^u}\frac{\partial}{\partial x^v} + \frac{\partial \hat{x}^u}{\partial x^k}\hat{\Gamma}^l_{tu}\frac{\partial x^v}{\partial \hat{x}^l}\frac{\partial}{\partial x^v}\right) \\
&= \left(\frac{\partial^2 \hat{x}^u}{\partial x^i \partial x^k}\frac{\partial x^v}{\partial \hat{x}^u} + \frac{\partial \hat{x}^t}{\partial x^i}\frac{\partial \hat{x}^u}{\partial x^k}\frac{\partial x^v}{\partial \hat{x}^l}\hat{\Gamma}^l_{tu}\right)\frac{\partial}{\partial x^v} \\
&= \Gamma^v_{ik}\frac{\partial}{\partial x^v},
\end{aligned}$$
where the last equality follows from the Christoffel symbol transformation rule. This proves (11). Thus, for any point $(p, q) \in M_1 \times M_2$, we can define $(\nabla_X Y)(p, q)$ in *any* local coordinates about (p, q) by (9), and our previous work shows that ∇ is a well defined connection on $M_1 \times M_2$.

We now verify that ∇ is metric compatible. Recall that the product metric is defined by
$$g_{(p,q)}((v_1, v_2), (w_1, w_2)) = g^1_p(v_1, w_1) + g^2_q(v_2, w_2),$$
for all $v_1, w_1 \in T_pM_1$ and $v_2, w_2 \in T_qM_2$. For simplicity, we denote all metrics by $\langle . \rangle$. To show that ∇ is metric compatible, we must show that for all $X, Y, Z \in \Gamma(T(M_1 \times M_2))$
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$
In local coordinates, we can write
$$\begin{aligned}
Y &= Y_1 + Y_2 = a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial y^j}, \\
Z &= Z_1 + Z_2 = c^k\frac{\partial}{\partial x^k} + d^l\frac{\partial}{\partial y^l}.
\end{aligned}$$
Now using the definition of the metric on $M_1 \times M_2$, we compute
$$X\langle Y, Z\rangle = X\left\langle a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial y^j}, c^k\frac{\partial}{\partial x^k} + d^l\frac{\partial}{\partial y^l}\right\rangle$$

$$= X\left(a^i c^k \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle + b^j d^l \left\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^l} \right\rangle\right) \tag{14}$$

Computing the first term gives

$$\begin{aligned}
X\left(a^i c^k \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle\right) &= \left\langle (Xa^i)\frac{\partial}{\partial x^i}, c^k \frac{\partial}{\partial x^k} \right\rangle + \left\langle a^i \frac{\partial}{\partial x^i}, (Xc^k)\frac{\partial}{\partial x^k} \right\rangle + a^i c^k X \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle \\
&= \left\langle (Xa^i)\frac{\partial}{\partial x^i}, c^k \frac{\partial}{\partial x^k} \right\rangle + \left\langle a^i \frac{\partial}{\partial x^i}, (Xc^k)\frac{\partial}{\partial x^k} \right\rangle \\
&\quad + \left\langle a^i \nabla_X \frac{\partial}{\partial x^i}, c^k \frac{\partial}{\partial x^k} \right\rangle + \left\langle a^i \frac{\partial}{\partial x^i}, c^k \nabla_X \frac{\partial}{\partial x^k} \right\rangle \\
&= \left\langle \nabla_X a^i \frac{\partial}{\partial x^i}, c^k \frac{\partial}{\partial x^k} \right\rangle + \left\langle a^i \frac{\partial}{\partial x^i}, \nabla_X c^k \frac{\partial}{\partial x^k} \right\rangle \\
&= \langle \nabla_X Y_1, Z_1 \rangle + \langle Y_1, \nabla_X Z_1 \rangle.
\end{aligned} \tag{15}$$

Similarly,

$$X\left(b^j d^l \left\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^l} \right\rangle\right) = \langle \nabla_X Y_2, Z_2 \rangle + \langle Y_2, \nabla_X Z_2 \rangle. \tag{16}$$

Combining (14), (15), and (16), we obtain the required result. Finally, we show that ∇ is torsion free. Since the torsion is tensorial, it suffices to prove the result for coordinate vector fields. Since the Lie bracket of a coordinate vector field vanishes, we have

$$\begin{aligned}
T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) &= \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} - \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} \\
&= 0 - 0 = 0,
\end{aligned}$$

$$\begin{aligned}
T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \\
&= \nabla^1_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla^1_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = 0,
\end{aligned}$$

$$\begin{aligned}
T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} - \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} \\
&= \nabla^2_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} - \nabla^2_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = 0,
\end{aligned}$$

where we used that ∇ satisfies (6) and that ∇^1 and ∇^2 are torsion free. This shows that ∇ is the Levi-Civita connection. ■

Lemma 3.16 *Suppose M_1 and M_2 are Riemannian manifolds. Let $v \in T_{(p,q)}(M_1 \times M_2)$, so $v = v_1 + v_2$ where $v_1 \in T_pM_1$ and $v_2 \in T_qM_2$. Then the geodesic with initial data v is the curve*

$$\tau = (\gamma_{v_1}(t), \gamma_{v_2}(t))$$

where γ_{v_i} is the M_i geodesic with initial data v_i.

Proof. Let γ_{v_i} be the M_i-geodesic with initial data v_i. Fix some time t_0. In some neighborhood of t_0, $\gamma'_{v_i}(t)$ can be extended to a local vector field X_i on M_i. In local coordinates, we write

$$\begin{aligned} X_1 &= f^i(x^1, \dots, x^k) \frac{\partial}{\partial x^i}\bigg|_{(x^1,\dots,x^k)}, \\ X_2 &= g^j(y^1, \dots, y^n) \frac{\partial}{\partial y^j}\bigg|_{(y^1,\dots,y^n)}, \end{aligned}$$

where $(x^1, \dots, x^k)$ are M_1-local coordinates about q and $(y^1, \dots, y^n)$ are M_2-local coordinates about q. The vector field X_1 can be extended to an open subset of (p, q) be defining

$$\begin{aligned} X_1 &= f^i(x^1, \dots, x^k) \frac{\partial}{\partial x^i}\bigg|_{(x^1,\dots,x^k,y^1,\dots,y^n)}, \\ X_2 &= g^j(y^1, \dots, y^n) \frac{\partial}{\partial y^j}\bigg|_{(x^1,\dots,x^k,y^1,\dots,y^n)}, \end{aligned}$$

where the f^i are independent of the $(y^1, \dots, y^n)$ coordinates and the g^j are independent of the $(x^1, \dots, x^n)$ coordinates. Let $\tau(t) = (\gamma_{v_1}(t), \gamma_{v_2}(t))$. Using (7) and the fact that X_1 is a local vector field extension of γ_1 and X_2 is a local vector field extension of γ_2, we have

$$(D_t\tau)(t) = (\nabla_{X_1+X_2}(X_1+X_2))(p, q) = (\nabla^1_{X_1}X_1)(p)+(\nabla^2_{X_2}X_2)(q) = (D^1_t(\gamma'_{v_1})+D^2_t(\gamma'_{v_2}))(t) = 0$$

in an open neighborhood of t_0. Since t_0 was arbitrary, τ is a geodesic with initial data $\tau'(0) = \gamma'_{v_1}(0) + \gamma'_{v_2}(0) = v_1 + v_2 = v$. ∎

Let us use this lemma to show that $\{p\} \times M_2$ is a totally geodesic submanifold of $M_1 \times M_2$.

Example 3.17 For every $p \in M_1$, the set

$$(M_2)_p = \{(p, q) \in M_1 \times M_2 \colon q \in M_2\}$$

is a totally geodesic submanifold of $M_1 \times M_2$.

Proof. Let τ be a geodesic of $\{p\} \times M_2$ with initial data $v \in T_{(p,q)}(\{p\} \times M_2) \cong T_qM_2$. By Lemma 3.16, $\tau = (p, \gamma_v)$ where γ_v is the M_2 geodesic with initial data v. Applying Lemma 3.16 again, we see that τ is the geodesic in $M_1 \times M_2$ with initial data $v \in T_qM_2 \subset T_{(p,q)}(M_1 \times M_2)$. ∎

Example 3.18 Consider a vector bundle $\pi : E \to M$ over a Riemannian manifold (M, g) where E has a connection ∇^E and fibre metric h, and let E be endowed with the induced Riemannian metric $\widehat{g}$ from section 2.4. Then the zero section Z from Example 2.2g inherits the metric $g \circ \pi$ from $(E, \widehat{g})$ and Z is a totally geodesic submanifold of $(E, \widehat{g})$.

Proof. We first show that g^Z is essentially just g, which means $i \colon Z \to E$ is a Riemannian immersion. Fix $p \in M$. Let $(x^1, \ldots, x^n)$ be a set of local coordinates for M about p such that $g_{ij}(p) = \delta_{ij}$ for $1 \le i, j \le n$. Let $\{s_1, \ldots, s_k\}$ be an $\mathbb{R}$-local frame for E such that that $h_{ab}(p) = \delta_{ab}$ for $1 \le a, b \le k$. As usual, let $(x^1, \ldots, x^n, y^1, \ldots, y^k)$ be the usual induced local coordinates for E in terms of the local coordinates $(x^1, \ldots, x^n)$ for M and the frame $\{s_1, \ldots, s_k\}$ for E.

Let $\vartheta \in Z$ be arbitrary. Then we have

$$\begin{aligned} g^Z_{ij}(\vartheta) &= \widehat{g}_\vartheta \left([\iota_*]_\vartheta \frac{\partial}{\partial x^i}\bigg|_\vartheta, [\iota_*]_\vartheta \frac{\partial}{\partial x^j}\bigg|_\vartheta \right) \\ &= \widehat{g}_\vartheta \left(\frac{\partial}{\partial x^i}\bigg|_\vartheta, \frac{\partial}{\partial x^j}\bigg|_\vartheta \right) \\ &= g_{ij}(\pi(\vartheta)) + \Gamma^l_{ia}(\pi(\vartheta)) \Gamma^m_{jb}(\pi(\vartheta)) y^a\big|_\vartheta y^b\big|_\vartheta h_{lm}(p) \\ &= g_{ij}(\pi(\vartheta)), \end{aligned}$$

where the last equality follows since each y^i is zero at a point of the zero section. So $g^Z = g \circ \pi$. Moreover, using Proposition 2.43 and the fact that the y^j are zero at a point of Z, we have

$$\begin{aligned} \widehat{g}\left(\frac{\partial}{\partial x^i}\bigg|_\vartheta, \frac{\partial}{\partial x^j}\bigg|_\vartheta \right) &= g_{ij}(\pi(\vartheta)), \\ \widehat{g}\left(\frac{\partial}{\partial x^i}\bigg|_\vartheta, \frac{\partial}{\partial y^j}\bigg|_\vartheta \right) &= 0, \\ \widehat{g}\left(\frac{\partial}{\partial y^i}\bigg|_\vartheta, \frac{\partial}{\partial y^j}\bigg|_\vartheta \right) &= h_{ij}(\pi(\vartheta)). \end{aligned}$$

Now write p in local coordinates as $p = (a^1, \ldots, a^n)$ and define $\vartheta = (a^1, \ldots, a^n, 0, \ldots, 0) \in Z$. In local coordinates, the inclusion map is given by $\iota(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$. So $\{\frac{\partial}{\partial x^1}\big|_\vartheta, \ldots, \frac{\partial}{\partial x^n}\big|_\vartheta\}$ is a basis for $T_\vartheta Z$. By choice of local coordinates, we have $\widehat{g}\left(\frac{\partial}{\partial x^i}\big|_\vartheta, \frac{\partial}{\partial x^j}\big|_\vartheta\right) = g_{ij}(p) = \delta_{ij}$, $\widehat{g}\left(\frac{\partial}{\partial x^i}\big|_\vartheta, \frac{\partial}{\partial y^j}\big|_\vartheta\right) = 0$, and $\widehat{g}\left(\frac{\partial}{\partial y^i}\big|_\vartheta, \frac{\partial}{\partial y^j}\big|_\vartheta\right) = h_{ij}(p) = \delta_{ij}$. Thus, the set $\{\frac{\partial}{\partial y^1}\big|_\vartheta, \ldots, \frac{\partial}{\partial y^k}\big|_\vartheta\}$ is a basis for $N_\vartheta Z$. Let ∇ be the Levi-Civita connection on E and let Ω^l_{ij} be the Christoffel symbols for ∇. Then

$$B\left(\frac{\partial}{\partial x^i}\bigg|_\vartheta, \frac{\partial}{\partial x^j}\bigg|_\vartheta \right) = \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)^\perp (\vartheta)$$

$$
\begin{aligned}
&= \left(\sum_{l=1}^{n} \left[\Omega_{ij}^{l} \frac{\partial}{\partial x^l} \right] + \sum_{l=1}^{k} \left[\Omega_{ij}^{n+l} \frac{\partial}{\partial y^l} \right] \right)^{\perp} (\vartheta) \\
&= \sum_{l=1}^{k} \Omega_{ij}^{n+l}(\vartheta) \frac{\partial}{\partial y^l}\bigg|_{\vartheta}.
\end{aligned}
$$

Now for any $1 \leq l \leq k$, we compute $\Omega_{ij}^{n+l}(\vartheta)$ as

$$
\begin{aligned}
\Omega_{ij}^{n+l} &= \frac{1}{2} \sum_{m=1}^{n+k} \widehat{g}^{m(l+n)} \left(\frac{\partial \widehat{g}_{jm}}{\partial x^i} + \frac{\partial \widehat{g}_{im}}{\partial x^j} - \frac{\partial \widehat{g}_{ij}}{\partial x^m} \right) \\
&= \frac{1}{2} \left(\frac{\partial}{\partial x^i}\bigg|_{\vartheta} \widehat{g} \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^l} \right) + \frac{\partial}{\partial x^j}\bigg|_{\vartheta} \widehat{g} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^l} \right) - \frac{\partial}{\partial y^l}\bigg|_{\vartheta} \widehat{g} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right) \\
&= \frac{1}{2} \left(\frac{\partial}{\partial x^i} \left(\Gamma_{ja}^{m} y^a h_{ml} \right) + \frac{\partial}{\partial x^j} \left(\Gamma_{ia}^{m} y^a h_{ml} \right) - \frac{\partial}{\partial y^l} \left(g_{ij} + \Gamma_{ia}^{b} \Gamma_{jc}^{d} y^a y^c h_{bd} \right) \right).
\end{aligned}
$$

So every term in Ω_{ij}^{n+l} contains a y^i term for some i. Thus, $\Omega_{ij}^{n+l}(\vartheta) = 0$, which implies $B\left(\frac{\partial}{\partial x^i}\big|_{\vartheta}, \frac{\partial}{\partial x^j}\big|_{\vartheta}\right) = 0$. Since B is a tensor, $B = 0$. That is, Z is a totally geodesic submanifold of E. ■

We have seen that the second fundamental form is a measure of the "extrinsic curvature" of M in $\widetilde{M}$. It is very reasonable, then, to expect the second fundamental form to relate the Riemann curvature tensors of $\widetilde{M}$ and M. This leads us to the *Gauss Equation*.

Theorem 3.19 (The Gauss Equation) *Let $\widetilde{R}$ be the curvature of $\widetilde{M}$ and R the curvature of M. Then for every $X, Y, Z, W \in \Gamma(TM)$,*

$$
\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle B(X,W), B(Y,Z) \rangle + \langle B(X,Z), B(Y,W) \rangle.
$$

Proof. We first note that

$$
\begin{aligned}
\langle \widetilde{\nabla}_X B(Y,Z), W \rangle &= X(\langle B(Y,Z), W \rangle) - \langle B(Y,Z), \widetilde{\nabla}_X W \rangle \\
&= X(0) - \langle B(Y,Z), (\widetilde{\nabla}_X W)^{\perp} \rangle \\
&= -\langle B(Y,Z), B(X,W) \rangle. \qquad (17)
\end{aligned}
$$

Similarly,

$$
\langle \widetilde{\nabla}_Y B(X,Z), W \rangle = -\langle B(Y,W), B(X,Z) \rangle. \qquad (18)
$$

From (17) and (18), we obtain

$$
\widetilde{R}(X,Y,Z,W) = \langle \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W \rangle
$$

$$
\begin{aligned}
&= \langle \widetilde{\nabla}_X(\nabla_Y Z + B(Y,Z)) - \widetilde{\nabla}_Y(\nabla_X Z + B(X,Z)) - \widetilde{\nabla}_{[X,Y]}Z, W\rangle \\
&= \langle (\widetilde{\nabla}_X(\nabla_Y Z))^\top - (\widetilde{\nabla}_Y(\nabla_X Z))^\top - (\widetilde{\nabla}_{[X,Y]}Z)^\top, W\rangle \\
&\quad + \langle \widetilde{\nabla}_X B(Y,Z), W\rangle - \langle \widetilde{\nabla}_Y B(X,Z), W\rangle \\
&= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, W\rangle \\
&\quad - \langle B(Y,Z), B(X,W)\rangle + \langle B(Y,W), B(X,Z)\rangle \\
&= R(X,Y,Z,W) - \langle B(X,W), B(Y,Z)\rangle + \langle B(X,Z), B(Y,W)\rangle.
\end{aligned}
$$

■

Corollary 3.20 *Let $\widetilde{sec}$ be the sectional curvature for $\widetilde{M}$ and let sec be the sectional curvature for M, and fix $p \in M$. Then for all linearly independent $X_p, Y_p \in T_pM$, we have*

$$\widetilde{sec}(X_p, Y_p) = sec(X_p, Y_p) + \frac{|B(X_p, Y_p)|^2 - \langle B(X_p, X_p), B(Y_p, Y_p)\rangle}{|X_p \wedge Y_p|^2}.$$

Remark 3.21 The previous corollary provides a geometric interpretation of the sectional curvature. Let Ω_p be a two-dimensional subspace of $T_p\widetilde{M}$, and let V be a star-shaped, open subset of $T_p\widetilde{M}$ such that the exponential map $\exp_p$ is a diffeomorphism. Since $\Omega_p \cap V$ is an embedded 2-submanifold of $T_p\widetilde{M}$, $M = \exp_p(\Omega_p \cap V)$ is an embedded 2-submanifold of $\widetilde{M}$. M consists of small geodesics passing through p with initial data in Ω_p. By construction, each $\widetilde{M}$ geodesic with initial data in T_pM lies in M for a short time. Proposition 3.14 together with Corollary 3.20 implies that

$$\widetilde{sec}(\Omega_p) = sec_M(p),$$

where $sec_M(p)$ is the sectional curvature of M at p. Thus, the Ω_p sectional curvature of a manifold is the curvature at p of the embedded submanifold formed by geodesics passing through p with initial data in Ω_p. //

For any $X, Y, Z \in \Gamma(TM)$, Theorem 3.19 allowed us to find the tangential components of $\widetilde{R}(X,Y)Z$ in terms of $R(X,Y)Z$ and the second fundamental form. We can also find the normal components of $\widetilde{R}(X,Y)Z$ in terms of covariant derivatives of the second fundamental form. To do this, we must first introduce the *normal connection*.

Definition 3.22 *Define the* normal connection $\nabla^\perp : \Gamma(TM) \times \Gamma(NM) \to \Gamma(NM)$ *by*

$$\nabla^\perp_X N = (\widetilde{\nabla}_X N)^\perp.$$

Proposition 3.23 *The normal connection is a metric compatible connection on NM in the sense that for $X \in \Gamma(TM)$ and $N, M \in \Gamma(NM)$ we have*

$$X\langle N, M\rangle = \langle \nabla^\perp_X N, M\rangle + \langle N, \nabla^\perp_X M\rangle.$$

Proof. That $\nabla^\perp$ is a connection follows an identical format to the beginning of Proposition 3.11. To show metric compatibility, let $X \in \Gamma(TM)$ and $N, L \in \Gamma(NM)$. Using the metric compatibility of $\widetilde{\nabla}$, we have

$$\begin{aligned} X\langle N, L\rangle &= \langle \widetilde{\nabla}_X N, L\rangle + \langle N, \widetilde{\nabla}_X L\rangle \\ &= \langle (\widetilde{\nabla}_X N)^\perp, L\rangle + \langle N, (\widetilde{\nabla}_X L)^\perp\rangle \\ &= \langle \nabla^\perp_X N, L\rangle + \langle N, \nabla^\perp_X L\rangle. \end{aligned}$$

■

Given that we now have connections on TM and NM, there is a naturally induced connection on $T^*M \otimes T^*M \otimes NM$. Now $B \in \Gamma(T^*M \otimes T^*M \otimes NM)$, and this connection satisfies

$$(\nabla_X B)(Y, Z) = \nabla^\perp_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. This definition of $\nabla_X B$ is clearly $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in B. It is a simple calculation to show that this $(\nabla_X B)(Y, Z, N)$ is tensorial in each of its three arguments and that $\nabla_X B$ satisfies the product rule.

Theorem 3.24 (The Codazzi Equation)
Let $X, Y, Z \in \Gamma(TM)$. Then

$$(\widetilde{R}(X, Y)Z)^\perp = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z).$$

Proof. Since the equation to prove is tensorial in X, Y and Z, we may assume $[X, Y] = 0$. Now calculating, we have

$$\begin{aligned} (\widetilde{R}(X, Y)Z)^\perp &= (\widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z)^\perp \\ &= (\widetilde{\nabla}_X(\nabla_Y Z + B(Y, Z)) - \widetilde{\nabla}_Y(\nabla_X Z + B(X, Z)))^\perp \\ &= (\widetilde{\nabla}_X(\nabla_Y Z))^\perp + \nabla^\perp_X(B(Y, Z)) - (\widetilde{\nabla}_Y(\nabla_X Z))^\perp - \nabla^\perp_Y(B(X, Z)) \\ &= B(X, \nabla_Y Z) + \nabla^\perp_X(B(Y, Z)) - B(Y, \nabla_X Z) - \nabla^\perp_Y(B(X, Z)) \end{aligned}$$

Rearranging terms, we have

$$\begin{aligned} (\widetilde{R}(X, Y)Z)^\perp &= \big(\nabla^\perp_X(B(Y, Z)) - B(Y, \nabla_X Z)\big) - \big(\nabla^\perp_Y(B(X, Z)) - B(X, \nabla_Y Z)\big) \\ &= (\nabla_X B)(Y, Z) + B(\nabla_X Y, Z) - (\nabla_Y B)(X, Z) - B(\nabla_Y X, Z) \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) - B([X, Y], Z) \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \end{aligned}$$

where the second to last equality used the fact that $[X, Y] = 0$. ■

When $\widetilde{M}$ has constant sectional curvature (in particular, when $\widetilde{M} = \mathbb{R}^{n+1}$), the Codazzi equation takes a simpler form.

Corollary 3.25 *Suppose $\widetilde{M}$ has constant sectional curvature C. Then*

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z).$$

Proof. By Proposition 2.32, for any normal vector field $N \in \Gamma(NM)$, we have

$$\begin{aligned}\widetilde{R}(X, Y, Z, N) &= C \cdot (g(X, N)g(Y, Z) - g(X, Z)g(Y, N)) \\ &= C \cdot (0 \cdot g(Y, Z) - g(X, Z) \cdot 0) \\ &= 0,\end{aligned}$$

so $(\widetilde{R}(X, Y)Z)^{\perp} = 0$. The result now follows immediately from Theorem 3.24. ∎

So far, we have investigated the relationship between the intrinsic and extrinsic curvature of TM. However, similar questions may be asked about the curvature of NM. To make this precise, we define the "normal curvature" of NM in terms of the normal connection.

Definition 3.26 *Define the* normal curvature *of NM to be the map $R^{\perp} \colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(NM) \to \Gamma(NM)$ given by*

$$R^{\perp}(X, Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N.$$

Similar to the standard curvature tensor, one can easily verify that $R^{\perp}$ is tensorial in each of its arguments. Thus, $R^{\perp} \in \Gamma(T^*M \otimes T^*M \otimes N^*M \otimes NM)$.

When $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(NM)$, we found that the second fundamental form, defined by $B(X, Y) = (\widetilde{\nabla}_X Y)^{\perp}$, related $R(X, Y)Z$ to $\widetilde{R}(X, Y)Z$. Reversing the tangential and normal roles in the second fundamental form, one may expect the map $(X, N) \mapsto (\widetilde{\nabla}_X N)^{\top}$ to relate $R^{\perp}(X, Y)N$ to $\widetilde{R}(X, Y)N$. We first fix $N \in \Gamma(NM)$ and consider the map $X \mapsto (\widetilde{\nabla}_N X)^{\top}$. The reason for fixing N will be come clearer when we consider the hypersurface case.

Definition 3.27 *Fix $N \in \Gamma(NM)$ and define the* shape operator *determined by N to be the map $S_N \colon \Gamma(TM) \to \Gamma(TM)$ defined by*

$$S_N(X) = -(\widetilde{\nabla}_X N)^{\top}.$$

S_N is clearly tensorial, and so $S_N \in \Gamma(T^*M \otimes TM)$. There is a natural relationship between the shape operator and the second fundamental form.

Proposition 3.28 *Let $X, Y \in \Gamma(TM)$ and $N \in \Gamma(NM)$. Then*

$$\langle S_N(X), Y \rangle = \langle B(X, Y), N \rangle.$$

Proof. Indeed, we have

$$\begin{aligned}\langle -(\widetilde{\nabla}_X N)^\top, Y\rangle &= -\langle \widetilde{\nabla}_X N, Y\rangle \\ &= \langle N, \widetilde{\nabla}_X Y\rangle - X\langle N, Y\rangle \\ &= \langle N, (\widetilde{\nabla}_X Y)^\perp\rangle - X(0) \\ &= \langle B(X,Y), N\rangle.\end{aligned}$$

■

Since B is symmetric, S_N is self-adjoint. This fact allows us to prove the relation between the intrinsic and extrinsic normal curvature of NM.

Theorem 3.29 (The Ricci Equation) *Let $X, Y \in \Gamma(TM)$ and $L, N \in \Gamma(NM)$. Then*

$$\langle \widetilde{R}(X,Y)L, N\rangle = \langle R^\perp(X,Y)L, N\rangle - \langle [S_L, S_N]X, Y\rangle$$

where $[S_L, S_N] = S_L \circ S_N - S_N \circ S_L$.

Proof. This is simply a calculation. Since the statement is tensorial in X and Y, we may assume $[X,Y] = 0$. Indeed, we have

$$\begin{aligned}\langle \widetilde{R}(X,Y)L, N\rangle &= \langle \widetilde{\nabla}_X \widetilde{\nabla}_Y L - \widetilde{\nabla}_Y \widetilde{\nabla}_X L, N\rangle \\ &= \langle (\widetilde{\nabla}_X(\nabla^\perp_Y L - S_L Y) - \widetilde{\nabla}_Y(\nabla^\perp_X L - S_L X))^\perp, N\rangle \\ &= \langle \nabla^\perp_X \nabla^\perp_Y L - \nabla^\perp_Y \nabla^\perp_X L, N\rangle + \langle (\widetilde{\nabla}_Y S_L X)^\perp, N\rangle - \langle (\widetilde{\nabla}_X S_L Y)^\perp, N\rangle \\ &= \langle R^\perp(X,Y)L, N\rangle + \langle B(Y, S_L X), N\rangle - \langle B(X, S_L Y), N\rangle \\ &= \langle R^\perp(X,Y)L, N\rangle + \langle S_N(S_L X), Y\rangle - \langle S_N(S_L Y), X\rangle \\ &= \langle R^\perp(X,Y)L, N\rangle + \langle S_N(S_L X), Y\rangle - \langle S_L(S_N X), Y\rangle \\ &= \langle R^\perp(X,Y)L, N\rangle - \langle [S_L, S_N]X, Y\rangle,\end{aligned}$$

where the second to last equality used that S_N and S_L are self-adjoint. ■

3.1 Hypersurfaces

We now suppose that M is an n-dimensional *hypersurface* in $\widetilde{M}^{n+1}$. In this case, NM is a rank-one vector bundle, so if $N, L \in \Gamma(NM)$ are any nonvanishing normal vector fields, then $S_N = fS_M$ for some nonvanishing $f \in C^\infty(M)$. Thus, to understand all the shape operators on M, it suffices to study S_N where N is a unit normal vector field. Such a vector field does not always exist globally. However, using an adapted orthonormal frame $\{E_1, \ldots, E_{n+1}\}$

it is easy to see that locally such a unit normal vector field exists and must be equal to $\pm E_{n+1}$. Throughout this subsection, we suppose that we are restricting ourselves to a sufficiently small open neighborhood of M such that there is a smooth unit normal vector field N.

In the case of a hypersurface with a smooth unit normal vector field, we are able to simplify our expressions for the second fundamental form and the shape operator.

Definition 3.30 *Suppose N is the local unit normal vector field for M. Then we define the* scalar second fundamental form $h\colon \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ *by*

$$h(X, Y) = \langle B(X, Y), N\rangle,$$

so $B(X, Y) = h(X, Y)N$, and h is well defined up to choice of sign.

Remark 3.31 Given a smooth unit normal vector field N, we denote by S the shape operator determined by N, which is also well defined up to a sign.

3.1.1 Gauss Formula, Gauss Equation, and Ricci Equation for Hypersurfaces

One can write the Gauss formula in terms of the scalar second fundamental form as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N.$$

Similarly, the Gauss formula over a curve becomes

$$\widetilde{D}_t V = D_t V + h(\gamma', V)N.$$

Using the *Kulkarni-Nomizu* product and *exterior covariant derivative*, one can also write the Gauss equation and Codazzi equation in terms of h. For more details, see [3]. Moreover, the normal component of $\widetilde{\nabla}_X N$ vanishes when M is a hypersurface. This leads to a simpler expression for the shape operator.

Proposition 3.32 *Suppose $M \subseteq \widetilde{M}$ is a hypersurface with local shape operator S determined by the local unit normal vector field N. Then*

$$(\widetilde{\nabla}_X N)^\perp = 0.$$

Proof. Since $X\langle N, N\rangle = X(1) = 0$, we have $\langle \widetilde{\nabla}_X N, N\rangle = \frac{1}{2} X\langle N, N\rangle = 0$. ■

Corollary 3.33 *Suppose $M \subseteq \widetilde{M}$ is a hypersurface with local shape operator S determined by the local unit normal vector field N. Then*

$$S(X) = -\widetilde{\nabla}_X N.$$

Remark 3.34 We can use the previous corollary to give a geometric interpretation of the shape operator when $\widetilde{M} = \mathbb{R}^{n+1}$. Let $M \subseteq \mathbb{R}^{n+1}$ be a hypersurface, and let $X_p \in T_pM$. Let $\alpha\colon I \to M$ be any curve with $\alpha'(0) = X_p$. Let $N = (N^1, \dots, N^{n+1})$ be a smooth unit normal vector field in a neighborhood of p. Using the previous corollary and the definition of the Euclidean connection, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (N \circ \alpha) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (N^1 \circ \alpha, \dots, N^{n+1} \circ \alpha) \\
&= \left.\frac{\mathrm{d}(N^j \circ \alpha)}{\mathrm{d}t}\right|_0 \frac{\partial}{\partial x^j} \\
&= \left.\frac{\mathrm{d}\alpha^i}{\mathrm{d}t}\right|_0 \left.\frac{\partial N^j}{\partial x^i}\right|_p \frac{\partial}{\partial x^j} \\
&= \overline{\nabla}_{X_p} N \\
&= -S_N(X_p).
\end{aligned}$$

Thus, we see that the shape operator measures the rate of change of the unit normal on M. //

The Ricci equation can be simplified when M is a hypersurface.

Proposition 3.35 *Let $M \subseteq \widetilde{M}$ be a hypersurface. Then the Ricci equation becomes*

$$((\widetilde{R}(X,Y)L)^{\perp} = R^{\perp}(X,Y)L$$

for all $X, Y \in \Gamma(TM)$, $L \in \Gamma(NM)$.

Proof. Since the statement to prove is tensorial in L, it suffices to prove the statement when L has unit norm. Applying Theorem 3.29 gives

$$((\widetilde{R}(X,Y)L)^{\perp} = \langle \widetilde{R}(X,Y)L, L\rangle L = \langle R^{\perp}(X,Y)L, L\rangle L = R^{\perp}(X,Y)L.$$

■

We can say even more about the shape operator and the normal component of $\widetilde{R}(X,Y)L$ if $\widetilde{M}$ has constant sectional curvature (in particular, if $\widetilde{M} = \mathbb{R}^{n+1}$).

Corollary 3.36 *Suppose $M \subseteq \widetilde{M}$ is a hypersurface and $\widetilde{M}$ has constant sectional curvature. Let $S = S_N$ be the shape operator, where N is a unit normal vector field. Then for all $X, Y \in \Gamma(TM)$ and $L \in \Gamma(NM)$, we have*

a. $(\nabla_X S)(Y) = (\nabla_Y S)(X)$, *b.* $\widetilde{R}(X,Y)L = R^{\perp}(X,Y)L$.

Proof. We first prove part a. Fix $X, Y, Z \in \Gamma(TM)$. Since $(\widetilde{\nabla}_X N)^{\perp} = 0$ by Proposition 3.32, this implies
$$\langle B(Y,Z), \nabla^{\perp}_X N\rangle = 0. \tag{19}$$
Moreover, recall that the covariant derivative of B satisfies
$$(\nabla_X B)(Y,Z) = \nabla^{\perp}_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{20}$$
Applying (19) and (20), we have
$$\begin{aligned}
\langle(\nabla_X S)(Y), Z\rangle &= \langle \nabla_X(S(Y)), Z\rangle - \langle S(\nabla_X Y), Z\rangle \\
&= (X\langle S(Y), Z\rangle - \langle S(Y), \nabla_X Z\rangle) - \langle S(\nabla_X Y), Z\rangle \\
&= X\langle B(Y,Z), N\rangle - \langle B(Y, \nabla_X Z), N\rangle - \langle B(\nabla_X Y, Z), N\rangle \\
&= (\langle \nabla^{\perp}_X B(Y,Z), N\rangle + \langle B(Y,Z), \nabla^{\perp}_X N\rangle) \\
&\quad - \langle B(Y, \nabla_X Z), N\rangle - \langle B(\nabla_X Y, Z), N\rangle
\end{aligned}$$

But $\nabla^{\perp}_X N = 0$ by Proposition 3.32, so
$$\begin{aligned}
\langle(\nabla_X S)(Y), Z\rangle &= \langle \nabla^{\perp}_X B(Y,Z), N\rangle - \langle B(Y, \nabla_X Z), N\rangle - \langle B(\nabla_X Y, Z), N\rangle \\
&= \langle(\nabla_X B)(Y,Z), N\rangle.
\end{aligned}$$

The proof of part a now follows immediately from Corollary 3.25.

We now prove part b. Since the statement to prove is tensorial, we may assume $L = N$, and we may assume $[X,Y] = 0$. By Proposition 3.35, it suffices to show that $(\widetilde{R}(X,Y)N)^{\top} = 0$. From Corollary 3.33, we have
$$\begin{aligned}
(\widetilde{R}(X,Y)N)^{\top} &= (\widetilde{\nabla}_X\widetilde{\nabla}_Y N - \widetilde{\nabla}_Y\widetilde{\nabla}_X N)^{\top} \\
&= -\nabla_X(S(Y)) + \nabla_Y(S(X)) \\
&= -(\nabla_X S)(Y) - S(\nabla_X Y) + (\nabla_Y S)(X) + S(\nabla_Y X).
\end{aligned}$$

Now applying part a, we have
$$\begin{aligned}
(\widetilde{R}(X,Y)N)^{\top} &= -(S(\nabla_X Y) - S(\nabla_Y X)) \\
&= -S([X,Y]) = 0.
\end{aligned}$$

■

Much of our time so far has been concerned with unit normal vector fields on M. The question arises: is there an easy way to find N? In general, we do not want to have to perform Gram-Schmidt to find such a unit normal vector field. However, if $f\colon \widetilde{M} \to \mathbb{R}$ is a smooth map and $M = f^{-1}(c)$ is a regular level set, then

$$N = \frac{\boldsymbol{\nabla} f}{\|\boldsymbol{\nabla} f\|}$$

is a smooth unit normal vector field for M. Indeed, if $Y \in \Gamma(TM)$, then

$$\langle \boldsymbol{\nabla} f, Y \rangle = Yf = 0$$

since f is constant on M.

Example 3.37 The unit sphere $\mathbb{S}^n$ is given by $f^{-1}(1)$ where $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is given by $f(x) = \|x\|^2$. The unit normal on $\mathbb{S}^n$ is given by

$$N = \frac{\nabla f}{\|\nabla f\|} = \frac{(2x^1, \ldots, 2x^{n+1})}{2\|x\|} = (x^1, \ldots, x^{n+1}) = x.$$

That is, the position vector field is normal to $\mathbb{S}^n$. By Corollary 3.33, the shape operator for $\mathbb{S}^n$ is given by $S_N(f^i \frac{\partial}{\partial x^i}) = -\overline{\nabla}_{f^i \frac{\partial}{\partial x^i}}(x^j \frac{\partial}{\partial x^j}) = -f^i \frac{\partial x^j}{\partial x^i}\frac{\partial}{\partial x^j} - f^i x^j \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = -f^i \frac{\partial}{\partial x^i}$, so $S_N = -\operatorname{Id}$. Thus, for any vector fields $X, Y, Z, W \in \Gamma(T\mathbb{S}^N)$, we have

$$\begin{aligned}
\langle B(X,Y), B(Z,W) \rangle &= \langle \langle B(X,Y), N \rangle N, \langle B(Z,W), N \rangle N \rangle \\
&= \langle B(X,Y), N \rangle \langle B(Z,W), N \rangle \\
&= \langle S_N X, Y \rangle \langle S_N Z, W \rangle \\
&= \langle X, Y \rangle \langle Z, W \rangle.
\end{aligned}$$

Applying Theorem 3.19, the curvature tensor of $\mathbb{S}^n$ is given by

$$R(X,Y,Z,W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

In particular, by taking an orthonormal basis of any two-dimensional subspace of T_pM, it is easy to see that the sectional curvature of $\mathbb{S}^n$ is 1 everywhere. //

3.1.2 Computing h and S for Euclidean Hypersurfaces

In this section, indices α, β, γ range between 1 and n, while indices i, j, k range between 1 and $n+1$. We now address how to compute the second fundamental form when $\widetilde{M} = \mathbb{R}^{n+1}$. Suppose $M \subseteq \mathbb{R}^{n+1}$ is a hypersurface with a local parametrization $\varphi\colon U \to \mathbb{R}^{n+1}$. That is, φ is an embedding of U whose image is an open subset of M. Then $(u^1, \ldots, u^n)$ are local coordinates for M, and identifying T_pM with the image of T_pM under $[\varphi_*]_p$, we have

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial \varphi^j}{\partial u^\alpha} \frac{\partial}{\partial x^j}. \tag{21}$$

Proposition 3.38 (From [4, Proposition 8.23]) *Let $M \subseteq \mathbb{R}^{n+1}$ be a hypersurface of Euclidean space, and consider the setup above. If N is a local unit normal vector field for M, then*

$$h\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = \left\langle \frac{\partial^2 \varphi}{\partial u^\alpha \partial u^\beta}, N \right\rangle$$

Proof. First note that

$$\overline{\nabla}_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u^\beta} = \overline{\nabla}_{\frac{\partial \varphi^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}} \frac{\partial \varphi^j}{\partial u^\beta}\frac{\partial}{\partial x^j} = \frac{\partial \varphi^i}{\partial u^\alpha}\frac{\partial^2 \varphi^j}{\partial x^i \partial u^\beta}\frac{\partial}{\partial x^j} = \frac{\partial^2 \varphi^j}{\partial u^\alpha \partial u^\beta}\frac{\partial}{\partial x^j} = \frac{\partial^2 \varphi}{\partial u^\alpha \partial u^\beta},$$

which implies

$$h\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = \left\langle B\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right), N \right\rangle = \left\langle \overline{\nabla}_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u^\beta}, N \right\rangle = \left\langle \frac{\partial^2 \varphi}{\partial u^\alpha \partial u^\beta}, N \right\rangle.$$

■

Example 3.39 Let $M \subseteq \mathbb{R}^3$ be the cylinder of radius one about the z-axis. A local parametrization for M is $\varphi\colon (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3$ given by $\varphi(\theta, z) = (\cos\theta, \sin\theta, z)$. The vector field $N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^3$ is normal to M at all points in M, and is unit at all points on M.

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial \theta^2} &= (-\cos\theta, -\sin\theta, 0), \\
\frac{\partial^2 \varphi}{\partial \theta \partial z} &= (0, 0, 0), \\
\frac{\partial^2 \varphi}{\partial z^2} &= (0, 0, 0).
\end{aligned}$$

We can extend these to vector fields on $\mathbb{R}^3$ by defining $\frac{\partial^2 \varphi}{\partial \theta^2} = (-x, -y, 0)$. Using the previous proposition, we have

$$\begin{aligned}
h\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) &= \langle(-x, -y, 0), (x, y, 0)\rangle = -x^2 - y^2 = -1, \\
h\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right) &= h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = 0. \quad /\!\!/
\end{aligned}$$

When we are given a local defining function, we can compute the scalar second fundamental form using the Hessian.

Proposition 3.40 *Let f be a local defining function for M, and let $N = \frac{\nabla f}{|\nabla f|}$. Then the scalar second fundamental form of M with respect to the unit normal N is given by*

$$h(X,Y) = -\frac{\operatorname{Hess}(f)(X,Y)}{|\nabla f|}.$$

Proof. Indeed, for any vector fields tangent to M, we have

$$\begin{aligned} h(X,Y) &= \langle \widetilde{\nabla}_X Y, N\rangle \\ &= \frac{\langle \widetilde{\nabla}_X Y, \nabla f\rangle}{|\nabla f|} \\ &= \frac{(\widetilde{\nabla}_X Y)(f)}{|\nabla f|}. \end{aligned}$$

Now Y is tangent and ∇f is normal, so $X\langle Y, \nabla f\rangle = X(0) = 0$. Thus,

$$\begin{aligned} h(X,Y) &= -\frac{X\langle Y, \nabla f\rangle - (\widetilde{\nabla}_X Y)(f)}{|\nabla f|} \\ &= -\frac{X(Yf) - (\widetilde{\nabla}_X Y)(f)}{|\nabla f|} \\ &= -\frac{\operatorname{Hess}(f)(X,Y)}{|\nabla f|}. \end{aligned}$$

■

Remark 3.41 (Shape Operator Expression in Local Coordinates)
Suppose $(E_1, \dots, E_n)$ is a local frame for TM. Then with respect to this basis, the matrix for S is given by

$$\begin{aligned} \langle S_i^l E_l, E_k\rangle &= \langle S(E_i), E_k\rangle \\ &= \langle B(E_i, E_k), N\rangle \\ &= h(E_i, E_k), \end{aligned}$$

so $S_i^l g_{lk} = h_{ik}$. Thus, $S_i^j = h_{ik} g^{kj}$. //

Example 3.42 Suppose $U \subseteq \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}$ is smooth. Let $M = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^{n+1}$ be the graph of f, endowed with the induced Riemannian metric and upward unit normal. Let us find the components of the shape operator in graph coordinates.

Let $F\colon U \times \mathbb{R} \to \mathbb{R}$ be given by $F(\mathbf{x}, y) = y - f(\mathbf{x})$. Notice that $M = F^{-1}(0)$, so an upward pointing unit normal for M is given by

$$N = \frac{\boldsymbol{\nabla} F}{\|\boldsymbol{\nabla} F\|} = \frac{(-\partial_1 f, \ldots, -\partial_n f, 1)}{\sqrt{1 + \|\boldsymbol{\nabla} f\|^2}}.$$

Now define $\varphi\colon U \to U \times \mathbb{R}$ to be the Monge patch for f, given by $\varphi(u) = (u, f(u))$. The pair (φ, U) is a global chart for f with local coordinates $(u^1, \ldots, u^n)$. By (21), we have

$$\frac{\partial}{\partial u^i} = \frac{\partial}{\partial x^i} + \frac{\partial f}{\partial u^i}\frac{\partial}{\partial x^{n+1}}, \tag{22}$$

and applying Proposition 3.38, we have

$$\begin{aligned} h_{ij} &= \left\langle \frac{\partial^2 \varphi}{\partial u^i \partial u^j}, N \right\rangle \\ &= \frac{1}{\sqrt{1 + \|\boldsymbol{\nabla} f\|^2}} \left\langle \left(0, \ldots, 0, \frac{\partial^2 f}{\partial u^i \partial u^j}\right), \left(-\frac{\partial f}{\partial u^1}, \ldots, -\frac{\partial f}{\partial u^n}, 1\right) \right\rangle \\ &= (1 + \|\boldsymbol{\nabla} f\|^2)^{-1/2} \frac{\partial^2 f}{\partial u^i \partial u^j}. \end{aligned} \tag{23}$$

Moreover, computing the g_{ij} in the $\left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\right)$ basis using (22), we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}.$$

By direct computation, it is easy to verify that

$$g^{ij} = \delta_{ij} - \frac{1}{1 + \|\boldsymbol{\nabla} f\|^2}\frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}. \tag{24}$$

Combining (23) and (24) gives

$$\begin{aligned} S_i^j &= h_{ik} g^{kj} \\ &= \sum_{k=1}^n (1 + \|\boldsymbol{\nabla} f\|^2)^{-1/2} \frac{\partial^2 f}{\partial u^i \partial u^k} \left(\delta_{kj} - \frac{1}{1 + \|\boldsymbol{\nabla} f\|^2}\frac{\partial f}{\partial u^k}\frac{\partial f}{\partial u^j} \right) \\ &= (1 + \|\boldsymbol{\nabla} f\|^2)^{-1/2} \frac{\partial^2 f}{\partial u^i \partial u^j} - \frac{1}{\left(1 + \|\boldsymbol{\nabla} f\|^2\right)^{3/2}} \frac{\partial f}{\partial u^j} \sum_{k=1}^n \frac{\partial^2 f}{\partial u^i \partial u^k}\frac{\partial f}{\partial u^k}. \end{aligned}$$

This is a fairly ugly looking expression, but it will allows us to compute the shape operator for many specific examples. //

Another fairly general class of surfaces are *surfaces of revolution*.

Example 3.43 Let C be an embedded smooth curve in the half-plane $H = \{(r, z) \colon r > 0\}$, and $S_C \subseteq \mathbb{R}^3$ be the surface of revolution determined by C.

Let $\gamma(t) = (a(t), b(t))$ be a local unit speed parametrization of C. A local parametrization of S_C is given by

$$\varphi(t, \theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t)).$$

We complete all the following computations in the general case and then specialize to the unit speed case at the end. The metric on S_C is given by

$$\begin{aligned}
g = \varphi^*\overline{g} &= d(a\cos\theta)^2 + d(a\sin\theta)^2 + d(b)^2 \\
&= (-a\sin(\theta)d\theta + \dot{a}\cos(\theta)dt)^2 + (a\cos(\theta)d\theta + \dot{a}\sin(\theta)dt)^2 + \dot{b}^2dt^2 \\
&= a^2\sin^2(\theta)d\theta^2 + \dot{a}^2\cos^2(\theta)dt^2 + a^2\cos^2(\theta)d\theta^2 + \dot{a}^2\sin^2(\theta)dt^2 + \dot{b}^2dt^2 \\
&= (\dot{a}^2 + \dot{b}^2)dt^2 + a^2d\theta^2
\end{aligned}$$

The components of the inverse matrix for g are given by

$$\begin{aligned}
g^{11} &= \frac{1}{\dot{a}^2 + \dot{b}^2} \\
g^{12} &= 0 \\
g^{22} &= \frac{1}{a^2}
\end{aligned}$$

We now compute the shape operator of S_C in terms of a and b, and we show that the principal directions (eigenspaces of S_C) at each point are tangent to the meridians and latitude circles.

To calculate the unit normal, we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial t} &= (\dot{a}\cos\theta, \dot{a}\sin\theta, \dot{b}) \\
\frac{\partial\varphi}{\partial\theta} &= (-a\sin\theta, a\cos\theta, 0),
\end{aligned}$$

which gives

$$\frac{\partial\varphi}{\partial t} \times \frac{\partial\varphi}{\partial\theta} = (-a\dot{b}\cos\theta, -a\dot{b}\sin\theta, a\dot{a}).$$

Thus, our unit normal is given by

$$N = \frac{\frac{\partial\varphi}{\partial t} \times \frac{\partial\varphi}{\partial\theta}}{|\frac{\partial\varphi}{\partial t} \times \frac{\partial\varphi}{\partial\theta}|} = \frac{1}{\sqrt{\dot{a}^2 + \dot{b}^2}}(-\dot{b}\cos\theta, -\dot{b}\sin\theta, \dot{a}).$$

Let $u^1 = t$, $u^2 = \theta$. Computing $h_{ij} = h\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$ using Proposition 3.38 gives

$$\begin{aligned}
h_{11} &= \left\langle \frac{\partial^2 \varphi}{\partial t^2}, N \right\rangle \\
&= \frac{1}{\sqrt{\dot{a}^2 + \dot{b}^2}} \left\langle (\ddot{a}\cos\theta, \ddot{a}\sin\theta, \ddot{b}), (-\dot{b}\cos\theta, -\dot{b}\sin\theta, \dot{a}) \right\rangle, \\
&= \frac{\dot{a}\ddot{b} - \ddot{a}\dot{b}}{\sqrt{\dot{a}^2 + \dot{b}^2}} \\
h_{12} &= \left\langle \frac{\partial^2 \varphi}{\partial t \partial \theta}, N \right\rangle \\
&= \frac{1}{\sqrt{\dot{a}^2 + \dot{b}^2}} \left\langle (-\dot{a}\sin\theta, \dot{a}\cos\theta, 0), (-\dot{b}\cos\theta, -\dot{b}\sin\theta, \dot{a}) \right\rangle \\
&= 0, \\
h_{22} &= \dot{a}\ddot{b} - \ddot{a}\dot{b} \left\langle \frac{\partial^2 \varphi}{\partial \theta^2}, N \right\rangle \\
&= \frac{1}{\sqrt{\dot{a}^2 + \dot{b}^2}} \left\langle (-a\cos\theta, -a\sin\theta, 0), (-\dot{b}\cos\theta, -\dot{b}\sin\theta, \dot{a}) \right\rangle \\
&= \frac{a\dot{b}}{\sqrt{\dot{a}^2 + \dot{b}^2}}.
\end{aligned}$$

Computing the components of the shape operator S using $S_i^j = h_{ik} g^{kj}$, we find

$$\begin{aligned}
S_1^1 &= h_{1k} g^{k1} = h_{11} g^{11} = \frac{\dot{a}\ddot{b} - \ddot{a}\dot{b}}{(\dot{a}^2 + \dot{b}^2)^{3/2}}, \\
S_1^2 &= h_{1k} g^{k2} = h_{12} g^{22} = 0, \\
S_2^1 &= h_{2k} g^{k1} = h_{22} g^{21} = 0, \\
S_2^2 &= h_{2k} g^{k2} = h_{22} g^{22} = \frac{\dot{b}}{a\sqrt{\dot{a}^2 + \dot{b}^2}}.
\end{aligned}$$

In the unit speed case, this becomes

$$\begin{aligned}
S_1^1 &= \dot{a}\ddot{b} - \ddot{a}\dot{b}, \\
S_1^2 &= S_2^1 = 0, \\
S_2^2 &= \frac{\dot{b}}{a}. \quad /\!\!/
\end{aligned}$$

Let us return to the torus one more time and compute its shape operator.

Example 3.44 Consider the immersion of the torus into $\mathbb{S}^3$ from Example 2.2e. Recall that in Example 3.2 we showed that

$$e_1 = (-\sin\theta, \cos\theta, 0, 0),$$
$$e_2 = (0, 0, -\sin\varphi, \cos\varphi)$$

form an orthonormal basis of the tangent space, and the vectors

$$n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi),$$
$$n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$$

form an orthonormal basis of the normal space, as an immersed submanifold of $\mathbb{R}^4$. Recall that the recall that $x\colon \mathbb{R}^2 \to \mathbb{R}^4$ is given by

$$x(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi).$$

So n_1 is the position vector at (θ, φ) and thus is normal to $S^3(1)$ by Example 3.37. Hence, n_2 is the normal vector for the torus as an immersed submanifold of $\mathbb{S}^3$. Let $\widetilde{\nabla}$ be the connection on $\mathbb{S}^3$ induced from the Euclidean connection $\overline{\nabla}$ on $\mathbb{R}^4$. Let us compute the shape operator S for the hypersurface $\mathbb{T}^2$ using the fact that

$$\langle S(e_i), e_j\rangle = -\langle \widetilde{\nabla}_{e_i} n_2, e_j\rangle = \langle \widetilde{\nabla}_{e_i} e_j, n_2\rangle = \langle \overline{\nabla}_{e_i} e_j, n_2\rangle.$$

We first need to extend e_1,e_2 to $\mathbb{R}^4$. Let $\{\partial_1, \ldots, \partial_4\}$ be the global frame for $\mathbb{R}^4$, and recall that $x\colon \mathbb{R}^2 \to \mathbb{R}^4$ is given by

$$x(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi).$$

Thus, we can extend e_1, e_2 to all of $\mathbb{R}^4$ by

$$\bar{e}_1 = \sqrt{2}(-x^2, x^1, 0, 0) = -\sqrt{2}x^2\partial_1 + \sqrt{2}x^1\partial_2,$$
$$\bar{e}_2 = \sqrt{2}(0, 0, -x^4, x^3) = -\sqrt{2}x^4\partial_3 + \sqrt{2}x^3\partial_4,$$

Using the product rule and the fact that $\overline{\nabla}_{\partial_i}\partial_j = 0$ for all i and j,

$$\begin{aligned}\overline{\nabla}_{e_1} e_1 &= 2(-x^2\overline{\nabla}_{\partial_1}(-x^2\partial_1 + x^1\partial_2) + x^1\overline{\nabla}_{\partial_2}(-x^2\partial_1 + x^1\partial_2))\\ &= 2(-x^2\partial_2 - x^1\partial_1)\end{aligned}$$

$$
\begin{aligned}
&= 2\left(-\frac{1}{\sqrt{2}}\cos\theta, -\frac{1}{\sqrt{2}}\sin\theta, 0, 0\right) \\
&= \left(-\sqrt{2}\cos\theta, -\sqrt{2}\sin\theta, 0, 0\right),
\end{aligned}
$$

where in the second to last equality we restricted $\overline{\nabla}_{e_1} e_1$ to the torus. Similarly, we have

$$
\begin{aligned}
\overline{\nabla}_{e_1} e_2 &= 2(-x^2\overline{\nabla}_{\partial_1}(-x^4\partial_3 + x^3\partial_4) + x^1\overline{\nabla}_{\partial_2}(-x^4\partial_3 + x^3\partial_4)) \\
&= 0 \\
\overline{\nabla}_{e_2} e_2 &= 2(-x^4\overline{\nabla}_{\partial_3}(-x^4\partial_3 + x^3\partial_4) + x^3\overline{\nabla}_{\partial_4}(-x^4\partial_3 + x^3\partial_4)) \\
&= 2(-x^4\partial_4 - x^3\partial_3) \\
&= \left(0, 0, -\sqrt{2}\cos\varphi, -\sqrt{2}\sin\varphi\right).
\end{aligned}
$$

Computing the components of S gives

$$
\begin{aligned}
\langle S_{n_2}(e_1), e_1\rangle &= \langle \overline{\nabla}_{e_1} e_1, n_2\rangle \\
&= \left\langle \left(-\sqrt{2}\cos\theta, -\sqrt{2}\sin\theta, 0, 0\right), \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)\right\rangle \\
&= \cos^2\theta + \sin^2\theta \\
&= 1, \\
\langle S_{n_2}(e_1), e_2\rangle &= \langle \overline{\nabla}_{e_1} e_2, n_2\rangle = 0, \\
\langle S_{n_2}(e_2), e_2\rangle &= \langle \overline{\nabla}_{e_2} e_2, n_2\rangle \\
&= \left\langle \left(0, 0, -\sqrt{2}\cos\varphi, -\sqrt{2}\sin\varphi\right), \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)\right\rangle \\
&= -\cos^2\varphi - \sin^2\varphi \\
&= -1,
\end{aligned}
$$

so $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. //

3.1.3 Gaussian and Mean Curvature

We have seen that the shape operator S is a tensorial, self-adjoint operator. Thus, for each $p \in M$, S_p has a basis of eigenvectors $\{e_1, \ldots, e_n\}$ with corresponding real eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. The eigenvectors of S_p are called the *principal directions* of M at p, and the eigenvalues of S_p are called the *principal curvatures* of M at p. We define the *Gaussian curvature* of M at p to be $K(p) = \det(S_p)$, and we define the *mean curvature* of M at p to be $H(p) = \mathrm{tr}(S_p)$. In terms of the eigenvalues of S at p, we have

$$
K(p) = \lambda_1 \cdots \lambda_n,
$$

$$H(p) = \lambda_1 + \cdots + \lambda_n.$$

If we change the sign of N, then K changes by a factor of $(-1)^n$ and H changes by a factor of -1.

Example 3.45 Let $M \subseteq \mathbb{R}^{n+1}$ be the n-dimensional paraboloid defined as the graph of $f(x) = \|x\|^2$. Let us compute the principal curvatures of M. By symmetry of the graph about the y-axis, it suffices to compute the principal curvatures at $(a, 0, \ldots, 0)$ and then substitute $\sqrt{(x^1)^2 + \cdots + (x^n)^2}$ for a at the end. Now we have

$$\begin{aligned} \frac{\partial f}{\partial x^i} &= 2x^i \\ \frac{\partial^2 f}{\partial x^i \partial x^j} &= 2\delta_{ij}. \end{aligned}$$

Applying Example 3.42 at the point $(a, 0, \ldots, 0)$, we have

$$\begin{aligned} S_i^j &= 2\delta_{ij}(1+4a^2)^{-1/2} - \frac{2\delta_{1j}a}{(1+4a^2)^{3/2}} \sum_{k=1}^n (2\delta_{ik})(2\delta_{1k}a), \\ &= 2\delta_{ij}(1+4a^2)^{-1/2} - \frac{8\delta_{1j}\delta_{1i}a^2}{(1+4a^2)^{3/2}}. \end{aligned}$$

The matrix (S_i^j) is diagonal and thus the principal curvatures are the diagonal entries. We have

$$\begin{aligned} S_1^1 &= \frac{2(1+4a^2) - 8a^2}{(1+4a^2)^{3/2}} = \frac{2}{(1+4a^2)^{3/2}}, \\ S_j^j &= \frac{2}{(1+4a^2)^{1/2}}, \qquad 2 \leq j \leq n. \end{aligned}$$

Substituting $\sqrt{(x^1)^2 + \cdots + (x^n)^2}$ for a as previously discussed, we obtain principal curvatures given by

$$\begin{aligned} \kappa_1 &= \frac{2}{[1+4((x^1)^2 + \cdots + (x^n)^2)]^{3/2}}, \\ \kappa_i &= \frac{2}{[1+4((x^1)^2 + \cdots + (x^n)^2)]^{1/2}}, \qquad 2 \leq i \leq n. \end{aligned}$$

In particular, the Gaussian curvature is $K = 2^n[1 + 4((x^1)^2 + \cdots + (x^n)^2)]^{-(n+2)/2}$. //

Example 3.46 For $\lambda > 0$, let $M_\lambda \subseteq \mathbb{R}^3$ be the surface of revolution obtained by revolving the curve $\gamma(t) = (\lambda \cosh(t/\lambda), 0, t)$ in the xz-plane around the z-axis, called a *catenoid*. Let us compute the mean curvature of the catenoid. We apply the results from Example 3.43 when $a(t) = \lambda \cosh(t/\lambda)$ and $b(t) = t$.

$$\begin{aligned}
H &= \frac{\dot{a}\ddot{b} - \ddot{a}\dot{b}}{(\dot{a}^2 + \dot{b}^2)^{3/2}} + \frac{\dot{b}}{a\sqrt{\dot{a}^2 + \dot{b}^2}} \\
&= \frac{1}{a(\dot{a}^2 + \dot{b}^2)^{3/2}}(a\dot{a}\ddot{b} - a\ddot{a}\dot{b} + \dot{b}(\dot{a}^2 + \dot{b}^2)) \\
&= \frac{1}{a(\dot{a}^2 + 1)^{3/2}}(-a\ddot{a} + (\dot{a}^2 + 1)) \\
&= \frac{1}{a(\dot{a}^2 + 1)^{3/2}}(-\lambda \cosh(t/\lambda)((1/\lambda)\cosh(t/\lambda)) + \sinh^2(t/\lambda) + 1) \\
&= \frac{1}{a(\dot{a}^2 + 1)^{3/2}}(-\cosh^2(t/\lambda) + \sinh^2(t/\lambda) + 1) \\
&= 0.
\end{aligned}$$

So a catenoid has *zero* mean curvature. Such surfaces are called *minimal surfaces*. //

Notice that the Gaussian and mean curvature are defined in terms of a particular embedding into an ambient manifold, so we should not expect them to be invariant under a local isometry. Indeed, the mean curvature is *not* invariant under local isometries, as the next example shows.

Example 3.47 Let $M_1 \subseteq \mathbb{R}^3$ be the xy plane and let $M_2 \subseteq \mathbb{R}^3$ be a cylinder of radius 1 about the z axis. Let g_1 be the metric for the plane and let g_2 be the induced metric for the cylinder. For the plane, we have

$$g_1 = dx^2 + dy^2.$$

Now local coordinates for the cylinder are given by (θ, z), where the relation to the standard Euclidean coordinates is

$$\begin{aligned}
x &= \cos\theta, \\
y &= \sin\theta, \\
z &= z.
\end{aligned}$$

Thus, the metric for M_2 is given by

$$\begin{aligned}
g_2 &= d(\cos\theta)^2 + d(\sin\theta)^2 + dz^2 \\
&= \sin^2(\theta)d\theta^2 + \cos^2(\theta)d\theta^2 + dz^2 \\
&= d\theta^2 + dz^2.
\end{aligned}$$

Thus, the local smooth map $(\theta(x, y), z(x, y)) = (x, y)$ is a local isometry from M_1 to M_2. Note that the matrix of the pushforward is the identity matrix, so we can use the Inverse

Function Theorem to obtain smoothness of the inverse map.

We now want to compute the shape operators for M_1 and M_2. First, notice that the local parametrization for M_1 is $X(x,y) = (x,y,0)$, and all second order derivatives of X vanish. Thus, the scalar second fundamental form h^1 for M_1 vanishes by Proposition 3.38. Calculating the entries of the shape operators for M_1 using Remark 3.41 gives

$$(S_1)_i^j = h^1_{ik} g_1^{kj} = 0 \cdot \delta^{kj} = 0.$$

A normal vector for M_2 is $N_2(\theta, z) = (\cos\theta, \sin\theta, 0)$. Calculating the entries of the shape operators for M_1 and M_2 using Remark 3.41 and Example 3.39, we have $(S_2)_\theta^\theta = -1$, $(S_2)_\theta^z = 0$, $(S_2)_z^z = 0$. Thus, $H_1 = K_1 = K_2 = 0$, $H_2 = -1$. The mean curvatures of M_1 and M_2 are not equal, so the mean curvature is not preserved by a local isometry. However, we see that the Gaussian curvatures are equal. //

In the previous example, we found that the Gaussian curvature was preserved by a local isometry. This makes us curious: is the Gaussian curvature of Euclidean hypersurface $M \subseteq \mathbb{R}^{n+1}$ invariant under local isometries? This would be very suprising, because $K(p)$ is defined in terms of a *particular* embedding of M into $\mathbb{R}^{n+1}$. And in fact, it is easy to see that this is not be true when n is odd, because changing the sign of the unit normal changes the Gaussian curvature by a factor of $(-1)^n = -1$. Amazingly, the Gaussian curvature *is* invariant under local isometries when n is even, and the absolute value of the Gaussian curvature is invariant when n is odd. We prove the $n = 2$ case, which is known as Gauss's Theorema Egregium.

Theorem 3.48 (Gauss's Theorema Egregium - proof from [2, Remark 2.7]) *Suppose $M \subseteq \mathbb{R}^3$ is a hypersurface. Then $K(p) = \frac{1}{2}Sc(p)$. Thus, the Gaussian curvature is preserved under local isometries.*

Proof. Fix $p \in M$, unit normal N, and let $\{e_1, e_2\}$ be an orthonormal basis of T_pM formed by eigenvectors of the shape operator S_p. Then $h(e_i, e_j) = \langle S(e_i), e_j\rangle = \lambda_i \delta_{ij}$. Now applying Corollary 3.20, we get

$$\begin{aligned} sec(p) = sec(e_1, e_2) &= \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)|^2 \\ &= \langle h(e_1,e_1)N, h(e_2,e_2)N\rangle - |h(e_1,e_2)N|^2 \\ &= \lambda_1\lambda_2\langle N, N\rangle - 0 \\ &= \lambda_1\lambda_2 \\ &= K(p). \end{aligned}$$

With respect to the basis $\{e_1, e_2\}$ the matrix entries for g are $g_{ij} = \delta_{ij}$ and $g^{ij} = \delta^{ij}$. Thus, $R(e_1,e_2,e_2,e_1) + R(e_2,e_1,e_1,e_2) = R_{ijkl}g^{il}g^{jk} = Sc(p)$. We conclude that

$$sec(p) = R(e_1, e_2, e_2, e_1)$$

$$= \frac{1}{2}(R(e_1, e_2, e_2, e_1) + R(e_2, e_1, e_1, e_2))$$
$$= \frac{1}{2} Sc(p).$$

■

4 Riemannian Submersions

Let $\pi : M^n \to B^m$ be a surjective submersion of Riemannian manifolds. We call M the *total space* and we call B the *base*. By Theorem 2.12, $\pi^{-1}(p)$ is a properly embedded submanifold of M.

By the Rank Theorem, there are local coordinates $(x^1, \ldots, x^n)$ for M and local coordinates $(y^1, \ldots, y^m)$ for B such that

$$(y^1, \ldots, y^m) = \pi(x^1, \ldots, x^n) = (x^1, \ldots, x^m).$$

We define the *vertical tangent space* at $p \in M$ as $V_p = \ker[\pi_*]_p$. By the above coordinate representation for π, we conclude $V_p = \operatorname{span}\{\frac{\partial}{\partial x^{m+1}}|_p, \ldots, \frac{\partial}{\partial x^n}|_p\}$, so V_p is an $(n-m)$-dimensional vector space. Using the metric on M, we define the *horizontal tangent space* as

$$H_p = (V_p)^{\perp},$$

so we we have

$$T_pM = V_pM \oplus H_pM.$$

We are now ready to define a Riemannian submersion.

Definition 4.1 *Let $\pi : M \to B$ be a submersion of Riemannian manifolds. Then π is called a* Riemannian submersion *if $[\pi_*]_p$ maps H_pM isometrically onto T_pB. That is, if $p \in M$, and $X_p, Y_p \in H_p$, then*

$$\langle X_p, Y_p \rangle = \langle [\pi_*]_p X_p, [\pi_*]_p Y_p \rangle.$$

Example 4.2 Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, and consider the product manifold $(M_1 \times M_2, g)$, where $g = g_1 + g_2$ is the product metric. Fix $(p, q) \in M_1 \times M_2$. Let $(x^1, \ldots, x^n)$ be local coordinates for M_1 about p and let $(y^1, \ldots, y^m)$ be local coordinates for M_2 about q. Then we have local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ for $M_1 \times M_2$. Let $\pi : M_1 \times M_2 \to M_1$ be the canonical projection onto M_1. In local coordinates, $\pi(x^1, \ldots, x^n, y^1, \ldots, y^m) = (x^1, \ldots, x^n)$. Now fix any $X_{(p,q)}, Y_{(p,q)} \in H_{(p,q)}$, say

$$X_{(p,q)} = a^i \frac{\partial}{\partial x^i}\bigg|_{(p,q)},$$

$$Y_{(p,q)} = c^i \frac{\partial}{\partial x^i}\bigg|_{(p,q)}.$$

Then

$$\begin{aligned}
[\pi_*]_{(p,q)} X_{(p,q)} &= a^i \frac{\partial}{\partial x^i}\bigg|_p, \\
[\pi_*]_{(p,q)} Y_{(p,q)} &= c^i \frac{\partial}{\partial x^i}\bigg|_p,
\end{aligned}$$

so

$$g\left(X_{(p,q)}, Y_{(p,q)}\right) = g_1\left(a^i \frac{\partial}{\partial x^i}\bigg|_p, c^i \frac{\partial}{\partial x^i}\bigg|_p\right) = g_1([\pi_*]_{(p,q)} X_{(p,q)}, [\pi_*]_{(p,q)} Y_{(p,q)}).$$

Thus, π is a Riemannian submersion. //

Example 4.3 Consider the vector bundle construction from section 2.4. We claim that $\pi : E \to M$ is a Riemannian submersion. Fix any $\vartheta \in E$, and write $\vartheta = (x^1, \ldots, x^n, V^1, \ldots, V^k)$ in local coordinates. Consider the horizontal vectors

$$\begin{aligned}
X_\vartheta &= \frac{\partial}{\partial x^a} - \Gamma^l_{aj} V^j \frac{\partial}{\partial y^l}, \\
Y_\vartheta &= \frac{\partial}{\partial x^b} - \Gamma^t_{bs} V^s \frac{\partial}{\partial y^t}.
\end{aligned}$$

Then

$$\begin{aligned}
\widehat{g}\left(X_\vartheta, Y_\vartheta\right) &= \widehat{g}\left(\frac{\partial}{\partial x^a} - \Gamma^l_{aj} V^j \frac{\partial}{\partial y^l}, \frac{\partial}{\partial x^b} - \Gamma^t_{bs} V^s \frac{\partial}{\partial y^t}\right) \\
&= \widehat{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) - \Gamma^l_{aj} V^j \widehat{g}\left(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial y^l}\right) - \Gamma^t_{bs} V^s \widehat{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^t}\right) \\
&\quad + \Gamma^l_{aj} \Gamma^t_{bs} V^j V^s \widehat{g}\left(\frac{\partial}{\partial y^l}, \frac{\partial}{\partial y^t}\right).
\end{aligned}$$

Now using Proposition 2.43, all the terms with Christoffel symbols cancel, and we obtain

$$\begin{aligned}
\widehat{g}\left(X_\vartheta, Y_\vartheta\right) &= g_{ab} \\
&= g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) \\
&= g\left([\pi_*]_p X_\vartheta, [\pi_*]_p Y_\vartheta\right),
\end{aligned}$$

so $\pi : E \to M$ is a Riemannian submersion. //

We have seen that T_pM can be decomposed into horizontal and vertical tangent spaces for every $p \in M$. The next proposition shows that this decomposition is smooth over M.

Proposition 4.4 *Let $X \in \Gamma(TM)$ be a smooth vector field. Then we can write $X = \mathcal{H}X + \mathcal{V}X$ where $\mathcal{H}X, \mathcal{V}X \in \Gamma(TM)$ are smooth horizontal and vertical vector fields, respectively.*

Proof. Recall that using local coordinates $(x^1, \dots, x^n)$ for M from the Rank Theorem, we have $V_p = \operatorname{span}\{\frac{\partial}{\partial x^{n-m}}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ for each p in the local chart. Performing Graham-Schmidt on the ordered frame$(\frac{\partial}{\partial x^{m+1}}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ we obtain a smooth orthonormal frame $(E_1, \dots, \dots, E_n)$ such that $V_p = \operatorname{span}\{E_1, \dots, E_{n-m}\}$ and $H_p = \operatorname{span}\{E_{n-m+1}, E_n\}$. Write X in this orthonormal frame as $X = f^i E_i$ for smooth functions f^i. Then

$$
\begin{aligned}
X &= (f^1 E_1 + \cdots, + f^{n-m} E_{n-m}) + (f^{n-m+1} E_{n-m+1} + \cdots + f^n E_n) \\
&= \mathcal{V}X + \mathcal{H}X
\end{aligned}
$$

and clearly $\mathcal{V}X$ is a smooth vertical vector field and $\mathcal{H}X$ is a smooth horizontal vector field on M. ■

4.1 Fubini Study Metric

Suppose G is a Lie group and M is a smooth manifold. A *smooth action* of G on M is a smooth map $G \times M \to M$ such that $a \cdot (b \cdot p) = (ab) \cdot p$ for all $a, b \in G$ and all $p \in M$, and $e \cdot p = p$ for all $p \in M$, where e is the identity element of G.

Now suppose that $\pi : M \to N$ is a submersion and G is a Lie group acting on M. The action is called *vertical* if $\pi(a \cdot p) = \pi(p)$ for all $a \in G$ and $p \in M$. The action is called *transitive on fibres* if whenever $\pi(p) = \pi(q)$, there is an $a \in G$ such that $a \cdot p = q$. If M is endowed with a Riemannian metric, then the action is said to be *isometric* if the map $p \to a \cdot p$ is an isometry for all $a \in G$.

Lemma 4.5 *Suppose G is a Lie group acting vertically and isometrically on $\widetilde{M}$ and fix $a \in G$. If $X_p \in H_p$ is horizontal, then so is $[a_*]_p X_p$.*

Proof. Let $W_{a\cdot p} \in V_{a\cdot p}$ be any vertical vector. We must show $\langle W_{a\cdot p}, [a_*]_p X_p \rangle = 0$. Since the map $p \mapsto a \cdot p$ is, in particular, a diffeomorphism, the linear map $[a_*]_p \colon T_pM \to T_{a\cdot p}M$ is an isomorphism. Thus, there is some $W_p \in T_pM$ such that $[a_*]_p W_p = W_{a\cdot p}$. Since the action is vertical, we have $\pi \circ a = \pi$, and so $[\pi]_p W_p = [\pi_*]_p [a_*]_p^{-1} W_{a\cdot p} = [\pi_*]_p [(a^{-1})_*]_{a\cdot p} W_{a\cdot p} = [(\pi \circ a^{-1})_*]_{a\cdot p} W_{a\cdot p} = [\pi_*]_{a\cdot p} W_{a\cdot p} = 0$, which shows W_p is vertical. Now using the fact that the map $p \mapsto a \cdot p$ is an isometry, we have $\langle W_{a\cdot p}, [a_*]_p X_p \rangle = \langle [a_*]_p W_p, [a_*]_p X_p \rangle = \langle W_p, X_p \rangle = 0$, as desired. ■

Theorem 4.6 *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and let M be a smooth manifold. Let $\pi : \widetilde{M} \to M$ be a smooth surjective submersion. Moreover, suppose that G is a Lie group acting vertically, transitively, and isometrically on $\widetilde{M}$. Then there is a unique Riemannian metric g on M such that π is a Riemannian submersion.*

Proof. We first show uniqueness. Consider an arbitrary element of M, say $\pi(p) \in M$ and $X_{\pi(p)}, Y_{\pi(p)} \in T_{\pi(p)}M$. Since $[\pi_*]_p|_{H_p}$ is a linear isomorphism from H_p to $T_{\pi(p)}M$, there are unique horizontal vectors $\widetilde{X}_p, \widetilde{Y}_p$ in H_p such that $[\pi_*]_p\widetilde{X}_p = X_{\pi(p)}$, and similarly for $\widetilde{Y}_p$. If g is any satisfactory Riemannian metric on M, then we must have $g_{\pi(p)}([\pi_*]_p\widetilde{X}_p, [\pi_*]_p\widetilde{Y}_p) = \widetilde{g}_p(\widetilde{X}_p, \widetilde{Y}_p)$. That is, $g_{\pi(p)}(X_{\pi(p)}, Y_{\pi(p)}) = \widetilde{g}_p(\widetilde{X}_p, \widetilde{Y}_p)$. So g is uniquely determined by $\widetilde{g}$.

We now show existence. Fix any $x \in \widetilde{M}$, and let $X_{\pi(p)}, Y_{\pi(p)} \in T_{\pi(x)}M$. Let $\widetilde{X}_p, \widetilde{Y}_p$ be the unique horizontal lifts of $X_{\pi(p)}, Y_{\pi(p)}$ to H_p, respectively. We want to define $g_{\pi(p)}(X_{\pi(p)}, Y_{\pi(p)}) = \widetilde{g}_p(\widetilde{X}_p, \widetilde{Y}_p)$. We must show that this is well defined. To do so, let $q \in \widetilde{M}$ be any other element in the fibre of $\pi(p)$. Again, let $\widetilde{X}_q, \widetilde{Y}_q$ be the unique horizontal lifts of $X_{\pi(p)}, Y_{\pi(p)}$ to H_q. To show that g_p is well defined, we must show that $\widetilde{g}_p(\widetilde{X}_p, \widetilde{Y}_p) = \widetilde{g}_q(\widetilde{X}_q, \widetilde{Y}_q)$.

Since the action is transitive, there is an $a \in G$ such that $a \cdot p = q$. Thus, we have $[\pi_*]_q[a_*]_p\widetilde{X}_p = [(\pi \circ a)_*]_p\widetilde{X}_p = [\pi_*]_p\widetilde{X}_p = X_{\pi(p)}$, where the first equality follows from the fact that $\pi \circ a = \pi$ since the action is vertical. But also $[\pi_*]_q\widetilde{X}_q = X_{\pi(p)}$, so $[\pi_*]_q[a_*]_p\widetilde{X}_p = [\pi_*]_q\widetilde{X}_q$. Now $[a_*]_p\widetilde{X}_p$ is horizontal by the previous lemma and $[\pi_*]_q|_{H_q} \colon H_q \to T_{\pi(p)}M$ is an isometry, so we conclude that $[a_*]_p\widetilde{X}_p = \widetilde{X}_q$. Similarly, $[a_*]_p\widetilde{Y}_p = \widetilde{Y}_q$. Now using the fact that G acts isometrically, we have $\widetilde{g}_p(\widetilde{X}_p, \widetilde{X}_p) = \widetilde{g}_q([a_*]_p\widetilde{X}_p, [a_*]_p\widetilde{Y}_p) = \widetilde{g}_q(\widetilde{X}_q, \widetilde{Y}_q)$, which is what we needed to show.

We now show that g is a Riemannian metric on M. Bilinearity and symmetry are clear. Consider any element of M, say $\pi(p) \in M$ for some $p \in \widetilde{M}$ and let $X_{\pi(p)} \in T_{\pi(p)}M$. Consider the vector space isomorphism $L = ([\pi_*]_p|_{H_p})^{-1} \colon T_{\pi(p)} \to H_p$. The horizontal lift to H_p is given by $LX_{\pi(p)}$, so

$$\begin{aligned} g_{\pi(p)}(X_{\pi(p)}, X_{\pi(p)}) = 0 &\iff \widetilde{g}_p(LX_{\pi(p)}, LX_{\pi(p)}) = 0 \\ &\iff LX_{\pi(p)} = 0 \\ &\iff X_{\pi(p)} = 0. \end{aligned}$$

Thus, g is symmetric, bilinear, and positive definite at every point. We now show that g is smooth. Fix $p \in M$. Let $X, Y \in \Gamma(TM)$ be smooth vector fields on M, and let $\mathcal{H}X, \mathcal{H}Y \in \Gamma(T\widetilde{M})$ be the corresponding horizontal lifts. Now since π is a submersion, there is an open neighborhood $U \subseteq M$ of p and a smooth section $\sigma \colon U \to \widetilde{M}$. On U, we have $g(X, Y) = \widetilde{g}(\mathcal{H}X, \mathcal{H}Y) \circ \sigma$ on U. Since the right hand side is smooth on U, $g(X, Y)$ is smooth in a neighborhood of p, and we conclude that $g(X, Y)$ is smooth. This holds for all $X, Y \in \Gamma(TM)$, so g is a smooth $(2,0)$-tensor. ■

Now consider the submersion $\pi\colon \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$ from Example 2.2d. Now since $\mathbb{S}^{2n+1}$ is an embedded submanifold of $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$, we can restrict π to a smooth map $P\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. The map P is clearly still surjective. It remains to show that P is a submersion. Fix $x \in \mathbb{S}^{2n+1}$. Since π is a submersion, there is a a neighborhood $U \subseteq \mathbb{CP}^n$ containing $\pi(x)$ and a local smooth section $\sigma\colon U \to \mathbb{C}^{n+1} \setminus \{0\}$ with $\sigma(\pi(x)) = x$. Now define $\tau\colon U \to \mathbb{C}^{n+1} \setminus \{0\}$ by

$$\tau(\zeta) = \frac{\sigma(\zeta)}{\|\sigma(\zeta)\|}.$$

Clearly this is a smooth map into $\mathbb{S}^{2n+1}$, it satisfies $\tau(P(x)) = \sigma(\pi(x))/\|\sigma(\pi(x))\| = x/\|x\| = x$, and for every $\zeta \in \mathbb{CP}^n$ we have $(P \circ \tau)(\zeta) = (\pi \circ \tau)(\zeta) = \pi(\frac{\sigma(\zeta)}{\|\sigma(\zeta)\|}) = \pi(\sigma(\zeta)) = \zeta$, τ is a smooth section. By Theorem 2.3, P is a submersion.

Now define the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$ by $\lambda \cdot (z_1, \ldots, z^{n+1}) = (\lambda z^1, \ldots, \lambda z^{n+1})$. It is easy to see that this action is vertical, and transitive on fibres. To see that it acts isometrically, identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via the coordinates $(x^1, y^1, \ldots, x^{n+1}, y^{n+1})$ given by $z^j = x^j + \sqrt{-1}y^j$ (we use $\sqrt{-1}$ instead of $i \in \mathbb{C}$ since the letter i is used for indices). The action is given in local coordinates by

$$\begin{aligned} e^{\sqrt{-1}\theta} \cdot (x^1, y^1, \ldots, x^{n+1}, y^{n+1}) = (&\cos(\theta)x^1 - \sin(\theta)y^1, \sin(\theta)x^1 + \cos(\theta)y^1, \\ &\ldots, \cos(\theta)x^{n+1} - \sin(\theta)y^{n+1}, \sin(\theta)x^{n+1} + \cos(\theta)y^1). \end{aligned}$$

Fix any $p \in \mathbb{S}^{2n+1}$ and let $X_p = a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i}$ be a vector tangent to $\mathbb{S}^{2n+1}$ at p. Write $\lambda = e^{\sqrt{-1}\theta}$ for some $\theta \in \mathbb{R}$. Then identifying λ with the map $q \to \lambda \cdot q$, we have

$$\begin{aligned} g([\lambda_*]_p X_p, [\lambda_*]_p Y_p) &= g\left([\lambda_*]_p\left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i}\right), [\lambda_*]_p\left(c^j \frac{\partial}{\partial x^j} + d^j \frac{\partial}{\partial y^j}\right)\right) \\ &= g\Bigg(\left((\cos(\theta)a^i - \sin(\theta)b^i)\frac{\partial}{\partial x^i} + (\sin(\theta)a^i + \cos(\theta)b^i)\frac{\partial}{\partial y^i}\right), \\ &\qquad \left((\cos(\theta)c^j - \sin(\theta)d^j)\frac{\partial}{\partial x^j} + (\sin(\theta)c^j + \cos(\theta)d^j)\frac{\partial}{\partial y^j}\right)\Bigg) \\ &= \sum_{i=1}^{n+1}\left(\cos^2(\theta)a^ic^i - \cos(\theta)\sin(\theta)(a^id^i + b^ic^i) + \sin^2(\theta)b^id^i\right) \\ &\qquad + \sum_{i=1}^{n+1}\left(\sin^2(\theta)a^ic^i + \cos(\theta)\sin(\theta)(a^id^i + b^ic^i) + \sin^2(\theta)b^id^i\right) \\ &= \sum_{i=1}^{n}\left(a^ic^i + b^id^i\right) \end{aligned}$$

$$= g\left(\left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i}\right), \left(c^j \frac{\partial}{\partial x^j} + d^j \frac{\partial}{\partial y^j}\right)\right) = g(X_p, Y_p),$$

so the action is isometric. By Theorem 4.6, there is a unique metric on $\mathbb{CP}^n$ making $P : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ into a Riemannian submersion. This metric is called the *Fubini Study metric.* For more on the Fubini-Study metric, see chapter 1 of [6].

4.2 The O'Neill Tensors

In this section, we will introduce the O'Neill tensors and derive the fundamental equations. These fundamental equations are similar to the Gauss, Ricci, and Codazzi equations for Riemannian immersions. Much of this section loosely follows [5]. However, we discuss the necessary preliminaries in more detail, expand/provide proofs that were skipped in the original paper, and discuss applications of the O'Neill tensors to new examples.

Definition 4.7 *A vector field $X \in \Gamma(TM)$ is called* horizontal *if $\mathcal{V}X = 0$. We say that X is the* horizontal lift *of a vector field $X_* \in \Gamma(TB)$ if X is horizontal and $[\pi_*]_p X_p = (X_*)_{\pi(p)}$ for all $p \in M$. We say that X is a* basic *vector field if it is the horizontal lift of a vector field X_* on B.*

Proposition 4.8 *There is a bijective correspondence between the vector fields on B and the basic vector fields on M.*

Proof. Let X_* be a vector field on B. Since $[\pi_*]|_{H_p}$ is an isometry for all $p \in M$, $[\pi_*]|_{H_p} \colon H_p \to T_{\pi(p)}B$ is an injective linear map between vector spaces of the same dimension and is thus an isomorphism. So X_* determines a unique horizontal section $X : M \to TM$. It remains to show that X is smooth. From the rank theorem with coordinates $(x^1, \ldots, x^n)$ for M and $(y^1, \ldots, y^m)$ for B, we can write $X_* = f^j \frac{\partial}{\partial y^j}$, where each f^j is a smooth function of the coordinates $(y^1, \ldots, y^m)$. We can extend each f^j to a local smooth function on M by defining

$$f^j(x^1, \ldots, x^n) = f^j(x^1, \ldots, x^m).$$

Now let $Y = \sum_{j=1}^m f^j \frac{\partial}{\partial x^j}$. By construction, this is a smooth local vector field whose pushforward is X_*. Subtracting the vertical component of Y gives a smooth horizontal vector field whose pushforward is X_*. By uniqueness, this smooth horizontal vector field is X which shows X is smooth. ∎

We denote the Levi-Civita connection on M by ∇ and the Levi-Civita connection on B by ∇^*. The correspondence between basic vector fields on M and arbitrary vector fields on B respects metrics, Lie brackets, and connections.

Lemma 4.9 (From [5, Lemma 1] *Let X and Y be basic vector fields on M and let X_* and Y_* be the vector fields on B corresponding to X and Y, respectively. The following hold.*

a. $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ \pi$,

b. $\mathcal{H}[X, Y]$ *is the basic vector field corresponding to* $[X_*, Y_*]$,

c. $\mathcal{H}\nabla_X Y$ *is the basic vector field corresponding to* $\nabla^*_{X_*} Y_*$.

Proof. Part a follows directly from the definition of a Riemannian submersion and the definition of a basic vector field. For part b, note that since X (respectively, Y) is π-related to X_* (respectively, Y_*), it follows that $[X, Y]$ is π-related to $[X_*, Y_*]$. That is, $[\pi_*]_p[X, Y]_p = [X_*, Y_*]_{\pi(p)}$, and thus $\mathcal{H}[X, Y]$ is a horizontal vector field and $[\pi_*]_p(\mathcal{H}[X, Y]_p) = [\pi_*]_p[X, Y]_p = [X_*, Y_*]_p$, as desired. For c, fix any horizontal vector field Y and recall that the Koszul formula is given by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ + \langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned}$$

Since this equation is tensorial in Z, we may assume Z is basic. From part a, we know that

$$\begin{aligned} (X\langle Y, Z\rangle)(p) &= (X\,(\langle Y_*, Z_*\rangle \circ \pi))(p) \\ &= [\pi_*]_p X_p \langle Y_*, Z_*\rangle \\ &= (X_*)_{\pi(p)}\langle Y_*, Z_*\rangle \\ &= (X_*\langle Y_*, Z_*\rangle \circ \pi)(p), \end{aligned}$$

so $X\langle Y, Z\rangle = X_*\langle Y_*, Z_*\rangle \circ \pi$. Similar formulas hold for $Y\langle X, Z\rangle$ and $Z\langle X, Y\rangle$. Using part b, we have $\langle [X, Y], Z\rangle = \langle \mathcal{H}[X, Y], Z\rangle = \langle [X_*, Y_*], Z_*\rangle \circ \pi$, and similar formulas hold for $\langle [X, Z], Y\rangle$ and $\langle [Y, Z], X\rangle$. Substituting into the Koszul formula gives

$$\langle \nabla^*_{X_*} Y_*, Z_*\rangle \circ \pi = \langle \nabla_X Y, Z\rangle = \langle \mathcal{H}\nabla_X Y, Z\rangle.$$

Invoking part a, the above gives $\langle \nabla_{X_*} Y_*, Z_*\rangle \circ \pi = \langle (\mathcal{H}\nabla_X Y)_*, Z_*\rangle \circ \pi$ for any basic vector field Z, which proves part c. ∎

We are now able to define the O'Neill tensors for the Riemannian submersion π. We will see that these tensors play an analogous role to the second fundamental form for an embedding.

Definition 4.10 (O'Neill Tensors)
For vector fields $E, F \in \Gamma(TM)$, we define

$$\begin{aligned} T_E F &= \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F), \\ A_E F &= \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F). \end{aligned}$$

We have called these the O'Neill *tensors*. The next proposition shows that the O'Neill tensors are indeed tensors, and establishes some basic properties of T and A.

Proposition 4.11 (From [5, Page 460] and [5, Lemma 2]) *Let E be an arbitrary vector field on M. Then T is a $(2,1)$-tensor, and the following properties of T hold:*

1. *At each point, T_E is a skew-symmetric and it reverses the horizontal and vertical subspaces.*
2. *T is vertical. That is, $T_E = T_{\mathcal{V}E}$.*
3. *If V, W are vertical vector fields, then $T_V W = T_W V$.*

Similarly, A is a (2,1) tensor and satisfies the following properties.

1'. *At each point, A_E is skew-symmetric and it reverses the horizontal and vertical subspaces.*

2'. *A is horizontal. That is, $A_E = A_{\mathcal{H}E}$.*

3'. *If X, Y are horizontal vector fields, then $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$. In particular, $A_X Y = -A_Y X$.*

Proof. Properties 2 and $2'$ are obvious, as is the reversal part of 1 and $1'$. We prove that T is a tensor, as well as properties 1, 3, and $3'$. The others are similar. Let $f \in C^\infty(M)$. Then for any $F \in \Gamma(TM)$, we have

$$\begin{aligned}
T_E(fF) &= \mathcal{H}\nabla_{\mathcal{V}E}(f\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(f\mathcal{H}F) \\
&= \mathcal{H}((\mathcal{V}E)(f)\mathcal{V}F + f\nabla_{\mathcal{V}E}\mathcal{V}F) \\
&\qquad + \mathcal{V}((\mathcal{V}E)(f)\mathcal{H}F + f\nabla_{\mathcal{V}E}\mathcal{H}F) \\
&= f(\mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}E) \\
&= fT_E F,
\end{aligned}$$

so T is tensorial in F. It is clearly tensorial in E. For property 1, let F_1, F_2 be two arbitrary vector fields on M. Then

$$\begin{aligned}
\langle T_E F_1, F_2\rangle &= \langle \nabla_{\mathcal{V}E}\mathcal{V}F_1, \mathcal{H}F_2\rangle + \langle \nabla_{\mathcal{V}E}\mathcal{H}F_1, \mathcal{V}F_2\rangle \\
&= -\langle \mathcal{V}F_1, \nabla_{\mathcal{V}E}\mathcal{H}F_2\rangle - \langle \mathcal{H}F_1, \nabla_{\mathcal{V}E}\mathcal{V}F_2\rangle \\
&= -\langle F_1, \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F_2\rangle - \langle F_1, \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F_2\rangle \\
&= \langle F_1, -T_E F_2\rangle,
\end{aligned}$$

which proves skew-symmetry. To prove 3, let V, W be vertical vector fields. Then $T_V W - T_W V = \mathcal{H}(\nabla_V W - \nabla_W V) = \mathcal{H}[V, W] = 0$ since the bracket of two vertical vector fields is vertical. Finally, we prove $3'$. Let X and Y be horizontal vector fields on M. Then $A_X Y - A_Y X = \mathcal{V}(\nabla_X Y - \nabla_Y X) = \mathcal{V}[X, Y]$. So it suffices to show that $A_X Y = -A_Y X$, or equivalently that $A_X X = 0$. Let V be a vertical vector field. Since A is tensorial, we may assume that X is basic. Fix two point $p, q \in M$ in the same fibre of π. Since π is a Riemannian submersion and X is basic with some corresponding vector field X_* on B, we have

$$\begin{aligned}
\langle X_p, X_p \rangle &= \langle [\pi_*]_p X_p, [\pi_*]_p X_p \rangle \\
&= \langle (X_*)_{\pi(p)}, (X_*)_{\pi(p)} \rangle \\
&= \langle (X_*)_{\pi(q)}, (X_*)_{\pi(q)} \rangle \\
&= \langle [\pi_*]_q X_q, [\pi_*]_q X_q \rangle \\
&= \langle X_q, X_q \rangle,
\end{aligned}$$

so $\langle X, X \rangle$ is constant along the fibres of π. Thus, $V\langle X, X \rangle = 0$. Now $[V, X] = \nabla_V X - \nabla_X V$ is vertical since V is π-related to $0 \in \Gamma(TB)$ and X is π-related to a smooth vector field on B. This implies that

$$\langle \nabla_X V, X \rangle = \langle \nabla_V X, X \rangle.$$

Putting it all together, we have

$$0 = V\langle X, X \rangle = 2\langle \nabla_V X, X \rangle = 2\langle \nabla_X V, X \rangle = -2\langle V, \nabla_X X \rangle = -2\langle V, A_X X \rangle.$$

Applying property 1', we conclude $A_X X = 0$. ■

We denote the Levi-Civita connection on an arbitrary fibre of M by $\widehat{\nabla}$. By Proposition 3.11, $\widehat{\nabla}_V W = \mathcal{V}\nabla_V W$ for all vector fields on the fibre (that is, for all vertical vector fields on M restricted to the fibre). Notice that when $V_p, W_p \in T_p M$ are vertical vectors, then $T_{V_p} W_p$ is just the second fundamental form of the fibre $\pi^{-1}(\pi(p))$ as an embedded submanifold of M. This leads to the following result.

Proposition 4.12 *Each fibre of π is a totally geodesic submanifold of M if and only if $T = 0$.*

Proof. By the discussion preceding the proposition and Proposition 3.14, each fibre of π is totally geodesic if and only iff $T_V W = 0$ for all vertical vector fields $V, W \in \Gamma(TM)$. Now suppose that $T_V W = 0$ for all vertical vector fields $V, W \in \Gamma(TM)$. To complete the proof, it suffices to show that $T = 0$. Let $V, W \in \Gamma(TM)$ be vertical vector fields, and let $X \in \Gamma(TM)$ be a a horizontal vector field. Then

$$\langle T_V X, W \rangle = -\langle X, T_V W \rangle = 0.$$

Since this holds for any vertical $W \in \Gamma(TM)$ and $T_V X$ is vertical, the above implies $T_V X = 0$. All together, we have $T_V F = 0$ for all vector fields F and all vertical vector fields V, which implies $T = 0$ since T is vertical. ■

Lemma 4.13 (From [5, Lemma 3]) *Let X, Y be horizontal vector fields, and V, W be vertical vector fields. Then*

1. $\nabla_V W = T_V W + \widehat{\nabla}_V W$
2. $\nabla_V X = \mathcal{H}\nabla_V X + T_V X$
3. $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$
4. $\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y$

Furthermore, if X is basic, $\mathcal{H}\nabla_V X = A_X V$.

Proof. Parts 2,3, and 4 are obvious. Part 1 is just Gauss's formula on the fibres of M. The last statement follows from part 3 and the fact that $[X, V]$ is vertical when X is basic and V is vertical. ■

We now compute the covariant derivatives of T and A.

Lemma 4.14 (From [5, Lemma 4]) *If X, Y are horizontal and V, W are vertical, then*

$$\begin{aligned} (\nabla_V A)_W &= -A_{T_V W}, \qquad & (\nabla_X T)_Y &= -T_{A_X Y}, \\ (\nabla_X A)_W &= -A_{A_X W}, \qquad & (\nabla_V T)_Y &= -T_{T_V Y}. \end{aligned}$$

Proof. Let E be an arbitrary vector field. Then

$$(\nabla_X T)_Y E = \nabla_X(T_Y E) - T_{\nabla_X Y}(E) - T_Y(\nabla_X E),$$

and the first and last terms vanish since T is vertical. By Lemma 4.13, we have

$$(\nabla_X T)_Y E = -T_{\nabla_X Y}(E) = -T_{\mathcal{V}\nabla_X Y}(E) = -T_{A_X Y}(E).$$

Similarly, we have

$$\begin{aligned} (\nabla_V T)_Y(E) &= \nabla_V(T_Y E) - T_{\nabla_V Y}(E) - T_Y(\nabla_V E) \\ &= -T_{\mathcal{V}\nabla_V Y}(E) \\ &= -T_{T_V Y}(E). \end{aligned}$$

The proof of the formulas for A are similar. ■

The next lemma shows that $(\nabla_F T)_E$ and $(\nabla_F A)_E$ do not reverse horizontal and vertical vectors.

Lemma 4.15 (From [5, Lemma 5]) *Let X, Y, Z be horizontal vector fields and let U, V, W be vertical vector fields. Then*

$$\begin{aligned}
\langle (\nabla_U A)_X V, W\rangle &= \langle T_U V, A_X W\rangle - \langle T_U W, A_X V\rangle,\\
\langle (\nabla_U A)_X Y, Z\rangle &= \langle T_U Y, A_X Z\rangle - \langle T_U Z, A_X Y\rangle,\\
\langle (\nabla_X T)_U Y, Z\rangle &= \langle A_X Y, T_U Z\rangle - \langle A_X Z, T_U Y\rangle,\\
\langle (\nabla_X T)_U V, W\rangle &= \langle A_X V, T_U W\rangle - \langle A_X W, T_U V\rangle.
\end{aligned}$$

Proof. We prove the first formula. The others are similar. First, note that

$$\langle (\nabla_U A)_X V, W\rangle = \langle \nabla_U (A_X V) - A_{\nabla_U X} V - A_X(\nabla_U V), W\rangle.$$

Since $A_{\nabla_U X} V$ is horizontal, $\langle A_{\nabla_U X} V, W\rangle = 0$. Using metric compatibility, Lemma 4.13, and Proposition 4.11, we have

$$\begin{aligned}
\langle (\nabla_U A)_X V, W\rangle &= \langle \nabla_U (A_X V) - A_X(\nabla_U V), W\rangle\\
&= -\langle A_X V, \nabla_U W\rangle + \langle \nabla_U V, A_X W\rangle\\
&= -\langle A_X V, T_U W\rangle + \langle T_U V, A_X W\rangle.
\end{aligned}$$

∎

Finally, we review some symmetry properties of T and A that we will use in the fundamental equations of T and A.

Lemma 4.16 (From [5, Lemma 6]) *If X, Y are horizontal, V, W vertical, and E, F are arbitrary vector fields, then*

1. $\langle (\nabla_E T)_V W, X\rangle$ *is symmetric in V and W, and*
2. $\langle (\nabla_E A)_X Y, V\rangle$ *is alternate in X and Y.*
3. $(\nabla_E T)_F$ *and* $(\nabla_E A)_F$ *are skew-symmetric* $(1,1)$*-tensors.*

Proof. We prove 2 and 3, since 1 is similar to 2. First, we prove 2. Using Proposition 4.11, we have

$$\begin{aligned}
\langle (\nabla_E A)_X Y, V\rangle &= \langle \nabla_E (A_X Y) - A_{\nabla_E X} Y - A_X(\nabla_E Y), V\rangle\\
&= \langle \nabla_E (A_X Y) - A_{\mathcal{H}\nabla_E X}(\mathcal{H}Y) - A_{\mathcal{H}X}(\mathcal{H}\nabla_E Y), V\rangle\\
&= -\langle \nabla_E (A_Y X) - A_{\mathcal{H}Y}(\mathcal{H}\nabla_E X) - A_{\mathcal{H}\nabla_E Y}(\mathcal{H}X), V\rangle\\
&= -\langle \nabla_E (A_Y X) - A_Y(\nabla_E X) - A_{\nabla_E Y} X, V\rangle
\end{aligned}$$

$$= -\langle (\nabla_E A)_Y X, V\rangle.$$

We now prove 3. Let $K, L \in \Gamma(TM)$ be arbitrary vector fields. Then $(\nabla_E T)_F K = \nabla_E(T_F K) - T_{\nabla_E F} K - T_F(\nabla_E K)$. Applying metric compatibility and the fact that T_F is skew-symmetric, we obtain

$$\begin{aligned}
\langle (\nabla_E T)_F K, L\rangle &= \langle \nabla_E(T_F K) - T_{\nabla_E F} K - T_F(\nabla_E K), L\rangle \\
&= (E\langle T_F K, L\rangle - \langle T_F K, \nabla_E L\rangle) - \langle T_{\nabla_E F} K, L\rangle - \langle T_F(\nabla_E K), L\rangle \\
&= -(E\langle K, T_F L\rangle - \langle K, T_F(\nabla_E L)\rangle - \langle K, T_{\nabla_E F} L\rangle - \langle \nabla_E K, T_F L\rangle) \\
&= -(\langle K, \nabla_E(T_F L)\rangle - \langle K, T_F(\nabla_E L)\rangle - \langle K, T_{\nabla_E F} L\rangle) \\
&= -\langle K, (\nabla_E T)_F L\rangle.
\end{aligned}$$

The proof that $(\nabla_E A)_F$ is skew-symmetric is similar. ■

4.2.1 The Fundamental Equations

Given a Riemannian submersion $\pi : M \to B$, we want to now find the fundmental equations relating the curvature tensor of M with the curvature tensors of B and the curvature tensors of the fibres. We will number these equations by $\{n\}$ to denote the number of horizontal vector fields on the left-hand side of the given fundamental equation. For a point $p \in M$, let $\widehat{R}$ denote the curvature tensor of the fibre $\pi^{-1}(\pi(p))$. Similarly, for horizontal vectors h_1, h_2, h_3, and h_4 in T_pM, define $\langle R^*_{h_2 h_2} h_3, h_4\rangle = \langle R^*_{h_{1*} h_{2*}} h_{3*}, h_{4*}\rangle$.

Recall that when $V_p, W_p \in T_pM$ are vertical vectors, we have $T_{V_p} W_p$ is simply the second fundamental form of the fibre $\pi^{-1}(\pi(p))$ as a submanifold of M. Thus, the first two fundamental equations are simply the Gauss and Codazzi equations from Theorem 3.19 and Theorem 3.24, respectively.

Theorem 4.17 From [5, Theorem 1] *If U, V, W, F are vertical vector fields on M and X is a horizontal vector field on M, then*

{0}. $\langle R_{UV} W, F\rangle = \langle \widehat{R}_{UV} W, F\rangle - \langle T_U F, T_V W\rangle + \langle T_U W, T_V F\rangle$,

{1}. $\langle R_{UV} W, X\rangle = \langle (\nabla_U T)_V W, X\rangle - \langle (\nabla_V T)_U W, X\rangle$.

We now prove the fundamental equations containing two, three, and four horizontal vector fields one at a time. We first prove a lemma.

Lemma 4.18 *Fix $p \in M$ and let $X_p, Y_p \in T_pM$ be horizontal vectors. Then there are basic fields $X, Y \in \Gamma(TM)$ such that $X\big|_p = X_p$ and $Y\big|_p = Y_p$ and $[X, Y]$ is vertical at p.*

Proof. Let $(X_*)_{\pi(p)} = [\pi_*]_p X_p$ and $(Y_*)_{\pi(p)} = [\pi_*]_p Y_p$. In a chart (U, φ) with local coordinates $(x^1, \ldots, x^n)$, we can write $(X_*)_{\pi(p)} = a^i \frac{\partial}{\partial x^i}\Big|_{\pi(p)}$ and $(Y_*)_{\pi(p)} = b^i \frac{\partial}{\partial x^i}\Big|_{\pi(p)}$ for some $a^i, b^i \in \mathbb{R}$. Let $\rho \in C^\infty(M)$ be a bump function on M with $\rho = 1$ in a neighborhood W of $\pi(p)$ and $\operatorname{supp}(\rho) \subseteq U$. On B we can extend these vectors to *global* vector fields X_* and Y_*, respectively by defining $X_* = \rho a^i \frac{\partial}{\partial x^i}$ and $Y_* = \rho b^i \frac{\partial}{\partial x^i}$. Now let $X, Y \in \Gamma(TM)$ be the basic vector fields on M which are π-related to X_* and Y_*, respectively. Since $[\pi_*]\big|_{H_p}$ is injective, we have $X\big|_p = X_p$ and $Y\big|_p = Y_p$.

Since ρ is constant on W, $[X_*, Y_*] = \rho^2 a^i b^j \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ on W. Since X is π-related to X_* and Y is π-related to Y_*, we have $[X, Y]$ is π-related to $[X_*, Y_*]$. Thus, $[X, Y]$ is vertical on the open set $\pi^{-1}(W)$, which contains p. ■

Theorem 4.19 (From [5, Theorem 3]) *Let V, W be vertical vector fields on M and let X, Y be horizontal vector fields on M. Then*

$$\{2\}.\ \langle R_{XV}Y, W\rangle = \langle T_V X, T_Y W\rangle - \langle A_X V, A_Y W\rangle - \langle (\nabla_X T)_V W, Y\rangle - \langle (\nabla_V A)_X Y, W\rangle.$$

Proof. Let $p \in M$. Using Lemma 4.18 and the fact that the equation is tensorial, we may assume that X and Y are basic with vertical bracket at p. We compute each term of $R_{XV}Y$.

The vector field $[X, V]$ is vertical since X is basic and V is vertical at p, so $[X, V]\big|_p = (\mathcal{V}\nabla_X V)(p) - (\mathcal{V}\nabla_V X)(p) = (\mathcal{V}\nabla_X V)(p) - (T_V X)(p)$. Using Lemma 4.13 and the fact that T is vertical, we have

$$(\mathcal{V}\nabla_{[X,V]}Y)(p) = (T_{\nabla_X V}Y - T_{T_V X}Y)(p). \tag{25}$$

Applying Lemma 4.13 twice, we have

$$\begin{aligned}\nabla_X \nabla_V Y &= \nabla_X(\mathcal{H}\nabla_V Y + T_V Y)\\ &= \mathcal{H}\nabla_X(\mathcal{H}\nabla_V X) + A_X(\mathcal{H}\nabla_V Y) + A_X(T_V Y) + \mathcal{V}\nabla_X(T_V Y),\end{aligned}$$

which implies

$$\mathcal{V}\nabla_X \nabla_V Y = A_X(\mathcal{H}\nabla_V Y) + \mathcal{V}\nabla_X(T_V Y). \tag{26}$$

Similarly,

$$\mathcal{V}\nabla_V \nabla_X Y = T_V(\mathcal{H}\nabla_X Y) + \mathcal{V}\nabla_V(A_X Y). \tag{27}$$

Combining (25), (26), and (27), we have

$$\begin{aligned}\langle R_{XV}Y, W\rangle(p) &= \langle \mathcal{V}\nabla_X\nabla_V Y - \mathcal{V}\nabla_V\nabla_X Y - \mathcal{V}\nabla_{[X,V]}Y, W\rangle(p)\\ &= \langle A_X(\mathcal{H}\nabla_V Y) + \mathcal{V}\nabla_X(T_V Y) - T_V(\mathcal{H}\nabla_X Y)\\ &\qquad - \mathcal{V}\nabla_V(A_X Y) - T_{\nabla_X V}Y + T_{T_V X}Y, W\rangle(p)\end{aligned}$$

$$
\begin{aligned}
&= \langle A_X(\nabla_V Y) + \nabla_X(T_V Y) - T_V(\nabla_X Y) \\
&\qquad - \nabla_V(A_X Y) - T_{\nabla_X V} Y + T_{T_V X} Y, W\rangle(p).
\end{aligned} \tag{28}
$$

Moreover, using Proposition 4.11 and the last part of Lemma 4.13, we have

$$
\begin{aligned}
\langle A_X(\nabla_V Y) - \nabla_V(A_X Y), W\rangle &= \langle -(\nabla_V A)_X Y - A_{\nabla_V X} Y, W\rangle \\
&= \langle -(\nabla_V A)_X Y - A_{\mathcal{H}\nabla_V X} Y, W\rangle \\
&= -\langle (\nabla_V A)_X Y, W\rangle - \langle A_{A_X V} Y, W\rangle \\
&= -\langle (\nabla_V A)_X Y, W\rangle + \langle A_Y A_X V, W\rangle \\
&= -\langle (\nabla_V A)_X Y, W\rangle - \langle A_X V, A_Y W\rangle.
\end{aligned} \tag{29}
$$

By definition of the induced connection on $(2,1)$-tensors, we have

$$
\nabla_X(T_V Y) - T_V(\nabla_X Y) - T_{\nabla_X V} Z = (\nabla_X T)_V Y. \tag{30}
$$

Again applying Proposition 4.11, we have

$$
\begin{aligned}
\langle T_{T_V X} Y, W\rangle &= -\langle Y, T_{T_V X} W\rangle \\
&= -\langle Y, T_W T_V X\rangle \\
&= \langle T_W Y, T_V X\rangle \\
&= \langle T_Y W, T_V X\rangle.
\end{aligned} \tag{31}
$$

Substituting (29), (30), and (31) into (28) gives

$$
\begin{aligned}
\langle R_{XV} Y, W\rangle(p) &= (\langle T_Y W, T_V X\rangle - \langle A_X V, A_Y W\rangle + \langle (\nabla_X T)_V Y, W\rangle - \langle (\nabla_V A)_X Y, W\rangle)(p) \\
&= (\langle T_V X, T_Y W\rangle - \langle A_X V, A_Y W\rangle - \langle (\nabla_X T)_V W, Y\rangle - \langle (\nabla_V A)_X Y, W\rangle)(p),
\end{aligned}
$$

where the final equality follows from Lemma 4.16. ■

We now present the final two fundamental equations. For simplicity, if $h_1, h_2, h_3, h_4 \in T_pM$ are horizontal vectors, we define $\langle R^*_{h_1 h_2} h_3, h_4\rangle = \langle R^*_{h_{1*} h_{2*}} h_{3*}, h_{4*}\rangle$, where $h_{i*} = [\pi_*]_p h_i$.

Theorem 4.20 (From [5, Theorem 2]*) Let X, Y, Z, H be horizontal vector fields and let V be a vertical vector field. Then*

{3}. $\langle R_{XY} Z, V\rangle = \langle (\nabla_X A)_Y Z, V\rangle - \langle (\nabla_Y A)_X Z, V\rangle - 2\langle T_V Z, A_X Y\rangle.$

{4}. $\langle R_{XY} Z, H\rangle = \langle R^*_{XY} Z, H\rangle + 2\langle A_X Y, A_Z H\rangle - \langle A_Y Z, A_X H\rangle + \langle A_X Z, A_Y H\rangle$

Proof. Since both equations are tensorial, we may again assume that X, Y, Z, and H are basic and that $[X,Y]$ is vertical at p. We begin by computing the horizontal and vertical components of $R_{XY}Z$. Using Lemma 4.13 twice gives

$$\begin{aligned}\nabla_X\nabla_Y Z &= \nabla_X(\mathcal{H}\nabla_Y Z + A_Y Z)\\ &= \mathcal{H}\nabla_X(\mathcal{H}\nabla_Y Z) + A_X(\mathcal{H}\nabla_Y Z) + A_X A_Y Z + \mathcal{V}\nabla_X(A_Y Z),\end{aligned} \tag{32}$$

and reversing the roles of X and Y gives a similar equation for $\nabla_Y\nabla_X Z$. Now using the fact that $[X,Y]$ is vertical at p, Lemma 4.11, and Lemma 4.13, we have

$$\begin{aligned}(\nabla_{[X,Y]}Z)(p) &= (\mathcal{H}\nabla_{[X,Y]}Z)(p) + (\mathcal{V}\nabla_{[X,Y]}Z)(p)\\ &= (A_Z([X,Y]))(p) + (T_{[X,Y]}Z)(p)\\ &= 2(A_Z A_X Y)(p) + 2(T_{A_X Y}Z)(p).\end{aligned} \tag{33}$$

From (32) and (33), we have

$$\begin{aligned}(\mathcal{H}R_{XY}Z)(p) = (&\mathcal{H}\nabla_X(\mathcal{H}\nabla_Y Z) + A_X A_Y Z\\ &- \mathcal{H}\nabla_Y(\mathcal{H}\nabla_X Z) - A_Y A_X Z - 2A_Z A_X Y)(p),\end{aligned} \tag{34}$$

$$\begin{aligned}(\mathcal{V}R_{XY}Z)(p) = (&A_X(\mathcal{H}\nabla_Y Z) + \mathcal{V}\nabla_X(A_Y Z) - A_Y(\mathcal{H}\nabla_X Z)\\ &- \mathcal{V}\nabla_Y(A_X Z) - 2T_{A_X Y}Z)(p).\end{aligned} \tag{35}$$

Let X_*, Y_*, H_* be the vector fields on B corresponding to X, Y, and H, respectively. Notice that $[X_*, Y_*]|_{\pi(p)} = [\pi_*]_p[X,Y]|_p = 0$. Now using Lemma 4.9 twice and the fact that $[\pi_*]_p$ maps H_p isometrically onto T_pB, we have

$$\begin{aligned}\langle \mathcal{H}\nabla_X(\mathcal{H}\nabla_Y Z) - \mathcal{H}\nabla_Y(\mathcal{H}\nabla_X Z), H\rangle(p) &= \langle \nabla^*_{X_*}\nabla^*_{Y_*}Z_* - \nabla^*_{Y_*}\nabla^*_{X_*}Z_*, H_*\rangle(\pi(p))\\ &= \langle \nabla^*_{X_*}\nabla^*_{Y_*}Z_* - \nabla^*_{Y_*}\nabla^*_{X_*}Z_* - \nabla_{[X_*,Y_*]}Z_*, H_*\rangle(\pi(p))\\ &= \langle R^*_{X_*Y_*}Z_*, H_*\rangle(\pi(p))\\ &= \langle R^*_{XY}Z, H\rangle(p).\end{aligned}$$

Now using the skew-symmetry of A_E and taking the inner product of (34) with H, we have

$$\begin{aligned}\langle R_{XY}Z, H\rangle(p) &= \langle \mathcal{H}\nabla_X(\mathcal{H}\nabla_Y Z) - \mathcal{H}\nabla_Y(\mathcal{H}\nabla_X Z), H\rangle(p)\\ &\qquad + (-2\langle A_Z A_X Y, H\rangle - \langle A_Y A_X Z, H\rangle + \langle A_X A_Y Z, H\rangle)(p)\\ &= (\langle R^*_{XY}Z, H\rangle + 2\langle A_X Y, A_Z H\rangle + \langle A_X Z, A_Y H\rangle - \langle A_Y Z, A_X H\rangle)(p)\\ &= (\langle R^*_{XY}Z, H\rangle + 2\langle A_X Y, A_Z H\rangle - \langle A_Y Z, A_X H\rangle - \langle A_Z X, A_Y H\rangle)(p),\end{aligned}$$

which proves {4}. To prove {3}, take the inner product of (35) with V. Recalling that A_E reverses horizontal and vertical subspaces allows us drop the $\mathcal{H}$ operators. This gives

$$\begin{aligned}\langle R_{XY}Z, V\rangle(p) = \langle &A_X(\nabla_Y Z) + \nabla_X(A_Y Z) - A_Y(\nabla_X Z)\\ &- \nabla_Y(A_X Z) - 2T_{A_X Y}Z, V\rangle(p).\end{aligned} \tag{36}$$

Using Proposition 4.11 gives

$$\begin{aligned}\langle T_{A_XY}Z, V\rangle &= -\langle Z, T_{A_XY}V\rangle \\ &= -\langle Z, T_V A_X Y\rangle \\ &= \langle T_V Z, A_X Y\rangle,\end{aligned} \tag{37}$$

and using the fact that A is horizontal and $\mathcal{H}[X,Y]|_p = 0$, we have

$$\begin{aligned}(\langle \nabla_X(A_Y Z), V\rangle - \langle \nabla_Y(A_X Z), V\rangle)(p) = (&\langle(\nabla_X A)_Y Z, V\rangle + \langle A_Y(\nabla_X Z), V\rangle \\ &- \langle(\nabla_Y A)_X Z, V\rangle - \langle A_X(\nabla_Y Z), V\rangle)(p).\end{aligned} \tag{38}$$

Substituting (37) and (38) into (36) gives the result. ■

4.2.2 Example: O'Neill Tensors for the Complex Projective Space

Consider the Riemannian submersion $P\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ from section 4.1. We want to compute the O'Neill tensors for P. As usual, we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ with the coordinates $(x^1, y^1, \ldots, x^{n+1}, y^{n+1})$ defined by $z^j = x^j + \sqrt{-1}y^j$.

Lemma 4.21 *The vector field $S = x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i}$ is tangent to $\mathbb{S}^{2n+1}$ and is a basis of the vertical space at each point of $\mathbb{S}^{2n+1}$.*

Proof. The vector field $N = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}$ on $\mathbb{C}^{n+1}$ is normal to $\mathbb{S}^{2n+1}$ everywhere. Using the Euclidean metric on $\mathbb{C}^{n+1}$, we have

$$\begin{aligned}\langle N, S\rangle &= \left\langle x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}, x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j}\right\rangle \\ &= \sum_{i=1}^{n+1} -x^i y^i + y^i x^i = 0.\end{aligned}$$

Since S is normal to N, S is tangent to $\mathbb{S}^{2n+1}$, as desired. We now show that S is a basis of the vertical space at every point. Fix $p = (x^1 + \sqrt{-1}y^1, \ldots, x^{n+1} + \sqrt{-1}y^{n+1}) \in \mathbb{S}^{2n+1}$. Define a curve $\alpha\colon \mathbb{R} \to \mathbb{S}^{2n+1}$ by $\alpha(t) = e^{\sqrt{-1}t}p$. In local coordinates,

$$\begin{aligned}\alpha(t) = (&\cos(t)x^1 - \sin(t)y^1, \sin(t)x^1 + \cos(t)y^1, \\ &\ldots, \cos(t)x^{n+1} - \sin(t)y^{n+1}, \sin(t)x^{n+1} + \cos(t)y^1).\end{aligned}$$

Hence, $\alpha'(0) = -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i} = S(p)$. But notice that $\alpha(t)$ remains in the same fibre of P for all $t \in \mathbb{R}$, so $\alpha'(0) = S_p$ is tangent to the fibres and thus vertical. That is, $[\pi_*]_p S_p = 0$.

Clearly S is nonzero at all points of $\mathbb{S}^{2n+1}$. In fact, on $\mathbb{S}^{2n+1}$ we have

$$\begin{aligned}\langle S,S\rangle &= \left\langle x^j\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial x^j}, x^j\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial x^j}\right\rangle \\ &= \sum_{i=1}^{n+1}\left(\left(x^i\right)^2 + \left(y^i\right)^2\right) = 1.\end{aligned}$$

So S is a unit vector field at all points on $\mathbb{S}^{2n+1}$ which is in the vertical space at each point. Since $\mathbb{S}^{2n+1}$ has dimension $2n+1$ and $\mathbb{CP}^n$ has dimension $2n$, so each vertical space has dimension 1. So S_p is an orthonormal basis of V_p at each $p \in \mathbb{S}^{2n+1}$. ∎

Let $J \colon T\mathbb{C}^{n+1} \to T\mathbb{C}^{n+1}$ be given by

$$J\left(a^j\frac{\partial}{\partial x^j} + b^j\frac{\partial}{\partial y^j}\right) = a^j\frac{\partial}{\partial y^j} - b^j\frac{\partial}{\partial x^j}.$$

Lemma 4.22 *For any $E, F \in \Gamma(T\mathbb{C}^{n+1})$, we have*

(i) $\|E\| = \|JE\|$,

(ii) $\langle E, JF\rangle = -\langle JE, F\rangle$. *In particular,* $\langle JE, E\rangle = 0$.

(iii) *If $X_p \in T_p\mathbb{S}^{2n+1}$ is horizontal, then so is JX_p.*

Proof. (i) and (ii) are simple calculations. We prove (iii). Using the fact that $S = JN$ and $J^2 = -\operatorname{Id}$, we have

$$\langle JX_p, N_p\rangle = -\langle X_p, JN_p\rangle = -\langle X_p, J^2 S_p\rangle = \langle X_p, S_p\rangle = 0.$$

Thus, JX_p is tangent to $\mathbb{S}^{2n+1}$, since it is orthogonal to the normal vector field N for $\mathbb{S}^{2n+1}$. Now JX_p is in the horizontal vector space at p because

$$\langle JX_p, S_p\rangle = -\langle X_p, JS_p\rangle = \langle X_p, N_p\rangle = 0.$$

∎

Using J, we can now describe the O'Neill tensors.

Theorem 4.23 *Let $X, Y \in \Gamma(\mathbb{S}^{2n+1})$ be horizontal vector fields. We have*

(i) $T = 0$,

(ii) $A_X Y = \langle X, JY \rangle S$,

(iii) $A_X S = JX$.

Proof. We first prove (i). Notice that for any $p \in \mathbb{S}^{2n+1}$, the fibre containing p is $\{\lambda p \colon \lambda \in \mathbb{S}^1\}$, which is a great circle of $\mathbb{S}^{2n+1}$. By Proposition 4.12, $T = 0$.

We now prove (ii). In local coordinates, write $X = f^i \frac{\partial}{\partial x^i} + g^i \frac{\partial}{\partial y^i}$. Then $\overline{\nabla}_X S = f^i \overline{\nabla}_{\frac{\partial}{\partial x^i}}(S) + g^i \overline{\nabla}_{\frac{\partial}{\partial y^i}}(S)$, and

$$\begin{aligned}
\overline{\nabla}_{\frac{\partial}{\partial x^i}} S &= \overline{\nabla}_{\frac{\partial}{\partial x^i}} \left(x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j} \right) \\
&= \left(\left(\frac{\partial}{\partial y^i} + x^j \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} \right) - \left(0 + y^j \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right) \\
&= \frac{\partial}{\partial y^i}.
\end{aligned}$$

Similarly, $\overline{\nabla}_{\frac{\partial}{\partial y^i}} S = -\frac{\partial}{\partial x^i}$. Thus, we have

$$\overline{\nabla}_X S = f^i \frac{\partial}{\partial y^i} - g^i \frac{\partial}{\partial x^i} = JX.$$

Taking the component of the above vector field which is tangent to $\mathbb{S}^{2n+1}$ and applying Proposition 3.11 and part (iii) of Lemma 4.22, we obtain

$$\nabla_X S = JX. \tag{39}$$

We have $A_X Y = \mathcal{V} \nabla_X Y = \langle \nabla_X Y, S \rangle S = -\langle Y, \nabla_X S \rangle S = -\langle Y, JX \rangle S = \langle X, JY \rangle S$, which proves (ii). Finally, we combine (39) and Lemma 4.22 to get $A_X S = \mathcal{H} \nabla_X S = \mathcal{H}(JX) = JX$, which proves (iii). ∎

We now use use the O'Neill tensors to compute the curvature operator R^* of $\mathbb{CP}^n$.

Proposition 4.24 *The curvature tensor R^* of $\mathbb{CP}^n$ satisfies*

$$R^*(x, y, z, h) = \langle x, h \rangle \langle y, z \rangle - \langle x, z \rangle \langle y, h \rangle - 2 \langle x, Jy \rangle \langle z, Jh \rangle + \langle y, Jz \rangle \langle x, Jh \rangle + \langle z, Jx \rangle \langle y, Jh \rangle$$

for all horizontal $x, y, z, h \in T_p \mathbb{S}^{2n+1}$ and for all $p \in \mathbb{S}^{2n+1}$.

Proof. From {4} in Theorem 4.20, we have

$$\langle R^*_{xy} z, h \rangle = \langle R_{xy} z, h \rangle - 2 \langle A_x y, A_z h \rangle + \langle A_y z, A_x h \rangle + \langle A_z x, A_y h \rangle$$

$$= \langle R_{xy}z, h\rangle - 2\langle\langle x, Jy\rangle S, \langle z, Jh\rangle S\rangle + \langle\langle y, Jz\rangle S, \langle x, Jh\rangle S\rangle + \langle\langle z, Jx\rangle S, \langle y, Jh\rangle S\rangle$$
$$= \langle R_{xy}z, h\rangle - 2\langle x, Jy\rangle\langle z, Jh\rangle + \langle y, Jz\rangle\langle x, Jh\rangle + \langle z, Jx\rangle\langle y, Jh\rangle,$$

where R is the curvature tensor of $\mathbb{S}^{2n+1}$. But from Example 3.37, we have $\langle R_{xy}z, h\rangle = \langle x, h\rangle\langle y, z\rangle - \langle x, z\rangle\langle y, h\rangle$, which completes the proof. ■

Proposition 4.25 *For orthonormal vectors $x, y \in T_p\mathbb{CP}^n$, the sectional curvature of the plane spanned by $\{x, y\}$ is*

$$sec(x, y) = 1 + 3\langle x, Jy\rangle^2.$$

Proof. The previous proposition gives

$$\begin{aligned} sec(x, y) &= \langle R^*_{xy}y, x\rangle \\ &= \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle\langle y, x\rangle - 2\langle x, Jy\rangle\langle y, Jx\rangle + \langle y, Jy\rangle\langle x, Jx\rangle + \langle y, Jx\rangle\langle y, Jx\rangle. \end{aligned}$$

Using the fact that $\{x, y\}$ is an orthonormal set and Lemma 4.22, this simplifies to

$$sec(x, y) = 1 - 0 + 2\langle Jx, y\rangle^2 + 0 + \langle Jx, y\rangle^2.$$

■

Corollary 4.26 *For all $n \geq 2$, the sectional curvatures at each point of $\mathbb{CP}^n$ take on all values between 1 and 4, inclusive.*

Proof. Let $n \geq 2$ and fix $p \in \mathbb{CP}^n$. $\mathbb{CP}^n$ has dimension at least four, and the vertical space has already been found to have dimension equal to one, so the horizontal vector space at p has dimension at least three. Applying the previous lemma, there is a set of three orthonormal vectors $\{x, Jx, y\}$ in the horizontal vector space at p. Define $z(\theta) = \cos(\theta)Jx + \sin(\theta)y$. Notice that $z(\theta)$ is a unit vector in the horizontal space at p and is perpendicular to x for all $\theta \in \mathbb{R}$. Thus,

$$\begin{aligned} sec(z(\theta), x) &= 1 + 3\langle \cos(\theta)Jx + \sin(\theta)y, Jx\rangle^2 \\ &= 1 + 3\left(\cos(\theta)\langle Jx, Jx\rangle + \sin(\theta)\langle y, Jx\rangle\right)^2 \\ &= 1 + 3\cos^2(\theta). \end{aligned}$$

Since $\cos^2(\theta)$ takes on values between 0 and 1, inclusive, the sectional curvatures at p take on all values between 1 and 4, inclusive. ■

Proposition 4.27 *The Gaussian curvature of $\mathbb{CP}^1$ is everywhere equal to 4.*

Proof. Since $\mathbb{CP}^1$ is a two dimensional manifold, the sectional curvature is constant and the Gaussian curvature at p is the sectional curvature at p. Let $x \in T_p\mathbb{S}^{2n+1}$ be a horizorizontal unit vector, so $\{x, Jx\}$ is an orthonormal basis for the horizontal vector space at p. Hence

$$K(p) = sec(x, Jx) = 1 + 3\langle x, Jx\rangle^2 = 1 + 3\langle x, -x\rangle^2 = 4.$$

■

Remark 4.28 In Example 3.37 we found that the sphere $\mathbb{S}^n$ equipped with the round metric $g_{\mathbb{S}^n}$ has sectional curvature everywhere equal to 1. But $\mathbb{S}^2$ is diffeomorphic to $\mathbb{CP}^1$, so the previous corollary shows that the Fubini-Study metric on $\mathbb{CP}^1$ is $\frac{1}{4}g_{\mathbb{S}^2}$.

4.2.3 O'Neill Tensors for Vector Bundle Submersion

We showed in section 2.4 and Example 4.3 that given a Riemannian manifold (M, g), a vector bundle $E \to M$ with connection ∇^E and fibre metric h, there is an induced Riemannian metric $\widehat{g}$ for E which makes the projection $\pi : E \to M$ into a Riemannian submersion. We now want to compute the O'Neill tensors for this submersion.

Lemma 4.29 *Suppose ∇^E is a connection on a vector bundle E. Then*

$$(\nabla^E_m h)_{ij} = \frac{\partial h_{ij}}{\partial x^m} - \Gamma^p_{mi}h_{pj} - \Gamma^p_{mj}h_{ip}$$

for all $1 \leq i, j \leq k$ and $1 \leq m \leq n$.

Proof. Let $\{s^1 \ldots, s^k\}$ be the dual frame for $\{s_1, \ldots, s_k\}$. In this frame, the fibre metric h has the form $h = h_{ij}s^i \otimes s^j$. By definition of the dual connection, we have

$$\nabla_{\frac{\partial}{\partial x^m}} s^i = -\Gamma^i_{mp}s^p.$$

Now computing, we have

$$\begin{aligned}
\nabla^E_m h &= \nabla_{\frac{\partial}{\partial x^m}}(h_{ij}s^i \otimes s^j) \\
&= \frac{\partial h_{ij}}{\partial x^m}s^i \otimes s^j + h_{ij}\nabla_{\frac{\partial}{\partial x^m}}(s^i \otimes s^j) \\
&= \frac{\partial h_{ij}}{\partial x^m}s^i \otimes s^j - h_{ij}\Gamma^i_{mp}s^p \otimes s^j - h_{ij}\Gamma^j_{mp}s^i \otimes s^p \\
&= \left(\frac{\partial h_{ij}}{\partial x^m} - h_{pj}\Gamma^p_{mi} - h_{ip}\Gamma^p_{mj}\right) s^i \otimes s^j,
\end{aligned}$$

which proves the result. ■

Proposition 4.30 *Fix a point $\vartheta \in E$. By choosing an appropriate local frame for E, we assume $\Gamma^l_{ij}(x_0) = 0$, where $x_0 = (x^1_0, \ldots, x^n_0)$ are coordinates in M for $\pi(\vartheta)$. Defining fibre coordinates in terms of this choice of local frame, we have ϑ is identified with a tuple $(x_0, y_0) = (x^1_0, \ldots, x^n_0, y^1_0, \ldots, y^k_0)$. In this set of local coordinates, the expressions for T and A are given by*

$$T_{\frac{\partial}{\partial y^p}|_{(x_0,y_0)}} \frac{\partial}{\partial y^q}\bigg|_{(x_0,y_0)} = -\frac{1}{2} g^{ml} \left(\nabla^E_l h\right)_{pq} \frac{\partial}{\partial x^m}\bigg|_{(x_0,y_0)},$$

$$A_{\frac{\partial}{\partial x_p}|_{(x_0,y_0)}} \frac{\partial}{\partial y^q}\bigg|_{(x_0,y_0)} = \frac{1}{2} g^{ml} (F_{pm})^b_a y^a h_{bq} \frac{\partial}{\partial x^l}\bigg|_{(x_0,y_0)}.$$

where all functions on the right hand side are evaluated at (x_0, y_0), and F is the curvature tensor of ∇^E

Proof. In the following, we *ignore* the Einstein convention for the indices l and m and write the sums explicitly. For other indices, we use the Einstein convention. For notational purposes, we define the local coordinates $(u^1, \ldots, u^{n+k}) = (x^1, \ldots, x^n, y^1, \ldots, y^k)$. Moreover, if s is an index between 1 and k, we define $\underline{l} = l + n$. Notice, for example, that $\frac{\partial}{\partial u^{\underline{l}}} = \frac{\partial}{\partial y^l}$. This will help keep notation clear us as we compute T and A in local coordinates.

$$\begin{aligned}
T_{\frac{\partial}{\partial y^p}} \frac{\partial}{\partial y^q} &= \mathcal{H}\left(\nabla_{\frac{\partial}{\partial y^p}} \frac{\partial}{\partial y^q}\right) \\
&= \mathcal{H}\left(\nabla_{\frac{\partial}{\partial u^{\underline{p}}}} \frac{\partial}{\partial u^{\underline{q}}}\right) \\
&= \sum_{m=1}^{n+k} \mathcal{H}\left(\Omega^m_{\underline{pq}} \frac{\partial}{\partial u^m}\right) \\
&= \mathcal{H}\left(\sum_{m=1}^{n}\left(\Omega^m_{\underline{pq}} \frac{\partial}{\partial x^m}\right) + \sum_{m=1}^{k}\left(\Omega^{\underline{m}}_{\underline{pq}} \frac{\partial}{\partial y^m}\right)\right) \\
&= \mathcal{H}\Bigg(\sum_{m=1}^{n}\left(\Omega^m_{\underline{pq}} \frac{\partial}{\partial x^m} - \Gamma^m_{ij} \Omega^i_{\underline{pq}} y^j \frac{\partial}{\partial y^m}\right) \\
&\qquad + \sum_{m=1}^{k}\left(\Gamma^m_{ij} \Omega^i_{\underline{pq}} y^j \frac{\partial}{\partial y^m} + \Omega^{\underline{m}}_{\underline{pq}} \frac{\partial}{\partial y^m}\right)\Bigg) \\
&= \sum_{m=1}^{n}\left(\Omega^m_{\underline{pq}} \frac{\partial}{\partial x^m} - \Gamma^m_{ij} \Omega^i_{\underline{pq}} y^j \frac{\partial}{\partial y^m}\right)
\end{aligned}$$

We need to compute $\Omega^m_{\underline{pq}}$ for any $1 \leq m \leq n$. Indeed, using Proposition 2.43, we have

$$\Omega^m_{\underline{pq}} = \frac{1}{2} \sum_{l=1}^{n+k} \widehat{g}^{ml} \left(\frac{\partial \widehat{g}_{\underline{q}l}}{\partial u^{\underline{p}}} + \frac{\partial \widehat{g}_{\underline{p}l}}{\partial u^{\underline{q}}} - \frac{\partial \widehat{g}_{\underline{pq}}}{\partial u^l}\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{l=1}^{n} \widehat{g}^{ml}\left(\frac{\partial}{\partial y^p}\widehat{g}\left(\frac{\partial}{\partial x^l},\frac{\partial}{\partial y^q}\right)+\frac{\partial}{\partial y^q}\widehat{g}\left(\frac{\partial}{\partial x^l},\frac{\partial}{\partial y^p}\right)-\frac{\partial}{\partial x^l}\widehat{g}\left(\frac{\partial}{\partial y^p},\frac{\partial}{\partial y^q}\right)\right)\\
&\quad+\frac{1}{2}\sum_{l=1}^{k} \widehat{g}^{m\underline{l}}\left(\frac{\partial}{\partial y^p}\widehat{g}\left(\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^q}\right)+\frac{\partial}{\partial y^q}\widehat{g}\left(\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^p}\right)-\frac{\partial}{\partial y^l}\widehat{g}\left(\frac{\partial}{\partial y^p},\frac{\partial}{\partial y^q}\right)\right)\\
&= \frac{1}{2}\sum_{l=1}^{n} \widehat{g}^{ml}\left(\Gamma_{lp}^a h_{aq}+\Gamma_{lq}^a h_{ap}-\frac{\partial h_{pq}}{\partial x^l}\right)\\
&\quad+0.
\end{aligned}
$$

Evaluating at (x_0, y_0) and applying the previous lemma, we obtain

$$
\Omega^m_{\underline{pq}}(x_0,y_0) = -\frac{1}{2}\sum_{l=1}^{n} g^{ml}(\nabla^E_l h)_{pq}.
$$

All together, we have

$$
\begin{aligned}
\left(T_{\frac{\partial}{\partial y^p}}\frac{\partial}{\partial y^q}\right)(x,y) &= \sum_{l=1}^{n}\left(\Omega^l_{\underline{pq}}\frac{\partial}{\partial x^l}-\Gamma^l_{ij}\Omega^i_{\underline{pq}}y^j\frac{\partial}{\partial y^l}\right)\Bigg|_{(x,y)}\\
&= -\frac{1}{2}\sum_{m,l=1}^{n} g^{ml}(\nabla^E_l h)_{pq}\frac{\partial}{\partial x^m}.
\end{aligned}
$$

We now compute the second O'Neill tensor.

$$
\begin{aligned}
A_{\frac{\partial}{\partial x^p}-\Gamma^l_{pj}y^j\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right) &= \mathcal{H}\left(\nabla_{\frac{\partial}{\partial x^p}-\Gamma^l_{pj}y^j\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right)\right)\\
&= \mathcal{H}\left(\nabla_{\frac{\partial}{\partial x^p}}\left(\frac{\partial}{\partial y^q}\right)\right)-\Gamma^l_{pj}y^j\mathcal{H}\left(\nabla_{\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right)\right)\\
&= \mathcal{H}\left(\Omega^l_{p\underline{q}}\frac{\partial}{\partial u^l}\right)-\Gamma^l_{pj}y^j\mathcal{H}\left(\nabla_{\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right)\right)\\
&= \mathcal{H}\left(\sum_{l=1}^{n}\Omega^l_{p\underline{q}}\frac{\partial}{\partial x^l}+\sum_{l=1}^{k}\Omega^{\underline{l}}_{p\underline{q}}\frac{\partial}{\partial y^l}\right)-\Gamma^l_{pj}y^j\mathcal{H}\left(\nabla_{\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right)\right)\\
&= \sum_{l=1}^{n}\left(\Omega^l_{p\underline{q}}\frac{\partial}{\partial x^l}-\Omega^l_{p\underline{q}}\Gamma^m_{lj}y^j\frac{\partial}{\partial y^m}\right)-\Gamma^l_{pj}y^j\mathcal{H}\left(\nabla_{\frac{\partial}{\partial y^l}}\left(\frac{\partial}{\partial y^q}\right)\right).
\end{aligned}
$$

We need to compute $\Omega^l_{p\underline{q}}(x)$ for $1 \le l \le n$. Indeed, we have

$$
\Omega^l_{p\underline{q}} = \sum_{m=1}^{n+k}\frac{1}{2}\widehat{g}^{ml}\left[\frac{\partial \widehat{g}_{\underline{q}m}}{\partial u^p}+\frac{\partial \widehat{g}_{pm}}{\partial u^{\underline{q}}}-\frac{\partial \widehat{g}_{p\underline{q}}}{\partial u^m}\right].
$$

Consider when $n+1 \leq m \leq n+k$. Since $1 \leq l \leq n$ and the Christoffel symbols vanish at x_0, we have $\widehat{g}^{ml}(x_0) = 0$. So the last k terms of the sum vanish at (x_0, y_0), giving

$$\Omega^l_{p\underline{q}}(x_0, y_0) = \left(\sum_{m=1}^{n} \frac{1}{2} \widehat{g}^{ml} \left[\frac{\partial \widehat{g}_{\underline{q}m}}{\partial u^p} + \frac{\partial \widehat{g}_{pm}}{\partial u^{\underline{q}}} - \frac{\partial \widehat{g}_{p\underline{q}}}{\partial u^m} \right] \right) \Bigg|_{(x_0, y_0)}.$$

Suppose $1 \leq m \leq n$. Then

$$\begin{aligned}
&\frac{1}{2} \widehat{g}^{ml} \left(\frac{\partial \widehat{g}_{\underline{q}m}}{\partial u^p} + \frac{\partial \widehat{g}_{pm}}{\partial u^{\underline{q}}} - \frac{\partial \widehat{g}_{p\underline{q}}}{\partial u^m} \right) \\
&= \frac{1}{2} \widehat{g}^{ml} \left(\frac{\partial}{\partial x^p} \widehat{g} \left(\frac{\partial}{\partial y^q}, \frac{\partial}{\partial x^m} \right) + \frac{\partial}{\partial y^q} \widehat{g} \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^m} \right) - \frac{\partial}{\partial x^m} \widehat{g} \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial y^q} \right) \right) \\
&= \frac{1}{2} \widehat{g}^{ml} \left(\frac{\partial}{\partial x^p} \left(\Gamma^b_{ma} y^a h_{bq} \right) + \frac{\partial}{\partial y^q} \left(g_{pm} + \Gamma^b_{pa} \Gamma^d_{mc} y^a y^c h_{bd} \right) - \frac{\partial}{\partial x^m} \left(\Gamma^b_{pa} y^a h_{bq} \right) \right) \\
&= \frac{1}{2} \widehat{g}^{ml} \Bigg(\frac{\partial \Gamma^b_{ma}}{\partial x^p} y^a h_{bq} + \Gamma^b_{ma} y^a \frac{\partial h_{bq}}{\partial x^p} \\
&\qquad + \Gamma^b_{pq} \Gamma^d_{mc} y^c h_{bd} + \Gamma^b_{pa} \Gamma^d_{mq} y^a h_{bd} - \frac{\partial \Gamma^b_{pa}}{\partial x^m} y^a h_{bq} - \Gamma^b_{pa} y^a \frac{\partial h_{bq}}{\partial x^m} \Bigg).
\end{aligned}$$

Evaluating at (x_0, y_0), we obtain

$$\begin{aligned}
\frac{1}{2} \widehat{g}^{ml} \left(\frac{\partial \widehat{g}_{\underline{q}m}}{\partial u^p} + \frac{\partial \widehat{g}_{pm}}{\partial u^{\underline{q}}} - \frac{\partial \widehat{g}_{p\underline{q}}}{\partial u^m} \right) &= \frac{1}{2} g^{ml} \left(\frac{\partial \Gamma^b_{ma}}{\partial x^p} - \frac{\partial \Gamma^b_{pa}}{\partial x^m} \right) y^a h_{bq} \\
&= \frac{1}{2} g^{ml} \left(F_{pm} \right)^b_a y^a h_{bq},
\end{aligned}$$

where the right hand side is assumed to be evaluated at (x_0, y_0), and F_{pm} is the matrix of local functions corresponding to the curvature tensor of the connection on E at p. Putting it all together at (x_0, y_0), we have

$$\begin{aligned}
\left(A_{\frac{\partial}{\partial x^p} - \Gamma^l_{pj} y^j \frac{\partial}{\partial y^l}} \left(\frac{\partial}{\partial y^q} \right) \right) (x_0, y_0) &= \left(\sum_{l=1}^{n} \Omega^l_{p\underline{q}} \frac{\partial}{\partial x^l} - \Omega^l_{p\underline{q}} \Gamma^m_{lj} y^j \frac{\partial}{\partial y^m} \right) \Bigg|_{(x_0, y_0)} \\
&= \sum_{l=1}^{n} \Omega^l_{p\underline{q}}(x_0, y_0) \frac{\partial}{\partial x^l} \bigg|_{(x_0, y_0)} \\
&= \sum_{l,m=1}^{n} \frac{1}{2} g^{ml} \left(F_{pm} \right)^b_a y^a h_{bq} \frac{\partial}{\partial x^l} \bigg|_{(x_0, y_0)}.
\end{aligned}$$

■

Corollary 4.31 *$T = 0$ iff $\nabla h = 0$, and $A = 0$ iff $F = 0$ where F is the curvature of ∇^E.*

Proof. This follows immediately from the previous proposition and the fact that T_X and A_V are skew symmetric operators for all vector fields X on E. ■

Corollary 4.32 *The fibres of π are all totally geodesic iff h is metric compatible.*

Proof. Follows from the previous corollary and Proposition 4.12. ■

References

[1] William M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Second, Pure and Applied Mathematics, vol. 120, Academic Press, Inc., Orlando, FL, 1986. MR861409

[2] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty. MR1138207

[3] John M. Lee, *Introduction to smooth manifolds*, Second, Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR2954043

[4] ______, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. Second edition of [MR1468735]. MR3887684

[5] Barrett O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469. MR200865

[6] Peter Petersen, *Riemannian geometry*, Third, Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. MR3469435