Derived Geometry and The Integrability Problem for G-Structures

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2017

AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we study the integrability problem for G-structures. Broadly speaking, this is the problem of determining topological obstructions to the existence of principal G-subbundles of the frame bundle of a manifold, subject to certain differential equations. We begin this investigation by introducing general methods from homological algebra used to obtain cohomological obstructions to the existence of solutions to certain geometric problems. This leads us to a precise analogy between deformation theory and the formal integrability properties of partial differential equations. Along the way, we prove the following differential-geometric analogue of a well-known result from derived algebraic geometry:

$$H^*(N^\bullet(S^1 \otimes C^\infty(M))) \cong \Omega^{-*}(M)$$

as well as the identification of the infinitesimal generator of the natural S^1-action corresponding to loop rotation with the de Rham differential. As a short corollary we obtain a natural isomorphism

$$H^*(\mathrm{Hom}(N^\bullet(S^1 \otimes C^\infty(M)), C^\infty(M))) \cong \Gamma(M, \Lambda^{-*}T_M)$$

leading to a comparison between the standard Gerstenhaber bracket on the left-hand-side of the above equation and the Schouten bracket on the right. These two results are well-known in derived algebraic geometry and are folk-lore in differential geometry, where we were unable to find an explicit proof in the literature. In the end, this machinery is used to provide what the author believes is a new perspective on the integrability problem for G-structures.

Acknowledgements

First and foremost I would like to thank my supervisor Spiro Karigiannis for spending so much time answering my questions every week, for going through this monstrosity of a document with a fine-tooth comb, and for running the Geometric Analysis Working Seminar in which I learned so much.

I also want to thank Anton Borissov, Henry Liu and Raymond Cheng for running the Geometry Working Seminar as well as the Mirror Symmetry Seminar in the summer of 2016. Much of this thesis would have been far more difficult for me without some of the intuition for geometry I gained in that seminar. Furthermore, I extend my gratitude to my friends and family for bearing with me in this crazy year. Especially my parents Angelina and Jim, and my sister Michelle. A special thanks to Grant and Ian for organizing boardgame nights, Brandon for organizing trivia nights, Sam and Ilia for Friday, Parham for the office antics, and Osman and Danny for making time spent in some of the grossest apartments bearable, if not enjoyable.

Finally, I'd like to thank Benoit Charbonneau for sparking my initial interest in differential geometry, Jason Bell for teaching me most of the algebra I know, and Andu Nica for showing me how to integrate in infinite dimensions (something which blows my mind to this day).

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Chapter 1

Preliminaries

Throughout this thesis all of our complexes will be cohomological. Given a complex A, we'll write $A[n]$ for the complex with $A[n]^i = A^{i+n}$ and differential multiplied by $(-1)^n$. Notational conventions are in the appendix.

1.1 Introduction

In this section I'll be describing the basic motivating problem for this thesis. Our main goal will be to obtain a thorough understanding of the *integrability problem for G-structures*. To do this, let's begin by developing an intuitive understanding of what a G-structure is.

Let's think about the universe we live in. A hypothetical high-school student, when asked what an appropriate mathematical model for space is, might respond with $\mathbb{R}^3$. Indeed, the 3-dimensional space we live in is often how we first think of the vector space $\mathbb{R}^3$. But this is inaccurate. For example, one could ask this high-school student: where is the point $(0,0,0) \in \mathbb{R}^3$ located in space? In which directions do the three standard coordinate axes point? This line of questioning leads us to posit that perhaps space is best represented by an *affine* $\mathbb{R}^3$. So one can choose any point in space and any three independent direction vector emanating from that point and from this one will obtain an identification of space with $\mathbb{R}^3$.

But this description also falls short. Indeed, using the very same thought process, one could arrive at the conclusion that the surface of the earth is modelled by an affine $\mathbb{R}^2$, i.e. that the earth is flat. In this modern era we know this to be false. The point being that while locally the earth does indeed appear to be flat, globally this is not the case. For example, we know that the surface of the earth is compact (a topological property) while affine $\mathbb{R}^2$ is not.

As such, we cannot assume that space is an affine $\mathbb{R}^3$ a priori. Locally it certainly seems to look like one but globally there may be topological non-trivialities. Another way in which space differs from affine $\mathbb{R}^3$ is *curvature*. One measures distance on an affine $\mathbb{R}^3$ using the standard Euclidean distance function. Returning to the example of the surface of the earth, we can recall that if one draws a sufficiently large triangle on the ground and sums the interior angles, it is possible to obtain a value larger than 180°. This is not true for affine $\mathbb{R}^3$. Although, if one is only allowed to draw triangles on the ground in such a way that by standing at any corner one can see the whole triangle, then the sum of the interior angles should be a reasonably good approximation of 180°. The point is, the distance function on the surface differs non-trivially from the usual Euclidean distance function. This difference is called the curvature of the distance function. In the same way that we should not have assumed the surface of the earth was flat (had no curvature) we shouldn't assume space is flat.

So we now have two potential ways in which space could differ non-trivially from affine $\mathbb{R}^3$: the topology and the curvature. It turns out that these two properties are distinct but are related in subtle ways. Let's now describe the type of mathematical object we will use to model space.

Definition 1.1.1. A **smooth manifold** is a Hausdorff second countable topological space M together with an open cover by U_i's and continuous maps $\varphi_i : U_i \to \mathbb{R}^{n_i}$ which are homeomorphisms onto their (open) images such that

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a diffeomorphism for all i, j such that $U_i \cap U_j \neq \emptyset$. Furthermore, we will assume that the collection of such pairs (U_i, φ_i) is maximal and we call this collection a (maximal) **atlas**. The pairs (U_i, φ_i) themselves are called **charts**.

A standard fact from differential geometry is that the n_i's appearing in the above definition are locally constant with respect to the topology on M. However, it is possible to have non-connected manifolds for which the n_i's tend to infinity since second-countability allows the number of connected components to be countably infinite.

We should mention a couple of things regarding the above definition. The assumption that our space be Hausdorff can be thought of as saying that it is possible to distinguish between any two points in space. The second countability assumption is equivalent, for topological spaces satisfying all of the other assumptions in our definition, to having countably many connected components each of which is metrizable.

The last point in our definition was that the transition functions between charts (i.e. changes of coordinates) were assumed to be diffeomorphisms. This is essentially saying that the rules of calculus work the same in any coordinate chart. An important advantage of this is that it allows one to perform multivariable calculus globally on manifolds by simply doing constructions in each chart.

One thing worth observing is that we defined manifolds in dimensions other than just 3. This is because other objects than just space end up being modelled well by manifolds. For example, the surface of the earth is a 2-dimensional manifold, space-time is a 4-dimensional manifold, and the space of possible configurations of a rigid body in space looks like (at least locally)

$$\mathbb{R}^3 \rtimes \mathrm{SO}(3).$$

This is a 6-dimensional manifold (and, in fact, a Lie group). The most important example of a higher dimensional manifold for us will be the *frame bundle* of a manifold. Recall that the whole point of thinking of space-time as a manifold is that it is, in some sense, the most general type of reasonable object for which one can choose a reference frame at each point. As it turns out, the space of all reference frames at any point is itself a manifold. We describe this object in general below.

Definition 1.1.2. For a smooth manifold M of dimension n, the **frame bundle** of M is the principal $\mathrm{GL}(\mathbb{R}^n)$-bundle $F_M \to M$ whose fibre above a point $p \in M$ is given by

$$(F_M)_p := \{f : T_pM \to \mathbb{R}^n \ : \ f \text{ is a linear isomorphism}\}.$$

The left action of $\mathrm{GL}(\mathbb{R}^n)$ on F_M is given by post-composition of functions.

In the case that M is the manifold describing space, the frame bundle F_M becomes important when attempting to describe notions such as inertia and the introduction of units of measurement. Indeed, after a bit of thought one can realize that it is nigh impossible to define a reasonable notion of a *stationary* reference frame. While we often think of ourselves as being "stationary" when lying in our beds, we know that with respect to the sun we are actually moving quite fast in a non-trivial way. The point is: being stationary is a *relation*. One can only say something is stationary with respect to something else.

Suppose I'm given a reference frame in space, thought of as a chart $\varphi : U \to \mathbb{R}^3$, and let $p \in U$ be the point corresponding to $0 \in \mathbb{R}^3$ (we say the chart is *centered* at p). The velocities of objects with respect to this reference frame in various directions are typically measured with respect to the tangent vectors $\varphi^{-1}_{*,p}(e_i) \in T_pM$ where the e_i are the standard basis vectors for $\mathbb{R}^3$. We intuitively want two reference frames at p to be called inertial with respect to one another if and only if the differential equations governing our measurements of the motion of point particles with respect to the above velocity vectors should be the same. In particular, measurements of speed made using the dot product on $\mathbb{R}^3$ and the vectors $\varphi^{-1}_{*,p}(e_i)$ should yield the same results.

In order to relate reference frames at various points in space one would need to make use of the *frame Lie groupoid* [42, 43, 55] however we won't be concerning ourselves with this. Despite this, I believe these objects are incredibly important towards progress in differential geometry as I hope will become clear in section 2.3.

Speed is supposed to be the length of the velocity but, as of yet, we have no coordinate-independent way of measuring the length of tangent vectors to M. However, speeds in reality can indeed be measured and so we will assume that our space manifold M comes equipped with a Riemannian metric

$$g \in \Gamma(M, \mathrm{Sym}^2\, T^*_M)$$

i.e. a smoothly varying inner product on each tangent space. Notice that by multiplying g by a positive globally defined function e^f on M we change the length scale (i.e. units) in which we are measuring speeds without affecting how angles between velocity vectors are measured.

With respect to the usual Fréchet topology, the space of all Riemannian metrics on M is an open cone in $\Gamma(M, \mathrm{Sym}^2\, T^*_M)$. Now, any reference frame $f \in (F_M)_p$ at p arises from a local chart centered at p by the inverse function theorem, the existence of solutions to ODEs, and our assumption that our atlases are maximal. Indeed, given a frame $f : T_pM \to \mathbb{R}^n$ we use local coordinates centered at p to locally extend the inverses images of the standard basis vectors $f^{-1}(e_i) \in T_pM$ to vector fields X_i on our coordinate patch with $[X_i, X_j] = 0$. Then one obtains our desired local chart by successively flowing along these vector fields. It's worth mentioning, however, that different charts at p will often yield the same frame at p.

Given two frames $f, g \in (F_M)_p$ it then follows that measurements performed at p using these local coordinates will agree if and only if they are related by an orthogonal transformation, i.e. an element of the group

$$\mathrm{O}(T_pM, g_p) = \{A \in \mathrm{GL}(T_pM) \ : \ g_p(AX_p, AY_p) = g_p(X_p, Y_p) \ \text{ for all } X_p, Y_p \in T_pM\}.$$

Reference frames φ with orthogonal velocity vectors $\varphi^{-1}_{*,p}(e_i)$ of unit length are distinguished in the sense that to obtain the correct results when performing computations in local coordinates we are required to work in them. Furthermore, by distinguishing a compatible collection of such frames one actually specifies a metric g given by using the standard dot product on $\mathbb{R}^3$ in each distinguished reference frame.

This is all related to the notion of inertia in the following way. First of all, a Riemannian metric g determines a notion of *parallel transport* along curves in space, replacing the notion of translation by vectors in $\mathbb{R}^3$. This allows us to relate choices of units at different points in space and is seen most naturally from the formalism of Lie groupoids, mentioned above. While this generalizes the action of $\mathbb{R}^3 \ltimes \mathrm{SO}(3)$ on space, the notion of intertia corresponds to an analogous generalization of the action of the *Galilean group*, whose connected component of the identity is

$$\mathbb{R}^4 \ltimes (\mathbb{R}^3 \ltimes \mathrm{SO}(3)),$$

on *space-time*. All of this generalizes further as follows.

This notion of a distinguished collection of "orthonormal frames" on space M is what is called an $\mathrm{O}(3)$-structre (since each of the groups $\mathrm{O}(T_pM, g_p)$ is isomorphic to $\mathrm{O}(3)$). We now list examples where structures on manifolds arising from Lie groups other than $G = \mathrm{O}(3)$ arise naturally.

1. In order to measure both translational and rotational speeds on the configuration space M of a rigid body (recall: this locally looks like $\mathbb{R}^3 \rtimes \mathrm{SO}(3)$) one needs a $\mathrm{O}(6)$-structure on M. This specifies units for both translational and angular speed;

2. Since the change of variables formula for integration involves an *absolute value* of the determinant of the Jacobian matrix, it follows that if one hopes to integrate top degree forms on a manifold (as opposed to the less-talked-about densities) one requires a $\mathrm{GL}^+(\mathbb{R}^n)$ structure.

3. Hamilton's equations in classical mechanics come from the existence of a globally defined pairing of sorts between the generalized position and momentum coordinates on the phase space of a classical mechanical system. Mathematically, such a pairing is precisely a $\mathrm{Sp}(2n, \mathbb{R})$-structure.

4. When compactifying string theories, initially defined on $\mathbb{R}^{10}$ or $\mathbb{R}^{11}$, down to $\mathbb{R}^4 \times M$ one must use a (possibly non-flat) metric on M together with the usual Lorenzian metric on $\mathbb{R}^4$ in order to write down the action functional. If one hopes the associated quantum field theory to admit a certain amount of supersymmetry (an extension of the canonically defined representation of the Poincaré algebra to some Lie superalgebra extension of it by Clifford modules) then the action functional, which is used to describe the inner product on the Hilbert space these Lie algebras are being represented as self-adjoint operators, needs to have certain symmetries. This amounts to the metric on M having holonomy contained in $\mathrm{U}(3)$ or G_2 respectively, which in turn gives rise to $\mathrm{U}(3)$ and G_2-structures on M. Furthermore, in the $\mathrm{U}(3)$ case, for the resulting theory to be renormalizable we end up actually needing the metric on M to have holonomy in $\mathrm{SU}(3)$, giving rise to a $\mathrm{SU}(3)$-structure. [3, 13, 15, 31, 44]

Now that we have provided (hopefully) sufficient motivation, I will present a definition of a G-structure. This formal definition will probably be fairly non-transparent and so I will follow it up with a discussion attempting to give more intuition.

Definition 1.1.3. Fix a Lie group G together with a faithful representation $G \to \mathrm{GL}(\mathbb{R}^n)$. Then a ***G*-structure** on a n-dimensional manifold M is a principal G-subbundle P of the frame bundle F (where the G-equivariance of the inclusion $P \subseteq F$ is described using our given injective homomorphism $G \to \mathrm{GL}(\mathbb{R}^n)$).

Earlier we mentioned two distinct, but related, ways in which the surface of the earth differed from affine $\mathbb{R}^2$. One of them was topological, namely the fact that the surface of the earth was compact, and the other was with regards to the metric, i.e. $\mathrm{O}(2)$-structure. Let's now see our first example of the interplay between these two properties. The claim is: the existence of a G-structure on a manifold imposes restrictions on the topology of the manifold.

Indeed, what is a G-structure? A principal G-bundle on M is a surjective submersion $P \to M$ together with a free and proper smooth left action of G on P which preserves the fibres and has $P/G \cong M$ via the submersion $P \to M$. Now, a G-structure is a principal G-subbundle P of the frame bundle F in such a way that the G-equivariance of the inclusion $P \subseteq F$ is given by a specified faithful representation $G \subseteq \mathrm{GL}(\mathbb{R}^n)$. In other words, G is not merely a Lie group but is in fact a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ together with a choice of inclusion $G \subseteq \mathrm{GL}(\mathbb{R}^n)$.

Now, given such a subgroup $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ we can look at a local trivialization of the frame bundle

$$\begin{tikzcd} F|_U \arrow[r] \arrow[d] & \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n) \arrow[d] \\ U \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

By pulling-back the inclusion $\mathbb{R}^n \times G \subseteq \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ we get a principal G-subbundle of $F|_U$ (only defined over U). In other words, via the action of $\mathrm{GL}(\mathbb{R}^n)$ on $F|_U$ we also have an action of G on $F|_U$. However, while the action of $\mathrm{GL}(\mathbb{R}^n)$ is transitive on each fibre, this is not the case for the action of G and so the quotient $F|_U/G$ will not be diffeomorphic to $U \subseteq M$ via the projection $F|_U \to U$. Instead, the quotient is given by the associated fibre bundle

$$F|_U/G \cong F|_U \times_{\mathrm{GL}(\mathbb{R}^n)} (\mathrm{GL}(\mathbb{R}^n)/G).$$

Notice that the quotient $\mathrm{GL}(\mathbb{R}^n)/G$ is indeed a smooth manifold since the action of G on $\mathrm{GL}(\mathbb{R}^n)$ by left multiplication is free (if $gh = h$ then we simply right-cancel h) and if we have sequences $g_i \in G$, $h_i \in \mathrm{GL}(\mathbb{R}^n)$ such that $h_i \to h$ and $g_i h_i \to a$ then by continuity of $(-)^{-1}$ we have $h_i^{-1} \to h^{-1}$ and furthermore by the continuity of multiplication we have

$$g_i = (g_i h_i) h_i^{-1} \to a h^{-1}$$

and hence g_i (and, a fortiori, some subsequence of it) converges. Thus the action is also proper since the connected components of a manifold are metrizable and on metric spaces compactness and sequential compactness agree. Thus the quotient $\mathrm{GL}(\mathbb{R}^n)/G$ is naturally a smooth manifold.

So, by pulling back $\mathbb{R}^n \times G \subseteq \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ to obtain a principal G-subbundle of $F|_U$, we have really chosen a global section of the above associated fiber bundle. Namely, a smoothly varying choice of G-orbit in each fiber of the frame bundle. In quotienting this submanifold by the action of G, we then obtain a diffeomorphism with $U \subseteq M$ where U is identified with the image of our globally chosen section of the associated bundle in the quotient of our submanifold. From this we obtain the following proposition which can be found in [51].

Proposition 1.1.4. *Let M be a smooth n-dimensional manifold and $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ a Lie subgroup. Then G-structures on M are in a natural bijective correspondence with global sections of the associated fiber bundle $F \times_{\mathrm{GL}(\mathbb{R}^n)} (\mathrm{GL}(\mathbb{R}^n)/G)$.*

Proof. Via the identification of the associated fiber bundle with F/G we see that given a global section of F/G, the preimage of its image in F/G through $F \to F/G$ is a principal G-subbundle of F. Conversely, given a principal G-subbundle $P \subseteq F$ we note that the fiber of P over any point of M is sent to a single point of F/G under the quotient map. This gives a globally defined section which can be shown to be smooth using an atlas of M which trivializes P locally. $\square$

Some first examples of Lie groups G for which there are topological obstructions to the existence of G-structures on a manifold would be any Lie subgroup of $\mathrm{GL}^+(\mathbb{R}^n) \subseteq \mathrm{GL}(\mathbb{R}^n)$. Any such structure determines, in particular, a $\mathrm{GL}^+(\mathbb{R}^n)$-structure and hence an orientation. Indeed, if $G \subseteq H \subseteq \mathrm{GL}(\mathbb{R}^n)$ is a chain of Lie subgroups then the quotient map

$$\mathrm{GL}(\mathbb{R}^n)/G \to \mathrm{GL}(\mathbb{R}^n)/H$$

yields a morphism of fiber bundles

$$\begin{tikzcd} F/G \arrow[r] \arrow[d] & F/H \arrow[dl] \\ M & \end{tikzcd}$$

sending global sections of F/G to global sections of F/H (see [9] for more details). So, returning to the case of a Lie subgroup of $\mathrm{GL}^+(\mathbb{R}^n)$, we have an obstruction

$$w_1(TM) \in H^1(M, \mathbb{Z}/2)$$

given by the first Stiefel-Whitney class [40]. Notice that since the determinant is multiplicative it follows that $\mathrm{GL}^+(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and the quotient is

$$\mathrm{GL}(\mathbb{R}^n)/\,\mathrm{GL}^+(\mathbb{R}^n) \cong \mathbb{Z}/2,$$

the same group for which the coefficients of the singular cohomology groups containing our obstructions live in! Indeed, we'll see later in section 2.3 that one can use that $\mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)/G$ is a principal fibre bundle with fibre G, and in particular a fibration, to obtain that the obstructions to the existence of G-structures on a manifold M will generally lie in

$$H^{m+1}(M, \pi_m(\mathrm{GL}(\mathbb{R}^n)/G)).$$

A less trivial example of a topological obstruction to the existence of a G-structure comes in the case of $G = G_2$. Here we have two obstructions [28], namely both the first and second Stiefel-Whitney classes

$$w_1(TM) \in H^1(M, \mathbb{Z}/2), \quad w_2(TM) \in H^2(M, \mathbb{Z}/2).$$

The fact that $w_1(TM)$ is an obstruction is obvious since $G_2 \subseteq \mathrm{SO}(7) \subseteq \mathrm{GL}^+(\mathbb{R}^7)$ (assuming one is comfortable with the fact that $G_2 \subseteq \mathrm{SO}(7)$). The w_2 obstruction is more difficult to see but comes from spin geometry [40].

We now begin a very long example: the case of a Riemannian metric. So, suppose we had a smooth manifold M together with a $\mathrm{O}(n)$-structure, expressed in terms of a Riemannian metric g on M. Now, take a point $p \in M$ and choose a coordinate chart $(U, x^1, \cdots, x^n)$ centered at p. Writing vector fields on U in terms of the frame $\partial/\partial x^i$ we can express g as a matrix of functions g_{ij} on U via

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Then, if vector fields X, Y on U are given by $X = X^i \partial/\partial x^i$ and $Y = Y^j \partial/\partial x^j$ (summation over repeated indices is implicit) we can express their g-inner products, again using Einstein summation notation, as

$$g(X, Y) = g_{ij} X^i Y^j.$$

Now, there is a second metric on U given by

$$h(X, Y) = \delta_{ij} X^i Y^j.$$

This is just the pull-back of the dot product on $\mathbb{R}^n$ to U via the chart. One question we can ask is: how do the functions g_{ij} compare to the Kronecker delta δ_{ij}? By choosing the x^i so that the $\partial/\partial x^i|_p$ form a g_p-orthonormal basis for T_pM it follows that, at p, there is a coordinate system so that $g_{ij}(p) = \delta_{ij}$. However, we cannot guarantee that this can be made to happen on an entire neighbourhood of p. Indeed, let's compute the Taylor expansion of the functions g_{ij} at p. In the quotient of the ring of germs $C^\infty_{M,p}/\ker(\mathrm{ev}_p)^3$ with respect to local coordinates x^i centered at p we have

$$g_{ij} = g_{ij}(p) + (\partial_k g_{ij})(p) x^k + \frac{1}{2}(\partial_\ell \partial_k g_{ij})(p) x^\ell x^k + \ker(\mathrm{ev}_p)^3.$$

By taking orthonormal coordinates we know that we can get $g_{ij}(p) = \delta_{ij}$, so let's assume for now that we have chosen such coordinates. For the first order terms, we let ∇ be the Levi-Civita connection associated to g (the unique torsion-free g-compatible affine connection on M) and write $\partial/\partial x^i|_p =: e_i$, $\nabla_{e_i} =: \nabla_i$ and

$$\nabla_i e_j = \Gamma^k_{ij} e_k.$$

The functions Γ^k_{ij} are called the **Christoffel symbols** and satisfy $\Gamma^k_{ij} = \Gamma^k_{ji}$ since the Levi-Civita connection is torsion-free. Now, one can show [20, 59] that any geodesic $\gamma : (-\epsilon, \epsilon) \to M$ (i.e. $(\gamma^*\nabla)\gamma' = 0$) with $\gamma(0) = p$ satisfies

$$\frac{d^2(x^k \circ \gamma)}{dt^2} + (\Gamma^k_{ij} \circ \gamma)\frac{d(x^i \circ \gamma)}{dt}\frac{d(x^j \circ \gamma)}{dt} = 0. \tag{1.1}$$

Our goal is to use this to demonstrate that the existence of a g-compatible torsion-free affine connection implies that at any point, we can always find a coordinate system in which the 0'th and 1'st order terms of the Taylor expansion of g_{ij} at the center all vanish. Indeed, let $U_p \subseteq T_pM$ be the open neighbourhood of $0 \in T_pM$ consisting of all those tangent vectors X_p for which there exists a geodesic $\gamma_{X_p} : (-\epsilon, \epsilon) \to M$ with $\gamma_{X_p}(0) = p$, $\epsilon > 1$ and

$$(T_p\gamma_{X_p})\left(\frac{d}{dt}\Big|_{t=0}\right) = X_p.$$

This subset $U_p \subseteq T_pM$ does indeed end up being open. The *exponential map*

$$\begin{aligned} \exp_p : U_p &\to M \\ X_p &\mapsto \gamma_{X_p}(1) \end{aligned}$$

ends up not only being smooth, but actually a diffeomorphism onto its image [51] (after potentially restricting its domain to a smaller open neighbourhood of $0 \in T_pM$). This allows us to obtain local coordinates centered at p by letting x^i be orthonormal coordinates centered at p and then considering the functions

$$y^i := dx^i_p \circ \exp_p^{-1} : M \to \mathbb{R}$$

where $dx^i : T_pM \to \mathbb{R}$ is the basis of 1-forms dual to the orthonormal basis $\partial/\partial x^i|_p \in T_pM$. By construction, this is an orthonormal coordinate system centered at p, but now given any $X_p \in T_pM$ the unique geodesic $\gamma_{X_p} : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = X_p$ has components

$$(y^i \circ \gamma)(t) = X^i_p t$$

in this coordinate system. Any coordinates y^i constructed in this way are called **normal coordinates** centered at p. In these coordinates, the second-order equation satisfied by geodesics which we wrote above (1.1) applied to γ_{X_p} takes the form

$$(\Gamma^k_{ij} \circ \gamma)X^i_p X^j_p = 0.$$

Since this holds for all X_p and $\Gamma^k_{ij} = \Gamma^k_{ji}$ we obtain

$$\Gamma^k_{ij}(p) = 0$$

in normal coordinates. Returning to our Taylor expansion and writing $e_i := \partial/\partial y^i|_p$ we can compute the following, also in normal coordinates,

$$\partial_k g_{ij} = \partial_k g(e_i, e_j) = \nabla_k g(e_i, e_j) = g(\nabla_k e_i, e_k) + g(e_i, \nabla_k e_j) = g(0, e_k) + g(e_i, 0) = 0$$

and so our Taylor expansion becomes

$$g_{ij} = \delta_{ij} + (\partial_\ell \partial_k g_{ij})(p) y^\ell y^k + \ker(\mathrm{ev}_p)^3.$$

Can we push this further? Can we get the second-order terms to vanish at p as well?

We have now shown that one is always able to find coordinates centered at p so that the zero'th and first order terms of the Taylor expansion of g at p agree with the pullback of the dot product on $\mathbb{R}^n$. We are still nowhere near to having g_{ij} agree with δ_{ij} on a neighbourhood of p since, even if we can get the entire Taylor expansion of g_{ij} at p to be δ_{ij} in some coordinate system, there's no guarantee that this will happen on any open neighbourhood of p since we're working with C^∞, not analytic, objects. Furthermore, we have a non-trivial obstruction in the second-order terms of our Taylor expansion: the Riemann curvature tensor.

Just as we were able to express our ability to find coordinates making the first order terms of the Taylor expansion vanish at any given point in terms of the Levi-Civita connection associated to the metric (namely that it is torsion-free and metric compatible), one can also describe the second-order terms using the Levi-Civita connection. Indeed, suppose we were given any affine connection ∇ on M. In other words, ∇ is a $\mathbb{R}$-linear morphism of sheaves

$$\nabla : T_M \to T_M^* \otimes_{C_M^\infty} T_M$$

satisfying the following version of the Liebniz rule

$$\nabla(fX) = df \otimes X + f\nabla X.$$

We can then extend ∇ to a $\mathbb{R}$-linear morphism of sheaves

$$d^\nabla : \Lambda^p T_M^* \otimes_{C_M^\infty} T_M \to \Lambda^{p+1} T_M^* \otimes_{C_M^\infty} T_M$$

for each $p \geq 0$ via the graded Liebniz rule

$$d^\nabla(\alpha \otimes X) = d\alpha \otimes X + (-1)^{\deg(\alpha)} \alpha \wedge \nabla X.$$

One might then hope (for reasons explained in section 1.3) that the sequence of maps

$$T_M \xrightarrow{d^\nabla} T_M^* \otimes T_M \xrightarrow{d^\nabla} \Lambda^2 T_M^* \otimes T_M \xrightarrow{d^\nabla} \Lambda^3 T_M^* \otimes T_M \to \cdots$$

forms a complex (i.e. $d^\nabla \circ d^\nabla = 0$) resolving the sheaf of solutions to the homogeneous PDE associated to ∇, i.e. $\ker(\nabla) \subseteq T_M$. The point is that such a resolution would allow us to compute the sheaf cohomology of $\ker(\nabla)$ and therefore express the global existence/uniqueness problems for the differential operator in terms of the local ones. Anyways, even if we forget about the fact that such a sequence of maps need not be exact a priori, we still run into the problem that this need not even be a complex! In fact, one can show that this is a complex if and only if

$$d^\nabla \circ \nabla = 0.$$

This is because d^∇ is a linear differential operator and so we can work locally where forms decompose as wedge products and then use the Liebniz rule. Now, notice that for any vector field X and smooth function f, if we write $\nabla X = \alpha_i \otimes Y^i$ then

$$\begin{aligned}
d^\nabla(\nabla(fX)) &= d^\nabla(df \otimes X + f\nabla X) \\
&= -(df \wedge \alpha_i) \otimes Y^i + d^\nabla(f\alpha_i \otimes Y^i) \\
&= -(df \wedge \alpha_i) \otimes Y^i + (df \wedge \alpha_i) \otimes Y^i + fd\alpha_i \otimes Y^i - f\alpha_i \otimes \nabla Y^i \\
&= fd^\nabla(\nabla X)
\end{aligned}$$

and so $d^\nabla \circ \nabla$ is in fact tensorial. That is, as a map $T_M \to \Lambda^2 T_M^* \otimes T_M$ it is in fact a morphism of C_M^∞-modules and therefore is given by a global section

$$F^\nabla := d^\nabla \circ \nabla \in \Gamma(M, \Lambda^2 T_M^* \otimes_{C_M^\infty} \mathcal{E}\mathrm{nd}_{C_M^\infty}(T_M)).$$

This tensor F^∇ is called the **curvature** of the connection.

Proposition 1.1.5. *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection of g and write $R := F^\nabla$ for its curvature tensor. Then in normal coordinates centered at $p \in M$ the Taylor expansion of g about p has the form:*

$$g_{ij} = \delta_{ij} - \frac{1}{3}R^\ell_{ikj}g_{\ell m}(p)x^k x^\ell + \ker(\mathrm{ev}_p)^3.$$

Furthermore, if R vanishes identically on a neighbourhood of p, then there exists a chart centered at p in which g takes the form

$$g_{ij} = \delta_{ij}$$

*on the entire coordinate chart. A metric g with $R = 0$ on all of M is called **flat**.*

Proof. Choose an arbitrary point $p \in M$ and local coordinates centered at p. For simplicity, let's denote $R_{ijk\ell} := R^\ell_{ijk}g_{\ell m}$, our local coordinates as x^i and the metric duals of dx^i as e_i. We'll also write g^{ij} for the inverse matrix to g_{ij}. Notice that if we had locally defined vector fields X, Y, Z with $[X, Y] = 0$ and $\nabla Z = \alpha \otimes W$ then:

$$\begin{aligned}
R(X,Y)Z &= (d^\nabla \circ \nabla)(X,Y)Z = (d^\nabla(\alpha \otimes W))(X,Y) \\
&= (d\alpha)(X,Y)W - (\alpha(X)\nabla_Y W - \alpha(Y)\nabla_X W) \\
&= (Y\alpha(X) - \alpha(X)\nabla_Y W) - (X\alpha(Y) - \alpha(Y)\nabla_X W)
\end{aligned}$$

and so we have

$$R^\ell_{kji} = dx^\ell(\nabla_k \nabla_j e_i - \nabla_j \nabla_k e_i).$$

Now, the first thing we must do from here is compute the following in these local coordinates:

$$\begin{aligned}
R^\ell_{kji} &= dx^\ell(\nabla_k \nabla_j e_i - \nabla_j \nabla_k e_i) \\
&= dx^\ell(\nabla_k(\Gamma^m_{ji}e_m) - \nabla_j(\Gamma^m_{ki}e_m)) \\
&= dx^\ell((\partial_k \Gamma^m_{ji} - (\partial_j \Gamma^m_{ki}))e_m + \Gamma^m_{ji}\Gamma^n_{km}e_n - \Gamma^m_{ki}\Gamma^n_{jm}e_n) \\
&= \partial_k \Gamma^\ell_{ji} - \partial_j \Gamma^\ell_{ki} + \Gamma^m_{ji}\Gamma^\ell_{km} - \Gamma^m_{ki}\Gamma^\ell_{jm}.
\end{aligned}$$

Seeing as the last two terms in the above will vanish in normal coordinates at p, our problem now becomes to relate the derivatives of the Christoffel symbols to the second derivatives of the metric. To do this, we use metric compatibility with the Levi-Civita connection:

$$0 = \nabla_i g_{jk} = \partial_i g_{jk} - g_{\ell k}\Gamma^\ell_{ji} - g_{j\ell}\Gamma^\ell_{ki}$$

to get

$$\Gamma^\ell_{ji} = \frac{1}{2}g^{\ell m}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ji}) \tag{1.2}$$

which we can then differentiate to obtain

$$\begin{aligned}
\partial_k \Gamma^\ell_{ji} &= \frac{1}{2}(\partial_k g^{\ell m})(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ji}) + \frac{1}{2}g^{\ell m}(\partial_k \partial_i g_{mj} + \partial_k \partial_j g_{mi} - \partial_k \partial_m g_{ji}) \\
&= -\frac{1}{2}g^{\ell a}(\partial_k g_{ab})g^{bm}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ji}) + \frac{1}{2}(\partial_k \partial_i g_{mj} + \partial_k \partial_j g_{mi} - \partial_k \partial_m g_{ji}) \\
&= -g^{\ell a}(\partial_k g_{ab})\Gamma^b_{ji} + \frac{1}{2}g^{\ell m}(\partial_k \partial_i g_{mj} + \partial_k \partial_j g_{mi} - \partial_k \partial_m g_{ji})
\end{aligned}$$

by (1.2). We now see that the first term in our above equation for $\partial_k \Gamma^\ell_{ji}$ vanishes at p when written using normal coordinates centered at p. So, plugging this back into our original equation for $R_{kji\ell}$

and working exclusively in normal coordinates with everything **implicitly evaluated** at p we get:

$$\begin{aligned} R^{\ell}_{kji} &= \frac{1}{2}g^{\ell m}(\partial_k\partial_i g_{mj} + \partial_k\partial_j g_{mi} - \partial_k\partial_m g_{ji}) - \frac{1}{2}g^{\ell m}(\partial_j\partial_i g_{mk} + \partial_j\partial_k g_{mi} - \partial_j\partial_m g_{ki}) \\ &= \frac{1}{2}g^{\ell m}(\partial_k\partial_i g_{mj} - \partial_j\partial_i g_{mk} - \partial_k\partial_m g_{ji} + \partial_j\partial_m g_{ki}) \end{aligned}$$

and so

$$R_{kji\ell} = \frac{1}{2}(\partial_k\partial_i g_{\ell j} - \partial_j\partial_i g_{\ell k} - \partial_k\partial_\ell g_{ji} + \partial_j\partial_\ell g_{ki}) \tag{1.3}$$

To obtain our desired result from here we notice that in normal coordinates the components of a geodesic are given by $(x^i \circ \gamma)(t) = v^i t$ for some fixed vector v and so, using (1.1), (1.2) and setting $y^k := g_{k\ell}x^\ell$ we get:

$$\begin{aligned} 0 &= \frac{1}{2}(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij})x^i x^j \\ &= (\partial_j g_{mi} - \frac{1}{2}\partial_m g_{ij})x^i x^j \\ &= (\partial_j y^m - g_{jm})x^j - \frac{1}{2}\sum_i(\partial_m y^i - g_{mi})x^i \\ &= (\partial_j y^m)x^j - \frac{1}{2}y^m - \frac{1}{2}\sum_i(\partial_m y^i)x^i \\ &= (\partial_j y^m)x^j - \frac{1}{2}\partial_m\left(\sum_i y^i x^i\right) \\ &= (\partial_j y^m)x^j - \frac{1}{2}\partial_m(g_{ij}x^i x^j) \\ &= (\partial_j y^m)x^j - \frac{1}{2}\partial_m\left(\sum_i (x^i)^2\right) \\ &= (\partial_j y^m)x^j - x^m. \end{aligned}$$

Therefore, along any geodesic:

$$\frac{d(y^m - x^m)}{dt} = (\partial_j y^m)\frac{dx^j}{dt} - \frac{dx^m}{dt} = 0.$$

Since these two maps y^m, x^m agree at p it then follows from the uniqueness theorem for ODEs that:

$$x^k = g_{k\ell}x^\ell \text{ (Gauss' lemma)}$$

Differentiating this twice, evaluating at p and comparing with (1.3), will eventually yield the identity

$$-2R_{ikj\ell}(p) = 3(\text{the coefficient of } x^k x^\ell \text{ in } R_{ikj\ell}x^k x^\ell)$$

The fact that one obtains full integrability if $R = 0$ comes from the fact that all of the remaining terms in our Taylor expansion can be shown to be expressable in terms of the covariant derivatives of R and, furthermore, by a result from [14] it follows that g is real analytic if the Ricci tensor vanishes. □

Let's now re-intepret our above discussion in terms of the $\mathrm{O}(n)$-structure corresponding to g. Suppose M is a smooth manifold with $\mathrm{O}(n)$-structure $P \subseteq F$ (here F is the frame bundle). A local trivialization of P then takes the form:

$$\begin{array}{ccc} P|_U & \longrightarrow & \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbb{R}^n \end{array}$$

Notice that our equivariant map upstairs takes values in $\mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$, not $\mathbb{R}^n \times \mathrm{O}(n)$! The point is: while the image of the map upstairs looks like $\mathbb{R}^n \times \mathrm{O}(n)$, we have fixed a representation $\mathrm{O}(n) \subseteq \mathrm{GL}(\mathbb{R}^n)$ in our definition of a G-structure and the copy of $\mathrm{O}(n)$ sitting inside of $\mathrm{GL}(\mathbb{R}^n)$ via the image of $P|_U$ need not be the same copy as our fixed one! Another way of saying this is that a trivialization $P|_U$ is precisely a global section of $P|_U$ since this is a principal bundle. Such a global section determines a choice of unit in the fibres of $P|_U$ making them into Lie groups. However, the map $P|_U \to \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ need only be equivariant and therefore need not take our newly defined identiy to the identity in the fibres on the right hand side.

This phenomenon in fact occurs for general G-structures. The image of $P|_U$ in $\mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ for P a general G-structure on M need not lie in $\mathbb{R}^n \times G$. For example, in the case of $G = \mathrm{O}(n)$, the existence of an atlas of trivializations whose upstairs maps actually have image in $\mathbb{R}^n \times G$, not just merely some isomorphic copy of it in $\mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$, corresponds to the metric being flat. But not all metrics are flat since, for example, it is possible to draw triangles on the surface of the earth whose interior angles sum to $3\pi/2 > \pi$. This leads to the following definition which is from [21].

Definition 1.1.6. A G-structure P on M is called **integrable** if and only if there is an atlas of local trivializations for P such that for each trivialization $\psi : U \to \mathbb{R}^n$ in this atlas the image $\psi_*(P|_U) \subseteq \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ actually lies in $\mathbb{R}^n \times G$, where $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ is our fixed representation from the data of a G-structure.

We've seen that a $\mathrm{O}(n)$-structure is integrable if and only if the underlying Riemannian metric is flat. Notice that for any Lie subgroup $G \subseteq \mathrm{O}(n)$ and any G-structure on a manifold, if that G-structure is integrable then the G-invariant metric coming from $G \subseteq \mathrm{O}(n)$ is necessarily flat.

Let's do another example: that of a $\mathrm{GL}(\mathbb{C}^n) \subseteq \mathrm{GL}(\mathbb{R}^{2n})$ structure. Here we are viewing $\mathrm{GL}(\mathbb{C}^n)$ as the collection of all matrices in $\mathrm{GL}(\mathbb{R}^{2n})$ which commute with the matrix J given by

$$J = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

The point being that under the identification of real vector spaces $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via $z^j = x^j + ix^{j+n}$, the matrix J corresponds to the complex matrix iI_n (so, in particular, $J^2 = -I$). Now, suppose $P \subseteq F$ was a $\mathrm{GL}(\mathbb{C}^n)$-structure on M. By writing J in the above form at the center (not on the entire open set!) of each local $\mathrm{GL}(\mathbb{C}^n)$-coordinate chart on M we obtain a tensor

$$J \in \Gamma(M, \mathcal{E}\mathrm{nd}(T_M))$$

which satisfies $J^2 = -\,\mathrm{id}$ and is typically refered to as an *almost complex structure* (conversely, such an endomorphism determines a $\mathrm{GL}(\mathbb{C}^n)$-structure by looking at all frames in which it takes the above form at the center of the frame). Now, given a local trivialization coming from a chart $\psi : U \to \mathbb{R}^{2n}$ we can see that the image $\psi_*(P|_U)$ lies in $\mathbb{R}^{2n} \times \mathrm{GL}(\mathbb{C}^n)$ if and only if ψ_* satisfies

$$\psi_* \circ J = J \circ \psi_*$$

where we are using J to denote both the almost complex structure on M and the matrix above describing our representation $\mathrm{GL}(\mathbb{C}^n) \subseteq \mathrm{GL}(\mathbb{R}^{2n})$. Now if we had another chart φ which also had the above property then we would necessarily have

$$\varphi_*^{-1} \circ \psi_* \circ J = \varphi_*^{-1} \circ J \circ \psi_* = J \circ \varphi_*^{-1} \circ \psi_*$$

since $J^{-1} = -J$. But $\varphi_*^{-1} \circ \psi_* = (\varphi^{-1} \circ \psi)_*$ is the Jacobian matrix for the transition functions between these coordinate charts. Thus, a $\mathrm{GL}(\mathbb{C}^n)$-structure being integrable actually implies that the manifold admits an atlas of charts into $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with transition functions whose components all satisfy the Cauchy-Riemann equations. i.e. it admits an atlas of charts with holomorphic transition functions and is therefore a complex manifold!

So we've now seen that full integrability for G-structures is a fairly strong assumption. In the case of a $\mathrm{O}(n)$-structure it is equivalent to the metric being flat and in the case of a $\mathrm{GL}(\mathbb{C}^n)$-structure it is equivalent to the manifold having an atlas of charts with holomorphic transition functions. As such, we will now try to find some sort of weaker "integrability" criterion, using our Taylor expansion of a Riemannian metric as motivation.

Our full integrability condition was that for each $p \in M$ there was a chart ψ, U centered at p such that $\psi_*(P|_U) \subseteq \mathbb{R}^n \times G$. In general, all we can guarantee is that the two submanifolds

$$\psi_*(P|_U),\ \mathbb{R}^n \times G \subseteq \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$$

intersect at some point of the form (p, g) where $g \in G$. In view of our Taylor expansion of a Riemannian metric, we then make the following definition also due to [21].

Definition 1.1.7. A G-structure P on M is said to be m**'th order formally integrable** if and only if for each $p \in M$ there is a chart ψ, U centered at p such that the submanifolds

$$\psi_*(P|_U), \mathbb{R}^n \times G \subseteq \mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$$

intersect at some point of the form $(p, g) \in \mathbb{R}^n \times G$ and have m'th order **contact** at this point.

What do we mean by having m'th order *contact* at the point (p, g)? We mean that there exists some coordinate chart $V, x^1, \cdots, x^{n+n^2}$ for $\mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ centered at (p, g) such that if

$$k := \dim(\psi_*(P|_U)) = \dim(\mathbb{R}^n \times G)$$

then we have

$$\psi_*(P|_U) \cap V = Z(x^{k+1}, \cdots, x^{n+n^2})$$

where $Z(x^{k+1}, \cdots, x^{n+n^2}) \subseteq V$ refers to the common zero set, and also

$$(\mathbb{R}^n \times G) \cap V = Z(x^{k+1} - f_{k+1}(x^1, \cdots, x^{n+n^2}), \cdots, x^{n+n^2} - f_{n+n^2}(x^1, \cdots, x^{n+n^2}))$$

for some smooth functions $f_{k+1}, \cdots, f_{n+n^2} : V \to \mathbb{R}$ whose derivatives satisfy

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}\Big|_{(p,g)} = 0 \text{ for all multi-indices } 0 \leq |\alpha| \leq m.$$

This definition immediately raises some questions. For example: is the notion of m'th order contact at (p, g) symmetric in the submanifolds $\psi_*(P|_U)$ and $\mathbb{R}^n \times G$? To what extent does the vanishing of all the derivatives up to m'th order of the f_i's depend on the coordinate system in which we're taking the derivatives? The answer to the second question is that it is in fact independent of the choice of coordinate system but, since this fact is not really important now, we will postpone the proof of this until we introduce the notion of a *jet* since this coordinate-independence is really part of the more general statement that the notion of a jet is well-defined. The first question of whether $\psi_*(P|_U)$ has m'th order contact at (p, g) with $\mathbb{R}^n \times G$ if and only if $\mathbb{R}^n \times G$ has m'th order contact at (p, g) with $\psi_*(P|_U)$ we will however answer now.

Proposition 1.1.8. *Let P be a smooth manifold, $p \in P$, and M, N two submanifolds of P with the same dimension which intersect at p. Then M has m'th order contact at p with N if and only if N has m'th order contact at p with M.*

Proof. First, notice that having 0'th order contact at p simply means that they intersect at p and so we will assume $m \geq 1$ for the rest of this proof.

Now, the statement is symmetric in M and N so it suffices to prove that if M has m'th order contact with N at p then N has m'th order contact with M at p. Indeed, if M had m'th order contact with N at p then there would exist local coordinates $x^1, \cdots, x^N$ defined on a neighbourhood U of p so that

$$U \cap M = U \cap Z(x^1, \cdots, x^k)$$

meanwhile there would exist smooth functions $f_1, \cdots, f_k$ whose m-jet (the coordinate-free version of m'th order Taylor expansion) at p vanishes and furthermore

$$U \cap N = U \cap Z(x^1 - f_1(x_1, \cdots, x_N), \cdots, x^k - f_k(x^1, \cdots, x^N)).$$

Now, on the same neighbourhood we define

$$y^i := x^i - f_i(x^1, \cdots, x_N)$$

for $1 \leq i \leq k$ meanwhile we set $y^j := x^j$ for all other j. By the inverse function theorem, this is a new coordinate system centered at p on a perhaps smaller neighbourhood V since the m'th jet of the f_i's all vanish at p. Rewriting the x^i's in terms of the y^i's yields new functions

$$-g_i(y^1, \cdots, y^N) = f_i(x^1, \cdots, x^N)$$

which satisfy

$$V \cap M = V \cap Z(y^1 - g_1(y^1, \cdots, y^N), \cdots, y^k - g_k(y^1, \cdots, y^N))$$

as well as

$$V \cap N = V \cap Z(y^1, \cdots, y^k).$$

All that remains is then to check that the m-jet of the g_i's at p vanishes. But this follows from the coordinate-independence of jets, as required. □

Let's now give some examples illustrating the usefulness of the notion of m'th order formal integrability for G-structures.

Example 1.1.9. We've seen that $\mathrm{O}(n)$-structures are always first-order formally integrable due to the existence of an atlas of normal coordinates. Second order formal integrability for $\mathrm{O}(n)$-structures corresponds to the vanishing of the Riemann curvature tensor, i.e. flatness. So, as we've seen, second order formal integrability of a $\mathrm{O}(n)$-structure actually implies full integrability.

Example 1.1.10. General $\mathrm{GL}(\mathbb{C}^n)$-structures need not even be first-order formally integrable. First order formal integrability here is equivalent to the vanishing of the *Nijenhuis tensor*

$$N_J \in \Gamma(M, \Lambda^2 T_M^* \otimes T_M)$$

associated to the corresponding almost complex structure J. The famous Newlander-Nirenberg theorem [58], [24] states that for $\mathrm{GL}(\mathbb{C}^n)$-structures, first order formal integrability is actually equivalent to full integrability. This may seem analogous to the $\mathrm{O}(n)$-case but, as we'll see later, the proof of the Newlander-Nirenberg is significantly more difficult than the corresponding result for $\mathrm{O}(n)$-structures.

Example 1.1.11. A $\mathrm{Sp}(2n, \mathbb{R})$-structure, determined by a non-degenerate 2-form $\omega \in \Gamma(M, \Lambda^2 T_M^*)$, is first order formally integrable if and only if

$$d\omega = 0.$$

As with the case of $\mathrm{GL}(\mathbb{C}^n)$-structures, the famous *Darboux theorem* [11] says that first order formal integrability of a $\mathrm{Sp}(2n, \mathbb{R})$-structure actually implies full integrability. This is why complex and symplectic geometry tends to have a very algebraic flavour, whereas Riemannian geometry (for which one typically does not assume that one's metrics are flat) involves more analysis.

Example 1.1.12. A $\mathrm{U}(m)$-structure, determined by a 2-form ω and an almost complex structure J such that $\omega(J-,-)$ defines a Riemannian metric on M, is first-order formally integrable if and only if both

$$N_J = 0 \text{ and } d\omega = 0.$$

This statement is often expressed by saying that the Kähler form ω "osculates to the standard form to order 2" [26]. Again, the obstruction to second order formal integrability here is the Riemann curvature tensor of $\omega(J-,-)$ and second order formal integrability for $\mathrm{U}(m)$-structures implies full integrability. All of this is essentially due to the identity

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{C})$$

together with the fact that $N_J = 0$ implies full integrability of the complex structure, $d\omega = 0$ implies full integrability of the symplectic structure and then all that remains is the Riemann curvature tensor of $\omega(J-,-)$.

Example 1.1.13. $\mathrm{GL}^+(\mathbb{R}^n)$-structures have no obstructions to full integrability and are always fully integrable when they exist. The reason for this is essentially that the orientation line bundle for a manifold is a flat line bundle (it has locally constant transition functions given by the sign of the determinant of the Jacobian matrix for the transition functions between charts). A $\mathrm{GL}^+(\mathbb{R}^n)$-structure corresponds to a choice of global trivialization of the orientation line bundle and is therefore fully integrable whenever it exists.

There are general criteria guaranteeing the full integrability of a G-structure once all of its obstructions to formal integrability vanish. Now, every $\mathrm{GL}^+(\mathbb{R}^n)$-structure gives rise to a $\mathrm{SL}(\mathbb{R}^n)$-structure via a choice of Riemannian metric which then yields a volume form. These general criteria are not satisfied by $\mathrm{SL}(\mathbb{R}^n)$ (we say it is of *infinite type*) as was shown in [61], however it is still the case that every $\mathrm{SL}(\mathbb{R}^n)$-structure is fully integrable. Indeed, given any point $p \in M$ we choose an oriented coordinate chart $x^1, \cdots, x^n$ centered at p and then $dx^1 \wedge \cdots \wedge dx^n$ yields the standard $\mathrm{SL}(\mathbb{R}^n)$-structure on $\mathbb{R}^n$ in a potentially smaller neighbourhood of p since invertibility is an open condition.

Example 1.1.14. First order formal integrability of a G_2 or $\mathrm{Spin}(7)$-structure on a 7 or 8-manifold respectively corresponds to the existence of torsion-free affine G_2 (respectively $\mathrm{Spin}(7)$) connections. This is why first order formally integrable G_2 and $\mathrm{Spin}(7)$-structures are often called *torsion-free* and this is actually part of a more general phenomenon, as we'll see. G_2-structures are determined by a certain type of 3-form $\varphi \in \Gamma(M, \Lambda^3 T^*_M)$ called a *positive* 3-form. Since $G_2 \subseteq \mathrm{SO}(7)$ this form determines a metric g_φ and orientation or, equivalently given the metric, a Hodge star operator $*_\varphi$. All of these are determined in a highly non-linear way [32]. First order formal integrability of the G_2-structure can then be proven to be equivalent to

$$d\varphi = 0 \text{ and } d *_\varphi \varphi = 0$$

as was done in [9].

Example 1.1.15. A *Calabi-Yau manifold* is a manifold together with a first-order formally integrable $\mathrm{SU}(m)$-structure. A $\mathrm{SU}(m)$-structure is determined by ω, J such that $\omega(J-,-)$ is a Riemannian metric (as in the $\mathrm{U}(m)$ case) together with the additional data of a nowhere vanishing global section

$$\Omega \in \Gamma(M, \Lambda^{m,0} T^*_M) \text{ satisfying } \frac{1}{m!}\omega^m = (-1)^{m(m-1)/2} \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}.$$

First order formal integrability for $\mathrm{SU}(m)$-structures is equivalent to

$$d\omega = 0, \ N_J = 0, \text{ and } \overline{\partial}\Omega = 0.$$

i.e. Calabi-Yau manifolds are Kähler manifolds together with a choice of a compatible holomorphic volume form [28]. In particular, the canonical bundle of a Calabi-Yau manifold, the bundle of holomorphic sections of $\Lambda^{m,0}T^*_M$, is topologically trivial and so $c_1(M) = 0$. The famous Calabi-Yau theorem, which we state below, provides a converse to this in the case one begins with a compact Kähler manifold (although one might need to modify the Kähler structure along the way).

There is something worth noticing about the above examples. A first-order formally integrable $\mathrm{U}(m)$-structure on M defines a cohomology class

$$[\omega] \in H^2(M, \mathbb{R}).$$

The same holds for general symplectic structures and, furthermore, a first-order formally integrable G_2-structure defines two cohomology classes

$$[\varphi] \in H^3(M, \mathbb{R}) \text{ and } [*_\varphi \varphi] \in H^4(M, \mathbb{R}).$$

Meanwhile, first order formal integrability of a $\mathrm{SU}(m)$-structure implied the vanishing of a cohomology class

$$0 = c_1(M) \in H^2(M, \mathbb{C}).$$

The point is that there is a relationship between the existence of G-structures on a manifold with certain amounts of formal integrability and the topology of the underlying manifold. A (highly non-trivial) example of this relationship is the following consequence of the Calabi-Yau theorem (we will simply refer to this corollary of the more general theorem as the Calabi-Yau theorem).

Theorem 1.1.16. The Calabi-Yau Theorem [28]
Let M be a compact Kähler manifold with Kähler form ω. If $c_1(M) = 0$ then there exists a possibly different Kähler form $\widetilde{\omega}$ with

$$[\omega] = [\widetilde{\omega}] \in H^2(M, \mathbb{R})$$

such that the newly defined $\mathrm{U}(m)$-structure reduces further to a first order formally integrable $\mathrm{SU}(m)$-structure.

The general "integrability problem" for G-structures can be described as follows:

given G a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$, find topological obstructions to the existence of m'th order formally integrable G-structures on n-manifolds.

While a large portion of this thesis will consist of a summary and re-interpretation of some of the progress made so far in this problem, we actually have a more concrete goal in mind to which this theory will be applied.

As mentioned earlier, a $\mathrm{GL}(\mathbb{C}^n)$-structure on M is determined by a tensor

$$J \in \Gamma(M, \mathcal{E}\mathrm{nd}(T_M)), \quad J^2 = -\,\mathrm{id}\,.$$

For notational convenience, whenever E is a vector bundle on M we will write

$$\Omega^p(M, E) := \Gamma(M, \Lambda^p T^*_M \otimes_{C^\infty_M} E_M).$$

In the case where E is the trivial bundle we will simply write $\Omega^p(M)$. For example, we now write $J \in \Omega^1(M, T_M)$. As we will see later, the sheaf

$$\Lambda^* T^*_M \otimes T_M := \bigoplus_{k \geq 0} \Lambda^k T^*_M \otimes T_M$$

whose global sections we write as $\Omega^*(M, T_M)$, is naturally a sheaf of graded Lie algebras when equipped with the so-called *Frölicher-Nijenhuis bracket* $[-,-]_{FN}$ [35]. We will discuss this in more detail later, but the point is that one can compute:

$$N_J = \frac{1}{2}[J, J]_{FN}.$$

The other classical application of the Frölicher-Nijenhuis bracket is to connections. A connection on a manifold M can be interpreted as a projection

$$P \in \Omega^1(M, T_M), \;\; P^2 = \mathrm{id}$$

whose kernel is thought of as the vertical bundle $V_M \subseteq T_M$. The way this corresponds to a connection is by declaring the horizontal bundle to be the image of P in T_M. Again, one can compute that

$$F^P = \frac{1}{2}[P, P]_{FN}$$

where F^P is the curvature tensor of the connection. More generally, if P is just an arbitrary projection then we also have the appearance of a *cocurvature* term characterizing the integrability of the subbundle $\ker(P)$ (integrability here is in the sense of the Fröbenius theorem, i.e. being closed under the Lie bracket of vector fields) [35]. We now have two examples where plugging-in tensors into the Frölicher-Nijenhuis bracket yields obstructions to formal integrability. Two examples may indeed be a coincidence, but as we will see there are more examples.

Let M be a 7-manifold with G_2-structure $\varphi \in \Omega^3(M)$. Since a G_2-structure determines a metric g_φ and orientation $*_\varphi$ we can apply the Hodge star to obtain $*_\varphi\varphi \in \Omega^4(M)$ and then raise the last index to get a vector-valued 3-form

$$\chi \in \Omega^3(M, T_M).$$

It was shown by Kawai, Lê and Schwachhöfer [33] that a G_2-structure is first-order formally integrable if and only if

$$[\chi, \chi]_{FN} = 0.$$

Similarly, for a $\mathrm{Spin}(7)$-structure $\Phi \in \Omega^4(M)$ on a 8-manifold one can raise an index since $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$ (via the imaginary octonions sitting inside the rest of the octonions) to get

$$Q \in \Omega^3(M, T_M)$$

and again first order formal integrability corresponds to $[Q, Q]_{FN} = 0$. The goal of this thesis is to investigate whether this is part of a more general phenomenon, and to find coordinate-free proofs of the above results. In order to do this, we begin with an in-depth analysis of the PDEs characterizing formal integrability via the *Spencer complex* [62]. In fact, there are several different Spencer complexes. As we'll see, they all arise naturally from the homological algebra governing the deformation/obstruction-theoretic properties of partial differential equations. One can then apply this general machinery to the specific case of the PDEs arising from the integrability problem for G-structures.

1.2 Topology and PDEs

Here we will explore the general connections between existence/uniqueness properties for linear PDEs and the topology of the spaces they're defined on. In order to discuss this, we should begin by defining what we mean by a linear partial differential operator between sections of vector bundles. Initially, our approach to this material will follow [40].

Definition 1.2.1. A **linear partial differential operator of order** r between vector bundles E_M, F_M of ranks k and ℓ respectively on a manifold M is a $\mathbb{R}$-linear morphism of sheaves

$$P : E_M \to F_M$$

such that for each $p \in M$ there exists a chart $U, x^1, \cdots, x^n$ centered at p trivializing both E_M and F_M, and functions $a_\alpha \in C^\infty(U, \mathbb{R}^{\ell \times k})$ for $0 \leq |\alpha| \leq r$ such that

$$P|_U = \sum_{0 \leq |\alpha| \leq r} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

and furthermore we require one of the a_α's with $|\alpha| = r$ in one of the charts to be non-zero.

Proposition 1.2.2. *The order of a differential operator is independent of the choice of charts and local trivializations in which it is given by a differential operator on* $\mathbb{R}^n$.

Proof. Arbitrarily select a linear partial differential operator $P : E_M \to F_M$ and a point $p \in M$. Choose $x^1, \cdots, x^n, a_\alpha$ as in Definition 1.2.1 and first notice that if we only change the coordinates $x^1, \cdots, x^n$ to $y^1, \cdots, y^n$ then

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial y^j}.$$

The Jacobian matrix $\partial x^i / \partial y^j$ is invertible on a neighbourhood of p and so whether the order one terms are zero or not remains unaffected. For the higher order terms $\partial^{|\alpha|} / \partial x^\alpha$ one can use induction together with the Liebniz rule to see that the terms with the highest order derivatives appearing are those for which the matrix $\partial x^i / \partial y^j$ is not further differentiated.

Next we notice that changing local trivializations on F_M only multiplies by invertible matrices on the left and therefore has no effect on the order at p.

Finally we consider the effect of changing the local trivialization of E_M. This involves multiplying by an invertible matrix on the right and again we notice that, after repeated applications of the Liebniz rule, the highest order terms are those for which this matrix is not differentiated.

It's worth noting that this argument also demonstrates that the *principal symbol*

$$\sigma(P) := \sum_{|\alpha| = r} a_\alpha \frac{\partial^r}{\partial x^\alpha}$$

is in fact a tensor

$$\sigma(P) \in \Gamma(M, \mathrm{Sym}^r\, T_M^* \otimes \mathcal{H}\mathrm{om}(E_M, F_M)).$$

□

An important subtlety in the above definition is that the notion of being a differential operator of order r is a *global* one and not a local one. This is because we require the order of the operator to be bounded above by r on the entire manifold M. We now give an example of a $\mathbb{R}$-linear morphism of sheaves which, despite admitting an atlas of charts in which it looks like a differential operator of finite order, the order of the operator tends to infinity globally.

Example 1.2.3. On $M = \mathbb{R}$ we choose a smooth bump function ρ_n for each $n \in \mathbb{N}$ with $\mathrm{supp}(\rho_n) \subseteq [n - 1/4, n + 1/4]$ and $\rho_n(n) = 1$. Consider the morphism of sheaves

$$\begin{aligned} P : C^\infty_{\mathbb{R}} &\to C^\infty_{\mathbb{R}} \\ f &\mapsto \sum_{n \in \mathbb{N}} \rho_n \frac{d^n f}{dx^n}. \end{aligned}$$

The open intervals $(m-1/3, m+1/3)$ for $m \in \frac{1}{2}\mathbb{Z}$ define an atlas of charts on $\mathbb{R}$ such that in each chart P appears to be a differential operator of finite order. However, as we move further along the positive real axis, the order of P tends to infinity and therefore P is not a differential operator (of finite order).

We'll now see, as a corollary of a famous theorem of Jaak Peetre, that for $\mathbb{R}$-linear morphisms of sheaves the above problem is all that can go wrong. The proof we place here is from the notes "Characterization of Differential Operators" by Paul Garrett on his website [16].

Theorem 1.2.4. The Peetre Theorem
Let E_M, F_M be vector bundles on a smooth manifold M and

$$P : \Gamma(M, E_M) \to \Gamma(M, F_M)$$

be a $\mathbb{R}$-linear map satisfying

$$\operatorname{supp}(Ps) \subseteq \operatorname{supp}(s)$$

for any $s \in \Gamma(M, E_M)$. Then P is locally a differential operator of finite order, although the order may tend to infinity globally.

Proof. We'll begin by reducing the problem to the case where M is an open neighbourhood of the origin in $\mathbb{R}^n$ and F_M is the trivial line bundle. First, arbitrarily select an open subset $U \subseteq M$ and a section $s \in \Gamma(U, E_M)$. For $x \in U$ we define $(Ps)(x) \in F_x$ in the following way. Choose a smooth bump function $0 \leq \rho \leq 1$ with compact support in U which is equal to one identically on a neighbourhood of x in U. We then set

$$(Ps)(x) := (P(\rho s))(x).$$

Indeed ρs is now a global section of E_M over M and given any two such bump functions we have

$$P(\rho_1 s - \rho_2 s)(x) = 0$$

since P is support non-increasing and $(\rho_1 - \rho_2)s$ vanishes identically on a neighbourhood of x. One can show that the newly defined map $P|_U$ with domain $\Gamma(U, E_M)$ takes values in $\Gamma(U, F_M)$ and gives rise to a $\mathbb{R}$-linear morphism of sheaves

$$P : E_M \to F_M$$

whose action on global sections is given by our original P. Our goal now becomes to show that for each $p \in M$ there is an open neighbourhood U of p in M such that

$$P : \Gamma(U, E_M) \to \Gamma(U, F_M)$$

is a differential operator of finite order. Due to the existence of local coordinates on M diffeomorphic to the open ball in Euclidean space and the fact that vector bundles on a contractible space are trivial we may now assume that M is an open ball $B \subseteq \mathbb{R}^n$ centered at the origin and that E_M, F_M are trivial vector bundles. But now P is a differential operator of finite order if and only if each of its components in F_M are thus we may assume that F_M is the trivial line bundle.

So now we are in the case where $M = B \subseteq \mathbb{R}^n$ is an open ball centered at the origin and P is a $\mathbb{R}$-linear morphism of sheaves

$$P : (C_B^\infty)^{\oplus k} \to C_B^\infty.$$

But $\operatorname{Hom}_{\mathbb{R}_M\text{-Mod}}(-, C_B^\infty)$ takes colimits to limits thus

$$\operatorname{Hom}_{\mathbb{R}_M\text{-Mod}}((C_B^\infty)^{\oplus k}, C_B^\infty) \cong \operatorname{Hom}_{\mathbb{R}_M\text{-Mod}}(C_B^\infty, C_B^\infty)^{\times k}$$

and so we may also assume that E_M is the trivial line bundle. If one feels uneasy using such "abstract-nonsense" then instead one can simply interpret sections of $(C_B^\infty)^{\oplus k}$ as vector-valued smooth functions and notice that the rest of the proof still works if the domain of P consists of vector-valued functions instead of scalar-valued functions.

We've now reduced ourselves to the classical Peetre theorem which concerns

$$P : C_B^\infty \to C_B^\infty.$$

Consider then the seminorms defining the Fréchet topology on spaces of smooth functions. In other words, for $K \subseteq B$ compact, $r \geq 0$ and f a smooth function defined on an open neighbourhood of K we denote

$$|f|_{K,r} := \sup_{1 \leq |\alpha| \leq r} \sup_{x \in K} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|$$

where the x^i are the standard coordinate functions on $\mathbb{R}^n$. Notice that if we require f to have compact support then the above makes sense even without assuming K is compact. Let's now suppose that for each $x \in B$ there existed an open neighbourhood $x \in U \subseteq B$, an integer $r \geq 0$ and a constant $C > 0$ such that

$$|Pf|_{U,0} \leq C|f|_{U,r} \text{ for all } f \in C_c^\infty(U).$$

One can show that if the above holds then we'll obtain our desired result. Indeed, let's do this now.

Arbitrarily select $f \in C_c^\infty(B)$ and define a function

$$t_f(y) := f(y) - \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_x (x-y)^\alpha.$$

The order r Taylor expansion of $t_f(y)$ at x is zero. Now, let $\epsilon > 0$ and choose $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ in such a way that there exists open subsets

$$x \in V_\epsilon \subseteq W_\epsilon \subseteq B$$

with

$$\varphi_\epsilon|_{V_\epsilon} = 0 \text{ and } \varphi_\epsilon|_{B \setminus W_\epsilon} = t_f|_{B \setminus W_\epsilon}$$

and furthermore

$$|t_f - \varphi_\epsilon|_{B,r} < \epsilon.$$

Taking $\epsilon \to 0$ and using the assumption from our claim we get that there is some open neighbourhood U of x in B such that

$$\lim_{\epsilon \to 0} |P(t_f - \varphi_\epsilon)|_{U,0} = 0$$

and, since P is support non-increasing, $P\varphi_\epsilon$ vanishes on a neighbourhood of x for all $\epsilon > 0$. Hence $(Pt_f)(x) = 0$ since $|(Pt_f)(x)| < \epsilon$ for all $\epsilon > 0$. Therefore if we set $q_{x,\alpha}(y) := (x-y)^\alpha$ then:

$$(Pf)(x) = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_x (Pq_{x,\alpha})(x).$$

Since $q_{x,\alpha}$ is a polynomial in x and P doesn't differentiate the variable x here we have that the right hand side of the above equality does indeed define a differential operator on B. Since x was arbitrary, it follows that P is equal to this differential operator, as required.

Let's now see why our desired estimate actually holds. The idea is that we will first prove the weaker version where we only require it to hold for compactly supported smooth functions on a punctured neighbourhood of x in B. Indeed this is a weaker assumption since, as one may recall, compactly supported sections form a *cosheaf* not a sheaf and so we have extension maps, not restrictions.

Suppose for contradiction that our estimate does not hold for some $x_0 \in B$. Then given any open neighbourhood $x_0 \in U_0 \subseteq B$ there exists a function $f_1 \in C_c^\infty(U_0 \setminus \{x_0\})$ such that

$$|Pf_1|_{U_0,0} > 2^2 |f_1|_{U_0,1}.$$

We then set $Z_1 := \mathrm{supp}(f_1)$ and $U_1 := U_0 \setminus Z_1$. It follows then that there exists a $f_2 \in C_c^\infty(U_1 \setminus \{x_0\})$ such that

$$|Pf_2|_{U_1,0} > 2^4 |f_2|_{U_1,2}.$$

Again, we set $Z_2 := \mathrm{supp}(f_2)$ and $U_2 := U_1 \setminus Z_2$. By induction we obtain open neighbourhoods of x_0:

$$U_0 \supseteq U_1 \supseteq \cdots$$

and functions $f_{i+1} \in C_c^\infty(U_i \setminus \{x_0\})$ with

$$\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \varnothing \ \text{ for all } i,j \geq 1, \ \ i \neq j$$

and

$$|Pf_{i+1}|_{U_i,0} > 2^{2i} |f_{i+1}|_{U_i,1}.$$

Now, since the f_i have compact support they actually define functions on all of U_0 and, since their supports are disjoint, so does the expression:

$$f := \sum_{i=1}^{\infty} \frac{1}{2^i |f_i|_{U_0,i}} f_i.$$

In fact, each f_i is a compactly supported function on $\mathbb{R}^n$ and so f also defines a function on $\mathbb{R}^n$. Furthermore, since the supports of the f_i's are contained in B it follows that

$$\mathrm{supp}(f) \subseteq \overline{B}$$

and so f is a compactly supported function on $\mathbb{R}^n$. In particular, it is bounded since it is also continuous. Since P is support non-increasing, we then also have that Pf is bounded.

But now, since P is $\mathbb{R}$-linear and support non-increasing we have

$$(Pf)|_{U_i} = \frac{1}{2^i |f_i|_{U,i}} (Pf_i)|_{U_i}$$

for all i. So, there exists $x_i \in U_i$ for each i such that

$$|(Pf)(x_i)| > 2^i,$$

contradicting that Pf was bounded. So, we now have that for any $x \in B$ there exists an open neighbourhood $x \in U \subseteq B$, an integer $r \geq 0$ and a constant $C > 0$ such that

$$|Pf|_{U,0} \leq C|f|_{U,r} \ \text{ for all } f \in C_c^\infty(U \setminus \{x\}).$$

Finally we prove our estimate for compactly supported smooth functions on an actual open neighbourhood of $x \in B$, not a punctured neighbourhood. Once we have this we'll be done.

To do this we arbitiraly select $x \in B$ and shrink the neighbourhood U of x from above so that $x \in U \subseteq \overline{U} \subseteq B$. By our proof that demonstrating our estimate on a (non-punctured) neighbourhood implies that P is a differential operator on that neighbourhood, it follows from a partition of unity argument that P is a differential operator on $U \setminus \{x\}$. But then, for $f \in C_c^\infty(U)$ we can define Pf by first defining

$$(Pf)|_{U\setminus\{x\}} := P(f|_{U\setminus\{x\}}) \in C_c^\infty(U \setminus \{x\})$$

and then noticing, by the explicit form we had for P above, that Pf extends to a smooth function on all of U. Again, by our explicit expression for P it follows that this extended operator is a differential operator of finite order, as required. □

Corollary 1.2.5. *Let E_M, F_M be vector bundles on M and $P : E_M \to F_M$ a $\mathbb{R}$-linear morphism of sheaves. Then P is locally a differential operator of finite order on M.*

Proof. Morphisms of sheaves are support non-increasing since they are natural transformations and therefore compatible with restrictions. □

Now, let's suppose we were given a differential operator $P : E_M \to F_M$. For each local section f of F_M we obtain a linear partial differential equation

$$Pu = f$$

for local sections u of E_M. This is, in some sense, the most general form of a classical system of linear partial differential equations (with potentially non-constant coefficients). The usual goal in the theory of PDEs is the following:

1. find easy to check necessary and sufficient conditions on f for local solutions u to exist (the existence problem);
2. supposing solutions do exist locally, understand the space of solutions. For example: given f, what is the dimension of the space of solutions to $Pu = f$? (the uniqueness problem)

For ODEs we have general existence and uniqueness theorems but this does not typically happen for PDEs. The closest results I know of are the Cauchy-Kowalevskaya theorem and the Cartan-Kähler theorem (see [10]) in the analytic case. There are also general existence/uniqueness results regarding unbounded operators on Hilbert spaces used by Hormander in, for example, [24]. The techniques of Hormander essentially boil down to the Riesz representation theorem for Hilbert spaces. Let's now illustrate how existence can fail using a simple example. Consider the following system of PDEs on $\mathbb{R}^2$:

$$\frac{\partial u}{\partial x} = f(x,y) \text{ and } \frac{\partial u}{\partial y} = g(x,y).$$

This can be written more succintly as a single equation using differential forms:

$$du = f dx + g dy.$$

One can also interpret the level sets of a solution u to the above (via the implicit function theorem) as locally defined solutions y to the ODE

$$f(x,y)\frac{dy}{dx} + g(x,y) = 0$$

and such equations are often called *exact equations*. Now, assuming we're looking for C^2-solutions u we obtain the following *compatibility condition* from the fact that partial derivatives of C^2-functions commute:

$$\frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial g}{\partial x}.$$

On the level of differential forms this corresponds to the fact that $d^2 = 0$ and so if $du = f dx + g dy$ then we necessarily have

$$0 = d(f dx + g dy) = df \wedge dx + dg \wedge dy = \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right) dy \wedge dx.$$

So, for general f, g it is not the case that solutions u need exist, contrary to the situation with ODEs. In fancier terminology, linear partial differential operators $P : E_M \to F_M$ (such as the exterior derivative d) need not be surjective and therefore may have non-trivial cokernel

$$E_M \xrightarrow{P} F_M \to \operatorname{coker}(P) \to 0.$$

The reason for writing the cokernel as a part of the above sequence is that the above sequence is *exact* (and any sheaf fitting into such an exact sequence in the position of $\operatorname{coker}(P)$ above is isomorphic to $\operatorname{coker}(P)$). Indeed, a sequence of morphisms

$$\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots$$

in an abelian category is called **exact at** A_i if and only if

$$\operatorname{im}(A_{i-1} \to A_i) \cong \ker(A_i \to A_{i+1})$$

via the natural map from the left-hand-side to the right-hand-side; and the entire sequence is called **exact** if and only if it is exact at each object.

Let's now discuss the second question we ask when studying PDEs: what does the space of solutions look like given that they exist? As mentioned, we can ask for its dimension as a real vector space or affine space (more precisely, the dimensions of the stalks of the sheaf of solutions) and this will describe to what extent solutions are unique.

By looking at the stalks of the sheaf of solutions we've excluded certain trivial non-uniqueness problems. For example, consider the ODE

$$\frac{du}{dx} = \frac{1}{x}$$

on $\mathbb{R} \setminus \{0\}$. In first year calculus, one is often told that the general solution to this equation has the form

$$u(x) = \log|x| + c$$

for arbitrary $c \in \mathbb{R}$ and so it appears that the space of global solutions has dimension one. This is however false since $\mathbb{R} \setminus \{0\}$ has two connected components and so we actually have a 2-dimensional space of solutions in which the general solution is given by

$$u(x) = \begin{cases} \log(x) + c_1 & \text{if } x > 0 \\ \log(-x) + c_2 & \text{if } x < 0 \end{cases}$$

for arbitrary $c_1, c_2 \in \mathbb{R}$. However, when looking at germs of solutions, i.e. on the stalks, we can recall that even non-connected manifolds are locally path-connected in the sense that every neighbourhood of every point contains a path-connected neighbourhood. Thus when looking at the stalks of the sheaf of solutions we are always working on connected domains.

Now, suppose we had a local section f of F_M, $p \in M$ and two solutions u_1, u_2 to $Pu = f$ both defined on some neighbourhood of p in M. Since

$$Pu_1 = f = Pu_2$$

it follows that

$$P(u_1 - u_2) = f - f = 0$$

by linearity and so $u_1 - u_2$ is a local section of $\ker(P)$. So any two locally defined solutions differ by a local section of $\ker(P)$. Conversely, given a locally defined solution u near p and a local section v of $\ker(P)$ near p it follows that

$$P(u+v) = Pu + Pv = f + 0 = f$$

and so $u+v$ is also a solution. Thus the space of germs of solutions to $Pu = f$ defined near p is an affine space modelled on the space of germs of sections of $\ker(P)$ defined near p. Therefore the local existence and uniqueness properties for a linear partial differential equation determined by P are modelled by the exact sequence of sheaves of real vector spaces

$$0 \to \ker(P) \to E_M \xrightarrow{P} F_M \to \operatorname{coker}(P) \to 0.$$

There is, however, an important problem with the above sequence. Notice how saying that a local section of F_M yields a solvable PDE if and only if its image in $\operatorname{coker}(P)$ is zero feels like cheating. That is because it is. Indeed, the cokernel is defined precisely so that this holds. Let's see what happens when we try to use the above exact sequence to describe the *global* existence and uniqueness properties of the PDEs $Pu = f$. i.e. we are now trying to find necessary and sufficient conditions on global sections $f \in \Gamma(M, F_M)$ for globally defined solutions $u \in \Gamma(M, E_M)$ to $Pu = f$ to exist, and furthermore understand the space of such solutions when they exist.

The obvious thing to do is to try and apply the global sections functor $\Gamma(M, -)$ to the above sequence. However, $\Gamma(M, -)$ is merely left-exact (indeed, by definition local uniqueness implies global uniqueness for sections of sheaves) and so while we obtain an exact sequence of real vector spaces of the form

$$0 \to \Gamma(M, \ker(P)) \to \Gamma(M, E_M) \to \Gamma(M, F_M) \to \Gamma(M, \operatorname{coker}(P))$$

the last linear map in this sequence need not be surjective. So, while our above sequence of sheaves does give rise to a description of the global uniqueness problem after an application of the global sections functor, this is not the case for the existence problem. The remedy for this is to split our sequence of sheaves into two short exact sequences

$$\begin{aligned} 0 \to \ker(P) \to & E_M \to \operatorname{im}(P) \to 0 \ \text{ and} \\ 0 \to \operatorname{im}(P) \to & F_M \to \operatorname{coker}(P) \to 0. \end{aligned}$$

Using the methods of homological algebra, one obtains long exact sequences in sheaf cohomology when applying the global sections functor to short exact sequences. As such, if we can compute the sheaf cohomology groups of the involved sheaves then one can understand the global existence problem in terms of these long exact sequences. In order to compute these groups, one needs injective, or at least $\Gamma(M, -)$-*acyclic resolutions* of the sheaves involved. As a first step, we can try to do this for $\ker(P)$. As we'll see in section 1.3, vector bundles are $\Gamma(M, -)$-acyclic and so if $\operatorname{coker}(P)$ was a vector bundle then

$$0 \to \ker(P) \to E_M \to F_M \to \operatorname{coker}(P) \to 0$$

would be an acyclic resolution of $\ker(P)$. More generally, we hope to find a sequence of vector bundles G_i and $\mathbb{R}$-linear morphisms of sheaves $G_i \to G_{i+1}$ and $\ker(P) \to G_0$, such that

$$0 \to \ker(P) \to G_0 \to G_1 \to \cdots$$

is exact. Then the i'th sheaf cohomology group of $\ker(P)$ is given by

$$H^i(R^+\Gamma(M, \ker(P))) = \ker(\Gamma(M, G_i) \to \Gamma(M, G_{i+1})) / \operatorname{im}(\Gamma(M, G_{i-1}) \to \Gamma(M, G_i)).$$

It turns out that there is a standard way of attempting to do this for the kernel of an arbitrary linear partial differential operator $P : E_M \to F_M$ and the resulting sequence is called the *Spencer complex*. In fact, this complex is constructed in such a way that the operators involved are differential operators of finite order and so one can apply the methods of functional analysis to study this PDE. For example, Hormander's methods using unbounded operators on Hilbert spaces apply only to those operators which are densely defined and closeable, which differential operators always are.

Before constructing the Spencer complex we will do two things:

1. prove the existence of the long exact sequence in sheaf cohomology and the fact that vector bundles are $\Gamma(M, -)$-acyclic;

2. perform the construction of the Spencer complex for the exterior derivative d.

The first of the above points is standard material from homological algebra and will be done in section 1.3. The second point is adressed in section 2.3. Indeed, the Spencer complex for the exterior derivative $d : C^\infty_M \to T^*_M$ ends up being the well known de Rham complex

$$0 \to \mathbb{R}_M \hookrightarrow C^\infty_M \xrightarrow{d} T^*_M \xrightarrow{d} \Lambda^2 T^*_M \to \cdots$$

where $\mathbb{R}_M$ denotes the sheaf of locally constant real-valued functions on M. This complex is proven to be an exact complex of sheaves by the Poincaré lemma and it is used in the construction of the general Spencer complexes. More importantly, however, there is a sense in which the de Rham complex is natural and this fact generalizes to the Spencer complex. Section 2.3 is devoted to understanding the "natural" construction of the de Rham complex. For example, we'll answer the question as to why the exterior products appear and why the de Rham differential takes its usual form before answering those same questions for the Spencer complex. Namely, we'll provide a natural explicit *construction* of the de Rham and Spencer complexes as opposed to merely defining them and proving their naturality afterwards.

Looking at the de Rham complex we can see another link between the existence/uniqueness of solutions to PDEs on a manifold and the topology of that manifold, as discussed at the end of the last section. Indeed, we have the following result whose proof can be found in [7, 8]..

Theorem 1.2.6. The de Rham Theorem
Let M be a smooth manifold and write $H^*_{dR}(M, \mathbb{R})$ for its de Rham cohomology ring (i.e. the cohomology ring of the global sections functor applied to the de Rham resolution). Then there is an isomorphism of graded algebras

$$H^*_{dR}(M, \mathbb{R}) \cong H^*_{sing}(M, \mathbb{R})$$

where $H^*_{sing}(M, \mathbb{R})$ is the singular cohomology of M with coefficients in $\mathbb{R}$.

Proof. This proof proceeds in two steps. The first follows from the Poincaré lemma which we'll discuss and prove in section 1.4. Namely, the de Rham complex

$$C^\infty_M \to T^*_M \to \Lambda^2 T^*_M \to \cdots$$

is an acyclic resolution of $\mathbb{R}_M$ and therefore computes the sheaf cohomology

$$H^*_{dR}(M, \mathbb{R}) \cong H^*(R^+\Gamma(M, \mathbb{R}_M)).$$

We now show that from the singular cochain complex we can naturally obtain a complex of flabby (hence acyclic) sheaves which also computes the sheaf cohomology of $\mathbb{R}_M$ but whose global sections are naturally quasi-isomorphic to the singular cochain complex.

Let $C^\bullet_{sing}(M,\mathbb{R})$ denote the singular cochain complex with coefficients in $\mathbb{R}$. For each open $U \subseteq M$ we get complexes $C^\bullet_{sing}(U,\mathbb{R})$ which assemble to presheaves on M. For U contractible, $C^\bullet_{sing}(M,\mathbb{R})$ is exact away from degree zero (where its cohomology is $\mathbb{R}_M$) since fixing a contraction of U gives us an appropriate chain homotopy. The degree-wise sheafification $C^\bullet$ of $C^\bullet_{sing}(-,\mathbb{R})$ is obtained by quotienting-out by the subcomplex consisting of those cochains satisfying: for each p there is a neighbourhood U of p such that the cochain vanishes on all simplices contained in U.

Since our sheafification is obtained through such a quotient it follows that our chain homotopy arising from contractible U descends to the sheafification and so the resulting complex $C^\bullet$ of sheaves is a flabby resolution of $\mathbb{R}_M$. Indeed, exactness follows from exactness at each stalk which is proven using our chain homotopy on contractible neighbourhoods. Being *flabby* means that each of the restriction maps coming from inclusions of open sets are surjective (i.e. we can always extend sections on an open set to any larger open set). Each of the C^k's is flabby since we can extend elements of $C^k(U)$ to $C^k(V)$ for $V \supseteq U$ by defining our cochains to be zero on all chains not contained in U. One can show that flabby sheaves are *soft* (see section 1.4) and hence acyclic.

All that remains is to show that the above sheafification is a quasi-isomorphism and that the wedge and cap products on cohomology coincide. This is done in [8]. The advantage of this proof over the one found in [7] is that this demonstrated that singular cohomology with coefficients in an abelian group A and the sheaf cohomology of A_X agree on any locally contractible space X. $\square$

After seeing the above theorem and knowing of the existence of the Spencer complex, one might hope to obtain similar topological information from the sheaf cohomology of the sheaves $\ker(P)$ for P a more general linear partial differential operator. Indeed, the de Rham cohomology is the sheaf cohomology of $\ker(d) = \mathbb{R}_M$ since vector bundles are acyclic:

$$H^*_{dR}(M,\mathbb{R}) = H^*(R^+\Gamma(M,\mathbb{R}_M)).$$

General differential operators are far worse behaved than the exterior derivative d and so we cannot hope to obtain results as nice as the de Rham theorem in general. However, some general results are known. For example, there is the celebrated (and highly non-trivial) Atiyah-Singer index theorem (which can be found in [40]) stated below.

Theorem 1.2.7. The Atiyah-Singer Index Theorem
Let M be a compact smooth manifold of dimension n, $P : E_M \to F_M$ a (non-overdetermined and non-underdetermined) elliptic operator between complex vector bundles on M, and let $\pi : TM \to M$ be the projection. Then we have:

$$\mathrm{ind}(P) = (-1)^n(\mathrm{ch}([\pi^*E,\pi^*F;\sigma(P)])\widehat{A}(M))[TM]$$

where $\mathrm{ind}(-)$ *is the Fredholm index of P on any of the Sobolev-space completions of* $\Gamma(M,E_M),\Gamma(M,F_M)$, $\mathrm{ch}(-)$ *is the Chern character and* $\widehat{A}(M)$ *is the $\widehat{A}$-genus. The object* $[\pi^*E,\pi^*F;\sigma(P)]$ *is an element of the K-theory of TM obtained using the Yoneda presentation of K-theory.*

After seeing the above nice theorems, we might hope to obtain topological obstructions to the formal integrability of a given G-structure by formulating the problem as a PDE and studying the Spencer complex. Notice that here I am assuming we are given a G-structure and are merely checking whether it is formally integrable to some order. We need to be careful here however, since the above theory only applies to *linear* partial differential operators. Meanwhile, given a G_2-structure $\varphi \in \Omega^3(M)$, the equations

$$d\varphi = 0 \ \text{ and } \ d *_\varphi \varphi = 0$$

are highly non-linear in φ (at least the second one is; the first one is linear) since the metric and orientation defining $*_\varphi$ are determinined non-linearly from φ. Similarly, in the case of a $\mathrm{O}(n)$-structure determined by a Riemannian metric g the equation $R^g = 0$ is non-linear in g. Even for almost complex structures J the equation $N_J = 0$ is non-linear in J. The point is that our linear equations for integrability will not be written in terms of φ, g, J, etc. and will in fact be obtained from these tensors in a non-linear way. Unfortunately, as mentioned in [9], the resulting linear equations are typically overdetermined and invariant under diffeomorphisms of the underlying manifold, making their study using standard methods from functional analysis difficult. In the end, if one hopes to find solutions to these equations it is probably simpler to just work with the non-linear versions.

1.3 Homotopical Algebra

At this point, it is probably time I described what I mean by "topological" obstructions. What sort of obstruction counts as being topological? In both the de Rham theorem and the Atiyah-Singer index theorem the topological objects lived in singular cohomology groups. What information is encoded in the singular cohomology? The point is that objects such as manifolds are topologically built up from simpler objects in Top living in $\mathbb{R}^n$. Indeed, this is necessary if we are to have any hope of considering differential equations on these objects (more generally, one could look at objects built up from locally convex spaces [38]).

Objects such as manifolds admit many continuous maps

$$\mathbb{R}^n \to M.$$

Furthermore, every point on a manifold admits a neighbourhood which appears (at least differentiably and, in particular, topologically) as an open ball in $\mathbb{R}^n$ centered at the origin. Singular (co)homology encodes the way in which, topologically, objects in a manifold can be built-up from closed unit balls in the various $\mathbb{R}^n$'s by gluing along portions of the boundary (which is a sphere). Since we're only working topologically, we can replace balls and spheres with the combinatorially more convenient *simplices*

$$\Delta^n := \{\vec{x} \in \mathbb{R}^{n+1} \ : \ 0 \leq x_i \leq 1 \ \forall i, \ \ x_0 + \cdots + x_n = 1\}.$$

By "combinatorially convenient" we mean that a n-simplex Δ^n has $n+1$ faces which are themselves simplices. We can then glue simplices together along these faces, something which is more difficult to do in the case of the ball and sphere (where a "face" would correspond to something like the equator).

Let's see how this gluing is done. The initial data we are given before gluing is a sequence of sets X_n with the elements of X_n each being thought of as a n-simplex. We are then given maps between the various X_n which essentially specify which $(n-1)$-simplices are the faces of the various n-simplices and hence which n-simplices are glued together (and along which faces). In more abstract terms, we're given a specific type of *diagram* in the category Set of sets and functions between them. The morphisms in this diagram should be such that the gluing they dictate be done is consistent.

A diagram in a category $\mathcal{C}$ is typically thought of as a functor $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is some small category describing the "shape" of the diagram. For example, if $\mathcal{I}$ is the category with three objects labelled $1, 2, 3$ and three non-identity morphisms $1 \to 2$, $2 \to 3$ and $1 \to 3$ such that

$$(2 \to 3) \circ (1 \to 2) = (1 \to 3)$$

then a functor $F : \mathcal{I} \to \mathcal{C}$ is precisely what we typically call a commutative triangle

$$\begin{tikzcd} C_1 \arrow[d] \arrow[dr] & \\ C_2 \arrow[r] & C_3 \end{tikzcd}$$

with $C_1 := F(1)$, $C_2 := F(2)$ and $C_3 := F(3)$. So, in order to specify which sequences of sets $X_n \in \text{Set}$ together with morphisms in between them can be interpreted as "instructions" for gluing simplices, we need to define a category Δ whose objects are $\text{Ob}(\Delta) = \mathbb{Z}_{\geq 0}$ and whose morphisms are simplicial, in some sense. We now present a category that works and is fairly standard in the literature [25, 48, 49, 60, 65].

Definition 1.3.1. We denote by Δ the **simplex category** whose objects are the ordered sets

$$[n] := \{0 < 1 < \cdots < n\}$$

and whose morphisms are non-decreasing functions.

Let's see why this category works. We begin by defining the so-called **face** and **degeneracy** morphisms in Δ. One should think of the elements of $[n]$ as representing the vertices of a n-simplex and the face maps ϵ_i, given by

$$\begin{aligned} \epsilon_i : [n-1] &\to [n] \\ x &\mapsto \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geq i \end{cases} \end{aligned}$$

as the inclusion of the face opposite the i'th vertex. Similarly the degeneracy maps η_i, given by

$$\begin{aligned} \eta_i : [n+1] &\to [n] \\ x &\mapsto \begin{cases} x & \text{if } x \leq i \\ x-1 & \text{if } x > i. \end{cases} \end{aligned}$$

should be thought of as the collapsing map down to the face opposite the i'th vertex which takes the i'th vertex to the $(i+1)$'st. These satisfy the *simplicial relations*:

$$\begin{aligned} \epsilon_i \circ \epsilon_j &= \epsilon_j \circ \epsilon_{i-1} \text{ if } i > j \\ \eta_i \circ \eta_j &= \eta_j \circ \eta_{i+1} \text{ if } i \geq j \\ \eta_i \circ \epsilon_j &= \begin{cases} \epsilon_j \circ \eta_{i-1} & \text{if } i > j \\ \text{id} & \text{if } i = j \text{ or } j-1 \\ \epsilon_{j-1} \circ \eta_i & \text{if } i < j+1. \end{cases} \end{aligned}$$

We now state the following important lemma.

Lemma 1.3.2. *[69, 25] Every morphism $f : [n] \to [m]$ in Δ factors uniquely as*

$$f = \epsilon_{i_s} \circ \cdots \circ \epsilon_{i_1} \circ \eta_{j_1} \circ \cdots \circ \eta_{j_t}$$

where $0 \leq i_1 \leq \cdots \leq i_s \leq m$ and $0 \leq j_1 < \cdots < j_t < n$.

This allows us to describe diagrams in categories modelled on the simplex category Δ by only defining how the functor acts on the objects, face morphisms and degeneracy morphisms, so long as the assigned morphisms satisfy the above composition relations. For example, the geometric simplices $\mathbb{A}^n$ described above naturally fit into a diagram in Top modelled on Δ, justifying our intuitive description of Δ. Indeed, as in [25, 69], we define a functor

$$\begin{aligned} \mathbb{A} : \Delta &\to \text{Top} \\ [n] &\mapsto \mathbb{A}^n \subseteq \mathbb{R}^{n+1} \end{aligned}$$

by

$$\begin{aligned}\mathbb{A}(\epsilon_i) : \mathbb{A}^{n-1} &\to \mathbb{A}^n \\ \vec{x} &\mapsto (x_0, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)\end{aligned}$$

(i.e. inclusion of the face opposite the i'th vertex) and

$$\begin{aligned}\mathbb{A}(\eta_i) : \mathbb{A}^{n+1} &\to \mathbb{A}^n \\ \vec{x} &\mapsto (x_1, \cdots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \cdots, x_{n+1}).\end{aligned}$$

This is called the **standard cosimplicial space** for reasons we are about to describe.

Definition 1.3.3. Let $\mathcal{C}$ be a category. Then a **simplicial object** in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. Similarly, a **cosimplicial object** in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. The category of all simplicial objects in $\mathcal{C}$ with natural transformations as morphisms is denoted by $s\mathcal{C}$.

Now, suppose we had a simplicial set, i.e. a functor $F : \Delta^{op} \to \mathrm{Set}$. Recall that we want to think of the set $F([n])$ as the set of n-simplices to be glued together, with the particulars of the glueing specified by the images of the morphisms in Δ. As such, we define the **geometric realization** of F to be the topological space

$$|F| := \left(\bigsqcup_{n \geq 0} F([n]) \times \mathbb{A}^n \right) / \sim$$

where sets are interpreted as discrete topological spaces and $\sim$ is defined by declaring

$$(F(f)(x), \vec{y}) \sim (x, \mathbb{A}(f)(\vec{y}))$$

for all morphisms $f : [n] \to [m]$ in Δ (and all n, m), all $x \in F([m])$ and all $\vec{y} \in \mathbb{A}^n$. In fact, given any sufficently nice category $\mathcal{C}$ (namely, $\mathcal{C}$ must be a *simplicial model category*) there is a notion of geometric realization from $s\mathcal{C}$ to $\mathcal{C}$ [17].

We can now make precise what we meant by the statement that the simplicial (co)homology encodes combinatorial information regarding the way in which the space in question is built up from simplices. Indeed, for a topological space X we can define the **singular simplicial set** of X to be

$$S(X) := \mathrm{Hom}_{\mathrm{Top}}(\Delta, X) \text{ (as a composition of functors).}$$

This is indeed a functor $S(X) : \Delta^{op} \to \mathrm{Set}$ since $\mathbb{A}$ is a functor $\Delta \to \mathrm{Top}$ and $\mathrm{Hom}_{\mathrm{Top}}(-, X)$ is contravariant. We now have the following result which can be found in [49] and [48].

Proposition 1.3.4. *The functor* $S(-) : \mathrm{Top} \to s\,\mathrm{Set}$ *is right-adjoint to the geometric realization functor* $|\cdot| : s\,\mathrm{Set} \to \mathrm{Top}$.

Proof. Following May's books we define two natural transformations

$$\Phi : \mathrm{Hom}_{\mathrm{Top}}(|-|, -) \to \mathrm{Hom}_{s\,\mathrm{Set}}(-, S(-))$$

and

$$\Psi : \mathrm{Hom}_{s\,\mathrm{Set}}(-, S(-)) \to \mathrm{Hom}_{\mathrm{Top}}(|-|, -)$$

as follows. Let K be a simplicial set and X a topological space. Given a continuous map $f : |K| \to X$ we define

$$\begin{aligned}\Phi(f)_n : K_n &\to S_n(X) \\ x &\mapsto (y \mapsto f([x, y]))\end{aligned}$$

where $[x,y]$ is the equivalence class of $(x,y) \in K_n \times \mathbb{\Delta}^n$ in $|K|$. More precisely, one makes this definition for non-degenerate $[x,y]$ and then proves that every element is equivalent to a unique non-degenerate element [48]. One can show that this is well-defined. Similarly, given a morphism $f : K \to S(X)$ of simplicial sets we define

$$\begin{aligned}\Psi(f) : |K| = \left(\amalg_{n\geq 0} K_n \times \mathbb{\Delta}^n\right) \Big/ \sim &\to X\\ [x,y] &\mapsto f(x)(y).\end{aligned}$$

Again one checks that this actually works and that Φ and Ψ are inverse to one another object-wise. In [48] it is then concluded that this is an adjunction by defining a suitable unit and counit for the monad and comonad that this adjunction should give rise to. $\square$

Before demonstrating how one can obtain the singular (co)homology of a space X from $S(X)$ we will attempt to answer to what extent X can be recovered from $S(X)$. By our adjunction proven above, we see that from the identity map

$$\mathrm{id} \in \mathrm{Hom}_{s\,\mathrm{Set}}(S(X), S(X))$$

we obtain a natural morphism

$$|S(X)| \to X.$$

The question we then ask is: when is $|S(X)| \to X$ a weak equivalence (i.e. induces an isomorphism on all homotopy groups)? It is worth noticing first that since the image of a compact set is compact and the $\mathbb{\Delta}^n$'s are all compact, we cannot hope to obtain much information from the singular simplicial set when the space X has few compact sets. For example, if many points aren't compact then there won't be many continuous maps $\Delta^0 \to X$. As such, we make the following definition.

Definition 1.3.5. A topological space X is called **weakly Hausdorff** if and only if the image of every continuous map $K \to X$ where K is compact Hausdorff, is closed. A weakly Hausdorff space X is called **compactly generated** if and only if subsets of X are closed if and only if their intersection with each compact subset of X is closed. Let $\mathcal{WC}$ denote the full subcategory of Top consisting of all weakly Hausdorff compactly generated topological spaces.

Let's now give some basic examples of topological spaces which end up being in $\mathcal{WC}$.

Example 1.3.6. All Hausdorff spaces are weakly Hausdorff since the continuous image of a compact set is compact and compact subsets of Hausdorff spaces are closed. Also, all weakly Hausdorff spaces are T_1 (points are closed) since for each $p \in X$ the inclusion $\{p\} \hookrightarrow X$ is continuous. So the weak Hausdorff property sits in between T_1 and T_2 (Hausdorff).

Example 1.3.7. All locally compact Hausdorff spaces are in $\mathcal{WC}$. Indeed, Hausdorff spaces are weakly Hausdorff as described above. Now, suppose we were given a locally compact Hausdorff space X and a subset $A \subseteq X$ whose intersection with each compact subset of X is closed.

Let $x \in X$ be such that every open neighbourhood of x in X intersects A non-trivially. Since X is locally compact, there exists an open neighbourhood U of x and a compact subset $K \subseteq X$ such that

$$x \in U \subseteq K \subseteq X.$$

But then $K \cap A$ is closed and $x \in K$ has the property that each open neighbourhood of x in K intersects A non-trivially. Hence $x \in K \cap A \subseteq A$ and so A is closed in X, as required.

As we've now seen, the category $\mathcal{WC}$ contains most of the objects we actually care about. In fact, this is an incredibly nice category from a categorical perspective, as we are about to see, however categorical constructions in $\mathcal{WC}$ don't always agree with those in Top. This ends up being a relatively small price to pay.

Proposition 1.3.8. *[49] The category $\mathcal{WC}$ is complete and cocomplete and, while the forgetful functor $\mathcal{WC} \to \mathrm{Top}$ need not preserve these limits and colimits, the underlying sets of the limits and colimits computed in $\mathcal{WC}$ and* Top *are the same.*

Now, recall our construction of the geometric realization $|F|$ of a simplicial set F. If we hope to work in $\mathcal{WC}$ instead then we should perform the operations used in the construction of $|F|$ categorically in $\mathcal{WC}$. Luckily, it is shown in [49] that for $X, Y \in \mathcal{WC}$ with X locally compact the product $X \times Y$ in $\mathcal{WC}$ agrees with the usual one in Top. Since for a simplicial set $X_\bullet$ we consider the sets $X_n \in \mathcal{WC}$ as discrete spaces it follows that the products $\mathbb{\Delta}^n \times X_n$ are the usual ones. The coproducts and quotient, however, have different topologies in general.

What did we mean earlier when we said that the differences between constructions in $\mathcal{WC}$ and Top are a small price to pay? What we meant was essentially the following proposition.

Proposition 1.3.9. *[49] Given $X, Y \in \mathcal{WC}$, if we equip $C^0(X,Y) =: Y^X$ with the strongest weakly Hausdorff compactly generated topology weaker than the compact-open topology, then $\mathcal{WC}$ becomes Cartesian-closed.*

This tells us, for example, that homotopies $h : I \times X \to Y$ in $\mathcal{WC}$ can also be thought of as continuous maps $h : X \to Y^I$ where Y^I is equipped with the topology described as above.

Now, how does homotopy theory work in $\mathcal{WC}$? Let's denote $I := [0,1]$ for the standard unit interval and recall that a homotopy between two continuous maps $f, g : X \to Y$ is a continuous map $h : X \times I \to Y$ such that $h(-,0) = f(-)$ and $h(-,1) = g(-)$. We then write

$$[X,Y] := C^0(X,Y)/\sim$$

for the set of continuous maps from X to Y modulo the relation of being homotopic. Now, recall that given $X \in \mathcal{WC}$ and $x \in X$ we can define the space of based loops in (X,x) to be

$$\Omega_x X := (X,x)^{(S^1,1)}$$

where we use the topology described above and require our continuous maps to preserve the basepoint (the category of based spaces in is typically denoted by $\mathcal{WC}_*$). Note that $\Omega_x X$ is itself a based space with basepoint the constant loop at x.

Now, recall that the **fundamental group** of (X,x) is given by

$$\pi_1(X,x) := [(S^1,1),(X,x)]$$

where the group multiplication is concatenation of paths. Similarly, we have a set

$$\pi_0(X,x) := [(S^0,1),(X,x)]$$

and for each $n \geq 2$ we can define the **homotopy groups** as

$$\pi_n(X,x) := \pi_1(\Omega_x^{n-1}X, c_x)$$

where c_x denotes the constant loop. These are all functors:

$$\begin{aligned}
&\pi_0 : \mathcal{WC}_* \to \mathrm{Set} \\
&\pi_1 : \mathcal{WC}_* \to \mathrm{Grp} \\
&\pi_n : \mathcal{WC}_* \to \mathrm{Ab} \quad \text{for } n \geq 2.
\end{aligned}$$

Now, a continuous map $f : X \to Y$ in $\mathcal{WC}$ induces morphisms

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

for all $n \geq 0$ and all $x \in X$. We call f a **weak equivalence** if and only if all of these are isomorphisms. At this point, one should be thinking of the notion of a quasi-isomorphism of chain complexes. As we'll see later, quasi-isomorphisms of chain complexes need not be homotopy equivalences in general. Similarly, weak equivalences of objects in $\mathcal{WC}$ need not be homotopy equivalences [49]. However, in both the cases of complexes and of topological spaces, there are special classes of objects for which weak equivalences/quasi-isomorphisms will always end up being homotopy equivalences.

The next two theorems can be found on pages 122-124 of Peter May's "A Concise Course in Algebraic Topology" [49]. He only provides a proof sketch there whereas the full proofs can be found in section 16 of his other book "Simplicial Sets in Algebraic Topology" [48], but I am of the opinion that it is incredibly valuable for one to read the proof-sketch first. Also, I don't intend on saying what a "CW-complex" is here since they are equivalent, in a precise sense described below, to simplicial sets and simplicial sets are much simpler and more intuitive in my opinion.

Theorem 1.3.10. *For any $X \in \mathcal{WC}$, $|S(X)|$ is a CW-complex and the cellular chain complex $C_\bullet(|S(X)|)$ is naturally isomorphic to the singular chain complex $C_\bullet(X)$.*

Theorem 1.3.11. *For any $X \in \mathcal{WC}$, the natural map $|S(X)| \to X$ is a weak equivalence. In particular, if X is a CW-complex then it is a homotopy equivalence.*

Let's now see how the singular simplicial set $S(X)$ of $X \in \mathcal{WC}$ can be used to obtain the singular (co)homology. Denote by F the free functor

$$F : \mathrm{Set} \to \mathrm{Ab}$$

which is left-adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$. Given any commutative ring R we obtain a simplicial R-module:

$$S_\bullet(X, R) := F(S(X)) \otimes_{\mathbb{Z}} R \in sR\text{-Mod}\,.$$

Similarly we can obtain a cosimplicial R-module via

$$S^\bullet(X, R) := \mathrm{Hom}_{\mathbb{Z}}(F(S(X)), R).$$

The claim now is that the categories of simplicial and cosimplicial objects in an abelian category $\mathcal{A}$ are precisely the same thing as $\mathrm{Ch}^{\leq 0}(\mathcal{A})$ (complexes concentrated in degrees ≤ 0) and $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ respectively. Our treatment of this result here essentially follows [69].

Theorem 1.3.12. The Dold-Kan Correspondence
Let $\mathcal{A}$ be any abelian category. Given any simplicial object $A \in s\mathcal{A}$ we define a complex $N(A) \in \mathrm{Ch}^{\leq 0}(\mathcal{A})$ via

$$N^{-n}(A) := \bigcap_{k=0}^{n-1} \ker(A(\epsilon_i) : A_n \to A_{n-1})$$

where the "intersection" is really the limit of the diagram consisting of the inclusions of the kernels in A_n, with differential

$$d^n := (-1)^n A(\epsilon_n).$$

Similarly, given a cosimplicial object A in $\mathcal{A}$ we define a complex $N(A) \in \mathrm{Ch}^{\geq 0}(\mathcal{A})$ via

$$N^n(A) := \bigcap_{k=0}^{n-1} \ker(A(\epsilon_i) : A^n \to A^{n+1})$$

with differential

$$d^n := (-1)^n A(\epsilon_n).$$

Then these constructions yield functors N from (co)simplicial objects in $\mathcal{A}$ to ($\mathrm{Ch}^{\geq 0}(\mathcal{A})$) $\mathrm{Ch}^{\leq 0}(\mathcal{A})$ *which are equivalences of categories, both admitting left adjoints.*

Proof. We will actually construct an explicit functor going in the other directions to yield our equivalences of categories. It's worth noticing that the case of cosimplicial objects can be obtained from the first case by applying the $(-)^{op}$ functor and so it suffices to prove the result for simplicial objects.

Now, let $C \in \mathrm{Ch}^{\leq 0}(\mathcal{A})$ and denote by $s(n,k)$ the set of all surjections $[n] \to [k]$ in Δ. We now define the following sequence of objects in $\mathcal{A}$:

$$\Gamma_n(C) := \bigoplus_{k \leq n} (C^{-k})^{\oplus s(n,k)}.$$

We hope to assemble these into a simplicial object in $\mathcal{A}$. To do this, we need to associate in a consistent way a morphism $\Gamma_n(C) \to \Gamma_m(C)$ in $\mathcal{A}$ to each morphism $\alpha : [m] \to [n]$ in Δ. By the universal property of coproducts it then suffices to construct appropriate morphisms

$$C^{-k} \to \Gamma_m(C)$$

for each $k \leq n$ and each surjective morphism $\eta : [n] \to [k]$ in Δ. So, given such a k and η we perform our epic-monic factorization:

$$\begin{tikzcd} {[m]} \arrow[r, "\eta\circ\alpha"] \arrow[d, "\beta"] & {[k]} \\ {[q]} \arrow[ur, "\epsilon"] & \end{tikzcd}$$

and then do the following. If $\epsilon = \mathrm{id}$ then we define $C^{-k} \to \Gamma_m(C)$ to simply be the identification of C^{-k} with its summand in $\Gamma_m(C)$ corresponding to β. If $\epsilon = \epsilon_k$ (so $q = k-1$) then we define $C^{-k} \to \Gamma_m(C)$ to be the composition

$$C^{-k} \xrightarrow{d} C^{-k+1} = C^{-(k-1)} \to \Gamma_m(C)$$

where $C^{-(k-1)} \to \Gamma_m(C)$ is the identification of $C^{-(k-1)}$ with the summand corresponding to β again. In all other cases, we let $C^{-k} \to \Gamma_m(C)$ simply be the zero morphism.

If one defines N and Γ on morphisms in the natural way then all that remains is to show that the functors

$$N : s\mathcal{A} \to \mathrm{Ch}^{\leq 0}(\mathcal{A})$$

and

$$\Gamma : \mathrm{Ch}^{\leq 0}(\mathcal{A}) \to s\mathcal{A}$$

form an adjoint equivalence of categories (with Γ the left-adjoint of N). This relies on the fact that if f is a morphism in $s\mathcal{A}$ such that $N(f)$ is an isomorphism then f was actually an isomorphism all along [69]. □

We should mention that the full statement of the Dold-Kan correspondence also says that the functor N, called the **normalized complex functor**, takes the *simplicial homotopy groups* isomorphically to the cohomology groups of the associated complex. However, we have yet to even define these since they are only defined for *fibrant* objects, which we also haven't defined.

The point is that most of the (co)homology theories we actually care about can be obtained by replacing the category $\mathcal{C}$ we happen to be working in with $s\mathcal{C}$. In $s\mathcal{C}$ there are notions of homotopy, weak equivalence, fibration and cofibration allowing us to make precise the relationship between the (co)homology theories obtained from homological algebra and the corresponding theories obtained from algebraic topology [25].

Furthermore, this provides a useful generalization of homological algebra. In this thesis we have already mentioned sheaf cohomology, which is a cohomology theory obtained by taking the *derived functors* of the global sections functor, which is an additive functor between abelian categories. On the other hand, singular (co)homology, as well as the cohomology theories relevant to deformation theory are defined most naturally on categories which are non-abelian. Nevertheless, they can be naturally extended to categories which sufficiently resemble $\mathcal{WC}$ so that homotopy/homology can be performed. This extension is typically done by replacing the category $\mathcal{C}$ in question with the category $s\mathcal{C}$.

Let's see how one performs homotopy theory in a category of simplicial objects $s\mathcal{C}$. To do things properly, we should really introduce the notion of a model category.

Definition 1.3.13. A **model category** is a complete and cocomplete category $\mathcal{C}$ together with three subcategories: the **weak equivalences**, the **fibrations** and the **cofibrations**, as well as functorial factorizations factoring morphisms into cofibrations follows by trivial fibrations and fibrations followed by trivial cofibrations satisfying the following:

1. all three subcategories are closed under retracts and weak equivalences satisfy 2-out-of-3. That is, if $f \circ g = h$ and two of f, g, h are weak equivalences then so is the third;
2. fibrations have the left lifting property with respect to trivial cofibrations. To be precise, if we had a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd} \qquad (2)$$

   with f both a cofibration and a weak equivalence and g a fibration then there exists a morphism $C \to B$ making everything commute;
3. cofibrations have the right lifting property with respect to trivial fibrations. That is, if we had a commutative diagram such as item 2 above where g was both a fibration and a weak equivalence and f a cofibration then there would exist a morphism $C \to B$ making everything commute.

Objects for which the morphism from the initial object to them is a cofibration are called **cofibrant** and objects for which the morphism from them to the terminal object is a fibration are called **fibrant**.

It's worth noting that our definition follows [25] which is slightly different from the original definition in that it requires the factorizations to not only be functorial, but to also be included in the data of the model category. Often, one instead proves they exist for what are called *cofibrantly generated* model categories (where one uses a weaker definition of "model category"). The reason Hovey includes them as a part of the data is that doing so allows one to assemble model categories into a 2-category in a nice way and, furthermore, functorial factorizations can be shown to exist in all of the cases we actually care about.

The point is that once we know something is a model category we are in a very good position and a lot can be done. The difficult part becomes proving that various things (simplicial sets, chain complexes, topological spaces, modules over Frobenius algebras, comodules over Hopf algebras, differential graded algebras, differential graded Lie algebras, etc.) actually are (cofibrantly generated) model categories. Performing these proofs often requires huge amounts of set theory since one can typically only prove that functorial factorizations exist and cannot actually write them down explicitly. All of this is done in gory detail in [25].

One can show that in a model category, the above right and left lifting properties for cofibrations and fibrations actually completely characterize these subcategories [25]. Thus to define a model structure on a category it suffices to specify the weak equivalences and either the fibrations or cofibrations.

Following Mark Hovey's book on model categories, we begin by introducing the simplicial sets

$$\Delta[n] := \mathrm{Hom}_{\Delta}(-, [n]).$$

These are natural objects to consider via the Yoneda embedding and their geometric realizations $|\Delta[n]|$ end up being the geometric simplices $\mathbb{A}^n$ [25, 48]. Similarly, if we set $\partial\Delta[n] \in s\,\mathrm{Set}$ to be the simplicial set whose non-degenerate k-simplices correspond to the non-identity monotonically increasing injective maps

$$[k] \to [n]$$

then the face and degeneracies of $\Delta[n]$ restrict to face and degeneracy morphisms on $\partial\Delta[n]$ making $\partial\Delta[n]$ into a subfunctor of $\Delta[n]$. As implied by the notation, the geometric realization of this simplicial set is the boundary of the standard n-simplex:

$$|\partial\Delta[n]| = \partial\mathbb{A}^n.$$

As we'll see, the inclusions $\partial\mathbb{A}^n \hookrightarrow \mathbb{A}^n$ are used to specify a sufficiently large class of trivial fibrations in $\mathcal{WC}$ so that the cofibrations become precisely the objects satisfying a lifting property with respect to these trivial fibrations. One should think of fibrations and cofibrations as up to homotopy versions of fibre bundles and the inclusions of fibres into a fibre bundle. Fibrations and cofibrations are useful in the construction of homotopy categories (since formal localizations of categories need not be locally small). The point of describing cofibrations in terms of the inclusions $\partial\mathbb{A}^n \hookrightarrow \mathbb{A}^n$ is that it will allow us to extend the notion of "cofibration" to the category of simplicial sets (in a way compatible with geometric realization).

It is more intuitive, however, to describe the fibrations in $s\,\mathrm{Set}$ first since the definition of these is very similar to that of a Serre fibration in algebraic topology. To do this, we denote by

$$\Lambda^r[n] \subseteq \partial\Delta[n]$$

the r'th **horn** of $\Delta[n]$ (for $n > 0$, $0 \leq r \leq n$). This is the exact same simplicial set as $\partial\Delta[n]$ only we have removed the face map

$$\epsilon_r : [n-1] \to [n]$$

from the collection of $n-1$ simplices in $\Lambda^r[n]$. One should think of $\Lambda^r[n]$ as the n-simplex $\mathbb{A}^n$ with the face opposite the r'th vertex removed (as well as the interior). We now make the following definition.

Definition 1.3.14. A **fibration** (also called a **Kan fibration**) in $s\,\mathrm{Set}$ is a morphism which has the right lifting property with respect to all of the inclusions

$$\Lambda^r[n] \hookrightarrow \Delta[n]$$

for $n > 0$ and $0 \leq r \leq n$.

Explicitly the above definition means the following. A morphism $f : X \to Y$ in $s\,\mathrm{Set}$ is a Kan fibration if and only if for every $0 \le k \le n+1$ and every $y \in Y_{n+1}$, if $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in X_n$ satisfy

$$\begin{aligned} Y(\epsilon_i)(y) &= f(x_i), \text{ and} \\ X(\epsilon_i)(x_j) &= X(\epsilon_{j-1})(x_i) \text{ for all } i < j,\ i \neq k \end{aligned}$$

then there exists $x \in X_{n+1}$ such that

$$f(x) = y \text{ and } X(\epsilon_i)(x) = x_i \text{ for } i \neq k.$$

This description is useful for explicitly checking when something is a Kan fibration. Let's give an example of such a computation.

Example 1.3.15. The underlying simplicial set of any simplicial group is fibrant. Indeed, let G be a simplicial object in the category of groups and suppose we had $x_i \in G_n$ for $i \neq k$ satisfying

$$G(\epsilon_i)(x_j) = G(\epsilon_{j-1})(x_i) \text{ for } i < j.$$

We will construct our element $y \in G_{n+1}$ recursively. Indeed, set $y_{-1} := 1 \in G_{n+1}$ and perform the following. Let $r \ge 0$ and suppose we had constructed y_{r-1}. If $r = k$ then we set $y_r := y_{r-1}$. Otherwise we set

$$y_r := y_{r-1}(G(\eta_k)(x_r^{-1}G(\epsilon_r)(y_{r-1})))^{-1}.$$

The claim is then that $y := y_n$ works and the proof of this can be found in [69].

Proposition 1.3.16. *[25] If f is a fibration in $s\,\mathrm{Set}$ then $|f|$ is a Serre fibration in $\mathcal{WC}$.*

This is probably an appropriate time to introduce the simplicial homotopy groups. These are defined intrinsically in [25] for fibrant simplicial sets and then are proven to agree with the usual homotopy groups of their geometric realizations. Since we won't be actually working with them explicitly, we'll simply *define* them to be the homotopy groups of their geometric realization (again, we only define them for fibrant objects).

Now, we know that cofibrations in $s\,\mathrm{Set}$ must be defined as those morphisms which have the left lifting property with respect to all trivial fibrations. As we are about to see, there are simpler characterizations.

Lemma 1.3.17. *[25] Cofibrations in $s\,\mathrm{Set}$ are precisely those morphisms which have the left lifting property with respect to every morphism which has the right lifting property with respect to the morphisms $\partial\Delta[n] \to \Delta[n]$ for all $n \ge 0$.*

Now, we've seen that simplicial sets and morphisms between them are nicer than general, even compactly generated, topological spaces and continuous maps. As such, the next result shouldn't be too surprising.

Proposition 1.3.18. *[25] In $s\,\mathrm{Set}$ the cofibrations are precisely those maps f for which $f[n]$ is injective for all $n \ge 0$.*

There is a similar characterization of cofibrations in $\mathcal{WC}$ assuming we have defined weak equivalences to be those continuous maps inducing isomorphisms on all homotopy groups and fibrations to be Serre fibrations.

Lemma 1.3.19. *[25] A morphism in $\mathcal{WC}$ has the left lifting property with respect to all trivial Serre fibrations if and only if it has the left lifting property with respect to all morphisms having the right lifting property with respect to the inclusions $S^n \hookrightarrow D^{n+1}$.*

Theorem 1.3.20. *[25] If we define the fibrations in $\mathcal{WC}$ to be the Serre fibrations, the weak equivalences to be the continuous maps inducing isomorphisms on the homotopy groups, and the cofibrations induced by our choice of fibrations and weak equivalences then $\mathcal{WC}$ becomes a model category.*

In fact, the above ends up being a symmetric monoidal model category (this is the primary advantage it has over Top). As we're about to see, $s\,\mathrm{Set}$ is also one of these and in fact $s\,\mathrm{Set}$ is what is called *Quillen equivalent* to $\mathcal{WC}$.

Theorem 1.3.21. *Suppose we define weak equivalences in $s\,\mathrm{Set}$ to be those morphisms f whose induced maps on the geometric realizations $|f|$ are weak equivalences in $\mathcal{WC}$. Then, together with our earlier definitions of fibrations and cofibrations, $s\,\mathrm{Set}$ is a model category.*

Localizing a model category $\mathcal{C}$ at its weak equivalences yields a category $h(\mathcal{C})$ called the associated **homotopy category**. In the case of chain complexes in a sufficiently nice abelian category $\mathcal{A}$ (by this we mean small enough that $s\mathcal{A}$ and $\mathcal{A}^{\Delta}$ end up being model categories with respect to our definition, i.e. cofibrantly generated) we write

$$D^{\leq 0}(\mathcal{A}) := h(s\mathcal{A})$$

and refer to this as the **derived category**. One can also define $D^{\geq 0}(\mathcal{A})$ using cosimplicial objects instead. The reason we need to say "sufficiently nice" is that not all categories of simplicial objects form model categories, however the categories sR-Mod and $s\,\mathrm{Sh}_{\mathcal{O}_X}$ both do. The point is that when localizing a category one may obtain too many morphisms. i.e. we can always construct $h(\mathcal{C})$ for a category with weak equivalences but in general the "hom-sets" of this category need not be sets! For model categories, however, the localization does indeed yield a category in the usual sense; namely the hom-sets are actual sets. Even in classical homological algebra, see section 1.4, one has this problem when constructing the derived categories.

The way in which homotopy/(co)homology theory is performed in a model category $\mathcal{C}$ is the following. First, we need to understand what the appropriate notion of morphism between model categories is. This is somewhat subtle and it turns out that the correct "category" of model categories only makes sense as a higher category [25]. We, however, won't really care about this as we only need to understand how to form the derived functors associated to a morphism between model categories. Let's define these morphisms now.

Definition 1.3.22. A **left Quillen functor** between model categories is a left adjoint that preserves cofibrations and trivial cofibrations. A **right Quillen functor** is a right adjoint which preserves fibrations and trivial fibrations. A **Quillen adjunction** is an adjoint pair for which the left adjoint is left Quillen and the right adjoint is right Quillen.

Morphisms between model categories are defined to be Quillen adjunctions. As is typical with "categories" whose morphisms are adjunctions, this actually only forms a 2-category (with natural isomorphisms as 2-morphisms).

We now define derived functors. The point of these is that left Quillen functors should really only be applied to cofibrant objects and similarly right Quillen functors should only be applied to fibrant objects. Essentially the idea is that pullbacks and pushouts of diagrams not consisting of fibrant objects and fibrations or cofibrant objects and cofibrations respectively need not be homotopically stable. Meanwhile, homology and cohomology theories tend to be invariant under perturbation by homotopies and so when performing functorial constructions with a view towards ariving at some sort of (co)homological invariant for a problem, one should perform these constructions in a homotopically stable way.

Definition 1.3.23. Let $F : \mathcal{C} \to \mathcal{D}$ be a left Quillen functor between model categories. Then F defines a **left total derived functor**

$$LF : h\mathcal{C} \to h\mathcal{D}$$

obtained by first replacing objects with their functorially associated cofibrant replacements. Similarly, a right Quillen functor $U : \mathcal{D} \to \mathcal{C}$ has a **right total derived functor**

$$RU : h\mathcal{D} \to h\mathcal{C}$$

obtained by first replacing objects with their functorially associated fibrant replacements. A **Quillen equivalence** is a Quillen adjunction whose corresponding total derived functors form an adjoint equivalence of categories.

Proposition 1.3.24. *[25] The adjunction between the geometric realization functor and the singular simplicial set functor is a Quillen equivalence between* $s\,\mathrm{Set}$ *and* $\mathcal{WC}$.

The next proposition is something we'll see more of in the next section and again later when we will have need of the so-called *monoidal* Dold-Kan correspondence.

Proposition 1.3.25. *[17, 25] The Dold-Kan correspondence induces model category structures on* $\mathrm{Ch}^{\geq 0}(\mathrm{Sh}_{\mathcal{O}_X})$ *and* $\mathrm{Ch}^{\leq 0}(R\text{-Mod})$ *for any commutative ring R and any locally ringed space* $(X, \mathcal{O}_X)$.

1.4 Homological Algebra

In the last section we saw the Dold-Kan correspondence between the categories of simplicial objects in an abelian category, and the category of complexes concentrated in non-positive degrees in that category. In this section we will discuss the case of abelian categories in more detail since here the notion of a derived functor is simpler and more computable. Most of the results here can be found in [69].

In section 1.2 we saw examples where an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories need not be exact coming from PDEs. In this way, information regarding objects is lost after the application of F. Namely, if we can understand an object $B \in \mathcal{A}$ as an extension of a better-understood object C by another well-understood object A, i.e. if

$$0 \to A \to B \to C \to 0$$

is exact, then in general we will lose this description after applying F, only recovering the not-necessarily-exact complex

$$0 \to F(A) \to F(B) \to F(C) \to 0.$$

The idea then is to encode the information that would have been lost when applying F to A, B, C by replacing A, B, C by objects in $\mathrm{Ch}(\mathcal{A})$ which are equivalent to A, B, C in some sense and are better behaved with respect to the application of F. The key thing to notice is the following.

Proposition 1.4.1. *Let* $A^\bullet$ *be a split exact complex in an abelian category* $\mathcal{A}$ *and* $F : \mathcal{A} \to \mathcal{B}$ *an additive functor between abelian categories. Then* $F(A^\bullet)$ *is a split exact complex in* $\mathcal{B}$.

Proof. Recall that a complex

$$\cdots \xrightarrow{d} A^{n-1} \xrightarrow{d} A^n \xrightarrow{d} A^{n+1} \to \cdots$$

is called split exact whenever there exist morphisms $s : A^{n+1} \to A^n$ for all n such that

$$d = dsd.$$

The significance of this definition comes from breaking up our complexes into a bunch of short exact sequences and then recalling the splitting lemma.

Applying an additive functor F to such a complex yields another complex since $F(0) = 0$ so $0 = F(d^2) = F(d)^2$. The morphisms $F(s)$ split this new complex by functoriality and so we have split exactness by the splitting lemma. □

All of the additive functors we encounter (or at least the ones we care about) will end up being either left or right exact. The general strategy then is to find classes of objects in an abelian category for which either bounded above or bounded below complexes of these objects are always split exact. Let's introduce these objects now.

Definition 1.4.2. An object P in an abelian category $\mathcal{A}$ is called **projective** if and only if

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Ab}$$

is exact. Dually, an object I in $\mathcal{A}$ is called **injective** if and only if

$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to \mathrm{Ab}$$

is exact.

Notice that for any object $M \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is left exact and $\mathrm{Hom}_{\mathcal{A}}(-, M) = \mathrm{Hom}_{\mathcal{A}^{op}}(M, -)$ so $\mathrm{Hom}_{\mathcal{A}}(-, M)$ takes colimits in $\mathcal{A}$ to limits in Ab. This tells us that the definitions of injectivity and projectivity can be written in terms of lifting properties for maps into and out of objects in short exact sequences or, more precisely, monomorphisms and epimorphisms respectively. One should make a mental note here regarding the analogy between this and the definition of fibrations and cofibrations in algebraic topology. Let's now give some examples of injective and projective objects in abelian categories.

Example 1.4.3. An R-module is projective if and only if it is a direct summand of a free module. $\mathbb{Q}$ is an injective abelian group. [69].

Proposition 1.4.4. *Every bounded above exact complex of projective objects is split exact. Similarly, every bounded below exact complex of injective objects is split exact.*

The proof of the above proposition follows by induction and application of the definitions of projectivity and injectivity via lifting properties.

Definition 1.4.5. Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$. A **projective resolution** is an epimorphism

$$P^\bullet \to A \to 0$$

in $\mathrm{Ch}(\mathcal{A})$ (where A is interpreted as being concentrated in degree zero) which is also a quasi-isomorphism of complexes with $P^i = 0$ for all $i > 0$ and P^i projective for all $i \leq 0$. An **injective resolution** of A is a monomorphism

$$0 \to A \to I^\bullet$$

in $\mathrm{Ch}(\mathcal{A})$ which is also a quasi-isomorphism of complexes with $I^i = 0$ for all $i < 0$ and I^i injective for all $i \geq 0$.

Proposition 1.4.6. *Let $\mathcal{A}$ be an abelian category such that for every object $A \in \mathcal{A}$ there exists an epimorphism $P \to A \to 0$ from some projective object P in $\mathcal{A}$ (we say that $\mathcal{A}$ has **enough projectives**). Then every object of $\mathcal{A}$ admits a projective resolution. Dually, if for each object $A \in \mathcal{A}$ there existed a monomorphism $0 \to A \to I$ into some injective object I in $\mathcal{A}$ (i.e. $\mathcal{A}$ has **enough injectives**) then every object in $\mathcal{A}$ admits an injective resolution.*

Proof. First suppose that for every object $A \in \mathcal{A}$ there exists an epimorphism $P^0 \to A \to 0$ from some projective object P^0. Then there also exists an epimorphism

$$P^{-1} \to \ker(P^0 \to A)$$

and composition then yields an exact sequence

$$P^{-1} \to P^0 \to A \to 0.$$

We now repeat this process on $\ker(P^{-1} \to P^0)$ and the result follows by induction.

Similarly, if for every object $A \in \mathcal{A}$ there existed a monomorphism $0 \to A \to I^0$ into some injective object I^0 then there also exists a monomorphism

$$\mathrm{coker}(A \to I^0) \to I^1$$

where I^1 is injective and the composition

$$0 \to A \to I^0 \to I^1$$

is exact. Again, proceeding inductively yields our desired result. $\square$

The most important applications for our purposes are to the categories of sheaves of $\mathcal{O}_X$-modules on a locally ringed space $(X, \mathcal{O}_X)$, and to the category of modules over a ring. Recall that a **locally ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$ whose stalks at each point are local rings.

Proposition 1.4.7. *Let $(X, \mathcal{O}_X)$ be a locally ringed space and R a ring. Then the categories $\mathrm{Sh}_{\mathcal{O}_X}$ of sheaves of $\mathcal{O}_X$-modules and R-Mod of left R-modules have enough injectives. The category R-Mod also has enough projectives.*

Proof. The proof for R-Mod can be found in Weibel's *An Introduction to Homological algebra* [69]. Let's construct an injective resolution of an arbitrary sheaf $\mathcal{F}_X$ of $\mathcal{O}_X$-modules on a locally ringed space. This construction is outlined in [22].

For each $p \in X$ the abelian group $\mathcal{F}_{X,p}$ is a $\mathcal{O}_{X,p}$-module and the category of $\mathcal{O}_{X,p}$-modules has enough injectives so there is an injective $\mathcal{O}_{X,p}$-module I_p together with a monomorphism $\mathcal{F}_{X,p} \to I_p$. We use the axiom of choice to chose such an injective object and monomorphism for each $p \in X$.

Now, let $j_p : \{p\} \hookrightarrow X$ be the inclusion and define the sheaf

$$\mathcal{I}_X := \prod_{p \in X} (j_p)_* I_p.$$

Now, since the covariant $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}$ preserves limits (it has a left adjoint) it follows that for every sheaf $\mathcal{G}_X$ of $\mathcal{O}_X$-modules we have

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{G}_X, \mathcal{I}_X) \cong \prod_{p \in X} \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{G}_X, (j_p)_* I_p) \cong \prod_{p \in X} \mathrm{Hom}_{\mathcal{O}_{\{p\}}\text{-Mod}}(j_p^* \mathcal{G}_X, I_p)$$

and furthermore

$$j_p^* \mathcal{G}_X \cong \mathcal{G}_{X,p}.$$

So, by taking the product of the monomorphisms $\mathcal{F}_{X,p} \to I_p$ we obtain a monomorphism of $\mathcal{O}_X$-modules

$$\mathcal{F}_X \to \mathcal{I}_X.$$

To see that $\mathcal{I}_X$ is an injective $\mathcal{O}_X$-module we simply notice that since each I_p is injective the functors $\mathrm{Hom}(-, I_p)$ are exact and since taking stalks is exact the injectivity of $\mathcal{I}_X$ follows from our natural isomorphism of functors

$$\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I}_X) \cong \prod_{p \in X} \mathrm{Hom}_{\mathcal{O}_{X,p}}((-)_p, I_p).$$

Hence $\mathrm{Sh}_{\mathcal{O}_X}$ has enough injectives. It is worth mentioning that in the case $X = \mathrm{Spec}(R)$ this result specializes to the statement that R-Mod has enough injectives, however we used this fact non-trivially in the proof (at least in the case R is a local ring). Also, the sheaves $\mathcal{I}_X$ we constructed are examples of sheaves of **discontinuous sections** (and are therefore quite difficult to work with). $\square$

Now, our original goal was to replace our objects by projective or injective resolutions in such a way that we can replace an exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ with an exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of chain complexes such that each of the chain complexes involved is either a projective or injective resolution of the original object in $\mathcal{A}$. We would then want to apply our additive functor to this exact sequence instead. In order to do this, we need to discuss the problem of functoriality for assigning projective and injective resolutions.

Definition 1.4.8. Let $\mathcal{A}$ be an abelian category and $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be two parallel morphisms in $\mathrm{Ch}(\mathcal{A})$. We call a morphism $s : A[1]^\bullet \to B^\bullet$ of graded objects in $\mathcal{A}$ (not a morphism in $\mathrm{Ch}(\mathcal{A})$ since we don't want to require it to commute with the differential) a **homotopy** from $f^\bullet$ to $g^\bullet$ if and only if

$$f - g = ds + sd$$

where we are abusing notation and writing d for the differential on both $A^\bullet$ and $B^\bullet$. We will write $K(\mathcal{A})$ ($K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$ are defined analogously) for the category whose objects are the same as $\mathrm{Ch}(\mathcal{A})$ but whose morphisms are instead given by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$$

where $\sim$ is homotopy equivalence. Notice that composition is indeed well-defined since composition is bilinear. It's worth noting that this definition can be phrased as saying that the difference $f - g$, which can be demonstrated to live in an enriched hom: $f - g \in \mathrm{Hom}^\bullet(A, B) \in \mathrm{Ch}(\mathrm{Ab})$, is exact with respect to the differential in $\mathrm{Ch}(\mathrm{Ab})$.

The motivation for calling the above a homotopy comes from the analogous statements of the next lemma and proposition for singular (co)homology. Namely, homotopic continuous maps induce homotopic morphisms between the singular chain complexes. Actually, it is also possible to motivate this definition through the definition of homotopy for (co)simplicial objects in an abelian category via the Dold-Kan correspondence.

Lemma 1.4.9. *Let f, g be two homotopic morphisms in $\mathrm{Ch}(\mathcal{A})$. Then $H(f) = H(g)$.*

Proof. Suppose we had two homotopic morphisms $f, g : A \to B$ in $\mathrm{Ch}(\mathcal{A})$ with homotopy s. Since H is additive it suffices to prove that $H(f - g) = 0$. But $H(f - g)$ acts only on elements of $\ker(d)$ and $f - g = ds + sd$ so

$$H(f - g) = ds$$

takes values in $\mathrm{im}(d)$. Thus $H(f - g) = 0$ as required. □

Proposition 1.4.10. *Let $f : A \to B$ be any morphism in an abelian category $\mathcal{A}$. Given any two projective resolutions $P \to A$ and $Q \to B$ there exists a unique up to homotopy morphism $P \to Q$ making our square*

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

in $\mathrm{Ch}(\mathcal{A})$ commute. Similarly, given any two injective resolutions $A \to I$ and $B \to J$ there exists a unique up to homotopy morphism $I \to J$ making our new square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ I & \longrightarrow & J \end{array}$$

in $\mathrm{Ch}(\mathcal{A})$ *commute.*

Proof. First, arbitrarily select a morphism $f : A \to B$ in an abelian category $\mathcal{A}$ and suppose we had projective resolutions $P \to A$ and $Q \to B$. Since these resolutions consist of projective objects we have

$$\begin{tikzcd} & P^0 \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ Q^0 \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where the vertical morphism is the composition $P^0 \to A \to B$. Now that we have our morphism $P^0 \to Q^0$ we proceed by induction. Consider the diagram

$$\begin{tikzcd} P^{-n-1} \arrow[r] & P^{-n} \arrow[r] \arrow[d] & P^{-n+1} \arrow[d] \\ Q^{-n-1} \arrow[r] & Q^{-n} \arrow[r] & Q^{-n+1} \end{tikzcd}$$

where we have set $P^1 := A$, $Q^1 := B$ (so we must take care to not assume that P^{-n+1} and Q^{-n+1} are projective in our proof). By the definition of projectivity, it suffices to demonstrate that we can factor the composition $P^{-n-1} \to P^{-n} \to Q^{-n}$ through $\ker(Q^{-n} \to Q^{-n+1})$. But this follows immediately from the commutativity of the square on the right and the fact that $P^\bullet$ is a complex.

Now, suppose instead that we had injective resolutions $A \to I$ and $B \to J$. Again, the morphism at level zero comes from the definition of injectivity:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[dl, dashed, "\exists"] \\ & J^0 & \end{tikzcd}$$

where the vertical arrow is the composition $A \to B \to J^0$. Now that we have our morphism $I^0 \to J^0$ we again proceed inductively and consider the diagram

$$\begin{tikzcd} I^{n-1} \arrow[r] \arrow[d] & I^n \arrow[r] \arrow[d] & I^{n+1} \\ J^{n-1} \arrow[r] & J^n \arrow[r] & J^{n+1} \end{tikzcd}$$

where we have set $I^{-1} := A$ and $J^{-1} := B$ (so we must be careful to not assume that I^{n-1}, J^{n-1} are injective in our proof). By the definition of injectivity, it suffices to show that we can factor the composition $I^n \to J^n \to J^{n+1}$ through the coimage of $I^n \to I^{n+1}$. But since the kernel of $I^n \to I^{n+1}$ is the image of $I^{n-1} \to I^n$ and the square on the left commutes it follows from the fact that $J^\bullet$ is a complex that the composition $I^n \to J^n \to J^{n+1}$ takes the kernel of $I^n \to I^{n+1}$ to zero and thus factors through the coimage, as required.

All that remains now is to show that these morphisms $P \to Q$ and $I \to J$ are unique up to homotopy (out of all morphisms making our squares in $\mathrm{Ch}(\mathcal{A})$ commute). First, suppose we had two lifts $g, h : P \to Q$ of $f : A \to B$. The difference $g^0 - h^0$ post-composes with $Q^0 \to B$ to yield zero and hence factors through

$$\ker(Q^0 \to B) = \mathrm{im}(Q^{-1} \to Q).$$

So, by the definition of projective objects, this lifts to a morphism

$$s^0 : P^0 \to Q^{-1}.$$

We then proceed inductively to construct our desired null-homotopy of $g - h$ by replacing $g^0 - h^0$ with $g^{-n} - h^{-n} - s^{-n+1} \circ d$. For the case of the injective resolutions, we again suppose we had two lifting morphisms $g, h : I \to J$ and notice that the difference $g^0 - h^0 : I^0 \to J^0$ pre-composes with $A \to I^0$ to yield zero and therefore factors through the coimage of $I^0 \to I^1$ yielding, by the definition of injective objects a morphism

$$s^1 : I^1 \to J^0.$$

To complete the homotopy we again proceed inductively, replacing $g^0 - h^0$ with $g^{n-1} - h^{n-1} - s^n d$. $\square$

Lemma 1.4.11. Horseshoe Lemma
Let $\mathcal{A}$ be an abelian category and

$$0 \to A \to B \to C \to 0$$

a short exact sequence in $\mathcal{A}$. Furthermore, suppose we had projective resolutions $P^\bullet \to A$ and $Q^\bullet \to C$. Then one obtains a naturally-defined projective resolution $P^\bullet \oplus Q^\bullet \to B$ making the following diagram in $K(\mathcal{A})$ commute and have exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P^\bullet \arrow[r] \arrow[d] & P^\bullet \oplus Q^\bullet \arrow[r] \arrow[d] & Q^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

There is an analogous result for injective resolutions although the direct sum $\oplus$ should be replaced with a product $\times$ (however they are naturally isomorphic in this case since $\mathcal{A}$ is abelian).

Proof. First suppose we had projective resolutions $P \to A$ and $Q \to C$. Since $B \to C$ is an epimorphism the map $Q^0 \to C$ lifts to $Q^0 \to B$ and together with the composition $P^0 \to A \to B$ assemble to a morphism

$$P^0 \oplus Q^0 \to B$$

by the universal property of the coproduct. Similarly, for injective resolutions $A \to I, C \to J$ the fact that $A \to B$ is a monomorphism tells us that $A \to I^0$ lifts to $B \to I^0$. Together with the composition $B \to C \to J^0$ these two morphisms assemble to a map

$$B \to I^0 \times J^0 \cong I^0 \oplus J^0$$

via the universal property of the product. As expected, we now complete the argument by induction. Consider the diagrams

$$\begin{tikzcd}
0 \arrow[r] & P^{-n-1} \arrow[d] & & Q^{-n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P^{-n} \arrow[r] & P^{-n} \oplus Q^{-n} \arrow[r] & Q^{-n} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & I^n \arrow[r] \arrow[d] & I^n \oplus J^n \arrow[r] & J^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1} & & J^{n+1} \arrow[r] & 0
\end{tikzcd}$$

In both cases, one obtains morphisms

$$P^{-n-1} \oplus Q^{-n-1} \to P^{-n} \oplus Q^{-n} \ \text{ and } \ I^n \oplus J^n \to I^{n+1} \oplus J^{n+1}$$

exactly as in the level zero case. All that needs to be checked is exactness but this follows from the fact that the rows are exact as are the left and right columns, using the 5-lemma on the kernels and images. $\square$

Proposition 1.4.12. *Exact complexes in $\mathcal{A}$ are replaced using projective resolutions by split exact complexes in $K^-(\mathcal{A})$. Similarly, exact complexes in $\mathcal{A}$ are replaced using injective resolutions by split exact complexes in $K^+(\mathcal{A})$.*

Proof. First suppose we had an abelian category $\mathcal{A}$ with enough projectives. Given any exact complex $A^\bullet \in \mathrm{Ch}(\mathcal{A})$ with differential d we can split it up into pieces, each of which is a short exact sequence, via

$$0 \to \ker(d^n) \to A^n \to \operatorname{im}(d^n) \cong \ker(d^{n+1}) \to 0.$$

The result then follows by choosing projective (respectively injective) resolutions of each $\ker(d^n)$ and then applying the Horseshoe lemma to each A^n. The resulting exact complex of complexes of projective objects (respectively injective objects) is unique up to homotopy and one can obtain splitting morphisms by applying the splitting lemma to each of the split short exact sequences of complexes obtained from the above short exact sequence and the Horseshoe lemma. $\square$

Finally we are prepared to apply our results to additive functors between abelian categories. To do this, given an abelian category $\mathcal{A}$ we'll write $K_I^+(\mathcal{A})$ for the full subcategory of $K^+(\mathcal{A})$ consisting of bounded below complexes of injective objects. Similarly, we'll write $K_P^-(\mathcal{A})$ for the full-subcategory of $K^-(\mathcal{A})$ consisting of bounded above complexes of projective objects.

Proposition 1.4.13. *If $\mathcal{A}$ is an abelian category with enough projectives then there exists a functor*

$$P : \mathrm{Ch}(\mathcal{A}) \to K_P^-(\mathcal{A})$$

together with a natural transformation

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{A}) \arrow[d, "P"] \arrow[dr] & \\ K_P^-(\mathcal{A}) \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & K(\mathcal{A}) \end{tikzcd}$$

which is a quasi-isomorphism on each object. On the other hand, if $\mathcal{A}$ has enough injectives then there exists a functor

$$I : \mathrm{Ch}(\mathcal{A}) \to K_I^+(\mathcal{A})$$

together with a natural transformation

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{A}) \arrow[d, "I"] \arrow[dr] & \\ K_I^+(\mathcal{A}) \arrow[r] \arrow[ur, phantom, "\Leftarrow"] & K(\mathcal{A}) \end{tikzcd}$$

which is a quasi-isomorphism on each object.

Proof. As is often the case in category theory, this proof is set-theoretically subtle. If $\mathcal{A}$ has enough projectives then our functor

$$P : \mathrm{Ch}(\mathcal{A}) \to K_P^-(\mathcal{A})$$

is obtained by choosing projective resolutions for each object in $\mathcal{A}$ as well as a lifting for each morphism, and then applying the functor $\mathrm{Tot}^{\oplus}$ constructed in section 2.2. This works since the liftings are unique up to homotopy, however we are technically applying the axiom of choice to what might be a proper class. The natural transformation between the functors is given by the quasi-isomorphisms

$$P(A) \to A$$

defining the projective resolutions after the application of $\mathrm{Tot}^{\oplus}$. Similarly, in the case $\mathcal{A}$ has enough injectives we choose injective resolutions for each of our objects and liftings of our morphisms, then apply the functor $\mathrm{Tot}^{\prod}$ from section 2.2. Our natural transformation again comes from the quasi-isomorphisms

$$A \to I(A)$$

defining the injective resolutions. This is done in more detail in [69] and the resolutions obtained via P and I are called the **Cartan-Eilenberg resolutions**. Their cohomology is typically computed using spectral sequences. □

Proposition 1.4.14. *Let $\mathcal{A}$ be an abelian category with enough injectives so that the localization $D^+(\mathcal{A})$ of $K^+(\mathcal{A})$ along quasi-isomorphisms exists. Then any left-exact additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories such that $D^+(\mathcal{B})$ exists has a right-derived functor*

$$R^+F : D^+(\mathcal{A}) \to D^+(\mathcal{B}).$$

Similarly, if $\mathcal{A}$ has enough injectives then any right-exact additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories such that $\mathcal{D}^-(\mathcal{B})$ exists has a left-derived functor

$$L^-F : D^-(\mathcal{A}) \to D^-(\mathcal{B}).$$

Proof. This follows immediately from the previous proposition and the fact, proven in [69], that quasi-isomorphisms in $K_I^+(\mathcal{A})$ and $K_P^-(\mathcal{A})$ are homotopy equivalences and hence these categories are naturally isomorphic to $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ respectively. Furthermore, since the objects of $K_I^+(\mathcal{A})$ and $K_P^-(\mathcal{A})$ are split exact complexes our left and right exact functors respectively preserve exactness and therefore extend to the derived categories. This is analogous to *Whitehead's theorem* for CW-complexes in [49]. □

Finally, we obtain our main computational tool.

Proposition 1.4.15. *Let $\mathcal{A}$ be an abelian category with enough injectives, $F : \mathcal{A} \to \mathcal{B}$ a left-exact additive functor and suppose*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{A}$. Then we obtain a long exact sequence in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(R^+F(A)) \arrow[r] & H^0(R^+F(B)) \arrow[r] & H^0(R^+F(C)) \arrow[dll, rounded corners] \\
& H^1(R^+F(A)) \arrow[r] & H^1(R^+F(B)) \arrow[r] & H^1(R^+F(C)) \cdots
\end{tikzcd}$$

Similarly, if $F : \mathcal{A} \to \mathcal{B}$ is a right-exact additive functor then we obtain a long exact sequence in cohomology:

$$\begin{tikzcd}
\cdots H^{-1}(L^-F(A)) \arrow[r] & H^{-1}(L^-F(B)) \arrow[r] & H^{-1}(L^-F(C)) \arrow[dll, rounded corners] & \\
H^0(L^-F(A)) \arrow[r] & H^0(L^-F(B)) \arrow[r] & H^0(L^-F(C)) \arrow[r] & 0
\end{tikzcd}$$

Proof. This follows from the fact that associated to any short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

in $\mathrm{Ch}(\mathcal{B})$ there is a long exact sequence in cohomology. Indeed, since our short exact sequence in $\mathcal{A}$ is replaced by a split exact sequence in either $K_P^-(\mathcal{A})$ or $K_I^+(\mathcal{A})$ then after the application of L^-F or R^+F respectively we will have split short exact sequences in $K(\mathcal{B})$. Now, given a short exact sequence in $\mathrm{Ch}(\mathcal{B})$ as above we can consider for each n the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^n/B^{n-1}(A^\bullet) \arrow[r] \arrow[d] & B^n/B^{n-1}(B^\bullet) \arrow[r] \arrow[d] & C^n/B^{n-1}(C^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) \arrow[r] & 0
\end{tikzcd}$$

where $B^{n-1}(-)$ denotes the image of the differential leaving the $n-1$'st spot and $Z^n(-)$ denotes the kernel of the differential leaving the n'th spot. Our desired connecting homomorphism then comes from applying the snake lemma to this diagram. ☐

In the above proof, we have seen one of the perhaps unforseen advantages of working with the category of chain complexes. Later in section 2.2 we'll see a more geometric interpretation of the snake lemma. Let's now give some examples of applications of the above proposition.

Example 1.4.16. Let M be a smooth manifold and write $C^\infty_{M,\mathbb{C}}$ for the sheaf of complex valued smooth functions on M, while $(C^\infty_{M,\mathbb{C}})^\times$ will denote the sheaf of nowhere-vanishing complex valued smooth functions on M. Taking the sheaf cohomology of the short exact sequence

$$0 \to \mathbb{Z}_M \to C^\infty_{M,\mathbb{C}} \to (C^\infty_{M,\mathbb{C}})^\times \to 0$$

of sheaves of abelian groups where the second-last map is post-composition with $z \mapsto e^{2\pi i z}$ yields the boundary map

$$H^1(R^+\Gamma(M,(C^\infty_{M,\mathbb{C}})^\times)) \to H^2(R^+\Gamma(M,\mathbb{Z}_M)) \cong H^2_{sing}(M,\mathbb{Z})$$

which is the first Chern class morphism $c_1(-)$ (for complex vector bundles on a smooth manifold) after we identify

$$H^1(R^+\Gamma(M,(C^\infty_{M,\mathbb{Z}})^\times)) \cong \mathrm{Pic}(M)$$

by comparing the Čech cohomology groups with the cocycle definition of a vector bundle [70].

Example 1.4.17. A **Spin structure** on a smooth oriented Riemannian n-manifold $(M,g,*)$ is a principal $\mathrm{Spin}(n)$-bundle P_{Spin} together with a morphism of principal bundles

$$P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$$

whose equivariance is governed by the covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ and is itself a double-cover. The topological obstruction to the existence of such structures can be obtained by considering the extension

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

of groups and looking at the associated long exact sequence in cohomology on M [40]. The relevant terms are

$$H^1(M,\mathrm{Spin}(n)) \to H^1(M,\mathrm{SO}(n)) \to H^2(M,\mathbb{Z}/2)$$

The principal $\mathrm{SO}(n)$-bundle P_{SO} defining the metric and orientation gives rise to a class in $H^1(M,\mathrm{SO}(n))$ and our goal is to see whether this arises as the image of some class in $H^1(M,\mathrm{Spin}(n))$. By exactness, the obstruction to this is the image of our class in $H^2(M,\mathbb{Z}/2)$ which is the second Steifel-Whitney class $w_2(TM)$ we ran into earlier (note that we already know that the first one vanishes since we have an orientation).

Example 1.4.18. Recall that the notion of a G-structure on a n-manifold M is equivalent to a global section of the fibre bundle
$$F \times_{\mathrm{GL}(\mathbb{R}^n)} (\mathrm{GL}(\mathbb{R}^n)/G)$$
where F is the principal frame bundle. Now, in the case $G = G_2$ we have that G_2 is the stabilizer of an element of the representation
$$\mathrm{GL}(\mathbb{R}^7) \to \mathrm{GL}(\Lambda^3(\mathbb{R}^7)^*),$$
namely the standard positive 3-form φ_0 on $\mathbb{R}^7$ [32]. Similarly, in the case $G = \mathrm{GL}(\mathbb{C}^n) \subseteq \mathrm{GL}(\mathbb{R}^{2n})$ we have that G is the stabilizer of an element J living in the adjoint representation
$$\mathrm{GL}(\mathbb{R}^{2n}) \to \mathrm{GL}(\mathfrak{gl}(\mathbb{R}^{2n})),$$
namely the standard complex structure on $\mathbb{R}^{2n}$. In both of these cases, the orbit stabilizer theorem allows us to identify
$$\mathrm{GL}(\mathbb{R}^7)/G_2 \cong \mathrm{GL}(\mathbb{R}^7)\varphi_0 \subseteq \Lambda^3(\mathbb{R}^7)^*$$
and
$$\mathrm{GL}(\mathbb{R}^{2n})/\mathrm{GL}(\mathbb{C}^n) \cong \mathrm{GL}(\mathbb{R}^{2n})J \subseteq \mathfrak{gl}(\mathbb{R}^{2n}).$$
This gives us explicit descriptions of the space of G_2-structures and almost complex structures respectively on a manifold M. More generally, we can ask whether a Lie subgroup $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ can be written as the stabilizer of an element of some representation of $\mathrm{GL}(\mathbb{R}^n)$. This problem turns out to be quite difficult.

For starters, let's consider the case that $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ is an algebraic subgroup, i.e. the zero set of some polynomial equations on $\mathrm{GL}(\mathbb{R}^n)$. While in some books [22] one only does algebraic geometry over algebraically closed fields, much of the very basic theory of algebraic groups still holds over the reals since they are a perfect field (a fortiori since they have characteristic zero) [54]. All of the examples we care about are such groups. Then one can let R be the $\mathbb{R}$-algebra of regular functions on $\mathrm{GL}(\mathbb{R}^n)$ and $I \trianglelefteq R$ be the ideal of those functions which vanish on G. The following argument is from [45].

Since R is Noetherian we can write $I = (f_1, \cdots, f_k)$ for some elements $f_1, \cdots, f_k \in R$. Now, the group multiplication
$$\mathrm{GL}(\mathbb{R}^n) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$$
dualizes to a coproduct
$$R \to R \otimes_{\mathbb{R}} R.$$
Let's then write the image of each f_i in this coproduct as
$$\sum_{j=1}^{m_i} g_{j,i} \otimes h_{j,i}.$$
Given any $g \in \mathrm{GL}(\mathbb{R}^n)$ we can then compute $L_g^{\#} f_i$ where $L_g^{\#} : R \to R$ denotes the dual of left multiplication as
$$L_g^{\#} f_i = \sum_{i=1}^{m_i} g_{j,i}(g) h_{j,i}$$
and $g_{j,i}(g) \in \mathbb{R}$ (since the $g_{j,i}$ are polynomial functions $\mathrm{GL}(\mathbb{R}^n) \to \mathbb{R}$). But then by the associativity of the product on $\mathrm{GL}(\mathbb{R}^n)$ we have coassociativity for the coproduct on R (in fact, R is a Hopf algebra) thus expanding the $h_{j,i}$'s in our expression for the coproduct of f_i is the same as expanding

the $g_{j,i}$'s. Hence the subrepresentation of the (potentially infinite dimensional) representation R of $\mathrm{GL}(\mathbb{R}^n)$ generated by $f_1, \cdots, f_k$ sits inside of

$$\mathrm{Span}_{\mathbb{R}}\{f_1, \cdots, f_k, h_{1,1}, \cdots, \cdots, h_{m_k,k}\}$$

and is therefore finite dimensional. Let's call this finite dimensional representation

$$\varphi : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(V).$$

Now, consider the subspace $U := V \cap I \subseteq V$. This subspace is G-invariant by the definition of I and furthermore, if $g \in G$ satisfies $\varphi(g)U = U$ then since U contains a generating set $f_1, \cdots, f_k$ for I it follows that

$$\varphi(g)I = \varphi(g)(UR) = (\varphi(g)U)(\varphi(g)R) = UR = I$$

and therefore

$$G = \{g \in G \;:\; \varphi(g)U = U\}.$$

So we've written G as a stabilizer of an element of a Grassmannian on which $\mathrm{GL}(\mathbb{R}^n)$ acts. By denoting

$$d := \dim_{\mathbb{R}} U, \;\; X := \Lambda^d V, \;\; W := \Lambda^d U \subseteq X$$

we obtain a new representation

$$\rho := \Lambda^d \varphi : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(X)$$

for which

$$G = \{g \in G \;:\; \rho(g)W = W\}$$

(this works because if $\rho(g)\Lambda^d U = \Lambda^d U$ then $\varphi(g)U = U$ since $\dim(U) = d$). But now W is a 1-dimensional subspace and so we've obtained a *projective representation*

$$\mathrm{GL}(\mathbb{R}^n) \to \mathrm{PGL}(X)$$

together with a point $[W] \in \mathrm{PGL}(X)$ for which G is the stabilizer. This is a special case of what is sometimes called Chevalley's Theorem.

Now comes the homological algebra. We want to write G as the stabilizer of an element of a linear representation of $\mathrm{GL}(\mathbb{R}^n)$. One attempt to do so that we could make is to try to lift our projective representation in $\mathrm{PGL}(X)$ back to a different linear representation than X, since G might act non-trivially on $W \subseteq X$ by scaling, for which G is the stabilizer of a point. Our current linear representation on X defines a group-cohomology class

$$[X] \in H^2(\mathrm{GL}(\mathbb{R}^n), \mathbb{R}^\times)$$

via the scaling action on W [68] and the other elements of the cohomology class $[X]$ define equivalent lifts of projective representations to linear ones.

One caveat regarding the above applications is that they are really only useful if we can compute the cohomology groups in question. As we have seen in the case of sheaves, injective objects can be quite complicated. As such, we will often have occasion to use the following computational results.

Definition 1.4.19. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor between abelian categories. We call an object $A \in \mathcal{A}$ F**-acyclic** if and only if

$$H^i(R^+F(A)) = 0 \;\text{ for all }\; i > 0.$$

Similarly, if $F : \mathcal{A} \to \mathcal{B}$ was right-exact then we would call an object $A \in \mathcal{A}$ F-acyclic if and only if

$$H_i(L^-F(A)) := H^{-i}(L^-F(A)) = 0 \;\text{ for all } i > 0.$$

The goal now is to prove that the cohomology groups of the complexes obtained using derived functors can be computed using the, usually much simpler, acyclic resolutions. Furthermore, we'll see that the sheaf of smooth functions C^∞_M on a smooth manifold is a *fine* sheaf and therefore any sheaf of C^∞_M-modules is $\Gamma(M, -)$-acyclic.

Proposition 1.4.20. *[69] Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories, $A \in \mathcal{A}$ and $B^\bullet \to A$ a quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})$ which is also an epimorphism and has $B^i = 0$ for $i > 0$ and B^i F-acyclic for $i \leq 0$. Then*

$$H^i(R^+F(A)) = \ker(F(B^i \to B^{i+1})) / \operatorname{im}(F(B^{i-1} \to B^i)) \ \ \textit{for all} \ \ i \in \mathbb{Z}.$$

There is an analogous result for right-exact F.

Definition 1.4.21. Let X be a topological space. A sheaf $\mathcal{F}$ of abelian groups on X is called a **soft** sheaf if and only if for any closed subset $Z \subseteq X$, the natural map

$$\Gamma(X, \mathcal{F}) \to \mathcal{F}(Z)$$

is surjective where by $\mathcal{F}(Z)$ we mean the collection of compatible local sections of $\mathcal{F}$ defined on any open cover of Z, defined up to agreeing on a common refinement.

If X is paracompact Hausdorff then we call a sheaf $\mathcal{F}$ of abelian groups **fine** if and only if for any locally finite open cover U_i of X there are endomorphisms $\eta_i : \mathcal{F} \to \mathcal{F}$ whose sum (this makes sense since the open cover U_i is locally finite) is $\mathrm{id}_{\mathcal{F}}$ and $\eta_i(\mathcal{F}_x) = 0$ for all x in some neighbourhood of $X \setminus U_i$ where $\mathcal{F}_x$ denotes the stalk.

Proposition 1.4.22. *Every sheaf of modules over a fine sheaf of commutative unital rings is fine. Also, the sheaf of smooth functions on a smooth manifold is fine.*

Proof. Given a fine sheaf $\mathcal{O}_X$ of rings on a paracompact Hausdorff space, a sheaf $\mathcal{F}_X$ of modules over $\mathcal{O}_X$, and a locally finite open cover U_i of X we can take endomorphisms $\eta_i : \mathcal{O}_X \to \mathcal{O}_X$ summing to one and vanishing on neighbourhoods of the complements of their respective U_i's since $\mathcal{O}_X$ is fine. Applying these endomorphisms to the units of the $\mathcal{O}_X(U)$'s and using left multiplication we obtain our desired endomorphisms of $\mathcal{F}_X$. The fact that C^∞_M is fine simply corresponds to the existence of smooth partitions of unity on smooth manifolds (use left multiplication by the functions defining the partition of unity to get our desired endomorphisms). $\square$

Proposition 1.4.23. *[70] Sheaves on a paracompact Haudorff topological space X which are either fine or soft are $\Gamma(X, -)$-acyclic.*

Proof. This follows from first noticing that all fine sheaves on paracompact Hausdorff spaces are soft, and then considering an arbitrary exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves on X with $\mathcal{F}$ soft. One shows that

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to 0$$

is exact by using the fact that the original sequence is exact at each stalk and then proving that one can patch some collection of local sections of $\mathcal{G}$ to a global one mapping to our arbitrarily chosen global section of $\mathcal{H}$ by contradiction and a Zorn's lemma argument. $\square$

The main reason we care about the above result is that the Spencer complex is a resolution of the sheaf of solutions to a homogeneous linear partial differential equation. More precisely, it is a resolution of the sheaf by vector bundles and differential operators between vector bundles. Since the sheaves of smooth sections of a vector bundle on M are sheaves of C^∞_M-modules, it follows that this is an acyclic resolution and can therefore be used to compute the sheaf cohomology. This was our first goal from the previous section. Let's now give the simplest application of this technique.

Proposition 1.4.24. The Poincaré Lemma
The de Rham complex

$$0 \to \mathbb{R}_M \to C_M^\infty \to T_M^* \to \Lambda^2 T_M^* \to \cdots$$

is an exact complex of sheaves and therefore a $\Gamma(M,-)$*-acyclic resolution of the sheaf* $\mathbb{R}_M$*. Hence*

$$H^i(M,\mathbb{R}) := H^i(R^+\Gamma(M,\mathbb{R}_M)) = \ker(\Omega^i(M) \to \Omega^{i+1}(M))/d\Omega^{i-1}(M) =: H^i_{dR}(M).$$

Proof. We've already shown that all of the sheaves appearing in the de Rham complex (except $\mathbb{R}_M$) are $\Gamma(M,-)$-acyclic since they are fine sheaves on M (which is paracompact Hausdorff). So, all that remains is to prove exactness. Since complexes of sheaves are exact if and only if they're exact at each stalk it suffices to show that at each point $p \in M$ there is a neighbourhood base U_i of p in M such that, after applying each $\Gamma(U_i,-)$ to the de Rham complex we obtain an exact complex of real vector spaces.

Thanks to the existence of local charts we've now reduced the problem to proving that for any open ball $B \subseteq \mathbb{R}^n$ centered at $0 \in \mathbb{R}^n$ the complex

$$C_B^\infty \to T_B^* \to \Lambda^2 T_B^* \to \cdots$$

is exact and that for any smooth manifold M the kernel of $d : C_M^\infty \to T_M^*$ is $\mathbb{R}_M$. This second statement simply follows from looking at the level of stalks and then noticing that $df = 0$ corresponds to all of the first partial derivatives of f vanishing on a neighbourhood of p, hence f is locally constant on a neighbourhood of p.

For the first statement, we begin by considering the radial vector field

$$X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}.$$

The flow of this vector field is given by

$$\Phi_s^X(x) = e^s x.$$

For simplicity, we set $t(s) := e^s$, $\Psi_t(x) := tx = \Phi_s^X(x)$. We then define the following operator using an improper integral, which does converge as the integrand admits a continuous extension by zero to $[0,1]$:

$$\begin{aligned} R : \Lambda^k T_B^* &\to \Lambda^k T_B^* \\ \varphi &\mapsto \int_0^1 t^{-1}(\Psi_t^*\varphi)dt. \end{aligned}$$

This definition indeed makes sense since the flow of X preserves the ball B for $0 < t \leq 1$. Now, using the definition of the flow of a vector field we can compute:

$$\begin{aligned} R(\mathcal{L}_X\varphi) &= \int_0^1 t^{-1}(\Psi_t^*\mathcal{L}_X\varphi)dt \\ &= \int_0^1 t^{-1}((\Phi_s^X)^*\mathcal{L}_X\varphi)dt \\ &= \int_0^1 t^{-1}\left(\frac{d}{ds}(\Phi_s^X)^*\varphi\right)dt \\ &= \int_0^1 \left(\frac{d}{dt}(\Phi_s^X)^*\varphi\right)dt \\ &= \Psi_1^*\varphi - \lim_{t\to 0}\Psi_t^*\varphi = \varphi. \end{aligned}$$

The point is that we can now define the homotopy operator

$$h := R \circ \iota_X : \Lambda^{k+1} T_B^* \to \Lambda^k T_B^*$$

and, since R commutes with d, it follows from Cartan's formula that

$$h \circ d + d \circ h = R \circ (\iota_X \circ d + d \circ \iota_X) = R \circ \mathcal{L}_X = \mathrm{id}$$

yielding our desired null-homotopy of the identity. Hence our complex is indeed exact. □

It is probably worth mentioning here that one cannot hope to obtain such exact sequences for general linear partial differential operators between vector bundles. The goal of the Spencer complex is to provide a natural construction of the next-closest-thing, which will end up being exact when it should. Typically, proving exactness requires functional analytic techniques involving elliptic operators. For example, elliptic regularity is what tells us that the complexes of Banach spaces obtained by completing the Spencer complex are quasi-isomorphic to it. The question we will be attempting to answer in section 2.3 is: where does the explicit construction of the de Rham complex above come from?

Chapter 2

Algebraic Methods in Differential Geometry

2.1 C^∞-Rings

In section 1.1 we saw the definition of a smooth manifold as a second-countable Hausdorff topological space together with a maximal atlas of charts with smooth transition functions. Meanwhile, we know that a smooth manifold M determines a locally ringed space (M, C^∞_M) since the stalks $C^\infty_{M,p}$ are the rings of germs of smooth functions at p, which is a local ring with residue field $\mathbb{R}$. This allows us to apply the techniques of sheaf cohomology from the last section to the study of the differential geometry and differential topology of M. We'll see early in this section that one can actually define smooth manifolds to be certain types of locally ringed spaces, analogous to the definition of a scheme in algebraic geometry.

On the other hand, the fact that C^∞_M is a fine sheaf means that the cohomological properties of sheaves on M behave very differently than their analogous properties in algebraic geometry. As we'll see, smooth manifolds are all to some extent "affine" in the sense of algebraic geometry. As such, much of the data of a smooth manifold is encoded in the $\mathbb{R}$-algebra $C^\infty(M)$ and we'll see another definition of smooth manifolds in this section in which there is no underlying topological space a priori. This will allow us to transport the algebraic techniques of Hochschild and cyclic (co)homology over to differential geometry, providing a natural (and, in fact, geometric) construction of the de Rham complex. This construction will end up generalizing to provide a natural construction and interpretation of the Spencer complex in the next chapter.

Proposition 2.1.1. Milnor's Exercise
Let M be a smooth manifold. Then the following map is a natural bijection

$$\begin{aligned} M &\to \mathrm{Hom}_{\mathbb{R}\text{-alg}}(C^\infty(M), \mathbb{R}) \\ p &\mapsto \mathrm{ev}_p \end{aligned}$$

where ev_p *denotes the homomorphism given by evaluation at* p.

Proof. An outline of how to do this proof can be found in [30], and it is this reference that we follow for our proof below.

First we notice that the map $p \mapsto \mathrm{ev}_p \in \mathrm{Hom}_{\mathbb{R}-alg}(C^\infty(M), \mathbb{R})$ (taking p the the evaluation map at p) is an injection since $C^\infty(M)$ separates points of M. Again, this follows from the existence of smooth bump functions. We will now show that it is a surjection.

Let $\psi : C^\infty(M) \to \mathbb{R}$ be an arbitrary homomorphism of $\mathbb{R}$-algebras and notice that since the constant functions on M are always smooth (and ψ sends 1 to 1 by assumption) it follows that ψ is surjective. Therefore, the first isomorphism theorem tells us that if we denote $I := \ker \psi$ then

$$C^\infty(M)/I \cong \mathbb{R}.$$

Now, write

$$Z_I = \bigcap_{f \in I} \{x \in M \ : \ f(x) = 0\} = \bigcap_{f \in I} Z_f.$$

This should be reminiscent of the zero-sets appearing in the Zariski topology in algebraic geometry. If $Z_f = \varnothing$ for some $f \in I$ then f is invertible so $I = C^\infty(M)$ contradicting $C^\infty(M)/I \cong \mathbb{R}$. Hence each Z_f is a non-empty closed subset of M and, furthermore,

$$Z_f \cap Z_g = Z_{f^2+g^2}.$$

Thus $\{Z_f \ : f \in I\}$ is downwards-directed and has the finite intersection property. So, if Z_f were compact for some f then $\{Z_f \cap Z_g \ : \ g \in I\}$ would be a non-empty collection (the kernel of an algebra homomorphism always contains at least zero) of non-empty closed subsets of a compact set with the finite intersection property, and therefore would have non-empty intersection. Since its intersection is Z_I, it follows that $Z_I \neq \varnothing$ if there exists some $f \in I$ with Z_f compact. This is useful since points in Z_I would be candidates for our evaluation homomorphism.

So, let's demonstrate that such a f exists. If M were compact then we'd have our f since each Z_f would then be a closed subset of a compact set and would therefore be compact. So, let's suppose M was non-compact. By the Whitney embedding theorem we can embed M as a closed submanifold of $\mathbb{R}^N$ for some N. Since M is then a closed non-compact subset of $\mathbb{R}^N$ it is necessarily unbounded and so the function $(x^1)^2 + \cdots + (x^N)^2$ (where the x^i's are the standard coordinates on $\mathbb{R}^N$) is smooth and unbounded on the image of M. In fact, the above argument actually shows that it is unbounded on any closed non-compact subset of M.

Hence, after pre-composing with our embedding and adding the constant function at 1, we obtain a smooth function $\varphi \in C^\infty(M)$ with $\varphi > 0$ on M satisfying

$$\sup_{x \in C} \varphi(x) = \infty$$

for every $C \subseteq M$ closed and non-compact.

Now, I has codimension 1 in $C^\infty(M)$ so $\{\varphi, \varphi^2, \varphi^3, \cdots\}$ is linearly dependent modulo I (because φ is not the constant function at 1) therefore there exists a non-zero polynomial $P(x) \in \mathbb{R}[x]$ such that $f := P(\varphi) \in I$. We claim that Z_f is compact.

Suppose for contradiction that Z_f was non-compact. It is a closed subset of M so, by our construction involving φ, there exists a sequence $x_n \in Z_f$ with

$$\lim_{n \to \infty} \varphi(x_n) = \infty.$$

But, by definition, $P(\varphi(x_n)) = 0$ for all n contradicting the continuity of P as a function $\mathbb{R} \to \mathbb{R}$ (since P is non-zero). Thus Z_f must have been compact all along and, since $f \in I$ by construction, we have $Z_I \neq \varnothing$ via our previous reasoning.

Now, denote by $C^\infty(Z_I)$ the $\mathbb{R}$-algebra of all $\mathbb{R}$-valued functions on Z_I which are restrictions of smooth functions on M and let

$$r : C^\infty(M) \to C^\infty(Z_I)$$

be the restriction homomorphism (which is surjective by construction). Since $Z_I \neq \emptyset$ and smooth functions separate points on M it follows from $I \subseteq \ker(r)$ that $\dim_{\mathbb{R}} C^\infty(Z_I) \geq 1$ and therefore

$$1 \leq \dim_{\mathbb{R}} C^\infty(Z_I) = \dim_{\mathbb{R}}(C^\infty(M)/\ker(r)) \leq \dim_{\mathbb{R}}(C^\infty(M)/I) = \dim_{\mathbb{R}} \mathbb{R} = 1.$$

Thus $\dim_{\mathbb{R}} C^\infty(Z_I) = 1$ and so $Z_I = \{p\}$ is a point with $r = \mathrm{ev}_p$, the evaluation map at p. Furthermore, we now know that $I \subseteq \ker(\mathrm{ev}_p)$ and both have codimension one in $C^\infty(M)$. If the containment was proper then the image of $\ker(\mathrm{ev}_p)$ in the quotient $C^\infty(M)/I \cong \mathbb{R}$ would have to be all of $\mathbb{R}$ and hence further quotienting by this image would yield $C^\infty(M)/\ker(\mathrm{ev}_p) \cong \{0\}$, a contradiction. Thus $I = \ker(\mathrm{ev}_p)$.

Let's now return to our ψ. Since the isomorphism $C^\infty(M)/\ker(\mathrm{ev}_p) \cong \mathbb{R}$ is given by evaluation at p and the only $\mathbb{R}$-algebra homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity it follows from factoring ψ through $C^\infty(M)/I = C^\infty(M)/\ker(\mathrm{ev}_p)$ that $\psi = \mathrm{ev}_p$, as required. □

There is a somewhat common misconception regarding the above proposition. Namely, it is not true in general that the right hand side (the set of all $\mathbb{R}$-algebra homomorphisms $C^\infty(M) \to \mathbb{R}$) corresponds to the set of maximal ideals of $C^\infty(M)$. While each such $\mathbb{R}$-algebra homomorphism determines a maximal ideal, there are generally more of them. In fact, we have the following.

Corollary 2.1.2. *Let M be a smooth manifold. Then M is compact if and only if all of the maximal ideals of M have codimension one.*

Proof. First suppose that M is non-compact. We then let $I \trianglelefteq C^\infty(M)$ denote the ideal of compactly supported smooth functions and, by Zorn's lemma, we can find a maximal ideal $\mathfrak{m} \trianglelefteq C^\infty(M)$ containing I. $\mathfrak{m}$ cannot be the kernel of an evaluation homomorphism since compactly supported smooth functions separate points of M and so $\mathfrak{m}$ does not have codimension one in $C^\infty(M)$.

Now, suppose instead that M was compact. We know that each $\ker(\mathrm{ev}_p)$ for $p \in M$ is a maximal ideal of $C^\infty(M)$ so it suffices to prove that every proper ideal is contained in $\ker(\mathrm{ev}_p)$ for some $p \in M$. We will do this by demonstrating that any ideal not contained in $\ker(\mathrm{ev}_p)$ for every $p \in M$ must equal the entire ring.

So, let $I \trianglelefteq C^\infty(M)$ be such an ideal and apply the axiom of choice to pick, for each $p \in M$, a $f_p \in C^\infty(M)$ such that $f_p(p) \neq 0$. As such, the open sets $f_p^{-1}(\mathbb{R} \setminus \{0\}) \subseteq M$ as p ranges through M form an open cover of M and so we can take a finite subcover given by $f_1 = f_{p_1}, \cdots, f_n = f_{p_n}$ since M is compact. But then

$$f = f_1^2 + \cdots + f_n^2 \in I$$

is a strictly positive smooth function on M and hence a unit. Thus $I = C^\infty(M)$ as required. □

Corollary 2.1.3. *Let M, N be smooth manifolds. Then pre-composition by smooth functions yields a natural bijection*

$$C^\infty(M, N) \cong \mathrm{Hom}_{\mathbb{R}\text{-alg}}(C^\infty(N), C^\infty(M)).$$

Proof. First suppose that $f, g \in C^\infty(M, N)$ satisfy $\varphi \circ f = \varphi \circ g$ for all $\varphi \in C^\infty(N)$. Then for all $\varphi \in C^\infty(N)$ and all $p \in M$ we have $\varphi(f(p)) = \varphi(g(p))$ and, since $C^\infty(N)$ separates points of N, $f(p) = g(p)$ for all $p \in M$. Thus $f = g$ and so the map sending $f \in C^\infty(M, N)$ to the map $C^\infty(N) \to C^\infty(M)$ given by pre-composition by f is injective.

For surjectivity, arbitrarily select some $\mathbb{R}$-algebra homomorphism $\Phi : C^\infty(N) \to C^\infty(M)$. Now, for each $p \in M$ it follows that $\mathrm{ev}_p \circ \Phi$ is a $\mathbb{R}$-algebra homomorphism $C^\infty(N) \to \mathbb{R}$ and so there exists a unique $q \in N$ such that $\mathrm{ev}_p \circ \Phi = \mathrm{ev}_q$. Now, denote by $f : M \to N$ the map given by $p \mapsto q$ (i.e. find the corresponding q for each $p \in M$). Since $C^\infty(M)$ separates points of M and functions are completely determined by their value at each point it follows that Φ is given by pre-composition by f. But then f is smooth since $\varphi \circ f$ is smooth for each $\varphi \in C^\infty(N)$, as required. □

One should notice the similarity between the above result and the fact that in algebraic geometry, given two k-algebras R and S the collection of morphisms of schemes over k from $\mathrm{Spec}(R)$ to $\mathrm{Spec}(S)$ is in natural bijection with the collection of k-algebra homomorphisms $S \to R$. This will actually lead us to an alternate definition of smooth manifolds using locally ringed spaces.

Lemma 2.1.4. Hadamard's Lemma [56]
Let M be a smooth manifold, $p \in M$ and write $\mathfrak{m}_{M,p} := \ker(\mathrm{ev}_p)$ for the unique maximal ideal of the local ring $C^\infty_{M,p}$. If $(U, x^1, \cdots, x^n)$ is any chart for M centered at p then we have

$$\mathfrak{m}_{M,p} = (x^1, \cdots, x^n).$$

In particular, $\mathfrak{m}_{M,p}$ is finitely generated as an ideal in $C^\infty_{M,p}$.

Proof. Let $p \in M$ and $U, x^1, \cdots, x^n$ a chart for M centered at p. For any smooth function f defined on a neighbourhood of $p \in M$ we can write $f = \widetilde{f}(x^1, \cdots, x^n)$ for a smooth function $\widetilde{f}$ defined on a convex neighbourhood of the origin in $\mathbb{R}^n$. Since we only care about the behavious of $\widetilde{f}$ on arbitrarily small neighbourhoods of the origin we may assume, using a smooth bump function, that it is defined on all of $\mathbb{R}^n$.

Now, by Taylor's theorem there exists smooth functions $h_\alpha : \mathbb{R}^n \to \mathbb{R}$ which tend towards zero as they approach the origin such that

$$\widetilde{f}(x) = \widetilde{f}(0) + \sum_{i=1}^{n} \frac{\partial \widetilde{f}}{\partial e^i}\Big|_0 e^i + \sum_{|\alpha|=2} h_\alpha(x) e^\alpha$$

for all $x \in \mathbb{R}^n$ where e^i is the i'th standard coordinate function on $\mathbb{R}^n$. Pre-composing this with our chart then yields

$$f(x^1, \cdots, x^n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_p x^i + \sum_{|\alpha|=2} h_\alpha(x^1, \cdots, x^n) x^\alpha$$

in our coordinate chart. Hence, if $f(p) = 0$ (i.e. $f \in \mathfrak{m}_{M,p}$) then $f \in (x^1, \cdots, x^n)$, as required. $\square$

Proposition 2.1.5. *The functors $M \mapsto (M, C^\infty_M)$ and $(X, \mathcal{O}_X) \mapsto X$ yield an equivalence of categories between the category* Man *of smooth manifolds and the full subcategory of the category of locally ringed spaces over* $\mathrm{Spec}(\mathbb{R})$ *whose objects consist of those $(X, \mathcal{O}_X)$ for which X is Hausdorff second-countable and admits a cover by open subsets $U_i \subseteq X$ together with isomorphisms of locally ringed spaces*

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (V_i, C^\infty_{V_i})$$

where V_i is an open subset of some $\mathbb{R}^n$.

Proof. First, suppose we had a locally ringed space $(X, \mathcal{O}_X)$ for which X was Hausdorff, second-countable and $\mathcal{O}_X$ was locally isomorphic to smooth functions on an open subset of $\mathbb{R}^n$. These isomorphisms

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (V_i, C^\infty_{V_i})$$

give rise to homeomorphisms $U_i \cong V_i$. One then can show that this is a smooth manifold by applying Milnor's exercise to obtain that the transition functions on the overlaps are smooth. The fact that (M, C^∞_M) is a locally ringed space follows from the stalks of C^∞_M being the germs of smooth functions at points (by definition) and these are local rings with maximal ideal being the kernel of evaluation at that point. As such, smooth functions are sent to morphisms of locally ringed spaces by $M \mapsto (M, C^\infty_M)$ and Milnor's exercise together with the fact that C^∞_M is a fine sheaf tells us that this functor is fully faithful. Our above demonstration that $(X, \mathcal{O}_X) \mapsto X$ does indeed take values in smooth manifolds demonstrated essential surjectivity and thus these form an equivalence of categories. $\square$

Now, let's make precise my above statement that smooth manifolds are "affine" in the sense of algebraic geometry. The point is that the above proposition yields an alternate definition of smooth manifolds in terms of locally-ringed spaces over $\operatorname{Spec}(\mathbb{R})$, i.e. the structure sheaf is a sheaf of $\mathbb{R}$-algebras. In that sense, smooth manifolds are not affine since the "affine" smooth manifolds in the above sense would be the open subsets of $\mathbb{R}^n$ for some n. Notice how few of these local models there are compared to the case of algebraic geometry (the category of affine schemes is equivalent to the opposite category of the category of commutative unital rings). This is because the category $\mathbb{R}$-alg is not the correct receptacle for the structure sheaf of a smooth manifold. Let's see what we mean by this.

What is a commutative k-algebra where k is some field? We'll write either k-alg or Comm_k for the category of **commutative** unital k-algebras. Now, given $A \in k$-alg we obtain for each n a natural evaluation map:

$$\begin{aligned} A^n \times k[x_1, \cdots, x_n] &\to A \\ (a_1, \cdots, a_n, p(x_1, \cdots, x_n)) &\mapsto p(a_1, \cdots, a_n). \end{aligned}$$

One can interpret this as assigning to each polynomial map $k^n \to k^m$ a function

$$A^n \to A^m$$

and furthermore the association of this function to such a polynomial map is functorial. More precisely, if Poly_k is used to denote the symmetric monoidal category whose objects are k^n for each $n \geq 0$, whose morphisms are all polynomial maps and whose symmetric monoidal structure is the standard one given by Cartesian product $\times$, then a k-algebra A determines a (strict) monoidal functor

$$\begin{aligned} A : \mathrm{Poly}_k &\to \mathrm{Set} \\ k^n &\mapsto A^n. \end{aligned}$$

Furthermore, the underlying ring structure of A can be determined from this functor via the polynomial maps

$$\begin{aligned} + : k^2 &\to k \\ \cdot : k^2 &\to k \\ - : k &\to k \\ 0 : k^0 &\to k \\ 1 : k^0 &\to k. \end{aligned}$$

In this way, the field k simply becomes the forgetful functor

$$k : \mathrm{Poly}_k \to \mathrm{Set}.$$

Now, if $A : \mathrm{Poly}_k \to \mathrm{Set}$ denotes the functor associated to a k-algebra A then the morphism $k \to A$ (this is injective if and only if $0 \neq 1$ in A) induces a natural transformation of functors

$$k \Rightarrow A.$$

In fact, every k-algebra homomorphism $A \to B$ induces a natural transformation of functors $A \Rightarrow B$. The converse to this is encapsulated in the following proposition.

Proposition 2.1.6. *The category of finite-product-preserving functors* $\mathrm{Poly}_k \to \mathrm{Set}$ *is equivalent to the category* k*-alg via the functors sending a k-algebra to its above-defined functor* $\mathrm{Poly}_k \to \mathrm{Set}$*, and the functor sending a functor* $A(-) : \mathrm{Poly}_k \to \mathrm{Set}$ *to the k-algebra* $A(k)$ *together with the operations specified above.*

Proof. We have already shown how given a commutative unital k-algebra A we obtain such a functor. Conversely, given a monoidal functor $F : \mathrm{Poly}_k \to \mathrm{Set}$ we notice that functoriality together with the morphisms $+, \cdot, -, 0, 1$ in Poly_k discussed above endows $F(k)$ with the structure of a commutative unital ring.

The k-algebra structure is recovered by considering the polynomials $cx \in k[x]$ for each $c \in k$. These assemble to form a natural transformation

$$k \times F \Rightarrow F$$

which, by pre-composing with the functor $\mathrm{id}_{\mathrm{Poly}_k} \times 1$ where 1 is the functor $\mathrm{Poly}_k \to \mathrm{Set}$ assigning to each object in Poly_k the set $\{1\}$ and all morphisms the unique map $\{1\} \to \{1\}$, yields a natural transformation

$$k \Rightarrow F.$$

If we can show that natural transformations $F \Rightarrow G$ between such functors define ring homomorphisms at the level of $F(k) \to G(k)$ then we'll be done since it will follow that $k \to F(k)$ is a k-algebra structure on $F(k)$ and furthermore, $k \Rightarrow F$ will end up being initial hence all natural transformations will give rise to k-algebra homomorphisms. But the fact that natural transformations between these functors define ring homomorphisms follows from naturality, the fact that the functors are monoidal, and the existence of the polynomials $+, \cdot, -, 0, 1$. $\square$

Now, let's consider the $\mathbb{R}$-algebra $C^\infty(M)$ where M is some smooth manifold. As a ring, $C^\infty(M)$ is poorly behaved. First of all, it isn't an integral domain whenever M isn't a point due to the existence of smooth bump functions (it is, however, always reduced). This ends up not being too much of a problem since one often considers reduced but non-integral objects in algebraic geometry. The next two results illustrate ways in which the ring $C^\infty(M)$ can have stranger behaviour.

Proposition 2.1.7. *[56] Let M be a smooth manifold of positive dimension. Then $C^\infty(M)$ is non-Noetherian.*

Proof. Recall that the Krull intersection theorem states that if $(R, \mathfrak{m})$ is a local Noetherian ring then

$$\bigcap_{i=1}^{\infty} \mathfrak{m}^i = (0).$$

Now, suppose for contradiction that $C^\infty(M)$ was Noetherian. Arbitrarily select $p \in M$ and denote by $\mathfrak{m}_p^g$ the ideal of all smooth functions which vanish identically on an open neighbourhood of p. Then

$$C^\infty(M)/\mathfrak{m}_p^g$$

would be Noetherian and so, by the existence of local charts, $C_0^\infty(\mathbb{R}^n)$ would be Noetherian as well. Thus, by the Krull intersection theorem we have

$$\bigcap_{i=1}^{\infty} \mathfrak{m}^i_{\mathbb{R}^n,0} = (0).$$

This contradicts the fact that if we set $f(x) = e^{-1/|x|^2}$ then $f \in \mathfrak{m}^i_{\mathbb{R}^n,0}$ for all i but f does not vanish identically on any open neighbourhood of the origin, as required. $\square$

Theorem 2.1.8. Borel's Theorem
Let M be a smooth manifold, $\dim(M) = n > 0$, and $p \in M$. Define

$$\mathfrak{m}^\infty_{M,p} := \bigcap_{m=1}^{\infty} \mathfrak{m}^m_{M,p}.$$

Then taking Taylor expansions yields $\mathbb{R}$-algebra isomorphisms

$$C^\infty(M)/\mathfrak{m}^\infty_{M,p} \cong C^\infty_{M,p}/\mathfrak{m}^\infty_{M,p} \cong \mathbb{R}[\![x_1, \cdots, x_n]\!].$$

Proof. We will omit this proof, but it can be found on page 18 of Moerdijk and Reyes' book "Models for Smooth Infinitesimal Analysis" [56]. □

The point is that $C^\infty(M)$ is poorly behaved as a $\mathbb{R}$-algebra because we are able to do more than simply plug-in its elements to polynomials. In fact, elements of $C^\infty(M)$ can be inserted in a functorial way into general smooth functions on $\mathbb{R}^n$ to yield new elements of $C^\infty(M)$. Functional analysts might phrase this by saying that $C^\infty(M)$ admits a smooth multivariate functional calculus. Let's make this precise.

Definition 2.1.9. Let CartSp denote the symmetric monoidal category whose objects are $\mathbb{R}^n$ for all $n \geq 0$, whose monoidal product is the usual Cartesian product $\times$, and whose morphisms are all smooth functions $\mathbb{R}^n \to \mathbb{R}^m$ (the notation CartSp stands for "Cartesian spaces"). A C^∞**-ring** is a finite-product-preserving functor

$$A : \mathrm{CartSp} \to \mathrm{Set}\,.$$

A morphism of C^∞-rings is a natural transformation of functors and we write C^∞-Ring for the category of all C^∞-rings.

Now, given a smooth manifold M be obtain a C^∞-ring $C^\infty(M)$ by defining

$$C^\infty(M)(\mathbb{R}^n) := C^\infty(M, \mathbb{R}^n)$$

and sending smooth maps $\mathbb{R}^n \to \mathbb{R}^m$ to the functions $C^\infty(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^m)$ given by post-composition. We now have the following corollary of Milnor's exercise.

Corollary 2.1.10. *The functor*

$$\begin{aligned} \mathrm{Man}^{op} &\to \mathrm{C}^\infty\text{-Ring} \\ M &\mapsto C^\infty(M) \end{aligned}$$

is a full and faithful embedding.

Proof. Since morphisms of C^∞-rings give rise, in particular, to $\mathbb{R}$-algebra homomorphisms by pre-composing our functors with the inclusion

$$\mathrm{Poly}_{\mathbb{R}} \to \mathrm{CartSp}$$

it follows from Milnor's exercise that the functor $\mathrm{Man}^{op} \to \mathrm{C}^\infty$-Ring is fully faithful. □

What other objects are there in C^∞-Ring? Do they too have geometric significance? As it turns out, many of them do as we shall see. The key result is that if $A(-) \in \mathrm{C}^\infty$-Ring and $I \trianglelefteq A(\mathbb{R})$ is any ideal of the underlying $\mathbb{R}$-algebra then $A(\mathbb{R})/I$ comes naturally equipped with the structure of a C^∞-ring. While this is a nice result, it is worth asking whether there is a more general notion of an ideal of a C^∞-ring $A(-)$ and to what extent this differs from the ideals of $A(\mathbb{R})$. For that we need to study the notion of a module over a C^∞-ring.

Given a category $\mathcal{C}$ with finite products and a terminal object $1 \in \mathcal{C}$ there is a notion of an *abelian group object* in $\mathcal{C}$. This is an object $C \in \mathcal{C}$ together with morpisms

$$m : C \times C \to C, \quad i : C \to C, \;\text{ and } e : 1 \to C$$

satisfying the abelian group axioms. More generally, if $\mathcal{C}$ doesn't have finite products or a terminal object, one could define an abelian group object to be $C \in \mathcal{C}$ whose functor of points takes values in the category Ab of abelian groups. Morphisms in $\mathcal{C}$ between abelian group objects which preserve this additional structure form the morphisms of a category which we denote $\mathrm{Ab}(\mathcal{C})$. This generalizes the category of abelian groups in the following way.

Proposition 2.1.11. *Let k be a commutative ring and R a k-algebra. Then there is an equivalence of categories*

$$\mathrm{Ab}(k\text{-alg}\,/R) \cong R\text{-Mod}\,.$$

In the case $k = \mathbb{Z}$ we obtain the well-known equivalence

$$\mathrm{Ab} \cong \mathbb{Z}\text{-Mod}\,.$$

Proof. First let's define a functor $R\text{-Mod} \to \mathrm{Ab}(k\text{-alg}\,/R)$. Given a R-module M we can define the *square-zero extension* $R \ltimes M$ of R by M in the following manner. As a R-module, the square-zero extension is given by the direct sum

$$R \oplus M.$$

The multiplication is defined by

$$(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$$

and we should think of this as if M was a non-unital algebra with $M^2 = 0$ and $R \ltimes M$ was the product of algebras (in fact, this is precisely what this is). This k-algebra comes naturally equipped with a morphism $R \ltimes M \to R$ given by the projection. The morphisms make this into an abelian group object in $k\text{-alg}\,/R$ are given by the inclusion $e : R \to R \ltimes M$ as well as the following morphisms of k-algebras over R:

$$\begin{aligned} i : R \ltimes M &\to R \ltimes M \\ (r, m) &\mapsto (r, -m) \end{aligned}$$

and

$$\begin{aligned} m : R \ltimes (M \oplus M) &\to R \ltimes M \\ (r, (m_1, m_2)) &\mapsto (r, m_1 + m_2). \end{aligned}$$

Notice that for the second morphism we do indeed have that $R \ltimes (M \oplus M)$ is the product of $R \ltimes M$ with itself in the category $k\text{-alg}\,/R$. From the above definitions, we see that indeed $R \ltimes M \in \mathrm{Ab}(k\text{-alg}\,/R)$ and that morphisms of R-modules naturally give rise to morphisms of their corresponding square-zero extensions in $\mathrm{Ab}(k\text{-alg}\,/R)$.

On the other hand, suppose we were given an abelian group object

$$(A, p_A, e, i, m) \in \mathrm{Ab}(k\text{-alg}\,/R)$$

where $p_A : A \to R$ is the k-algebra morphism making A into a k-algebra over R. Then since e, i, m are all required to commute with p_A it follows that $\ker(p_A)$ is naturally endowed with the structure of a R-module and that morphisms of abelian group objects yield morphisms of these corresponding R-modules.

To see that these functors $M \mapsto R \ltimes M$ and $A \mapsto \ker(p_A)$ yield an equivalence of categories we first note that

$$\ker(M \ltimes R \to R) \cong M$$

naturally as R-modules. On the other hand, we have a natural isomorphism

$$A \cong R \oplus \ker(p_A)$$

coming from the fact that the morphism p_A pre-composes with e_A to yield

$$p_A \circ e_A = \mathrm{id}_R$$

The final thing to check is that the notions of "morphism" coincide naturally in these two categories. Morphisms in $\mathrm{Ab}(k\text{-alg}\,/R)$ do indeed yield morphisms of R-modules since the R-module structure on $\ker(p_A)$ comes from the abelian group-object structure on A. For the other direction, we simply note that a morphism $f : M_1 \to M_2$ yields

$$\begin{aligned} R \ltimes M_1 &\to R \ltimes M_2 \\ (r, m) &\mapsto (r, f(m)) \end{aligned}$$

which is a morphism in $\mathrm{Ab}(k\text{-alg}/R)$ since f is R-linear. □

The above result in fact has a geometric interpretation. Given a category $\mathcal{C}$ with finite limits, we will think of $\mathcal{C}$ as some sort of space whose points are the objects of $\mathcal{C}$ and whose morphisms are continuous/smooth paths between the points. If we let $\mathcal{I}$ denote the category with two objects $0, 1$ and one non-identity morphism $0 \to 1$ then the category of all paths in $\mathcal{C}$ is given by

$$\mathcal{C}^{\mathcal{I}} := \text{ the category of all functors } \mathcal{I} \to \mathcal{C}.$$

It's worth recalling our up-to-homotopy description of CW-complexes using simplicial sets at this point. Indeed, any small category $\mathcal{C}$ defines a simplicial set $N(\mathcal{C})$ called the **nerve** of $\mathcal{C}$ by declaring

$$N(\mathcal{C})([n]) := N(\mathcal{C})_n$$

to be the set of all length n composable sequences of morphisms in $\mathcal{C}$ and defining the face and degeneracy maps through composition of morphisms in $\mathcal{C}$ and the insertion of identity morphisms respectively. This simplicial set determines $\mathcal{C}$ up to equivalence and this is in fact the starting point of *infinity category theory*, where one of the definitions of an infinity category is a simplicial set satisfying a weakened version of the Kan property. Anyways, in this scenario our category of paths $\mathcal{C}^{\mathcal{I}}$ corresponds to $N(\mathcal{C})_1$.

Now, interpreting the category $\mathrm{Ab}(\mathcal{C})$ as a linearization of $\mathcal{C}$ we can perform the fibrewise linearization of the so-called *codomain fibration*

$$\begin{aligned} \mathcal{C}^{\mathcal{I}} &\to \mathcal{C} \\ (A \to B) &\mapsto B \end{aligned}$$

to obtain what is called the *tangent category* of $\mathcal{C}$. This is a category fibered over $\mathcal{C}$ whose fibre over an object $C \in \mathcal{C}$ is given by

$$\mathrm{Ab}(\mathcal{C}/C),$$

which we interpret as the category of "modules" over C, via our above result for algebras over a commutative ring. More precisely, $T_{\mathcal{C}}$ is the category whose objects are the objects in $\mathrm{Ab}(\mathcal{C}/C)$ as C ranges through all of $\mathcal{C}$, and whose morphisms

$$(A \to C) \to (B \to D)$$

are given by morphisms in $\mathcal{C}^{\mathcal{I}}$, say $f : C \to D$, $g : A \to B$, such that the induced diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & f^*B \arrow[dl] \\ C & \end{tikzcd}$$

is a morphism in $\mathrm{Ab}(\mathcal{C}/C)$ where f^*B is the pullback in $\mathcal{C}$.

The first thing worth noticing about the tangent category $T_{\mathcal{C}}$ in the case $\mathcal{C} = k\text{-alg}$, k a field, it that it has a "zero section" $k\text{-alg} \to T_{k\text{-alg}}$ given by associating to a k-algebra the zero module over that algebra. There is also a forgetful functor $T_{k\text{-alg}} \to k\text{-alg}^{\mathcal{I}}$. This forgetful functor has a left-adjoint whose specific form explains the name "tangent category".

Proposition 2.1.12. *The forgetful functor $U : T_{k\text{-alg}} \to k\text{-alg}^{\mathcal{I}}$ admits a left-adjoint Ω which, when restricted to a fibre $\mathrm{Ab}(k\text{-alg}/R)$ over any $R \in k\text{-alg}$ is given by*

$$\Omega_{R/k} : k\text{-alg}/R \to \mathrm{Ab}(k\text{-alg}/R) \cong R\text{-Mod}$$

and whose value on the k-algebra R is

$$\Omega_{R/k}(R) = \Omega_{R/k}, \ \textit{the module of Kähler differentials.}$$

Proof. First let's define define Ω on $k\text{-alg}/R$. Given a k-algebra over R, say $\rho : A \to R$, we let $\Omega_{R/k}(A)$ denote the quotient of the free R-module on the generators

$$da, \ a \in A$$

modulo the relations

$$d(sa+b) = sda + db \ \text{ for all } a, b \in A,\ s \in k$$

and

$$d(ab) = \rho(b)da + \rho(a)db \ \text{ for all } a, b \in A.$$

Now, let M be any R-module and write $R \ltimes M$ for its associated square-zero extension of R, interpreted as a k-algebra over R. Morphisms of k-algebras over R of te form

$$A \to R \ltimes M$$

correspond to k-linear maps $f : A \to M$ such that

$$f(ab) = \rho(b)f(a) + \rho(a)f(b).$$

Therefore such maps factor uniquely through module homomorphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"] & M \\ \Omega_{R/k}(A) \arrow[ur, dashed] & \end{tikzcd}$$

This yields our desired adjunction on the fiber

$$\mathrm{Hom}_{R\text{-Mod}}(\Omega_{R/k}(A), M) \cong \mathrm{Hom}_{k\text{-alg}/R}(A, R \ltimes M).$$

All that remains then is to consider the case of general morphisms in $k\text{-alg}^{\mathcal{I}}$ and $T_{k\text{-alg}}$. This is done in [69]. $\square$

In view of all of this, given a C^∞-ring $A(-)$ we define a **module** over $A(-)$ to be an object in the category of abelian group objects in $\mathrm{C}^\infty\text{-Ring}/A(-)$. i.e.

$$A(-)\text{-Mod} := \mathrm{Ab}(\mathrm{C}^\infty\text{-Ring}/A(-)).$$

In the case of general C^∞-rings, I feel that the next proposition justifies this as the "correct" definition. However, when using C^∞-rings to do differential geometry one typically works instead in the categories of sheaves on the opposite categories of finitely generated and *closed* or finitely generated and *germ-determined* C^∞-rings [56]. The first question to ask is then whether defining modules over these C^∞-rings by "Yonedaing" them up to the categories of sheaves and then using the tangent category construction there yields the same notion of module that we have here. For example, closed finitely generated C^∞-rings admit a natural Fréchet-space topology and I wonder whether one recovers Peter Michor's more general notion of module over a C^∞-ring [29] by looking at the tangent category of the category of sheaves over the opposite category of finitely generated closed C^∞-rings.

Also, for C^∞-rings we will only work fibre-wise with the tangent category for now. This is because we have not yet proven that $\mathrm{C}^\infty\text{-Ring}$ is complete (which it is [56]).

Proposition 2.1.13. *The forgetful functor*

$$T_{\mathrm{C}^\infty\text{-Ring}} \to \mathrm{C}^\infty\text{-Ring}^{\mathcal{I}}$$

admits a left-adjoint Ω^1 which, for the C^∞-ring $C^\infty(M,-)$ associated to a smooth manifold M yields the module of differential 1-forms:

$$\Omega^1_{C^\infty(M,-)}(C^\infty(M,-)) \cong \Omega^1(M) = \Gamma(M, T^*_M).$$

Proof. As mentioned above, we will restrict ourselves to working on the fibres here since we have not yet discussed how to take limits of C^∞-rings. Now, let A be a C^∞-ring and suppose that B was a C^∞-ring over A. We'll write $\eta : B \Rightarrow A$ for the corresponding natural transformation. We'll begin by defining $\Omega^1_A(B)$ as an $A(\mathbb{R})$-module (recall that $A(\mathbb{R})$ is, in particular, a $\mathbb{R}$-algebra). Let $\Omega^1_A(B)$ denote the quotient of the free $A(\mathbb{R})$-module generated by the symbols

$$db \;\text{ for } b \in B(\mathbb{R})$$

modulo the relations

$$d(f(b_1, \cdots, b_n)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\eta_{\mathbb{R}}(b_1), \cdots, \eta_{\mathbb{R}}(b_n))db_i$$

for all smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. It's worth noticing that we could have made the same definition in the algebraic case and we would have gotten the same answer: the Kähler differentials. However, as we'll see in the next proposition, the definition we used in the algebraic case would not work here. Also, by the naturality of the exterior derivative in differential geometry, we immediately obtain that this module is $\Omega^1(M)$ in the case of $A = B = C^\infty(M)$ and our morphism $B \to A$ is the identity.

We now claim that the square-zero extension $A(\mathbb{R}) \ltimes \Omega^1_A(B)$ naturally becomes the C^∞-rings over A that we want. Since we want our functors to be product-preserving it suffices to define our functor $F : \mathrm{CartSp} \to \mathrm{Set}$ with

$$F(\mathbb{R}) = A(\mathbb{R}) \ltimes \Omega^1_A(B)$$

by its action on morphisms of the form $f : \mathbb{R}^n \to \mathbb{R}$, f smooth. As such, we set

$$f((a_1, a_1' db_1), \cdots, (a_n, a_n' db_n)) := \left(f(a_1, \cdots, a_n), \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a_i') db_i \right).$$

Let's now check that this works. First notice that f yields the expected results when we have $f(x,y) = xy$, $f(x,y) = x+y$, $f(0) = 1$, $f(0) = 0$ and $f = \mathrm{id}_{\mathbb{R}}$. So all that remains is to check that our assignment preserves compositions of smooth functions. But if f is as above and $g : \mathbb{R} \to \mathbb{R}$ is smooth then

$$\sum_{i=1}^{n} \frac{\partial (g \circ f)}{\partial x^i}(a_i') db_i = g' \left(\frac{\partial f}{\partial x^i}(a_i') \right) db_i$$

as required. Now, let's write $\mathcal{A}$ for the category $\mathrm{Ab}(\mathrm{C}^\infty\text{-Ring}\,/\,A)$ and check that

$$\mathrm{Hom}_{\mathcal{A}}(A(\mathbb{R}) \ltimes \Omega^1_A(B), C) \cong \mathrm{Hom}_{\mathrm{C}^\infty\text{-Ring}\,/\,A}(B, C)$$

for every $C \in \mathrm{Ab}(\mathrm{C}^\infty\text{-Ring}\,/\,A)$. This is the same as in the case of Kähler differentials for algebras over a field only now derivations are replaced by C^∞-*derivations*, namely those derivations which differentiated the multivariate smooth functions used in the smooth functional calculus defining C^∞-rings. The point is that the map $d : B \to \Omega^1_A(B)$ is the universal C^∞-derivation for B as a C^∞-ring over A. □

There is a very important subtlety we should discuss here. This was the subject of a long discussion on MathOverflow and the n-category café which finally culminated in the following result due to David Speyer [63].

Proposition 2.1.14. *Let M be a smooth manifold. Then*

$$\begin{aligned}\mathrm{Hom}_{C^\infty(M)\text{-Mod}}(\Omega^1(M), C^\infty(M)) &\cong \Gamma(M, T_M)\\ &\cong \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))\\ &\cong \mathrm{Hom}_{C^\infty(M)\text{-Mod}}(\Omega_{C^\infty(M)/\mathbb{R}}, C^\infty(M)).\end{aligned}$$

Therefore, we have an isomorphism of $C^\infty(M)$-modules

$$\Omega^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathrm{Hom}_{C^\infty(M)}(\Omega_{C^\infty(M)/\mathbb{R}}, C^\infty(M)), C^\infty(M)).$$

Furthermore, the natural morphism

$$\Omega_{C^\infty(M)/\mathbb{R}} \to \mathrm{Hom}_{C^\infty(M)}(\mathrm{Hom}_{C^\infty(M)}(\Omega_{C^\infty(M)/\mathbb{R}}, C^\infty(M)), C^\infty(M))$$

*is surjective. But, if $\dim(M) > 0$ then it is **not** injective and so $\Omega_{C^\infty(M)/\mathbb{R}} \neq \Omega^1(M)$.*

In view of the above proposition, these next two results may seem surprising. As we are about to see, modules over $C^\infty(M, -)$ are really the same thing as modules over $C^\infty(M)$, so long as we use the definition involving the fiberwise abelianization of the tangent category. The point is that while the fibres of the codomain fibration

$$T_{\mathrm{C}^\infty\text{-Ring}} \to \mathrm{C}^\infty\text{-Ring}$$

end up being the same as the fibres of

$$T_{\mathbb{R}\text{-alg}} \to \mathbb{R}\text{-alg}$$

when restricted to objects in the image of the forgetful functor

$$\begin{aligned}\mathrm{C}^\infty\text{-Ring} &\to \mathbb{R}\text{-alg}\\ A(-) &\mapsto A(\mathbb{R}),\end{aligned}$$

the forgetful functor $T_{\mathrm{C}^\infty\text{-Ring}} \to \mathrm{C}^\infty\text{-Ring}^{\mathcal{I}}$ is different than the corresponding forgetful functor for $\mathbb{R}$-algebras. In fact, the below proposition can be interpreted as some sort of "global" version of the fact that vector fields on a manifold are "algebraic" derivations of the ring $C^\infty(M)$ over $\mathbb{R}$.

Proposition 2.1.15. *The forgetful functor $\mathrm{C}^\infty\text{-Ring} \to \mathbb{R}\text{-alg}$ gives rise to an equivalence of categories*

$$A(-)\text{-Mod} := \mathrm{Ab}(\mathrm{C}^\infty\text{-Ring}\,/\,A(-)) \cong \mathrm{Ab}(\mathbb{R}\text{-alg}\,/\,A(\mathbb{R})) \cong A(\mathbb{R})\text{-Mod}$$

for each C^∞-ring $A(-)$. In particular, $A(-)\text{-Mod}$ is an abelian category.

Proof. Given a $A(\mathbb{R})$-module M we endow the square-zero extension $A(\mathbb{R}) \oplus M$ with the structure of a C^∞-ring over A just as we did with $M = \Omega^1_A(B)$ in the proof of the existence of the adjoint functor Ω^1. This is a functorial assignment and defines the above equivalence of categories. $\square$

Corollary 2.1.16. *[56] Given any C^∞-ring $A(-)$ and any ideal $I \trianglelefteq A(\mathbb{R})$ the $\mathbb{R}$-algebra $A(\mathbb{R})/I$ comes naturally equipped with the structure of a C^∞-ring and an epimorphism*

$$A \to A(\mathbb{R})/I$$

of C^∞-rings yielding a short exact sequence in $A(-)$-Mod.

Proof. Let A be a C^∞-ring and $I \trianglelefteq A(\mathbb{R})$ an ideal. It suffices to show that if $a_i, b_i \in A(\mathbb{R})$ satisfy $a_i + I = b_i + I$ for all $i = 1, \cdots, n$ then

$$f(a_1, \cdots, a_n) = f(b_1, \cdots, b_n) \mod I$$

for all $f \in C^\infty(\mathbb{R}^n)$ (and all n). This follows from Hadamard's lemma. $\square$

It's worth mentioning that while defining an ideal of a C^∞-ring A to simply be an A-submodule of A sounds reasonable, there remain some problems with this definition. For example, we'll see that the notion of being "finitely generated" gets fixed when one interprets $C^\infty(M)$ as a C^∞-ring as opposed to merely a $\mathbb{R}$-algebra but one cannot say the same for Noetherianity. There is a recent preprint [5] which might address this (in fact, it address other important related problems). It's also worth mentioning that as far as I know there is no analog Krull dimension or transendance degree that works well for C^∞-rings. Intuitively, if we wanted to define a reasonable notion of dimension for C^∞-rings it should take values in $\mathbb{R}_{\geq 0}$ not $\mathbb{Z}_{\geq 0}$ and assign the Hausdorff dimension of a closed subset $Z \subseteq \mathbb{R}^n$ to $C^\infty(Z)$ (the algebra of germs of smooth functions along Z). Perhaps looking at the Gelfand-Kirillov dimension of the algebra of differential operators corresponding to a C^∞-ring might work?

Let's now give a useful example of C^∞-rings which are not simply $C^\infty(M)$ for M a smooth manifold.

Example 2.1.17. The $\mathbb{R}$-algebras $\mathbb{R}[\![x_1, \cdots, x_n]\!]$ are all C^∞-rings. As such, if $I \trianglelefteq \mathbb{R}[\![x_1, \cdots, x_n]\!]$ is any ideal containing some power of the maximal ideal $(x_1, \cdots, x_n)$ then

$$\mathbb{R}[\![x_1, \cdots, x_n]\!]/I$$

is a local C^∞-ring. Local C^∞-rings of the above form are called **Weil algebras**.

A good example of how Weil algebras are used is the following.

Proposition 2.1.18. *For any smooth manifold M there is a natural bijection*

$$TM \cong \mathrm{Hom}_{\mathbb{R}\text{-alg}}(C^\infty(M), \mathbb{R}[\epsilon]/(\epsilon^2)).$$

Proof. First recall that TM consists of derivations (over $\mathbb{R}$) from $C^\infty_{M,p} \to \mathbb{R}$ for all $p \in M$. Each such derivation $X_p : C^\infty_{M,p} \to \mathbb{R}$ defines a $\mathbb{R}$-algebra homomorphism $C^\infty(M) \to \mathbb{R}[\epsilon]/(\epsilon^2)$ via

$$f \mapsto f(p) + X_p(f_p)\epsilon$$

where f_p denotes the germ of f at p. Conversely, every $\mathbb{R}$-algebra homomorphism $\varphi : C^\infty(M) \to \mathbb{R}[\epsilon]/(\epsilon^2)$ gives rise, by post composition with the quotient map, to a $\mathbb{R}$-algebra homomorphism $C^\infty(M) \to \mathbb{R}$ which is given by ev_p for some unique $p \in M$ by Milnor's exercise. Writing $\varphi = \mathrm{ev}_p + X_p\epsilon$ where $X_p := \varphi - \mathrm{ev}_p$ is a $\mathbb{R}$-linear map $C^\infty(M) \to \mathbb{R}$ we obtain that X_p is a derivation of algebras over $\mathbb{R}$ through ev_p by comparing $\varphi(fg)$ and $\varphi(f)\varphi(g)$ using that $\epsilon^2 = 0$. The homomorphism φ then factors through $C^\infty_{M,p}$ by the universal property of localization since ev_p sends elements of $C^\infty(M) \setminus \ker(\mathrm{ev}_p)$ to units in $\mathbb{R}$ and $C^\infty_{M,p}$ is naturally isomorphic to the localization of $C^\infty(M)$ away from the ideal $\ker(\mathrm{ev}_p)$. $\square$

One might be concerned that homomorphisms of $\mathbb{R}$-algebras were used above as opposed to homomorphisms of C^∞-rings. As the next result demonstrates, in the cases of quotients of rings of formal power series it is usually alright to work with $\mathbb{R}$-algebra homomorphisms.

Proposition 2.1.19. *[56] Let A be a C^∞-ring and B a quotient of $\mathbb{R}[\![x_1, \cdots, x_n]\!]$ for some $n \geq 0$. Then the forgetful functor $\mathrm{C^\infty\text{-}Ring} \to \mathbb{R}\text{-alg}$ induces a natural bijection*

$$\mathrm{Hom}_{\mathrm{C^\infty\text{-}Ring}}(A, B) \cong \mathrm{Hom}_{\mathbb{R}\text{-alg}}(A, B).$$

Proposition 2.1.20. *[56] Let A be any C^∞-ring and B a Weil algebra. Then the forgetful functor $\mathrm{C^\infty\text{-}Ring} \to \mathbb{R}\text{-alg}$ induces a natural bijection*

$$\mathrm{Hom}_{\mathrm{C^\infty\text{-}Ring}}(B, A) \cong \mathrm{Hom}_{\mathbb{R}\text{-alg}}(B, A).$$

Our next goal is a characterization of those C^∞-rings which are of the form $C^\infty(M)$ for some smooth manifold M. We also want to understand, to some extent, what other types of C^∞-rings are there? Are all of them useful? Is there a nicely characterized full subcategory of $\mathrm{C^\infty\text{-}Ring}$ consisting of all of the C^∞-rings we actually care about? The first step towards answering these questions is to formulate and prove a C^∞-version of the Nullstellensatz.

Let's recall a simple version of the Nullstellensatz from algebraic geometry. It says that if k is an algebraically closed field then the map taking radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$ to their common zero sets $Z(I) \subseteq k^n$ and the map taking a Zariski-closed subset of k^n to the ideal of all polynomial functions vanishing on it yield an inclusion-reversing bijection

$$(\text{Zariski-closed subsets of } k^n) \longleftrightarrow (\text{radical ideals in } k[x_1, \ldots, x_n]).$$

In other words, the Nullstellensatz says that finitely generated reduced k-algebras are precisely the k-algebras of polynomial functions on Zariski-closed subsets of k^n for some $n \geq 0$ (so long as k is algebraically closed). So, if we hope to have a C^∞-ring version of this then we'll first need to describe the notion of a finitely generated C^∞-ring.

Proposition 2.1.21. *[56] The forgetful functor*

$$\begin{aligned} \mathrm{C^\infty\text{-}Ring} &\to \mathrm{Set} \\ A &\mapsto A(\mathbb{R}) \end{aligned}$$

has a left-adjoint $F : \mathrm{Set} \to \mathrm{C^\infty\text{-}Ring}$ which sends a set X to the C^∞-ring of all functions $\mathbb{R}^X \to \mathbb{R}$ which depend on only finitely many variables and are smooth in those variables.

In particular, the above proposition implies that $C^\infty(\mathbb{R}^n)$ is the free C^∞-ring on n generators. We should probably note that the construction of the free C^∞-ring $F(X)$ associated to $X \in \mathrm{Set}$ can be described, at least as a set, as the directed colimit

$$F(X) = \varinjlim_{Y \subseteq X, |Y| < \infty} C^\infty(R^Y).$$

This is due to the more general phenomenon described in the next proposition.

Proposition 2.1.22. *[56] The forgetful functor $\mathrm{C^\infty\text{-}Ring} \to \mathrm{Set}$ preserves directed colimits (it also preserves all limits since it has a left adjoint).*

An important observation regarding the above proposition is that coproducts are not *directed* colimits! Indeed, recall that for two k-algebras A, B, their coproduct in the category $k\text{-alg}$ is given by the tensor product

$$A \amalg_{k\text{-alg}} B = A \otimes_k B$$

over k. The underlying set of this algebra is not the same as the disjoint union

$$A \amalg_{\mathrm{Set}} B \neq A \otimes_k B.$$

Thus one should not expect the forgetful functor C^∞-Ring $\to$ Set to preserve coproducts. However, one could still ask whether the forgetful functor C^∞-Ring $\to$ $\mathbb{R}$-alg preserves coproducts. As we are about to see, it does not in general.

Proposition 2.1.23. *Let M, N be smooth manifolds. Then the coproduct of $C^\infty(M)$ and $C^\infty(N)$ in* C^∞-Ring *exists and is given by*

$$C^\infty(M) \otimes_\infty C^\infty(N) := C^\infty(M) \amalg_{\mathrm{C}^\infty\text{-Ring}} C^\infty(N) = C^\infty(M \times N)$$

where the morpshisms $C^\infty(M) \to C^\infty(M \times N)$ and $C^\infty(N) \to C^\infty(M \times N)$ are given by pre-composition with the projections $M \times N \to M$ and $M \times N \to N$ respectively.

It's worth mentioning that one can also obtain $C^\infty(M \times N)$ from $C^\infty(M)$ and $C^\infty(N)$ as a completed tensor product. Namely, these algebras are naturally nuclear Fréchet spaces so all natural choices of topologies on the tensor product $C^\infty(M) \otimes_{\mathbb{R}} C^\infty(N)$ yield isomorphic completed tensor products

$$C^\infty(M) \overline{\otimes} C^\infty(N) \cong C^\infty(M \times N).$$

More details can be found in [67]. Returning to C^∞-rings, we have the following more general result.

Proposition 2.1.24. *[56] The category* C^∞-Ring *is complete, admits all directed colimits and admits all finite coproducts. In fact, it is cocomplete although not all colimits are computed via their underlying sets (or even their underlying $\mathbb{R}$-algebras).*

Now, let's return to our C^∞-Nullstellensatz.

Definition 2.1.25. A C^∞-ring is called **finitely generated** if and only if it is isomorphic to a quotient of a free C^∞-ring on finitely many generators. The full subcategory of C^∞-Ring consisting of finitely generated C^∞-rings is denoted $\mathbb{L}$. If this quotient can be taken to be by a finitely generated ideal then we call the C^∞-ring **finitely presented**.

One might ask why we used the above definition as opposed to defining finitely generated C^∞-rings to be those $A \in \mathrm{C}^\infty$-Ring for which an epimorphism of the form

$$C^\infty(\mathbb{R}^n) \to A$$

exists for some n. The following proposition, which is really the first isomorphism theorem for C^∞-rings, shows that these two definitions are equivalent.

Proposition 2.1.26. *If $f : A \to B$ is an epimorphism in* C^∞-Ring *then it induces an isomorphism $A/\ker(f) \cong B$ in* C^∞-Ring.

Proof. As f is an epimorphism in C^∞-Ring it is also an epimorphism in $\mathrm{Comm}_{\mathbb{R}}$ and so we have $A/\ker(f) \to B$ is a natural isomorphism of $\mathbb{R}$-algebras. We have previously shown that it is a morphism of C^∞-rings and, furthermore, the inverse $B \to A/\ker(f)$ in $\mathrm{Comm}_{\mathbb{R}}$ given by sending $b \in B$ to any representative of any element of its preimage in $A/\ker(f)$ under $A/\ker(f) \to B$ is also a morphism of C^∞-rings by Hadamard's lemma. $\square$

Now, suppose we had a finitely generated C^∞-ring $C^\infty(\mathbb{R}^n)/I$. As usual, we will denote

$$Z(I) := \{x \in \mathbb{R}^n \ : \ f(x) = 0 \text{ for all } f \in I\}.$$

The notion of a radical ideal is no longer appropriate for the purposes of formulating a C^∞-version of the Nullstellensatz since it is possible for smooth functions to vanish to all orders (i.e. have vanishing Taylor series) at a point without vanishing identically on any neighbourhood of that point. So, we instead considering the following construction (which appears to be due to Joyce [27]).

Definition 2.1.27. Let A be a C^∞-ring. We define the topological space $\mathrm{C^\infty Spec}(A)$ to be the set of real codimension one ideals of A together with the weakest topology so that the elements of A define continuous functions $\mathrm{C^\infty Spec}(A) \to \mathbb{R}$.

It's worth mentioning the functoriality of $\mathrm{C^\infty Spec}(-)$. In algebraic geometry, one is forced to take all prime ideals of a ring instead of merely the maximal ideals since the preimage of a maximal ideal under a ring homomorphism need not be maximal (for example, consider $\mathbb{Z} \hookrightarrow \mathbb{Q}$). On the other hand, for $\mathrm{C^\infty Spec}(-)$ one can notice that if $A \to B$ is a morphism of C^∞-rings and $\mathfrak{m} \trianglelefteq B$ a codimension one ideal then the kernel of the composition

$$A \to B \to B/\mathfrak{m}$$

is again a codimension one ideal since $B/\mathfrak{m} \cong \mathbb{R}$ and the morphisms of the underlying $\mathbb{R}$-algebras are required to take 1 to 1. So this gives us a function

$$\mathrm{C^\infty Spec}(B) \to \mathrm{C^\infty Spec}(A)$$

which is indeed continuous since we're using the weak topology. Hence we have a functor

$$\mathrm{C^\infty Spec}(-) : \mathrm{C^\infty\text{-}Ring}^{op} \to \mathrm{Top}\,.$$

One can actually promote this to a functor taking values in locally C^∞-ringed spaces but we won't need this here.

Proposition 2.1.28. *Let M be a smooth manifold. Then there is a natural homeomorphism*

$$M \cong \mathrm{C^\infty Spec}(C^\infty(M))$$

of topological spaces.

Proof. We know from Milnor's exercise that there is a natural bijection $M \cong \mathrm{C^\infty Spec}(C^\infty(M))$ (in fact, we even know precisely what this bijection is). To show that this is a homeomorphism we need to demonstrate that the topology on M is precisely the weak topology generated by the functions in $C^\infty(M)$. First of all, since all closed subsets of M can be written as the common zero sets of finitely many smooth functions (due to the existence of partitions of unity and smooth bump functions) it follows that the weak topology contains the usual topology. But then we have equality since smooth functions are all continuous, as required. $\square$

In particular, we can see that the topology on $\mathrm{C^\infty Spec}(A)$ is Hausdorff when A is the algebra of functions on a smooth manifold. If our smooth Nullstellensatz is going to characterize the C^∞-rings of smooth functions defined on open neighbourhoods of closed subsets of the $\mathbb{R}^n$'s then our replacement for the notion of a radical ideal should be the ideals I such that $\mathrm{C^\infty Spec}(C^\infty(\mathbb{R}^n)/I)$ is Hausdorff.

Proposition 2.1.29. *Let A be a C^∞-ring. Then $\mathrm{C^\infty Spec}(A)$ is Hausdorff if and only if*

$$\bigcap_{\mathfrak{m} \in \mathrm{C^\infty Spec}(A)} \mathfrak{m} = (0) \trianglelefteq A.$$

*Such C^∞-rings are called **point determined**.*

We are now prepared to state our promised smooth Nullstellensatz (after the following lemma).

Lemma 2.1.30. *[56] Let $I \trianglelefteq C^\infty(\mathbb{R}^n)$ be an ideal. Then $C^\infty(\mathbb{R}^n)/I$ is point-determined if and only if we have*

$$f|_{Z(I)} = 0 \implies f \in I.$$

Theorem 2.1.31. The C^∞-Nullstellensatz [56]
Let $\mathbb{E}$ denote the category of closed subsets of $\mathbb{R}^n$ for any n whose morphisms are smooth maps defined on open neighbourhoods of the closed set in question, modulo agreeing on possibly smaller open neighbourhoods. Write $\mathbb{L}_{pd}$ for the category of point-determined finitely generated C^∞-rings. Then $\mathrm{C}^\infty\mathrm{Spec}$ *extends to a functor*

$$\mathrm{C}^\infty\mathrm{Spec} : \mathbb{L}_{pd}^{op} \to \mathbb{E}.$$

Together with the functor

$$\begin{aligned} \mathbb{E} &\to \mathbb{L}_{pd}^{op} \\ Z &\mapsto C^\infty(Z, \mathbb{R}), \end{aligned}$$

$\mathrm{C}^\infty\mathrm{Spec}$ *yields an equivalence of categories* $\mathbb{E} \cong \mathbb{L}_{pd}^{op}$.

It's worth mentioning that there are alternative versions of the above result using "germ-determined" C^∞-rings instead of point-determined ones. These lead to natural generalizations of smooth manifolds which can be used to produce well-behaved categories in which one can perform cobordism naturally without perturbing things to be tranverse [6].

We are almost prepared to state and prove the characterization of which C^∞-rings come from smooth manifolds. To do this, however, we need to develop a notion of localization for C^∞-rings. We'll see that the localization of the underlying algebra does indeed yield a C^∞-ring, however it is still unknown to me whether there is a more general notion of localization for C^∞-rings, or if any reasonable notion of localization for C^∞-rings simply corresponds to localizing the underlying algebra.

Proposition 2.1.32. *[56] Let A be a C^∞-ring and $S \subseteq A(\mathbb{R})$ a multiplicative system. Then the localization of the underlying $\mathbb{R}$-algebra $S^{-1}A(\mathbb{R})$ comes naturally equipped with the structure of a C^∞-ring in such a way that the localization map*

$$A(\mathbb{R}) \to S^{-1}A(\mathbb{R})$$

extends to a morphism of C^∞-rings

$$A \to S^{-1}A$$

such that if $A \to B$ is any morphism of C^∞-rings such that the induced $\mathbb{R}$-algebra map $A(\mathbb{R}) \to B(\mathbb{R})$ takes elements of S to units in $B(\mathbb{R})$, then there exists a unique morphism of C^∞-rings $S^{-1}A \to B$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ S^{-1}A \arrow[ur] & \end{tikzcd}$$

Proof. This proof proceeds as follows. Knowing, from commutative algebra, which properties we would hope any reasonable notion of localization to have, we reduce this problem to a simple case where we can define localization explicitly. For more general C^∞-rings, we define localization by asserting that these properties of localization from commutative algebra still hold. Indeed, let's do this now.

For general S we can write the localization as a directed colimit over all finite subsets of S, reducing to the case of S finite. But then we should also have

$$\{a, b\}^{-1}A \cong \{ab\}^{-1}A$$

and so we may assume that $S = \{a\}$ is a singleton. Writing A as a directed colimit of its finitely generated sub-C^∞-rings we can also assume that A is finitely generated:

$$A \cong C^\infty(\mathbb{R}^n)/I.$$

But then, since we'll require localization to be exact, we've reduced the problem to the case where $A = C^\infty(\mathbb{R}^n)$ and $S = \{f\}$ is some smooth function. But now we can simply define

$$\{f\}^{-1}C^\infty(\mathbb{R}^n) := C^\infty(f^{-1}(\mathbb{R} \setminus \{0\}))$$

and this will work. $\square$

Proposition 2.1.33. *Let M be a smooth manifold and $p \in M$. Then the localization of $C^\infty(M)$ outside of the maximal ideal $\mathfrak{m}_{M,p} := \ker(\mathrm{ev}_p)$ is naturally isomorphic to the C^∞-ring $C^\infty_{M,p}$ of germs, which is in turn isomorphic to $C^\infty_{\mathbb{R}^n,0}$ where $n = \dim(M)$.*

Proof. By the universal property of localization we get a natural map

$$C^\infty(M)_{\mathfrak{m}_{M,p}} \to C^\infty_{M,p}$$

since any smooth function not vanishing at p also doesn't vanish on a neighbourhood of p by continuity and is thus invertible in $C^\infty_{M,p}$. The inverse to this map is a map

$$C^\infty_{M,p} \to C^\infty(M)_{\mathfrak{m}_{M,p}}$$

obtained from the universal property of colimits and the existence of smooth bump functions. $\square$

Lemma 2.1.34. *Let M be a smooth manifold. Then $C^\infty(M)$ is a finitely presented C^∞-ring.*

Proof. This follows immediately from the Whitney embedding theorem. $\square$

Lemma 2.1.35. *[56] Every C^∞-ring which is finite dimensional as a real vector space and admits at most one morphism leaving it into $\mathbb{R}$ is isomorphic to a quotient of a C^∞-ring of the form $C^\infty_{\mathbb{R}^n,0}$ for some $n \geq 0$.*

Theorem 2.1.36. Michor–Vanžura [53]
A C^∞-ring A is isomorphic to $C^\infty(M)$ for some smooth manifold M if and only if the following hold:

1. *A is finitely generated;*
2. *A is point-determined;*
3. *for each $\mathfrak{m} \in C^\infty\mathrm{Spec}(A)$ there exists $f \in A \setminus \mathfrak{m}$ such that the localization A_f is free.*

Proof. One direction is clear since $C^\infty(M)$ does indeed satisfy all three of the above properties (for the third one, take a chart and invert a smooth bump function centered at the desired point with support in that chart). For the other direction, we use the C^∞-Nullstellensatz to obtain $A \cong C^\infty(Z(I))$ for some point-determined ideal $I \trianglelefteq C^\infty(\mathbb{R}^n)$. The third assumption we made above then tells us that $Z(I)$ is in fact a smooth embedded submanifold of $\mathbb{R}^n$, as required. $\square$

We now end this section with a brief discussion of vector bundles. First, we recall the following important theorem.

Theorem 2.1.37. The Smooth Serre–Swan Theorem [64, 57]
Let M be a smooth manifold with finitely many connected components. Then the functor taking vector bundles to their modules of global sections yields an equivalence of closed symmetric monoidal categories between the category of vector bundles on M and the category of projective modules of finite rank over $C^\infty(M)$.

Proof. We present an outline of the proof in [57]. The first step is to show that morphisms

$$\Gamma(M, E_M) \to \Gamma(M, F_M)$$

of $C^\infty(M)$-modules are the same thing as morphisms $E_M \to F_M$ of vector bundles. This is done by reducing to the case of trivial bundles via a partition of unity and bump function argument.

One then demonstrates that free modules over $C^\infty(M)$ of finite rank do indeed correspond to trivial vector bundles. This is easy. Next, one of the trickier parts, one proves that the modules of sections of vector bundles are finitely generated. This uses the fact that every connected smooth manifold has a finite atlas (not necessarily consisting of connected charts) mentioned in [52] and that we're assuming our manifold has finitely many connected components.

To get that $\Gamma(M, E_M)$ is projective we first use that it is finitely generated to obtain an epimorphism

$$C^\infty(M)^{\oplus k} \to \Gamma(M, E_M)$$

and then use that since the dimension of the kernel of the associated morphism of bundles is locally constant on M (because both the dimension of the cokernel and the rank of E_M are and so we can use the rank-nullity theorem) it follows that the kernel is in fact a subbundle. By choosing a smooth fiber metric on our trivial bundle (these always exist via partitions of unity but we don't need this since we have a trivial bundle) it then follows that this kernel is in fact a summand of $C^\infty(M)^{\oplus k}$. Let F_M then denote a choice of complement so that

$$M \times \mathbb{R}^k \cong \ker \oplus F.$$

The final step is to then show that $F_M \cong E_M$ and thus $\Gamma(M, E_M)$ is (isomorphic to) a direct summand of the free module $C^\infty(M)^{\oplus k}$ and is hence projective of finite rank. This concludes the difficult part of the proof. The fact that the global sections functor preserves the tensor product of internal hom of vector bundles (as well as the direct sum actually) is routine. □

From a (co)homological perspective this says that projective modules over general C^∞-rings are a good replacement for the notion of a vector bundle. Indeed, as a consequence of the above theorem we have obtained a proof that every bounded above exact sequence of smooth vector bundles is split-exact. Hence such complexes are very amenable to the application of additive functors.

As a final remark we mention that the forgetful functor

$$A / \mathrm{C}^\infty\text{-Ring} \to A\text{-Mod}$$

from C^∞-A-algebras to modules over the C^∞-ring A has a left adjoint

$$\mathrm{C}^\infty\mathrm{Sym}_A : A\text{-Mod} \to A / \mathrm{C}^\infty\text{-Ring}$$

analogous to the functor Sym_A for A-algebras (where A is just a $\mathbb{R}$-algebra). This is constructed for $M \in A\text{-Mod}$ by first taking the free A-algebra $\mathrm{Sym}_A(M)$ and then letting $\mathrm{C}^\infty\mathrm{Sym}_A(M)$ be the free C^∞-ring on the underlying $\mathbb{R}$-algebra of $\mathrm{Sym}_A(M)$.

2.2 Cofiber and Fiber Sequences

One of the main technical tools for performing computations in homological algebra is the long exact sequence in cohomology associated to any short exact sequence in $\mathrm{Ch}(\mathcal{A})$ for $\mathcal{A}$ an arbitrary abelian category. Similarly, in algebraic topology one has homotopy fiber and cofiber sequences associated to fibrations and cofibrations.

Other important computational results in algebraic topology are the van Kampen theorem for computing the fundamental group/groupoid, the higher van Kampen theorem, and the van Kampen spectral sequence due to Artin and Mazur [2, 49]. One also has (co)homological versions of these given by the Mayer-Vietoris theorem and the fact that Čech cohomology agrees with singular cohomology on spaces homotopy equivalent to CW-complexes [7].

The above results in algebraic topology also apply to differential geometry via de Rham's theorem and even to algebraic geometry since Čech cohomology and sheaf cohomology agree for quasi-coherent sheaves with respect to the Zariski topology on a Noetherian separated scheme [22].

What do all of these theorems have in common? They all express how one can compute, up to weak equivalence, derived functors of sufficiently nice limits and colimits in a model category. More precisely, they can all be viewed as explicit realizations of the following fact:

For any pointed model category $\mathcal{C}$, $h\mathcal{C}$ (the category obtained from $\mathcal{C}$ be localizing at all weak equivalences) is naturally a closed $hs\,$Set-module.

Our goal is to provide an explicit description of how this can be applied to C^∞-rings and hence to differential geometry.

Let's start with some examples. First, recall the category $\mathcal{WC}_*$. Classically, in algebraic topology, one declared a continuous map $f : X \to Y$ to be a fibration or cofibration (these are closely related to the model category theoretic definitions, but slightly less general) if and only if it respectively satisfied one of the following lifting properties:

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "(\mathrm{id}_Z,\{1\})"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z^I \arrow[d, "\mathrm{ev}_1"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists"] & Z \end{tikzcd}$$

Let's see how we can associate "long exact sequences" to fibrations and cofibrations in $\mathcal{WC}_*$. For this, we'll need the following constructions. First, equip both $I, S^1 \in \mathcal{WC}_*$ with basepoint 1. For $(X, x_0), (Y, y_0) \in \mathcal{WC}_*$ we then define:

1. $X \vee Y := \{(x,y) \in X \times Y \ : \ x = x_0 \text{ or } y = y_0\}$ with the subspace topology and basepoint (x_0, y_0),
2. the **smash product** $X \wedge Y := X \times Y / X \vee Y$ with the quotient topology and basepoint the equivalence class of anything in $X \vee Y$,
3. the **reduced cone** $CX := X \wedge I$ together with the morphism $i : X \to CX$ given by $x \mapsto [x, 0]$,
4. the **reduced suspension** $\Sigma X := X \wedge S^1$,
5. the **reduced path space** $PX := \mathrm{Hom}_{\mathcal{WC}_*}(I, X)$ together with $\mathrm{ev}_0 : PX \to X$,
6. the **reduced loop space** $\Omega X := \mathrm{Hom}_{\mathcal{WC}_*}(S^1, X)$.

The reduced path and loop spaces are then used to construct our so-called fiber sequences associated to a based fibration. Dually, one uses the reduced cone and suspensions to construct cofiber sequences associated to cofibrations. Indeed, suppose we had an arbitrary morphism $f : X \to Y$ in $\mathcal{WC}_*$. We can then respectively define the **homotopy fiber** and **homotopy cofiber** of f to be:

$$Ff := X \times_{f,Y,\mathrm{ev}_0} PY \ \text{ and } \ Cf := Y \amalg_{f,X,i} CX.$$

The following inclusion and projection

$$\Omega Y \to Ff \ \text{ and } \ Cf \to Cf/Y \cong \Sigma X$$

then induce the following sequences of morphisms in $\mathcal{WC}_*$:

$$\cdots \to \Omega Ff \to \Omega X \to \Omega Y \to Ff \to X \to Y \text{ and}$$
$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to \Sigma Cf \to \cdots$$

In May's book [49], the following proposition is then proved.

Proposition 2.2.1. *Given any $Z \in \mathcal{WC}_*$, applying the functor $[Z, -]$ to the first of the above sequences or $[-, Z]$ to the second of the above sequences yields an exact sequence of pointed sets.*

The analogy with the long exact sequence in cohomology associated to a short exact sequence of chain complexes in an abelian category becomes more apparent once we take f to be either a fibration or a cofibration. Indeed, the following is also proven in May's book [49].

Proposition 2.2.2. *If $f : X \to Y$ is a cofibration then the natural map $Cf \to Cf/CX \cong Y/X$ is a homotopy equivalence. Dually, if $f : X \to Y$ is a fibration then the natural inclusion $f^{-1}(y_0) \hookrightarrow Ff$ is a homotopy equivalence.*

So, up to homotopy, we can interpret these long exact sequences as coming from the "short exact sequences"

$$* \to f^{-1}(y_0) \cong Ff \to X \xrightarrow{f} Y \to *$$

if f is a fibration and

$$* \to X \xrightarrow{f} Y \to Y/X \cong Cf \to *$$

if f is a cofibration. In fact, in the case where $f : X \to Y$ is a fibration with Y path connected and we take $Z = S^1$, then using the identity $S^n \wedge S^m \cong S^{n+m}$ yields the long exact sequence in homotopy associated to a fibration:

$$\cdots \to \pi_1(f^{-1}(y_0)) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(f^{-1}(y_0)) \to \pi_0(X) \to \pi_0(Y).$$

The claim of this section is that the above constructions hold in far more generality than just $\mathcal{WC}_*$.

Now, consider the category $\mathrm{Ch}(R\text{-Mod})$ for R a commutative ring. Since R-Mod is an abelian category with both enough projectives and enough injectives (slightly more than this may be needed when R-Mod is replaced by a more general abelian category) it follows that $\mathrm{Ch}(R\text{-Mod})$ admits two natural model category structures [25]:

	Injective Model Structure	**Projective Model Structure**
weak equivalences	quasi-isomorphisms	quasi-isomorphisms
fibrations	right lifting property with respect to trivial cofibrations	degree-wise surjections
cofibrations	degree-wise injections	left lifting property with respect to trivial fibrations

Once one knows that these do indeed determine model structures, one can then prove the following.

Proposition 2.2.3. *[25] With respect to the injective model structure on* $\mathrm{Ch}(R\text{-Mod})$ *bounded above complexes of injective modules are fibrant. Dually, with respect to the projective model structure on* $\mathrm{Ch}(R\text{-Mod})$ *bounded below complexes of projective modules are cofibrant.*

Now, suppose we had a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathrm{Ch}(R\text{-Mod})$. Then the morphism f is a fibration with respect to the injective model structure while the morphism g is a cofibration with respect to the projective model structure. Furthermore, our projective and injective resolutions can be seen as fibrant and cofibrant replacement functors respectively.

The natural thing to ask at this point is: can the long exact sequence in (co)homology associated to our short exact sequence be seen as arising from a cofiber sequence associated to f and/or a fiber sequence associated to g? As it turns out, not only can this be done but we even have analogues of the cone, cylinder, path space and loop space constructions in $\mathrm{Ch}(R\text{-Mod})$ which will be used to construct them.

To see this, first recall that the singular cohomology groups of S^1 and I with coefficients in R are given by

$$\begin{aligned} H^*(S^1, R) &= R[0] \oplus R[-1] \in \mathrm{Ch}(R\text{-Mod}) \text{ and} \\ H^*(I, R) &= R[0] \in \mathrm{Ch}(R\text{-Mod}). \end{aligned}$$

However, since we previously worked in $\mathcal{WC}_*$ with based spaces we should do the analogous thing in $\mathrm{Ch}(R\text{-Mod})$. Namely, we should really be working with complexes over $H^*_{sing}(*, R) = R[0]$ (since $H^*(-, R)$ is contravariant). Another way of saying this is that we should be using *reduced* singular cohomology. A perhaps simpler way to do this is to work with the kernels of the morphisms $M^\bullet \to R[0]$ as this identifies $\mathrm{Ch}(R\text{-Mod})/R[0]$ with $\mathrm{Ch}(R\text{-Mod})$. Under this identification, we need to replace $H^*(S^1, R)$ and $H^*(I, R)$ with the following:

$\mathcal{WC}_*$	$\mathrm{Ch}(R\text{-Mod})$
S^1	$(R[-1], 0)$
I	$(R[0] \oplus R[-1], d(1[0]) = 1[-1])$

Here we have used the notation $(M^\bullet, d)$ for an object of $\mathrm{Ch}(R\text{-Mod})$ and have replaced the zero complex 0 corresponding to the based space $(I, 1)$ with the homotopy-equivalent complex $R[0] \oplus R[-1]$ with differential $d : R[0] \to R[-1]$ the natural isomorphism taking $1[0]$ to $1[-1]$. The reason for doing this is so that the exact sequence of pointed spaces

$$* \to \{0, 1\} \to I \to S^1 \to *$$

dualizes to

$$0 \to R[-1] \to R[0] \oplus R[-1] \to R[0] \to 0.$$

Now that we have our analogues of S^1, I in $\mathrm{Ch}(R\text{-Mod})$ (interpreting everything as being pointed) we can define our reduced path spaces, loop spaces, cones and suspensions. For these definitions, the smash product $\wedge$ will be replaced with the total Hom-complex bifunctor while the space of pointed continuous maps will be replaced by the total tensor product. The reason for the "swap" that occurred here is again the contravariance of $H^*(-, R)$.

Let $A, B \in \mathrm{Ch}(R\text{-Mod})$. The **total tensor product** complex of A and B is constructed by first forming the double complex

$$A_p \otimes_R A_q$$

with horizontal and vertical differentials

$$\begin{aligned} d^h : A^p \otimes_R B^q &\to A^{p+1} \otimes_R B^q \\ a \otimes b &\mapsto (d_A a) \otimes b \end{aligned}$$

and

$$\begin{aligned} d^v : A^p \otimes_R B^q &\to A^p \otimes_R B^{q+1} \\ a \otimes b &\mapsto (-1)^p a \otimes d_B b. \end{aligned}$$

The total tensor product of A and B is then the complex

$$\mathrm{Tot}^{\oplus}(A \otimes_R B) \in \mathrm{Ch}(R\text{-Mod})$$

given by

$$\mathrm{Tot}^{\oplus}(A \otimes_R B)^n := \bigoplus_{p+q=n} A^p \otimes_R B^q$$

with differential $d = d^v + d^h$. Indeed, $d^2 = 0$ since

$$\begin{aligned} d(d(a \otimes b)) &= d((d_A a) \otimes b + (-1)^p a \otimes d_B b) \\ &= (-1)^{p+1}(d_A a) \otimes (d_B b) + (-1)^p (d_A a) \otimes (d_B b) \\ &= 0. \end{aligned}$$

Similarly, we construct the **Total Hom-complex** by first forming the double complex

$$\mathrm{Hom}_R(A^{-p}, B^q)$$

with horizontal and vertical differentials given by

$$\begin{aligned} d^h : \mathrm{Hom}_R(A^{-p}, B^q) &\to \mathrm{Hom}_R(A^{-(p+1)}, B^q) \\ f &\mapsto f \circ d_A \end{aligned}$$

and

$$\begin{aligned} d^v : \mathrm{Hom}_R(A^{-p}, B^q) &\to \mathrm{Hom}_R(A^{-p}, B^{q+1}) \\ f &\mapsto (-1)^{p+q+1} d_B \circ f. \end{aligned}$$

The total Hom-complex of A and B is then the complex

$$\mathrm{Tot}^{\prod}(\mathrm{Hom}_R(A, B)) \in \mathrm{Ch}(R\text{-Mod})$$

given by

$$\mathrm{Tot}^{\prod}(\mathrm{Hom}_R(A, B))^n := \prod_{p+q=n} \mathrm{Hom}_R(A^{-p}, B^q)$$

with differential $d = d^v + d^h$. Again, we have $d^2 = 0$ since

$$\begin{aligned} d(df) &= d(f \circ d_A + (-1)^{p+q+1} d_B \circ f) \\ &= (-1)^{p+q+2} d_b \circ f \circ d_A + (-1)^{p+q+1} d_B \circ f \circ d_A \\ &= 0. \end{aligned}$$

We then define the following:

1. The**reduced cone** of A is
$$CA := \mathrm{Tot}^{\prod}(\mathrm{Hom}_R(I, A)),$$
2. the **reduced suspension** of A is
$$\Sigma A := \mathrm{Tot}^{\prod}(\mathrm{Hom}_R(S^1, A)),$$
3. the **reduced path space** of A is
$$PA := \mathrm{Tot}^{\oplus}(A \otimes_R I),$$

4. and the **reduced loop space** of A is

$$\Omega A := \mathrm{Tot}^{\oplus}(A \otimes_R S^1).$$

Using our explicit presentations of I as $R[0] \oplus R[-1]$ with differential $d1[0] = 1[-1]$ and S^1 as $R[-1]$ with trivial differential, a short computation shows that

$$(CA)^n \cong A^n \oplus A^{n+1}$$

with differential

$$\begin{aligned} d : A^n \oplus A^{n+1} &\to A^{n+1} \oplus A^{n+2} \\ (a, b) &\mapsto ((-1)^{n+1} d_A a - b, (-1)^{n+2} d_A b), \end{aligned}$$

as well as

$$\Sigma A \cong A[1].$$

Similarly, for the path and loop spaces, we have

$$(PA)^n \cong A^n \oplus A^{n-1}$$

with differential

$$\begin{aligned} d : A^n \oplus A^{n-1} &\to A^{n+1} \oplus A^n \\ (a, b) &\mapsto (d_A a, d_A b + (-1)^n a), \end{aligned}$$

as well as

$$\Omega A \cong A[-1].$$

It's worth noticing that here we have $\Sigma\Omega A \cong A \cong \Omega\Sigma A$. This is not something that always occurs in more general contexts and is a reflection of the fact that the homotopy categories of $\mathrm{Ch}(R\text{-Mod})$ with respect to either model structure are naturally *triangulated categories*. Another way of saying this is that $\mathrm{Ch}(R\text{-Mod})$ is a *stable model category* [25].

Now, suppose we had a morphism $f : A \to B$ in $\mathrm{Ch}(R\text{-Mod})$. Analogous to the case of $\mathcal{WC}_*$ we can define the **homotopy cofiber** of f to be

$$Cf := B \amalg_{f,A,i} CA$$

where $i : A \to CA$ is the inclusion coming from $(CA)^n \cong A^n \oplus A^{n+1}$. Explicitly this is given by

$$(Cf)^n \cong B^n \oplus A^{n+1}$$

with differential

$$\begin{aligned} d : B^n \oplus A^{n+1} &\to B^{n+1} \oplus A^{n+2} \\ (b, a) &\mapsto ((-1)^{n+1} d_B b - f(a), (-1)^{n+2} d_A a). \end{aligned}$$

Similarly, we define the **homotopy fiber** of f to be

$$Ff := A \times_{f,B,\pi} PB$$

where $\pi : PB \to B$ is the projection coming from $(PB)^n \cong B^n \oplus B^{n-1}$. Explicitly we have

$$(Ff)^n \cong A^n \oplus B^{n-1}$$

with differential

$$\begin{aligned} d : A^n \oplus B^{n-1} &\to A^{n+1} \oplus B^n \\ (a,b) &\mapsto (d_A a, d_B b + (-1)^n f(a)). \end{aligned}$$

The point being, as is shown in Weibel's book for the case of the homotopy cofiber (he calls it the *mapping cone*), if f is a degreewise injection then the natural map $B \to Cf$ is homotopy-equivalent to the cokernel:

$$0 \to A \to B \to Cf \to 0.$$

Furthermore, via $(Cf)^n \cong B^n \oplus A^{n+1}$ and $\Sigma A \cong A[1]$ (so $(\Sigma A)^n \cong A^{n+1}$) we have cofiber sequence

$$0 \to A \to B \to Cf \to \Sigma A \to \Sigma B \to \Sigma Cf \to \cdots$$

which yields our long exact sequence in cohomology after applying H^0. Dually, if $f : A \to B$ is a degreewise surjection then the natural map $Ff \to A$ is homotopy-equivalent to the kernel:

$$0 \to Ff \to A \to B \to 0$$

and since $(Ff)^n \cong A^n \oplus B^{n-1}$ while $\Omega A \cong A[-1]$ we have a fiber sequence

$$\cdots \to \Omega Ff \to \Omega A \to \Omega B \to Ff \to A \to B \to 0$$

which again yields our long exact sequence in cohomology, this time concentrated in negative degrees, after applying H^0.

We'll now state the analogous results for general model categories and pointed model categories, proven in Mark Hovey's book [25]. For us, these results are merely "moral" justification for the techniques to follow since we will be primarily working in model categories enriched in $s\,\mathrm{Set}$ or $s\,\mathrm{Set}_*$ in a way compatible with the model structure. In this case our constructions are greatly simplified.

Definition 2.2.4. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. We call $\mathcal{C}$ a **closed monoidal category** if it is equipped with the additional structure of functors

$$\mathrm{Map}_\ell, \mathrm{Map}_r : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$$

and natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, \mathrm{Map}_r(B,C)) \cong \mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}(B, \mathrm{Map}_\ell(A,C)).$$

Example 2.2.5. The monoidal category $(s\,\mathrm{Set}, \times, \{*\})$ is in fact a closed symmetric monoidal category with

$$\mathrm{Map}_\ell(X,Y)_n = \mathrm{Map}_r(X,Y)_n = \mathrm{Hom}_{s\,\mathrm{Set}}(X \times \Delta[n], Y).$$

More precisely, the simplicial sets $\Delta[n]$ assemble to form a cosimplicial simplicial set $[n] \mapsto \Delta[n]$ which, after post-composition with the contravariant functor

$$\mathrm{Hom}_{s\,\mathrm{Set}}(X \times (-), Y) : (s\,\mathrm{Set})^{op} \to \mathrm{Set}$$

yields a simplicial set

$$[n] \mapsto \mathrm{Hom}_{s\,\mathrm{Set}}(X \times \Delta[n], Y)$$

which we call $\mathrm{Map}_\ell(X,Y) = \mathrm{Map}_r(X,Y) = \mathrm{Map}(X,Y)$.

Example 2.2.6. [17] Let R be a commutative ring. Then the category $\mathrm{Ch}(R\text{-Mod})$ is also closed symmetric monoidal. The unit is $R[0]$ and the monoidal structure comes from the total tensor product:

$$A \otimes B := \mathrm{Tot}^{\oplus}(A \otimes_R B).$$

Similarly, the closedness comes from the total Hom-bifunctor:

$$\mathrm{Map}_\ell(A, B) = \mathrm{Map}_r(A, B) = \mathrm{Map}(A, B) := \mathrm{Tot}^{\prod}(\mathrm{Hom}_R(A, B)).$$

The symmetric structure comes from the symmetric structure on R-Mod as expected.

Example 2.2.7. As mentioned earlier, $\mathcal{WC}$ is a closed symmetric monoidal category while Top is not. $\mathcal{WC}_*$ is also a closed symmetric monoidal category but now the monoidal structure comes from the smash product $X \wedge Y$.

Let's now make precise why replacing S^1, I with their (reduced) singular cohomology complexes worked for obtaining the fiber and cofiber sequences in $\mathrm{Ch}(R\text{-Mod})$. For this, we need the notion of a closed module over a closed monoidal category.

Definition 2.2.8. Let $\mathcal{B}$ be a category and $(\mathcal{C}, \otimes, 1, \mathrm{Map}_\ell, \mathrm{Map}_r)$ be a closed monoidal category. We call $\mathcal{B}$ a **closed $\mathcal{C}$-module** if and only if it is equipped with functors

$$\begin{aligned} \otimes : \mathcal{C} \times \mathcal{B} &\to \mathcal{B} \\ \mathrm{Map} : \mathcal{C}^{op} \times \mathcal{B} &\to \mathcal{B} \\ \mathrm{Hom} : \mathcal{B}^{op} \times \mathcal{B} &\to \mathcal{C} \end{aligned}$$

and natural isomorphisms of functors into cC:

$$\mathrm{Hom}(C \otimes B_1, B_2) \cong \mathrm{Hom}(B_1, \mathrm{Map}(C, B_2)) \cong \mathrm{Map}_\ell(C, \mathrm{Hom}(B_1, B_2))$$

which satisfy all of the usual associativity and unit conditions for modules over monoidal categories. We also require $\mathcal{B}$ to be equipped with the version of the above natural isomorphisms, only into Set instead of $\mathcal{C}$, so we can use $\mathrm{Hom}_{\mathcal{B}}$ and $\mathrm{Hom}_{\mathcal{C}}$.

Example 2.2.9. Let $\mathcal{C}$ be a category with all limits and colimits and consider the category $s\mathcal{C}$ of all simplicial objects in $\mathcal{C}$. In the paper by Goerss and Schemmerhorn [17] it is shown that $s\mathcal{C}$ is naturally a closed $s\,$Set-module. Closed $s\,$Set-modules are called **simplicial categories**. Let's see how our above functors are defined.

First we need a functor $\otimes : s\,\mathrm{Set} \times s\mathcal{C} \to s\mathcal{C}$. Given $K \in s\,\mathrm{Set}$ and $C \in s\mathcal{C}$ we set

$$(K \otimes C)_n := \amalg_{K_n} C_n$$

and to a morphism $[n] \to [m]$ in Δ we associate a morphism $(K \otimes C)_m \to (K \otimes C)_n$ in $\mathcal{C}$ in the following way. By the universal property of the coproduct it suffices to specify a morphism $C_m \to (K \otimes C)_n$ for each element of K_m. The morphism $K_m \to K_n$ applied to a given element of K_m specifies an element of K_n which in turn specifies an inclusion

$$C_n \to (K \otimes C)_n.$$

Our morphism is then the composision

$$C_m \to C_n \to (K \otimes C)_n$$

where the morphism $C_n \to (K \otimes C)_n$ is as above and the morphism $C_m \to C_n$ is the one assigned to $[n] \to [m]$ by the simplicial object C. This makes $K \otimes C$ into a simplicial object in $\mathcal{C}$ and this construction is functorial in both $s\,\mathrm{Set}$ and $s\mathcal{C}$.

The functor $\mathrm{Hom} : (s\mathcal{C})^{op} \times s\mathcal{C} \to s\,\mathrm{Set}$ is defined by recalling that $[n] \mapsto \Delta[n]$ is a cosimplicial simplicial set and so given $B, C \in s\mathcal{C}$ we can define $\mathrm{Hom}(B, C)$ to be the simplicial set

$$[n] \mapsto \mathrm{Hom}_{s\mathcal{C}}(\Delta[n] \otimes B, C)$$

given by post-composing the cosimplicial simplicial set $[n] \mapsto \Delta[n]$ with the contravariant functor

$$\mathrm{Hom}_{s\mathcal{C}}((-) \otimes B, C).$$

The construction of $\mathrm{Map} : (s\,\mathrm{Set})^{op} \times s\mathcal{C} \to s\mathcal{C}$ in this level of generality is a bit trickier. Namely, one proves that given $K \in s\,\mathrm{Set}$ the functor $\mathrm{Map}(K, -) : s\mathcal{C} \to s\mathcal{C}$ exists and is a right-adjoint to $K \otimes (-)$ by the adjoint functor theorem. This requires some set-theoretic trickery, as does showing that this then extends to our desired bifunctor. In all of the examples we will need later, we will give an explicit construction of this functor unique to the specific category $\mathcal{C}$ we're working in.

We are now prepared to state the results in Mark Hovey's book [25] which justify our use of certain homotopical techniques in the constructions of the de Rham and Spencer complexes. For this, we need to describe what it means for a closed monoidal structure or closed module structure to be compatible with the model structure on a model category.

Definition 2.2.10. Let $(\mathcal{C}, \otimes, 1, \mathrm{Map}_\ell, \mathrm{Map}_r)$ be a closed monoidal category which is also a model category. We then call $\mathcal{C}$ a **closed monoidal model category** if and only if the following holds. Given any two cofibration $A \to B$ and $C \to D$ in $\mathcal{C}$ the induced map

$$(B \otimes C) \amalg_{A \otimes C} (A \otimes D) \to B \otimes D$$

is also a cofibration and it is a trivial cofibration if either $A \to B$ or $C \to D$ is.

Given a closed $\mathcal{C}$-module $\mathcal{B}$ which is also a model category we call $\mathcal{B}$ a **$\mathcal{C}$-model category** if and only if the corresponding functor $\otimes : \mathcal{C} \otimes \mathcal{B} \to \mathcal{B}$ has the same property as the functor $\otimes$ on $\mathcal{C}$ above, only now $C \to D$ is a cofibration in $\mathcal{B}$ and the above morphism is in $\mathcal{B}$. There are equivalent definitions of these types of model categories made using fibrations and cofibrations with the other functors $\mathrm{Map}_\ell, \mathrm{Map}_r, \mathrm{Map}$ and Hom. If $\mathcal{C}$ is the closed monoidal model category $s\,\mathrm{Set}$ then we call $\mathcal{B}$ a **simplicial model category**.

Theorem 2.2.11. *[25] Let $\mathcal{C}$ be a model category. Then $h\mathcal{C}$ is naturally a closed $h(s\,\mathrm{Set})$-module. Analogously, if $\mathcal{C}_*$ is a pointed model category then $h\mathcal{C}_*$ is naturally a closed $h(s\,\mathrm{Set}_*)$-module. Furthermore, if $\mathcal{C}$ was a simplicial model category (respectively a pointed $s\,\mathrm{Set}_*$-model category) to begin with then the induced closed $h(s\,\mathrm{Set})$ (respectively $h(s\,\mathrm{Set}_*)$) module structure on the homotopy category is naturally isomorphic to the natural one which exists by virtue of $\mathcal{C}$ being a (pointed) model category.*

The point of this is that in the homotopy category of a pointed model category $\mathcal{C}_*$ one has naturally defined fiber and cofiber sequences arising from the $h(s\,\mathrm{Set}_*)$-action. Indeed, one can define the (reduced) **derived loop space**, **derived path space**, **derived suspension** and **derived cone** using our Quillen equivalence

$$s\,\mathrm{Set}_* \cong \mathcal{WC}_*$$

from before to obtain objects $S^1, I \in h(s\,\mathrm{Set}_*)$ which then yield for $X \in h\mathcal{C}$:

$$\begin{aligned} \Omega X &:= \mathbb{R}\,\mathrm{Map}(S^1, X) \\ PX &:= \mathbb{R}\,\mathrm{Map}(I, X) \\ \Sigma X &:= S^1 \wedge^h X \\ CX &:= I \wedge^h X \end{aligned}$$

where in the case of closed $s\,\mathrm{Set}_*$ or $h(s\,\mathrm{Set}_*)$-modules we usually write $\wedge$ for the functor $s\,\mathrm{Set}_* \times \mathcal{C} \to \mathcal{C}$. The superscript h on $\wedge^h$ indicates that we have taken the derived functor. Furthermore, if $\mathcal{C}$ is any model category with initial object $0 \in \mathcal{C}$ and terminal object $1 \in \mathcal{C}$ we can obtain two pointed model categories:

$$\mathcal{C}/1 \text{ and } 0/\mathcal{C}.$$

The pointed model category $\mathcal{C}/1$ is most often used if we want to think of the objects of $\mathcal{C}$ as algebras of functions on spaces and, in this case, 1 is thought of as the algebra of functions on a point. The pointed model category $0/\mathcal{C}$ is used when we want to think of objects of $\mathcal{C}$ as spaces themselves and so 0 corresponds to the point space.

Now, $h\mathcal{C}$ has a natural $h(s\,\mathrm{Set})$-module structure which allows us to define **free** (i.e. non-reduced) versions of the path and loop spaces, as well as the suspensions and cones. These are again constructed using the derived functors $S^1 \otimes^{\mathbb{L}}$, $I \otimes^{\mathbb{L}}$, $\mathbb{R}\,\mathrm{Map}(S^1, -)$ and $\mathbb{R}\,\mathrm{Map}(I, -)$ and are only well-defined in the homotopy category. An important remark is that if we want to interpret objects of $\mathcal{C}$ as algebras of functions on spaces, then we should dualize our constructions and, for example, for $A \in \mathcal{C}$ define the (algebra of functions on) the **derived free loop space** to be

$$S^1 \otimes^{\mathbb{L}} A, \text{ and } \textit{not } \mathbb{R}\,\mathrm{Map}(S^1, A)$$

since $\mathbb{R}\,\mathrm{Map}(S^1, A)$ should then intuitively be the algebra of functions on the product $S^1 \times M$ where M is the space for which A is the algebra of functions. In this setting, if $A \to 1$ is in $\mathcal{C}/1$ then since $h(\mathcal{C}/1) \cong (h\mathcal{C})/1$ the reduced loop space of $A \to 1$ can be computed from the free loop space as the homotopy pushout

$$\Omega A \simeq (S^1 \otimes A) \amalg^h_A 1 \to 1.$$

In the next section we will compute this explicitly in the case $\mathcal{C} = s\,\mathrm{C}^\infty\text{-Ring}$. As we'll see, S^1 is naturally an abelian group object in $s\,\mathrm{Set}$ and the group multiplication morphism

$$S^1 \times S^1 \to S^1$$

gives rise to a morphism

$$S^1 \otimes (S^1 \otimes A) \simeq (S^1 \times S^1) \otimes A \to S^1 \otimes A$$

which then corresponds via our adjunction to a morphism

$$S^1 \otimes A \to \mathrm{Map}(S^1, S^1 \otimes A)$$

which should be intuitively thought of as the S^1-action

$$S^1 \times \mathrm{Spec}(S^1 \otimes A) \to \mathrm{Spec}(S^1 \otimes A)$$

given by loop rotation (however we haven't even defined "Spec" for a simplicial C^∞-ring). At the level of cohomology, when $A = C^\infty(M)$, $S^1 \otimes A$ will end up being $\Omega^{-*}(M)$ and the S^1-action will correspond to the de Rham differential. To do this, however, we need to understand what "at the level of cohomology" means.

From the above discussion, we can see that $s\,\mathrm{C}^\infty\text{-Ring}$ and $\mathrm{C}^\infty\text{-Ring}^\Delta$ are natural categories worth working in when developing (co)homology theories for functors leaving or going into the category C^∞-Ring. Indeed, as we saw with the Dold-Kan correspondence, these categories are to C^∞-Ring what $\mathrm{Ch}^{\leq 0}(R\text{-Mod})$ and $\mathrm{Ch}^{\geq 0}(R\text{-Mod})$ are to R-Mod. Let's use this analogy to understand what it means to "take cohomology" of an object in $s\,\mathrm{C}^\infty$-Ring. If one wants more evidence of the usefulness of this analogy, one need only look at rational homotopy theory [60].

Let k be a field of characteristic zero and write cdga_k for the category of $\mathbb{Z}$-graded-commutative algebras A over k together with a degree 1 graded derivation. The categories $\mathrm{cdga}_k^{\leq 0}$ and $\mathrm{cdga}_k^{\geq 0}$ will then be the full subcategories of those cdga's which are concentrated in non-positive and non-negative degrees respectively. If we hope to make use of our analogy between simplicial objects and complexes concentrated in non-positive degrees, then an intuitive first step would be to understand how the functors in the Dold-Kan correspondence relate objects in cdga_k to objects in $s\,\mathrm{Comm}_k$ and Comm_k^{Δ}, as opposed to simply $\mathrm{Ch}(k\text{-Mod})$ and $sk\text{-Mod}$, $k\text{-Mod}^{\Delta}$.

Lemma 2.2.12. *[17, 65, 66] By declaring weak equivalences, fibrations and cofibrations in cdga_k to be those morphisms which are weak equivalences, fibrations and cofibrations respectively after applying the forgetful functor $U : \mathrm{cdga}_k \to \mathrm{Ch}(k\text{-Mod})$ and using either the projective or injective model structure on $\mathrm{Ch}(k\text{-Mod})$, cdga_k becomes a model category.*

The full subcategories $\mathrm{cdga}_k^{\leq 0}$ and $\mathrm{cdga}_k^{\geq 0}$ with weak equivalences, fibrations and cofibrations defined in the same way, but now using the projective model structure in the case $\mathrm{cdga}_k^{\leq 0}$ and the injective model structure for the case of $\mathrm{cdga}_k^{\geq 0}$, also become model categories.

Let's now see what happens if we restrict the normalized complex functor from $sk\text{-Mod}$ to $s\,\mathrm{Comm}_k$. Does it preserve the product structure? For this, we'll need the notion of a *shuffle*. A (p,q)**-shuffle** is a permutation

$$(\sigma,\tau) = (\sigma_1, \cdots, \sigma_p, \tau_1, \cdots, \tau_q)$$

of $(0, 1, \cdots, p+q-1)$ such that $\sigma_1 < \cdots < \sigma_p$ and $\tau_1 < \cdots < \tau_q$. The set of all (p,q)-shuffles will be denoted by $\mathrm{shuf}(p,q)$. One then has the following result.

Lemma 2.2.13. *[69] Let k be a field. For $A, B \in sk\text{-Mod}$ we let $A \otimes B \in sk\text{-Mod}$ denote the level-wise tensor product of A and B. Then the map*

$$\begin{aligned} \nabla_{A,B} : \mathrm{Tot}^{\oplus}(N^\bullet(A) \otimes_k N^\bullet(B)) &\to N^\bullet(A \otimes B) \\ A_p \otimes_k B_q \ni a \otimes b &\mapsto \sum_{(\sigma,\tau)\in \mathrm{shuf}(p,q)} \mathrm{sign}(\sigma,\tau)(A(\eta_{\tau_q} \circ \cdots \circ \eta_{\tau_1})(a)) \otimes_k (B(\eta_{\sigma_p} \circ \cdots \circ \eta_{\sigma_1})(b)) \end{aligned}$$

defines a natural transformation which is symmetric with respect to $\mathrm{Tot}^{\oplus}(- \otimes_k -)$ and $\otimes$. This is called the ***Eilenberg-Zilber map****. Also, if $\epsilon^f_{p,p+q} : [p] \to [p+q]$ denotes the map $i \mapsto i$ and $\epsilon^b_{q,p+q} : [q] \to [p+q]$ denotes the map $i \mapsto i+p$ then the* ***Alexander-Whitney map***

$$\begin{aligned} \Delta_{A,B} : N^\bullet(A \otimes B) &\to \mathrm{Tot}^{\oplus}(N^\bullet(A) \otimes N^\bullet(B)) \\ A_n \otimes B_n \ni a \otimes b &\mapsto \bigoplus_{p+q=n} (A(\epsilon^f_{p,p+q})(a)) \otimes (B(\epsilon^b_{q,p+q})(b)) \end{aligned}$$

also defines a natural transformation which is a left inverse to the Eilenberg-Zilber map, i.e. the composition $\Delta_{A,B} \circ \nabla_{A,B}$ is the identity. Furthermore, the composition in the other direction $\nabla_{A,B} \circ \Delta_{A,B}$ is a homotopy equivalence. Unfortunately, however, the Alexander-Whitney map $\Delta_{A,B}$ is not symmetric with respect to the monoidal structures.

The importance of the above results is that they essentially prove the monoidal Dold-Kan correspondence so long as one doesn't care about commutativity (so it isn't a *symmetric* monoidal Dold-Kan correspondence). Since we primarily care about commutative things, some more work is needed. However, the Eilenberg-Zilber map is still useful.

Indeed, suppose we had $A \in s\,\mathrm{Comm}_k$. This can be thought of as $A \in sk\text{-Mod}$ together with morphisms

$$A \otimes A \to A \text{ and } k \to A$$

where k is the constant simplicial k-module at k, satisfying the usual relations that we'd require a commutative unital ring multiplication to satisfy. Since

$$N^\bullet(k) = k[0]$$

it follows that our unit map $k \to A$ gives rise to

$$k[0] \to N^\bullet(A)$$

whilst out multiplication map $A \otimes A \to A$ yields

$$N^\bullet(A \otimes A) \to N^\bullet(A).$$

Together with the Eilenberg-Zilber map we obtain

$$\mathrm{Tot}^{\oplus}(N^\bullet(A) \otimes_k N^\bullet(A)) \xrightarrow{\nabla_{A,A}} N^\bullet(A \otimes A) \to N^\bullet(A)$$

making $N^\bullet(A)$ into a differential graded k-algebra concentrated in non-positive degrees. Since the Eilenberg-Zilber map preserves the *symmetric* monoidal structure this is in fact a commutative differential graded algebra and so we have a functor

$$N^\bullet : s\,\mathrm{Comm}_k \to \mathrm{cdga}_k^{\leq 0}.$$

As mentioned above, the Alexander-Whitney map does not preserve the symmetric part of the symmetric monoidal structure and so our usual functor from the original Dold-Kan correspondence need not take values in *commutative* differential graded algebras. Luckily for us, Quillen was still able to prove the following result.

Theorem 2.2.14. The Symmetric Monoidal Dold-Kan Correspondence [60]
The functor

$$N^\bullet : s\,\mathrm{Comm}_k \to \mathrm{cdga}_k^{\leq 0}$$

admits a left adjoint

$$\Gamma^{sym} : \mathrm{cdga}_k^{\leq 0} \to s\,\mathrm{Comm}_k.$$

Furthermore, using the standard model structure on $s\,\mathrm{Comm}_k$ *as well as the (projective) model structure on* $\mathrm{cdga}_k^{\leq 0}$ *described above, the functor* $N^\bullet$ *takes weak equivalences to weak equivalences and the functor* Γ^{sym} *admits a left derived functor. Together, these yield an adjoint equivalence of categories*

$$h(\mathrm{cdga}_k^{\leq}) \underset{N^\bullet}{\overset{\mathbb{L}\Gamma^{sym}}{\rightleftarrows}} h(s\,\mathrm{Comm}_k)$$

Proof. Let's construct the functor Γ^{sym}. We know that $N^\bullet$ is an equivalence after restricting the domain and codomain of $N^\bullet$ to sk-Mod and $\mathrm{Ch}^{\leq 0}(k\text{-Mod})$ respectively by the Dold-Kan theorem. Let's write

$$\Gamma : \mathrm{Ch}^{\leq 0}(k\text{-Mod}) \to sk\text{-Mod}$$

for the "inverse" functor going in the other direction which we constructed in the proof of the Dold-Kan theorem. Now, we have a free-forgetful adjunction

$$k\text{-Mod} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Comm}_k$$

and so we have a functor

$$\mathrm{cdga}_k^{\leq 0} \hookrightarrow \mathrm{Ch}^{\leq 0}(k\text{-Mod}) \xrightarrow{F \circ \Gamma} s\,\mathrm{Comm}_k$$

where the functor $F : k\text{-Mod} \to \mathrm{Comm}_k$ is extended level-wise to

$$F : sk\text{-Mod} \to s\,\mathrm{Comm}_k\,.$$

It's worth mentioning that the functor F can be realized explicitly as the symmetric algebra functor applied levelwise

$$F(M) = \mathrm{Sym}_k^*(M).$$

Now, suppose we were given $A \in \mathrm{cdga}_k^{\leq 0}$ and a homogeneous element $a \in A$ of degree n. Recall then that the simplicial k-module $\Gamma(A)$ applied to $[n]$ was given by

$$\Gamma(A)_n = \bigoplus_{[n]\twoheadrightarrow[k]} (A^{-n})^{\oplus k}.$$

But we had a natural isomorphism $N^\bullet(\Gamma(A)) \cong A$ in $k\text{-Mod}$ which then identifies one of the copies of A^{-n} in $\Gamma(A)_n$ with the original A^{-n}. We can then define a map from the homogeneous elements of A to $\Gamma(A) \subseteq F(\Gamma(A))$ by sending a to the element $a \in A^{-n} \subseteq \Gamma(A)_n$ where this copy of A^{-n} is the one identified with $A^{-n} \subseteq A$ via $N^\bullet \circ \Gamma \cong \mathrm{id}$. Let's denote this map by f.

Now, let $I \trianglelefteq F(\Gamma(A))$ be the ideal generated by all elements of the form

$$f(a)f(b) - f(ab)$$

for $a, b \in A$ homogeneous elements. We define

$$\Gamma^{sym}(A) := F(K(A))/I \ \text{ levelwise.}$$

One then shows that this works. □

Another important result due to [60] is that the cohomology groups one obtains by applying $N^\bullet$ to a fibrant object agree with the simplicial homotopy groups of the original object.

There is a difficulty in trying to do this for simplicial C^∞-rings since C^∞-rings aren't monoid objects in some suitable category in any obvious way. However, the answer to this might lie in the preprint [5]. However, we still have the forgetful functor

$$s\,\mathrm{C}^\infty\text{-Ring} \to s\,\mathrm{Comm}_{\mathbb{R}}$$

and theorem 3.6 of [17] together with the fact that the adjoint of the above functor preserves directed colimits tells us that we have an induced (cofibrantly generated) model structure on $s\,\mathrm{C}^\infty$-Ring. This is the model structure used in [6]. We also have the Dold-Kan functor

$$N^\bullet : s\,\mathrm{Comm}_{\mathbb{R}} \to \mathrm{cdga}_{\mathbb{R}}^{\leq 0}\,.$$

Let's denote the composition of these two functors by $N^\bullet$ as well. Since the model structure on $s\,\mathrm{C}^\infty$-Ring is transfered from $s\,\mathrm{Comm}_{\mathbb{R}}$ it follows then that the homotopy groups of fibrant objects in $s\,\mathrm{C}^\infty$-Ring agree with the homotopy groups one obtains after applying the forgetful functor to $s\,\mathrm{Comm}_{\mathbb{R}}$ and therefore agree with the cohomology groups in $\mathrm{cdga}_{\mathbb{R}}^{\leq 0}$. Let's summarize what we have so far.

- We have a (cofibrantly generated) model structure on $s\,\mathrm{C}^\infty$-Ring.
- Therefore the localization $hs\,\mathrm{C}^\infty$-Ring at weak equivalences exists and is naturally a closed $hs\,$Set-module. This allows us to perform homological algebra in $hs\,\mathrm{C}^\infty$-Ring.
- We have a functor $N^\bullet : s\,\mathrm{C}^\infty\text{-Ring} \to \mathrm{cdga}_{\mathbb{R}}^{\leq 0}$ which takes the homotopy groups of fibrant objects in $s\,\mathrm{C}^\infty$-Ring to the cohomology groups of the corresponding commutative differential graded algebra.

In Chapter 3, we'll use the above *hs* Set-action together with the functor $N^\bullet$ to associate a differential graded Lie algebra to each C^∞-ring. The Maurer-Cartan equations of this differential graded Lie algebra will then classify infinitesimal deformations of the corresponding C^∞-ring. Furthermore, in the case of the C^∞-ring $C^\infty(M)$ this differential graded Lie algebra will be the Schouten algebra of polyvector fields. This leads us to the ultimate goal of this thesis: making precise the analogy between deformation theory and the prolongations of partial differential equations.

2.3 Chern-Weil Theory and the Derived Loop Space

We now make our long-awaited return to the theory of G-structures. Recall that for $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ a Lie subgroup and M a smooth n-dimensional manifold, G-structures on M correspond to smooth sections of the associated fiber bundle

$$F \times_{\mathrm{GL}(\mathbb{R}^n)} (\mathrm{GL}(\mathbb{R}^n)/G) \cong F/G$$

where $F \to M$ is the principal frame bundle of M. We mentioned previously that topological obstructions to the existence of global sections of the fiber bundle live in the cohomology groups

$$H^{m+1}(M, \pi_m(\mathrm{GL}(\mathbb{R}^n)/G)).$$

Let's explain this now. Our treatment essentially follows that of [49] and [37]. This result comes from a part of algebraic topology called *obstruction theory*. Namely, consider the following general set-up. Suppose we had a (topological) fiber bundle $f : E \to B$ in $\mathcal{WC}$ with typical fiber F. Indeed, we can suppose that the fiber over each point of B is F by noting that the problem of the existence of global sections can be solved separately on the connected components of B, and so we may assume B is connected.

Now, let's suppose that B was a CW-complex. All smooth manifolds are CW-complexes but if we really want to do this properly in the smooth setting we should be working with *handle decompositions* of manifolds. This, however, involves Morse theory and Morse theory is something I am not yet familiar with so I'll be brushing this (important) technicality under the rug.

Recall what it means for B to be a CW-complex. It means that we can write B as a directed colimit

$$B = \varinjlim_n B^n$$

of a sequence of maps in $\mathcal{WC}$ such that B^0 is a discrete set and for each n there is another discrete set J_{n+1} whose elements are continuous maps

$$S^n \to B^n$$

such that if $D^{n+1} \subseteq \mathbb{R}^{n+1}$ denotes the unit ball then the following commutative diagram is a pushout in $\mathcal{WC}$:

$$\begin{array}{ccc} J_{n+1} \times S^n & \longrightarrow & J_{n+1} \times D^{n+1} \\ \downarrow & & \downarrow \\ B^n & \longrightarrow & B^{n+1} \end{array}$$

In other words, B^{n+1} is obtained as the quotient

$$B^n \amalg (J_{n+1} \times D^{n+1}))/\sim$$

by the relation

$$J_{n+1} \times S^n \ni (j, x) \sim j(x) \in B^n.$$

Now, for all $j \in J_{n+1}$ we have

$$D^{n+1} \cong \{j\} \times D^{n+1} \hookrightarrow J_{n+1} \times D^{n+1}$$

which then descends to a map

$$D^{n+1} \to B^{n+1} \to B$$

which we call a $(n+1)$**-cell**. The space B^n is called the n**-skeleton** of B and we define $B^{-1} := \emptyset$.

Let's now construct our obstructions to the existence of global sections of $E \to B$. First notice that there is always a (unique) continuous map

$$\emptyset = B^{-1} \to E$$

and, by uniqueness of maps leaving the empty set, the following diagram commutes

$$\begin{tikzcd} B^{-1} \arrow[r] \arrow[dr] & E \arrow[d] \\ & B \end{tikzcd}$$

So, if we denote by $\Gamma(B^k, E)$ the collection of all continuous maps $B^k \to E$ such that the post-composition with $E \to B$ is equal to the natural map $B^k \to B$, then $\Gamma(B^{-1}, E) \neq \emptyset$.

Now, suppose inductively that we had an element $s \in \Gamma(B^k, E)$ for some $k \geq -1$. What is the obstruction to extending this to an element of $\Gamma(B^{k+1}, E)$ which pre-composes with $B^k \to B^{k+1}$ to yield s? For starters, let's suppose that $E \to B$ was the trivial bundle $B \times F \to B$. We'll use this case to handle the general one.

Take then a $(n+1)$-cell j' with attaching map j:

$$j' : D^{n+1} \to B, \quad j : S^n \to B^n.$$

Then $s \circ j : S^n \to F$ defines an element of $\pi_n(F)$ (with respect to a chosen basepoint). If we had a homotopy

$$h : S^n \to F^I$$

from the constant map at this basepoint (at time 0) to $s \circ j$ (at time 1) then since $S^n \hookrightarrow D^{n+1}$ is a cofibration and $F^I \to F$ is a trivial fibration we obtain the following lift from the left lifting property

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d] & F^I \arrow[d, "\mathrm{ev}_0"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & F \end{tikzcd}$$

where the map $D^{n+1} \to F$ is the constant map at our basepoint. This then gives us a map $D^{n+1} \to F$ extending $s \circ j$ via the composition

$$D^{n+1} \to F^I \xrightarrow{\mathrm{ev}_1} F.$$

If these null homotopies exist for each $(n+1)$-cell then they assemble, using the expression of B^{n+1} as a pushout, to a map

$$B^{n+1} \to F$$

lifting s. In particular, if $s \in \Gamma(B^n, B \times F)$ and $\pi_n(F) = 0$ then there exists an extension of s in $\Gamma(B^{n+1}, B \times F)$.

Now, let's handle the case of a general fiber bundle $E \to B$. If $\pi_0(F) \neq 0$ then we have obstructions arising from the possibility that there might not exist a globally defined continuous map from B into the set of connected components of the fibers. So, let's assume $\pi_0(F) = 0$. Our first goal is to show that for $n \geq 2$ the collections

$$\{\pi_n(E_x)\}_{x \in B}$$

are local systems of coefficients for B. To see this, we take an arbitrary path $\gamma : I \to B$ and recall that for all $x \in B$ we had

$$E_x \cong E \times_{\pi, B, \mathrm{ev}_0} P_x B.$$

Now, γ defines a morphism $P_{\gamma(0)}B \to P_{\gamma(1)}B$ which then, through the universal property of pullbacks, gives rise to

$$E_{\gamma(0)} \cong E \times_{\pi, B, \mathrm{ev}_0} P_{\gamma(0)}B \to E \times_{\pi, B, \mathrm{ev}_0} P_{\gamma(1)}B \cong E_{\gamma(1)}$$

and, through this, maps $\pi_n(E_{\gamma(0)}) \to \pi_n(E_{\gamma(1)})$ for all $n \geq 0$.

If $n = 1$ and $\pi_1(F)$ is abelian, or if $n \geq 2$ and $\pi_k(F) = 0$ for all $k < n$ then it follows that these maps determined from γ are group homomorphisms. Furthermore, they only depend on the homotopy class of γ and if γ is the constant path then the induced morphism is the identity. Hence $\{\pi_n(E_x)\}_{x \in B}$ is indeed a local system of coefficients on B. It is the cohomology of B with coefficients in this local system that the obstructions lie.

Now, the above discussion was purely topological. One of my hopes is that there is a smooth version of it coming from handle decomposition of manifolds using Morse theory.

A related topological description of G-structures can be seen using the notion of a *classifying space*. These in fact describe general principal G-bundles, and unlike our above discussion, I know how one can specialize to the smooth setting from the topological one. Furthermore, we can actually obtain explicit descriptions of these for general topological groups G in $\mathcal{WC}$ via our simplicial set machinery [49, 48].

Indeed, give a topological group $G \in \mathcal{WC}$ we define simplicial spaces $BG, EG \in s\mathcal{WC}$ together with a morphism of simplicial spaces $EG \to BG$ as follows. Set

$$(BG)_n := G^n$$

with degeneracies coming from the insertion of the identity elements and the $n+1$ face maps $G^n \to G^{n-1}$ as

$$\begin{aligned}
(BG)(\epsilon_0)(g_1, \cdots, g_n) &= (g_2, \cdots, g_n) \\
(BG)(\epsilon_i)(g_1, \cdots, g_n) &= (g_1, \cdots, g_i g_{i+1}, \cdots, g_n) \ \text{ for } \ 1 \leq i \leq n-1 \\
(BG)(\epsilon_n)(g_1, \cdots, g_n) &= (g_1, \cdots, g_{n-1}).
\end{aligned}$$

Similarly, we set

$$(EG)_n := G^{n+1}$$

with degeneracies coming from the insertion of the identity elements and the $n+1$ face maps $G^{n+1} \to G^n$ given by

$$\begin{aligned}
(EG)(\epsilon_0)(g_1, \cdots, g_{n+1}) &= (g_2, \cdots, g_{n+1}) \\
(EG)(\epsilon_i)(g_1, \cdots, g_{n+1}) &= (g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) \ \text{ for } \ 1 \leq i \leq n.
\end{aligned}$$

The morphism $EG \to BG$ is then given simply by forgetting the final component. Initially these definitions do not seem very transparent. However, let's interpret G as a one object category whose morphisms are given by the elements of G and composition rule by the group multiplication. This is a category enriched in $\mathcal{WC}$ (its hom-set is the topological group G) and so its nerve is a simplicial space

$$N_\bullet(G) \in s\mathcal{WC}.$$

In fact, by definition we have

$$N_\bullet(G) = BG.$$

The simplicial space EG is constructed in a similar way. One considers the category with one object for each element of G and precisely one morphism between any pair of objects. The nerve of this category is then the contractible (the category has an initial object) simplicial space EG. Furthermore, notice that the group G acts freely on the set of objects of the category whose nerve is EG. This should make the following proposition seem reasonable.

Proposition 2.3.1. *[49, 48] Denote by $EG, BG \in \mathcal{WC}$ the geometric realizations of the simplicial spaces $EG, BG \in s\mathcal{WC}$ respectively. This is defined in the exact same way as the geometric realization of the underlying simplicial sets, only the objects $(EG)_n, (BG)_n$ are considered as the topological spaces they are, not as discrete spaces. Then the induced map*

$$EG \to BG$$

in $\mathcal{WC}$ is a (topological) principal G-bundle with EG contractible. Furthermore, EG and BG are both naturally CW-complexes via the geometric realization functor.

Now, denote by CW the full subcategory of $\mathcal{WC}$ consisting of all CW-complexes. By Whitehead's theorem, given any weak equivalence

$$f : X \to Y$$

between CW-complexes X and Y there is a continuous map $g : Y \to X$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity. Thus the category hCW obtained by localizing at weak equivalences exists as a locally small category and is presented by simply taking the category CW and quotienting the hom-sets by the homotopy relation.

We now arrive at the entire point of the principal G-bundle $EG \to BG$. Consider the functor

$$\mathcal{P}_G : hCW^{op} \to \mathrm{Set}$$

that assigns to a CW-complex X the set $\mathcal{P}_G(X)$ of isomorphism classes of principal G-bundles on X. To a morphism $f : X \to Y$ in hCW (i.e. a homotopy class of continuous maps $X \to Y$) we define a map

$$\mathcal{P}_G(Y) \to \mathcal{P}_G(X)$$

going in the other direction by pulling back principal G-bundles on Y to principal G-bundles on X via any representative for f. One can show that this is well-defined.

Theorem 2.3.2. *[37] The functor $\mathcal{P}_G : hCW^{op} \to \mathrm{Set}$ is representable and, in fact, we have a natural isomorphism of functors*

$$[-, BG] \implies \mathcal{P}_G$$

given by sending a homotopy class of continuous maps $f : X \to BG$ to the pullback bundle

$$f^*EG \to X.$$

As we're about to see, one can occasionally obtain a smooth refinement of the above result. I know that, at the very least, this can be done for $G = \mathrm{O}(n)$, $G = \mathrm{SO}(n)$ and $G = \mathrm{U}(n)$ but I don't have any references for more general Lie groups G. The difficulty is that in order to properly do this one requires some infinite dimensional differential geometry.

Let's perform the construction for $G = \mathrm{O}(n)$. For this, we let $\mathrm{O}_n(\mathbb{R}^{\oplus \mathbb{N}})$ denote the set of linear maps $f : \mathbb{R}^n \to \mathbb{R}^{\oplus \mathbb{N}}$ so that for all $x, y \in \mathbb{R}^n$ we have

$$\langle f(x), f(y) \rangle = \langle x, y \rangle$$

where on both sides, $\langle -, - \rangle$ denotes the dot product. Note that the dot product on $\mathbb{R}^{\oplus \mathbb{N}}$ only induces an injection

$$\mathbb{R}^{\oplus \mathbb{N}} \hookrightarrow (\mathbb{R}^{\oplus \mathbb{N}})'$$

but not a surjection (here $(\mathbb{R}^{\oplus \mathbb{N}})'$ denotes the collection of bounded linear functionals with respect to the locally convex topology from the direct sum). We also denote by $\mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}})$ the set of all k-dimensional linear subspaces of $\mathbb{R}^{\oplus \mathbb{N}}$ (an infinite Grassmannian). One can view this as a quotient

$$\mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}}) \cong \mathrm{O}_n(\mathbb{R}^{\oplus \mathbb{N}}) / \mathrm{O}(n)$$

by the action given by pre-composition. We then have the following result from Kriegl and Michor's book [38].

Theorem 2.3.3. *The sets $\mathrm{O}_n(\mathbb{R}^{\oplus \mathbb{N}})$ and $\mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}})$ come naturally equipped with the structure of infinite dimensional smooth, in fact real analytic, manifolds (in the sense of Kriegl and Michor). They are both smoothly paracompact and the quotient map*

$$\mathrm{O}_n(\mathbb{R}^{\oplus \mathbb{N}}) \to \mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}})$$

makes them into a real analytic principal $\mathrm{O}(n)$-bundle. Furthermore, they can be written as directed colimits, both in the smooth and real analytic categories:

$$\mathrm{O}_n(\mathbb{R}^{\oplus \mathbb{N}}) \cong \varinjlim_{k \geq n} \mathrm{O}(\mathbb{R}^n, \mathbb{R}^k) \text{ and } \mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}}) \cong \varinjlim_{k \geq n} \mathrm{Gr}_n(\mathbb{R}^k).$$

One nice remark is that for any (possibly infinite dimensional) smooth manifold M in the sense of Kriegl and Michor, the set of smooth functions $C^\infty(M)$ from M to $\mathbb{R}$ is naturally a point-determined, but not necessarily finitely generated, C^∞-ring. However, many of the properties we would expect to hold, having seen the finite dimensional case, fail to hold unless M is smoothly paracompact.

For example, smoothly paracompact infinite dimensional manifolds still admit smooth partitions of unity (by definition) and are therefore both smoothly normal and smoothly regular. This implies, for example, that the germ of any smooth function along a closed subset of M has a representative given by a globally defined smooth function on M. It also implies that M is naturally a *Frölicher space* since a map $N \to M$ for N any other smooth manifold is smooth if and only if the post-compositions $g \circ f$ for $g \in C^\infty(M)$ are all smooth (this fact actually only requires M to be smoothly regular). Unfortunately, one can notice that our proof of Milnor's exercise doens't carry over to the infinite dimensional setting since we used the existence of a smooth function on M which is unbounded on every closed but not compact subset. I am not sure whether such a function exists on an infinite dimensional (smoothly paracompact) manifold.

One of the advantages of Kriegl and Michor's infinite dimensional differential geometry is the following refinement of our previous topological result.

Theorem 2.3.4. *[38] Given any finite-dimensional smooth manifold M there is a natural bijective correspondence between homotopy classes of smooth maps $M \to \mathrm{Gr}_n(\mathbb{R}^{\oplus \mathbb{N}})$ and isomorphism classes of smooth* *principal* $\mathrm{O}(n)$*-bundles on* M *given by pulling back the bundle* $\mathrm{O}_n(\mathbb{R}^{\oplus\mathbb{N}}) \to \mathrm{Gr}_n(\mathbb{R}^{\oplus\mathbb{N}})$*. Furthermore, the horizontal morphisms in hCW arising from the pullback diagram*

$$\begin{tikzcd} \mathrm{O}_n(\mathbb{R}^{\oplus\mathbb{N}}) \arrow[r] \arrow[d] & E\,\mathrm{O}(n) \arrow[d] \\ \mathrm{Gr}_n(\mathbb{R}^{\oplus\mathbb{N}}) \arrow[r] & B\,\mathrm{O}(n) \end{tikzcd}$$

yield isomorphisms of (topological) principal $\mathrm{O}(n)$*-bundles. As such, we will denote these infinite dimensional smooth manifolds as* $E\,\mathrm{O}(n)$ *and* $B\,\mathrm{O}(n)$ *from now on.*

Now, it's worth recalling that a G-structure on a smooth manifold M is not simply a principal G-bundle on M. This is because $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ is a closed Lie subgroup and we only allow principal G-subbundles of the frame bundle. It turns out there there is a formalism describing G-structures in a way analogous to the way $EG \to BG$ classifies principal G-bundles and, furthermore, one can encode the obstructions to formal integrability for G-structures using a formalism analogous to Chern-Weil theory [1]. For intuition, let's briefly describe the usual Chern-Weil theory.

Recall that every vector bundle on a finite dimensional smooth manifold admits a smooth fiber metric. A choice of such a metric then yields a principal $\mathrm{O}(k)$-bundle on the base manifold (where k is the rank of the vector bundle) whose fiber over a point it the collection of orthogonal frames for the fiber of the vector bundle over that point. This tells us that, up to isomorphism, rank k smooth vector bundles on M are classified by homotopy classes of smooth maps into $B\,\mathrm{O}(k)$

$$\begin{tikzcd} & E\,\mathrm{O}(k) \times_{\mathrm{O}(k)} \mathbb{R}^k \arrow[d] \\ M \arrow[r] & B\,\mathrm{O}(k) \end{tikzcd}$$

again by pulling back the universal bundle $E\,\mathrm{O}(k) \times_{\mathrm{O}(k)} \mathbb{R}^k$. In the study of vector bundles, one often makes use of certain topological invariants called *characteristic classes*. These are natural transformations of functors

$$\mathcal{P}_{\mathrm{O}(k)} \cong [-, B\,\mathrm{O}(k)] \implies H^n_{sing}(-, A)$$

for $n \geq 0$ and A an abelian group. As it turns out, the functors $H^n_{sing}(-, A)$ are also representable by CW-complexes. Again, our simplicial-set technology allows us to construct them easily.

Interpret A as a discrete topological abelian group and recall that the total space of $EA \to BA$ is contractible. Thus by the long exact sequence in homotopy associated to a fibration we have

$$\pi_{n+1}(A) \cong \pi_n(BA)$$

for all $n \geq 0$. In particular, BA is an **Eilenberg-Maclane space**:

$$BA \cong K(A, 1)$$

since

$$\pi_n(BA) \cong \begin{cases} A & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can show that "B" is a functor from topological groups to spaces which preserves products up to natural homeomorphism. As such, BA is again an abelian topological group and so we can form $B^2A := B(BA)$. This gives us an iterative construction of the Eilenberg-Maclane spaces as:

$$K(A, n) = B^n A.$$

Furthermore, we can see from the long exact sequence in homotopy associated to a fibration that in hCW we have natural isomorphisms

$$\Omega K(A, n+1) \cong K(A, n)$$

for each $n \geq 0$ where $\Omega K(A, n+1)$ denotes the based loop space and $K(A, n+1)$ has basepoint the identity in $B^{n+1}A$. The significance of these spaces is the following theorem.

Theorem 2.3.5. *[49] For $X \in \mathcal{WC}$ with the homotopy type of a CW-complex we have a natural isomorphism*

$$H^n_{sing}(X, A) \cong [X, K(n, A)]$$

for all $n \geq 0$. Thus the functor $H^n_{sing}(-, A)$ is represented by $[-, K(n, A)]$.

Now, by Yoneda's lemma the set of natural transformations

$$[-, B\,\mathrm{O}(k)] \implies H^n_{sing}(-, A) \cong [-, K(n, A)]$$

is in natural bijection with elements of

$$[B\,\mathrm{O}(k), K(n, A)] \cong H^n_{sing}(B\,\mathrm{O}(k), A).$$

Thus all characteristic classes for rank k vector bundles are given by pulling back elements of $H^*_{sing}(B\,\mathrm{O}(k), A)$ along the classifying maps. Furthermore, notice that the same thing happens if we replace the topological (in fact, Lie) group $\mathrm{O}(k)$ with any other topological group G.

Important to us will be the case $A = \mathbb{R}$. Recall that the de Rham theorem told us that on finite dimensional smooth manifolds we had a natural isomorphism

$$H^*_{dR}(-) \cong H^*_{sing}(-, \mathbb{R}).$$

It is shown by Kriegl and Michor that for, possibly infinite dimensional, smooth manifolds which are smoothly paracompact there is a well-behaved notion of de Rham cohomology which again agrees with both the singular cohomology and the cohomology of the locally constant sheaf valued in $\mathbb{R}$. In particular:

$$H^*_{dR}(B\,\mathrm{O}(k)) \cong H^*_{sing}(B\,\mathrm{O}(k), \mathbb{R}).$$

If we want to know what the characteristic classes with coefficients in $\mathbb{R}$ are for rank k vector bundles on finite dimensional smooth manifolds then it suffices to compute $H^*_{sing}(B\,\mathrm{O}(k), \mathbb{R})$. The computation of this follows easily from the Chern-Weil theorem.

Theorem 2.3.6. Chern-Weil [37]
Let G be any Lie group and write $\mathfrak{g}$ for its Lie algebra. Then we have a natural isomorphism

$$H^*_{sing}(BG, \mathbb{R}) \cong \mathrm{Sym}^*(\mathfrak{g}^*[-2])^G.$$

Furthermore, given a principal G-bundle P over a smooth manifold M and an element $p \in \mathrm{Sym}^*(\mathfrak{g}^*)^G$, the associated cohomology class in $H^*_{sing}(BG, \mathbb{R})$ is obtained by choosing any principal G-connection on P and inserting its curvature into the polynomial p.

Now, consider the case $G = \mathrm{GL}(\mathbb{R}^n)$. Here we have an inclusion of sets $\mathrm{GL}(\mathbb{R}^n) \subseteq \mathfrak{gl}(\mathbb{R}^n)$ and so elements of

$$H^*_{sing}(B\,\mathrm{GL}(\mathbb{R}^n), \mathbb{R}) \cong \mathrm{Sym}^*(\mathfrak{gl}(\mathbb{R}^n)^*[-2])^{\mathrm{GL}(\mathbb{R}^n)}$$

define functions $\mathrm{GL}(\mathbb{R}^n) \to \mathbb{R}$ invariant under conjugation. In other words, cohomology classes in $H^*_{sing}(B\,\mathrm{GL}(\mathbb{R}^n), \mathbb{R})$ define class functions on $\mathrm{GL}(\mathbb{R}^n)$ and, perhaps more importantly, any subgroup of $\mathrm{GL}(\mathbb{R}^n)$.

Unfortunately, to make this next argument precise one should really work with some version of smooth stacks. In fact, since we will want our "algebras of functions" to be something like a differential graded C^∞-ring we'll really need a proper notion of derived C^∞-stack. Since we do not have time to develop this here the next argument will be for intuition only.

One can show that the "stacky" quotient $[G/_{Ad}G] \simeq EG \times^h_{Ad} G$ of a Lie group G by its self-action via conjugation is given by

$$[G/_{Ad}G] \simeq \mathcal{L}(BG)$$

where $\mathcal{L}(-)$ is what is called the (derived) *free loop space functor*. Now, any cohomology class $p \in H^*_{sing}(B\,\mathrm{GL}(\mathbb{R}^n), \mathbb{R})$ defines a class functions $p : G \to \mathbb{R}$ as described above and thus a map

$$\mathcal{L}(BG) \simeq [G/_{Ad}G] \xrightarrow{p} \mathbb{R} = K(\mathbb{R}, 0)$$

where we are treating $\mathbb{R}$ as a group only (i.e. as a discrete group). Thus we have a map

$$H^*_{sing}(B\,\mathrm{GL}(\mathbb{R}^n), \mathbb{R}) \to H^0(\mathcal{L}(BG), \mathbb{R}).$$

Now, given a principal G-bundle on a manifold M defined by a smooth map

$$f : M \to BG$$

we can apply the functor $\mathcal{L}(-)$ to obtain a map

$$\mathcal{L}(f) : \mathcal{L}(M) \to \mathcal{L}(BG)$$

and, via $p \in H^0(\mathcal{L}(BG), \mathbb{R})$, a cohomology class

$$\mathcal{L}(f)^*(p) \in H^0(\mathcal{L}(M), \mathbb{R}).$$

An example of such a construction is given by replacing $\mathrm{GL}(\mathbb{R}^n)$ with $\mathrm{GL}(\mathbb{C}^n)$ and letting $p : G \to \mathbb{C}$ be the trace map. This gives rise to the *Chern character*.

The point is that when we are looking at subgroups of $\mathrm{GL}(\mathbb{R}^n)$ or $\mathrm{GL}(\mathbb{C}^n)$ there are additional invariants living in the collection $H^0(\mathcal{L}(M), \mathbb{R})$ of globally defined "functions" on the derived free loop space. To make our claim that the Chern character arises in this way, let's make this precise in the special case of smooth manifolds.

The way a derived smooth stack would be defined is by taking the category of simplicial-set-valued presheaves on $s\,\mathrm{C}^\infty$-Ring and formally inverting (in a derived, or simplicial, sense) all morphisms which induce homotopy equivalences locally. Namely, one can define homotopy groups levelwise using the usual definition in $s\,\mathrm{Set}$ to obtain set-valued preasheaves of homotopy groups associated to any presheaf of simplicial sets. Morphisms which induce isomorphisms on the sheafifications of these presheaves (with respect to a natural Grothendieck topology on $h(s\,\mathrm{C}^\infty$-Ring)) are then called *local weak equivalences* and it is these with respect to which we perform a simplicial localization. This is described in more detail in Toen and Vezossi's *Homotopical Algebraic Geometry I* [65]. Although, there are some problems with $s\,\mathrm{C}^\infty$-Ring that must be dealt with before this can be done properly in differential geometry [6].

For us, we'll stick to simply working with $s\,\mathrm{C}^\infty$-Ring itself. The next two results are proven in [4] in the context of derived algebraic geometry. The proofs carry over verbatim to simplicial C^∞-rings and are only sketched here due to time constraints. As far as I know, these results are well-known in the folklore but no proof has ever been written-up formally. One can also prove these results by a slight modification of the analogous proof for commutative algebras over a field of characteristic zero which appears in [41]. One should note that these provide a new, more algebraic, proof of a result due to Connes [12] which says that the Hochshild homology groups of the topological algebra $C^\infty(M)$ are given by $\Omega^{-*}(M)$.

To do all of this, we'll need a nice description of the simplicial set S^1. In particular, we'll want a description which is compatible with the group structure on S^1. This is done as follows in [41].

Definition 2.3.7. The **cyclic category** Λ has the same objects $[n]$ as Δ and also the same face and degeneracy maps ϵ_i, σ_j. However, we also include the morphisms

$$\tau_n : [n] \to [n]$$

given by the cyclic permutation sending i to $i+1$ for $0 \le i \le n-1$ and sending $n \mapsto 0$.

Proposition 2.3.8. *The morphisms in Λ satisfy the following relations for $1 \le i \le n$:*

$$\begin{aligned} \tau_n^{n+1} &= \mathrm{id} \\ \tau_n \circ \epsilon_i &= \epsilon_{i-1} \circ \tau_{n-1} \\ \tau_n \circ \epsilon_0 &= \epsilon_n \\ \tau_n \circ \sigma_j &= \sigma_{j-1} \circ \tau_{n+1} \\ \tau_n \circ \sigma_0 &= \sigma_n \circ \tau_{n+1}^2. \end{aligned}$$

The key thing to notice is that while the objects of Δ had trivial automorphism groups, the objects of Λ have automorphism groups given by the cyclic groups

$$\mathrm{Aut}_\Lambda([n]) \cong \mathbb{Z}/(n+1).$$

These automorphisms allow the following.

Proposition 2.3.9. *There is a natural equivalence of categories $\Lambda \cong \Lambda^{op}$.*

Thus, unlike the case of the simplex category, the notions of a **cyclic object** $\Lambda^{op} \to \mathcal{C}$ and a **cocyclic object** $\Lambda \to \mathcal{C}$ coincide. Nonetheless, it is still useful to distinguish between them since we will often want to forget down to simplicial sets where there is still a distinction.

Let's now see how the non-triviality of the automorphism groups in Λ give rise to S^1-actions on the geometric realizations of the simplicial spaces they determine.

Lemma 2.3.10. *Every morphism $f : [n] \to [m]$ in Λ factors uniquely as an element of $\mathrm{Aut}_\Lambda([n])$ followed by an element of $\mathrm{Hom}_\Delta([n],[m])$.*

Definition 2.3.11. Let C_n denote the set $\mathrm{Aut}_{\Lambda^{op}}([n])$ and for each morphism $f : [n] \to [m]$ in Λ^{op} define a map

$$C(f) : \mathrm{Aut}_{\Lambda^{op}}([n]) \to \mathrm{Aut}_{\Lambda^{op}}([m])$$

by sending an automorphism g of $[n]$ to the part of $f \circ g$ which is an automorphism of $[m]$.

Proposition 2.3.12. *$C_\bullet$ as defined above is a cyclic set and its underlying simplicial set has geometric realization S^1.*

Proof. Functoriality follows from the uniqueness of our factorization in the above lemma. To see that $|C_\bullet| \cong S^1$ we notice that for $1 \le i \le n$ we have

$$\begin{aligned} C(\epsilon_i)(\tau_n) &= \tau_{n-1} \\ C(\sigma_j)(\tau_n) &= \tau_{n+1} \\ C(\epsilon_0)(\tau_n) &= \mathrm{id} \\ C(\sigma_0)(\tau_n) &= \tau_{n+1}^2. \end{aligned}$$

Now, $C_0 = \mathrm{Aut}_{\Lambda^{op}}([0])$ has only one element $* := \tau_0$ and it is non-degenerate and there is only one degeneracy

$$C(\sigma_0) : C_0 \to C_1$$

which satisfies $C(\sigma_0)(*) = \tau_1^2$. Thus in C_1 there is also only one non-degenerate cell τ_0. But then, using our above relations we see that through combinations of $C(\sigma_j)$ for $1 \leq j \leq n$ and $C(\sigma_0)$ all of the elements of C_n for $n \geq 2$ are degenerate. Thus the geometric realization of the underlying simplicial set of $C_\bullet$ is obtained by gluing a single one simplex $[0,1]$ to itself along $0, 1 \in [0,1]$. Thus

$$|C_\bullet| \cong S^1$$

as required. □

The next step is to demonstrate that all cyclic sets come equipped with a natural S^1-action arising from the non-trivial automorphism groups.

Proposition 2.3.13. *The geometric realization of a cyclic set is equipped with a natural S^1-action where by "natural" we mean that morphims of cyclic sets give rise to S^1-equivariant continuous maps and the S^1-action on S^1 is precisely the group multiplication.*

Proof. Let X be a cyclic set and consider the auxiliary cyclic set

$$\begin{aligned} X' : \Lambda^{op} &\to \mathrm{Set} \\ [n] &\mapsto \mathrm{Aut}_{\Lambda^{op}}([n]) \times X_n \\ f &\mapsto ((z,x) \mapsto (f_*(z), (z^*(f))_*(x)). \end{aligned}$$

It then follows [41] that

$$\begin{aligned} \mathrm{ev} : X' &\to X \\ (z,x) &\mapsto z_* x \end{aligned}$$

is a morphism of cyclic spaces. The point of introducing X' is that its geometric realization ends up still being $S^1 \times |X|$ (up to natural homeomorphism) however X' has the added advantage over $C_\bullet \times X$ that the S^1-action is here given by a morphism of cyclic sets. Indeed, let

$$\pi_1 : X' \to C_\bullet$$

be the projection onto the first coordinate and

$$\begin{aligned} \pi_2 : X' &\to X \\ (z,x,y) &\mapsto (x, z^*(y)) \end{aligned}$$

be the twisted projection to the second coordinate (twisted by the S^1-action). Notice that indeed the cosimplicial space $\mathbb{\Delta}^\bullet$ is naturally a cosimplicial set via the permutation of the vertices. But now, one can show that

$$(|\pi_1|, |\pi_2|) : |X'| \to S^1 \times |X|$$

is a homeomorphism. We then define our S^1-action to be

$$\mathrm{ev} \circ (|\pi_1|, |\pi_2|)^{-1} : S^1 \times |X| \to |X|.$$

We omit the proof that this induces the usual group multiplication on S^1 when $X = C_\bullet$, but one can find this proof in [41]. □

Proposition 2.3.14. *Let M be a smooth manifold. Then*

$$H^*(N^\bullet(S^1 \otimes C^\infty(M))) \cong \Omega^{-*}(M).$$

Proof. We sketch this proof only. Letting $C_\bullet$ be the simplicial set modelling S^1 we see that

$$C_n \cong \mathbb{Z}/(n+1)\mathbb{Z}$$

and so

$$(S^1 \otimes C^\infty(M))_n \cong \amalg_{C_n} C^\infty(M) \cong C^\infty(M^{\times(n+1)}).$$

The normalized chain complex can be descibed by first setting (for $n \geq 2$)

$$\begin{aligned} &\partial_i : C^\infty(M^{\times(n+1)}) \to C^\infty(M^{\times n}) \\ &(\partial_i f)(p_1, \cdots, p_n) := \begin{cases} f(p_1, \cdots, p_{i-1}, p_i, p_i, p_{i+1}, \cdots, p_n) & \text{for } 1 \leq i \leq n \\ f(p_n, p_1, \cdots, p_n) \text{ for } i = n+1 \end{cases} \end{aligned}$$

and then computing, using our explicit formulae for the face and degeneracy maps on $C_\bullet$:

$$N^{-n}(S^1 \otimes C^\infty(M)) = \bigcap_{i=1}^{n} \ker(\partial_i) \subseteq C^\infty(M^{\times(n+1)})$$

and

$$\begin{gathered} b : N^{-n}(S^1 \otimes C^\infty(M)) \to N^{-n+1}(S^1 \otimes C^\infty(M)) \\ b = (-1)^n \partial_{n+1}. \end{gathered}$$

For $n = 1$ we have $\partial_1 = \Delta^* : C^\infty(M \times M) \to C^\infty(M) = \partial_2$ and so

$$H^0(N^*(S^1 \otimes C^\infty(M))) = C^\infty(M).$$

Meanwhile, by Hadamard's lemma,

$$H^{-1}(N^*(S^1 \otimes C^\infty(M))) \cong \ker(\Delta^*)/\ker(\Delta^*)^2 \cong \Omega^1(M).$$

We now illustrate the idea of how to prove the result for the remaining H^{-i}. This is done by induction on i and we'll sketch how it is done for general i by outlining how the $-i = 2$ case works. Here, we're essentially looking at functions $h(x, y, z)$ with

$$h(x, x, z) = 0 = h(x, y, y)$$

modulo functions of the form $f(z, x, y, z)$ where $f(x, y, z, w)$ satisfies

$$f(x, x, z, w) = f(x, y, y, w) = f(x, y, z, z) = 0.$$

By Hadamard's lemma we can write such h's as

$$h(x, y, z) = (x - y)(y - z)a(x, y, z)$$

for some function $a(x, y, z)$. It also tell's us that we can write such f's as

$$f(x, y, z, w) = (x - y)(y - z)(z - w)b(x, y, z, w)$$

for some function $b(x, y, z, w)$. So, we have

$$f(z, x, y, z) = (z - x)(x - y)(y - z)b(z, x, y, z).$$

But now, write

$$a(x, y, z) = (a(x, y, z) - a(z, y, x)) + a(z, y, x).$$

By Hadamard we can write

$$a(x,y,z) - a(z,y,x) = (x-z)c(x,y,z)$$

and so, in our cohomology group

$$[(x-y)(y-z)a(x,y,z)] = [-(x-y)(y-z)a(z,y,x)].$$

Thus, again in our cohomology group we have

$$[h(x,y,z)] = [-h(z,y,x)].$$

But then we can use $\ker(\Delta^*) / \ker(\Delta^*)^2 \cong \Omega^1(M)$ to obtain:

$$H^{-2}(N^\bullet(S^1 \otimes C^\infty(M))) \cong \Lambda^2_{C^\infty(M)}\Omega^1(M).$$

But then, by the fact that the isomorphism given by the smooth Serre-Swan theorem preserves the symmetric monoidal structure given by tensor-product of vector bundles it follows that

$$\Lambda^2_{C^\infty(M)}\Omega^1(M) \cong \Omega^2(M)$$

as required. □

The proof of the next result is identical to its algebro-geometric analogue in [4].

Proposition 2.3.15. *The natural map $S^1 \otimes C^\infty(M) \to \mathrm{Map}(S^1, S^1 \otimes C^\infty(M))$ arising from the group multiplication $S^1 \times S^1 \to S^1$ gives rise to the de Rham differential d at the level of cohomology via*

$$H^*(N^\bullet(\mathrm{Map}(S^1, S^1 \otimes C^\infty(M)))) \cong \Omega^{-*}(M) \otimes_{\mathbb{R}} H^*(S^1, \mathbb{R}) \cong \Omega^{-*}(M) \oplus \Omega^{-*}(M)[\eta]$$

with $\eta^2 = 0$ the generator of $H^1(S^1, \mathbb{R})$.

Proof. The point here is that the generator $\eta \in H^1(S^1, \mathbb{R})$ has degree -1 and squares to zero. It has degree -1 due to our grading conventions. Thus our map

$$\Omega^{-*}(M) \to \Omega^{-*}(M) \oplus \Omega^{-*}(M)[\eta]$$

is a degree -1 derivation of $\Omega^{-*}(M)$. The point is then that by the naturality of our S^1-action as well as the functoriality of $S^1 \otimes (-)$ it follows that this derivation commutes with all pullbacks along smooth functions. As such, it is (up to a constant multiple) given by the exterior derivative [34]. □

So, we conclude this chapter by answering the question: what was the point of this? The key point is that we now have a good idea of what the analogue of the de Rham complex should be for a general C^∞-ring A. Namely we should have

$$\Omega^*_A := H^{-*}(N^\bullet(S^1 \otimes A))$$

with de Rham differential given by the infinitesimal generator of the S^1-action via loop rotation. In the next chapter, we will make use of this computation to construct a differential graded Lie algebra suitable for the study of deformation functors arising from C^∞-rings.

Chapter 3

Applications

3.1 Prolongation

The contents of this section can be found in [62].

The goal here is to apply our results from the previous sections to the study of linear PDEs. The idea is the following. Suppose we had a linear partial differential operator

$$P : C^\infty_U \to C^\infty_U$$

of order r where U is some open neighbourhood of the origin in $\mathbb{R}^n$. Let's write

$$P = \sum_{0\leq|\alpha|\leq r} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

with respect to the standard coordinates on $\mathbb{R}^n$. Our goal is to determine whether solutions exist to the equation $Pu = f$ given f. In fact, we're going to be greedy and ask for even more. Suppose we were given numbers

$$b_\alpha \in \mathbb{R}$$

satisfying the linear equation

$$\sum_{0\leq|\alpha|\leq r} a_\alpha(0) b_\alpha = 0$$

and suppose we were given a smooth function f defined on some neighbourhood of $0 \in U$. We want to know whether there exists a smooth function u defined on some neighbourhood of $0 \in U$ such that

$$Pu = f$$

and

$$\left.\frac{\partial^{|\alpha|}u}{\partial x^\alpha}\right|_0 = b_\alpha \text{ for all } 1 \leq |\alpha| \leq r.$$

We know that in general such solutions need not exist since, for example, locally defined solutions to $du = \alpha$ exist if and only if $d\alpha = 0$. i.e. we need some compatibility conditions on f. Furthermore, we'd rather not simply say that the image of f in $\operatorname{coker}(P)$ is zero since that is cheating, as explained earlier. Notice that for the equation $du = \alpha$, the compatibility condition $d\alpha = 0$ came from differentiating the original equation. i.e.

$$d\alpha = d^2u = 0.$$

Similarly, if a smooth solution u to our general equation $Pu = f$ were to exist then for all $0 \leq |\beta| < \infty$ we would necessarily have

$$\frac{\partial^{|\beta|}(Pu)}{\partial x^{\beta}} = \frac{\partial^{|\beta|} f}{\partial x^{\beta}}.$$

For each m this gives us a new system of equations consisting of the above equations for all $0 \leq |\beta| \leq m$. By the product rule these end up also being linear partial differential equations and the system of all such equations for $0 \leq |\beta| \leq k$ is called the k'th **prolongation** of our original differential equation. Furthermore, just as our original equation $Pu = f$ imposed linear constraints on the first r partial derivatives of u at zero:

$$\sum_{0 \leq |\alpha| \leq r} a_{\alpha}(0) \frac{\partial^{|\alpha|} u}{\partial x^{\alpha}}\Big|_0 = 0,$$

these new equations impose *algebraic* constraints on the full Taylor expansion

$$\sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} u}{\partial x^{\alpha}}\Big|_0 x^{\alpha} \in \mathbb{R}[\![x_1, \cdots, x_n]\!].$$

This breaks up our problem into two steps:

1. understand the solvability of the algebraic, or *formal*, problem in the sense of finding formal power series satisfying the constraints coming from all prolongations;

2. understand which formal solutions come from actual solutions using techniques from functional analysis.

The second step will not be considered in this thesis. We'll content ourselves with the formal theory. Since we really need to work with differential operators

$$P : E_M \to F_M$$

between vector bundles, we should formalize the notion of a Taylor expansion of a section of a vector bundle on a manifold. This is precisely the notion of a *jet*.

Intuitively, one defines the r-jet of a smooth function $f : M \to N$ at $x \in M$ to be the equivalence class of smooth functions $g : M \to N$ modulo their Taylor expansions agreeing with that of f to r'th order at x in any coordinate chart centered at x (it turns out to suffice for this to happen in a single coordinate chart). Writing $J^r_x(M, N)_y$ for the set of all r-jets of smooth functions $f : M \to N$ at x with $f(x) = y$ one can see from Taylor's theorem that

$$J^r_x(M, \mathbb{R})_0 \cong C^{\infty}_{M,x} / \mathfrak{m}^{r+1}_{M,x}.$$

This motivates the following, less general but better suited for our purposes, definition.

Definition 3.1.1. Let M be a smooth manifold and $\Delta : M \to M \times M$ the diagonal map. We will denote by

$$\mathcal{J}_{\Delta} := \ker(\Delta^{\#}) \trianglelefteq C^{\infty}_{M \times M}$$

the sheaf of ideals defining the diagonal in $M \times M$. We then define the sheaf of r**-jets** of smooth functions leaving M to be:

$$J^r_M := \Delta^{-1}(C^{\infty}_{M \times M} / \mathcal{J}^{r+1}_{\Delta}).$$

This is a sheaf of $\mathbb{R}$-algebras on M. Writing $\pi_1, \pi_2 : M \times M \to M$ for the projections onto the first and second components respectively we obtain two sheaf maps

$$\pi_1^{\#} : \pi_1^{-1} C^{\infty}_M \to C^{\infty}_{M \times M} \text{ and } \pi_2^{\#} : \pi_2^{-1} C^{\infty}_M \to C^{\infty}_{M \times M}.$$

Quotienting the codomain by $\mathcal{J}_\Delta^{r+1}$ and applying the functor Δ^{-1} while using that $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}$ we obtain morphisms of sheaves of $\mathbb{R}$-algebras

$$\Delta^{-1}(\pi_1^\#), \Delta^{-1}(\pi_2^\#) : C_M^\infty \to J_M^r$$

defining two C_M^∞-module structures on J_M^r, the so-called **left and right**-module structures.

We now claim that J_M^r is actually the sheaf of smooth sections of a vector bundle over M. As we will be dealing with vector bundles identified with their sheaves of smooth sections throughout the rest of this thesis, it is probably prudent to describe some of the subtleties involved.

Proposition 3.1.2. *[39] Let M be a smooth manifold and $E \to F \to G$ be morphisms of vector bundles over M. The following are equivalent:*

1. *$E \to F \to G$ is an exact sequence of vector bundles over M;*
2. *$\Gamma(M, E) \to \Gamma(M, F) \to \Gamma(M, G)$ is an exact sequence of projective $C^\infty(M)$-modules;*
3. *$E_M \to F_M \to G_M$ is an exact sequence of sheaves of C_M^∞-modules which is split-exact in each stalk.*

Its worth mentioning now that locally free sheaves of C_M^∞-modules of finite rank are indeed the same thing as vector bundles. One direction of this is due to the softness of C_M^∞ while the other direction follows from the fact that projective modules over local rings are free.

Proposition 3.1.3. *[39] With respect to either the left or right C_M^∞-module structures on J_M^r, it is a locally free sheaf of finite rank and hence a vector bundle on M.*

An important notational convention we will adopt is that when tensoring other sheaves $\mathcal{F}_M$ of C_M^∞-modules with J_M^r we will write

$$\mathcal{F}_M \otimes_{C_M^\infty} J_M^r \ \text{ and } \ J_M^r \otimes_{C_M^\infty} \mathcal{F}_M$$

for the tensor products with respect to the left and right C_M^∞-module structure respectively. We will also often omit the subscript C_M^∞ on the tensor product when we believe it is implicit from the context (in order to avoid notational clutter).

Definition 3.1.4. Let M be a smooth manifold and E_M be a vector bundle on M. We define the bundle of ***r*-jets of sections** of E_M to be

$$J^r E_M := J_M^r \otimes E_M.$$

This is a vector bundle when equipped with the left C_M^∞-module structure on J_M^r. Furthermore, we denote

$$j^r := \Delta^{-1}(\pi_2^\#) \otimes \mathrm{id}_{E_M} : E_M \to J^r E_M$$

and notice that this is not a morphism of C_M^∞-modules with respect to the left C_M^∞-module structure.

If one looks at j^r in local coordinates then the definition becomes more illuminating (by applying Hadamard's lemma one can see that it is just the morphism taking the r'th order Taylor expansion). At this point we are ready to provide our first link between our new-found formalism and differential equations.

Proposition 3.1.5. *[39] A $\mathbb{R}$-linear morphism of sheaves $P : E_M \to F_M$ between vector bundles is a differential operator of r'th order if and only if it factors as*

$$\begin{tikzcd} J^r E_M \arrow[dr, "\phi_P"] & \\ E_M \arrow[u, "j^r"] \arrow[r, "P"] & F_M \end{tikzcd}$$

where $\phi_P : J^r E_M \to F_M$ is a morphism of vector bundles and such that there is no such factorization through $J^s E_M$ with $s < r$. Furthermore, the morphism ϕ_P satisfying the above is unique.

This allows us to provide a global description of the prolongation of a differential operator, independent of choices of coordinates.

Proposition 3.1.6. *[39] Let $P : E_M \to F_M$ be a differential operator of order r. Then for each $k \geq 0$ there exists a unique morphism of vector bundles $p_k(\phi_P) : J^{r+k} E_M \to J^k F_M$ called the k'th **prolongation** of ϕ_P making the following diagram commute*

$$\begin{array}{ccc} J^{r+k}E_M & \xrightarrow{p_k(\phi_P)} & J^k F_M \\ {\scriptstyle j^{r+k}}\uparrow & & {\scriptstyle j^k}\uparrow \\ E_M & \xrightarrow{\quad P \quad} & F_M \end{array}$$

For our purposes, it will end up sufficing to consider the case of homogeneous partial differential equations. As such, given a linear partial differential operator $P : E_M \to F_M$ of order r we introduce the following notation. We will write $p_0(\phi_P) := \phi_P$, $J^0 F_M := F_M$, $R^k := \ker(p_k(\phi_P))$. As mentioned near the start of this section, we will be studying here the existence of formal power series "solutions" to the homogeneous equation $\ker(P)$. i.e. we will want a sequence of local sections of the R^k's for each k which are compatible in some sense. Indeed, let's describe how to obtain local sections of R^k from local sections of R^{k+1}.

For $k \geq \ell$ we have $\mathcal{J}^k_\Delta \subseteq \mathcal{J}^\ell_\Delta$ and so there is a quotient map

$$J^k_M \xrightarrow{\pi^{k,\ell}} J^\ell_M \to 0.$$

Since exact sequences of vector bundles and morphisms in between them are split exact, it follows that by tensoring on the right with E_M we obtain

$$J^k E_M \xrightarrow{\pi^{k,\ell}} J^\ell E_M \to 0.$$

The idea to obtain sections of R^ℓ from R^k for $k \geq \ell$ is to then simply apply $\pi^{k,\ell}$ to the sections in R^k. Of especial importance is the case $k = \ell + 1$.

Proposition 3.1.7. *[39] The kernel of the map $J^{r+1}E_M \to J^r E_M$ is naturally isomorphic to $\mathrm{Sym}^{r+1} T^*_M \otimes E_M$ and the sequence*

$$0 \to \mathrm{Sym}^{r+1} T^*_M \otimes E_M \to J^{r+1} E_M \to J^r E_M \to 0$$

is an exact sequence of vector bundles.

An important remark which follows from the exactness of the above sequence is that not all sections of $J^r E_M$ come from E_M. This is intuitively obvious since a section of $J^r E_M$ could be non-zero while having its projection down to E_M being zero. This cannot happen for sections of E_M since if the values of such a section vanish on a neighbourhood then so do all of its derivatives. I felt obliged to mention this, however, since it was a point of confusion for me when I was first learning this.

Using the above exact sequence together with the prolongations of ϕ_P together with the *symbols* of ϕ_P (the analogues of ϕ_P acting on the $\mathrm{Sym}^{r+1} T^*_M \otimes E_M$-part) we'll obtain obstructions to extending sections of R^k to R^{k+1}. As we'll see, this is related to the notion of an overdetermined system of linear PDEs.

Definition 3.1.8. Let $P : E_M \to F_M$ be a differential operator of order r. Then for $k \geq 0$ we define the k'th **symbol** of P to be the unique morphism $\sigma_k(\phi_P)$ of vector bundles making the following diagram with exact rows commute:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Sym}^{r+k} T_M^* \otimes E_M \arrow[r] \arrow[d, "\sigma_k(\phi_P)"] & J^{r+k}E_M \arrow[r] \arrow[d, "p_k(\phi_P)"] & J^{r+k-1}E_M \arrow[r] \arrow[d, "p_{k-1}(\phi_P)"] & 0 \\
0 \arrow[r] & \operatorname{Sym}^{k} T_M^* \otimes F_M \arrow[r] & J^{k}F_M \arrow[r] & J^{k-1}F_M \arrow[r] & 0
\end{tikzcd}$$

where $p_{-1}(\phi_p) := 0$. Often, $\sigma_0(\phi_P) =: \sigma(P)$ is called the **principal symbol** of P.

From the above diagram we can obtain our desired obstruction to the surjectivity of the morphism $R^{k+1} \to R^k$ by the Snake lemma. Indeed, if we denote

$$g^{k+1} := \ker(\sigma_{k+1}(\phi_P))$$

then the Snake lemma yields a morphism $R^k \to \operatorname{coker}(\sigma_{k+1}(\phi_P))$ making the following sequence exact

$$g^{k+1} \to R^{k+1} \to R^k \to \operatorname{coker}(\sigma_{k+1}(\phi_P)) \to \operatorname{coker}(p_{k+1}(\phi_P)) \to \operatorname{coker}(p_k(\phi_P)).$$

This hints at the fact that the obstruction to surjectivity is one of a (co)homological nature. The cohomology theories governing obstructions such as the one above are given by the various Spencer complexes which we will describe in due time.

For now, we will take notice of the following "coincidence". Let $p \in M$, E_M a vector bundle on M and $r \geq 0$. Since the sequence

$$0 \to \operatorname{Sym}^{r+1} T_M^* \otimes E_M \to J^{r+1}T_M^* \to J^r T_M^* \to 0$$

is an exact sequence of vector bundles it induces an exact sequence of vector spaces upon looking at the fibres over p (i.e. taking stalks at p and then applying $\mathbb{R} \otimes_{C^\infty_{M,p}} (-)$):

$$0 \to \operatorname{Sym}^{r+1} T_p^* M \otimes E_p \to J_p^{r+1}(E) \to J_p^r(E) \to 0.$$

If we were to suppose that E_M was the trivial line bundle C^∞_M then the above sequence becomes

$$0 \to \mathfrak{m}_{M,p}^{r+1}/\mathfrak{m}_{M,p}^{r+2} \to C^\infty_{M,p}/\mathfrak{m}_{M,p}^{r+2} \to C^\infty_{M,p}/\mathfrak{m}_{M,p}^{r+1} \to 0.$$

If we then assume M is n-dimensional then we saw earlier (Borel's theorem) that $C^\infty_{M,p}/\mathfrak{m}^\infty_{M,p} \cong \mathbb{R}[\![x_1, \cdots, x_n]\!]$ and since $\mathfrak{m}^\infty_{M,p} \subseteq \mathfrak{m}^r_{M,p}$ for all r this is precisely the sequence

$$0 \to (x_1, \cdots, x_n)^{r+1}/(x_1, \cdots, x_n)^{r+2} \to \mathbb{R}[\![x_1, \cdots, x_n]\!]/(x_1, \cdots, x_n)^{r+2} \\ \to \mathbb{R}[\![x_1, \cdots, x_n]\!]/(x_1, \cdots, x_n)^{r+1} \to 0.$$

But since quotienting $\mathbb{R}[\![x_1, \cdots, x_n]\!]$ by these ideals yields the same result as if we had quotiented $\mathbb{R}[x_1, \cdots, x_n]$ by these ideals, it follows that we have an example of what is called a *thickening* of Weil algebras. These are the replacements of short exact sequences in the category of $\mathbb{R}$-algebras when studying deformations of $\mathbb{R}$-algebras.

Now, in deformation theory one would typically apply a *deformation functor* to the above "exact sequence" of $\mathbb{R}$-algebras and then, by taking the associated derived deformation functor, one would obtain a long exact sequence in cohomology describing obstructions to lifting deformations over Spec of the right-most term in the sequence to deformations over Spec of the middle term. For us, these thickenings form the fibres of a short exact sequence of vector bundles and instead of applying a deformation functor, we intersect our sequence down to the sheaves of formal solutions to our homogeneous PDE obtaining

$$g^{r+1} \to R^{r+1} \to R^r$$

and the obstructions to lifting r'th order formal solutions to $(r+1)$'st order ones are again obtained by looking at a "long exact sequence in cohomology".

Again, due to time constraints we merely state the following results.

Proposition 3.1.9. *Let A be a germ-determined C^∞-ring and E a projective A-module. Define*

$$J^\infty E := \left(A \amalg A / \bigcap_{k=1}^{\infty} \ker(\nabla)^k \right) \otimes_A E$$

where $\nabla : A \amalg A \to A$ is the fold map. Then the extension of $J^\infty E$ to $S^1 \otimes A$ via $A \to S^1 \otimes A$ is naturally S^1-equivariant and the induced flat connection is precisely the Spencer differential from [62]. Similarly, the extension of $\mathrm{Sym}_A \Omega^1_A \otimes_A E$ to $S^1 \otimes A$ is S^1-equivariant and the result flat connection is again the Spencer differential. The resulting complexes of sheaves on $\mathrm{C}^\infty\mathrm{Spec}(A)$ arising from the flat connection are resolutions of the sheaf associated to E.

3.2 Deformation Theory

The results of this section are essentially adaptations of [46, 47].

In this section we'll study deformations of three different types of objects:

1. algebras over a field k of characteristic 0;
2. C^∞-rings;
3. compact complex manifolds.

The reason for including this section is as follows. Firstly, it presents an application of some of the high-powered machinery we've developed in Chapters 1 and 2. Secondly, it provides us with an excuse to introduce the Schouten-Nijenhuis bracket, also simply known as the *Schouten bracket*. As we'll see in section 3.4, the Schouten bracket will subsume both the Frölicher-Nijenhuis bracket and the Nijenhuis-Richardson bracket. As discussed at the end of section 1.1 the Frölicher-Nijenhuis bracket is involved in several mysterious "coincidences" related to the integrability problem for G-structures.

To begin, we ask: what is a deformation of a k-algebra A? The trick is to think of A geometrically as

$$\mathrm{Spec}(A) \in \mathrm{Comm}_k^{op}.$$

Namely, we'll interpret A as the affine scheme $\mathrm{Spec}(A)$ over $\mathrm{Spec}(k)$. A **deformation** of A over a **base** or **parameter space** with distinguished point

$$\mathrm{Spec}(k) \to \mathrm{Spec}(B)$$

should then be thought of intuitively as a fiber bundle of sorts $\mathrm{Spec}(C)$ over $\mathrm{Spec}(B)$ with fiber $\mathrm{Spec}(A)$ over the distinguished point. More precisely, we want a sequence of morphisms

$$\mathrm{Spec}(A) \to \mathrm{Spec}(C) \to \mathrm{Spec}(B)$$

such that the induced map into the fibered product

$$\mathrm{Spec}(A) \to \mathrm{Spec}(k) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(C)$$

is an isomorphism. Dualizing this back to the category Comm_k we see that a deformation of A should be a sequence of k-algebra homomorphisms

$$B \to C \to A$$

such that the induced map out of the pushout

$$k \otimes_B C \to A$$

is an isomorphism.

Example 3.2.1. Let $p(\vec{z}, \lambda) \in \mathbb{C}[\vec{z}, \lambda]$ be a polynomial and consider the composition

$$\mathbb{C}[\lambda] \to \mathbb{C}[\vec{z}, \lambda] \to \mathbb{C}[\vec{z}, \lambda]/(p(\vec{z}, \lambda)) =: C.$$

If we consider $\mathbb{C}[\lambda]$ as a pointed algebra via

$$\mathbb{C}[\lambda] \to \mathbb{C}[\lambda]/(\lambda) \cong \mathbb{C}$$

then we have

$$C \otimes_{\mathbb{C}[\lambda]} \mathbb{C} \cong \mathbb{C}[\vec{z}]/(p(\vec{z}, 0)) =: A.$$

In this way, C becomes a deformation of A over $\mathbb{C}[\lambda]$. Geometrically, we're considering the family of hypersurfaces $p(\vec{z}, \lambda) = 0$, indexed by te parameter $\lambda \in \mathbb{C}$, as a deformation of the hypersurface $p(\vec{z}, 0) = 0$.

The next key observation is that we shouldn't really distinguish between two deformations that agree on a neighbourhood of the distinguished point in $\mathrm{Spec}(B)$. This is the difference between a deformation of an algebra and a *family* of algebras. Related to this is the assumption we make that the distinguished point of $\mathrm{Spec}(B)$ is a *closed point*.

But what does it mean for two deformations to agree on a neighbourhood of a closed point of $\mathrm{Spec}(B)$? For this we need to understand what morphisms of deformations of a fixed algebra A are. Before that, we must first notice that the pullback of a deformation is again a deformation.

Indeed, suppose $f : \mathrm{Spec}(B_1) \to \mathrm{Spec}(B_2)$ was a morphism of pointed spaces and $\mathrm{Spec}(A) \to \mathrm{Spec}(C) \to \mathrm{Spec}(B_2)$ a deformation. The pullback of $\mathrm{Spec}(C)$ along f:

$$f^* \mathrm{Spec}(C) = \mathrm{Spec}(B_1) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(C)$$

is given algebraically by

$$B_1 \otimes_{B_2} C.$$

The fiber of this over the distinguished point in B_1 is then

$$k \otimes_{B_1} (B_1 \otimes_{B_2} C) \cong (k \otimes_{B_1} B_1) \otimes_{B_2} C \cong k \otimes_{B_2} C \cong A.$$

So indeed this is isomorphic to A but we don't yet have an obvious natural choice of morphism

$$B_1 \otimes_{B_2} C \to A$$

realizing this isomorphism. Luckily, we do have morphisms

$$(\mathrm{Spec}(C) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(B_1)) \times_{B_1} \mathrm{Spec}(k) \to \mathrm{Spec}(C) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(B_1) \to \mathrm{Spec}(C)$$

and

$$(\mathrm{Spec}(C) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(B_1)) \times_{\mathrm{Spec}(B_1)} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$$

which, when respectively post-composed with

$$\mathrm{Spec}(C) \to \mathrm{Spec}(B_2) \text{ and } \mathrm{Spec}(k) \to \mathrm{Spec}(B_2)$$

yield the exact same morphism! Thus, by our universal property we have a natural map

$$(\mathrm{Spec}(C) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(B_1)) \times_{\mathrm{Spec}(B_1)} \mathrm{Spec}(k) \to \mathrm{Spec}(C) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(k).$$

This dualizes to a morphism

$$C \otimes_{B_2} k \to (C \otimes_{B_2} B_1) \otimes_{B_1} k.$$

Pre-composing this with the inverse of our isomorphism

$$C \otimes_{B_2} k \to A$$

yields our natural map

$$A \to (C \otimes_{B_2} B_1) \otimes_{B_1} k$$

realizing our isomorphism. As such, we have now proven the following result.

Proposition 3.2.2. *Let $f : \mathrm{Spec}(B_1) \to \mathrm{Spec}(B_2)$ be a morphism of pointed affine k-schemes. Then the pullback of a deformation over $\mathrm{Spec}(B_2)$ is a deformation over $\mathrm{Spec}(B_1)$.*

With this proposition in hand, we can now define a morphism of deformations.

Definition 3.2.3. Let $\mathrm{Spec}(A) \to \mathrm{Spec}(C_1) \to \mathrm{Spec}(B_1)$ and $\mathrm{Spec}(A) \to \mathrm{Spec}(C_2) \to \mathrm{Spec}(B_2)$ be two deformations of a k-algebra A. A **morphism** from the deformation C_1 to the deformation C_2 is a morphism of pointed affine k-schemes

$$\mathrm{Spec}(B_1) \to \mathrm{Spec}(B_2)$$

such that the induced map

$$\mathrm{Spec}(C_1) \to \mathrm{Spec}(B_1) \times_{\mathrm{Spec}(B_2)} \mathrm{Spec}(C_2)$$

is an isomorphism.

There is one problem with the above definition. Namely, we only wanted to consider deformations up to agreeing on some open neighbourhood of the distinguished point in the parameter space. As such, we really only want to consider germs of morphisms $\mathrm{Spec}(B_1) \to \mathrm{Spec}(B_2)$ around the distinguished point. Luckily for us, there is an easy way to do this:

> from now on, our parameter spaces $\mathrm{Spec}(B)$ will be assumed to be given by local k-algebras $(B, \mathfrak{m}_B)$ with residue field k and distinguished point given by the morphism $B \to B/\mathfrak{m}_B \cong k$.

Deformations of C^∞-rings over local C^∞-rings with residue field $\mathbb{R}$ can then be defined in exactly the same way, only now we assume $k = \mathbb{R}$ and all of our pullbacks are computed in $\mathrm{C}^\infty\text{-Ring}^{op}$ instead of Comm_k^{op}.

As for deformations of compact complex manifolds we do something slightly different. Recall that we wanted to think of our morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ as a fiber bundle of sorts. Well, a holomorphic map $Y \to B$ between complex manifolds is certainly not a fiber bundle in general. Furthermore, we also want the fibers to be compact. This motivates the following definition.

Definition 3.2.4. A **deformation** of a compact complex manifold X over a pointed complex manifold (B, b) is a sequence of holomorphic maps

$$X \to Y \to B$$

such that $Y \to B$ is a proper holomorphic submersion and the induced map

$$X \to \{b\} \times_B Y = Y_b$$

is a biholomorphism.

Properness is included in our assumptions to ensure the compactness of the fibers while we assume $Y \to B$ is a submersion to ensure, via *Ehressman's theorem* (see [46]), that $Y \to B$ is a locally trivial fibration with complex manifolds as fibers. Again, we only consider deformations up to (biholomorphic) isomorphism on a neighbourhood of the distinguished point in the base.

Let's now describe the key problem in deformation theory. Let $\mathcal{B}$ be some suitable category of base/parameter spaces. For example, $\mathcal{B}$ could be any of the following:

- the category of local k-algebras with residue field k and local homomorphisms as morphisms;
- the category of local C^∞-rings with residue field $\mathbb{R}$ and local C^∞-ring morphisms;
- the category of germs of pointed complex manifolds and germs of pointed holomorphic maps.

We then let $\mathcal{C}$ denote some category containing the non-pointed versions of the objects of $\mathcal{B}$. For example, $\mathcal{C}$ could be any of the following:

- the category of commutative unital k-algebras;
- the category of C^∞-rings;
- the category of compact complex manifolds.

Given an object $X \in \mathcal{C}$ we then encode the deformations of X in a functor

$$F_X : \mathcal{B} \to \mathrm{Set}$$

assigning to an object of $\mathcal{B}$ the set of isomorphism classes of deformations of X over that base. Sometimes, if we don't wish to quotient-out by isomorphism, we will make F_X groupoid-valued or simplicial-set valued.

The general problem is now the following:

can we find some object T of $\mathcal{B}$ (or some generalization of the category $\mathcal{B}$) together with an object S of $\mathcal{C}$ (or, again, some generalization thereof) and a morphism $S \to T$ such that

$$F_X \cong \mathrm{Hom}(-, T)$$

as functors with the natural isomorphism given by pullback of S.

In other words: is there a *universal* or *classifying* deformation? The first step we take towards understanding this problem is by asking ourselves what *infinitesimal* or *formal* deformations can tell us.

An **infinitesimal deformation** of a k-algebra A should simply be a deformation of A over a parameter space of dimension zero. Since the notion of dimension for algebras is best-behaved when things are Noetherian (see [22]) we will, from now on, assume that the algebras determining our base spaces are Noetherian.

Proposition 3.2.5. *Any commutative local Noetherian k-algebra of Krull dimension zero with residue field k is a Weil algebra, i.e. a quotient of $k[x_1, \cdots, x_n]$ by an ideal containing some power of the irrelevant ideal $(x_1, \cdots, x_n)$.*

Proof. Let $(A, \mathfrak{m})$ be any commutative local Noetherian k-algebra of Krull dimension zero with residue field k. A standard result from commutative algebra tells us that a commutative unital Noetherian ring has Krull dimension zero if and only if it is Artinian hence A is Artinian. But then the descending chain of ideals

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \cdots$$

necessarily stabilizes. Hence there is some n such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. But $\mathfrak{m} = J(A)$ is the Jacobson radical and

$$\mathfrak{m}^n = \mathfrak{m}^{n+1} = \mathfrak{m}\mathfrak{m}^n = J(A)\mathfrak{m}^n$$

so by Nakayama's lemma we have $\mathfrak{m}^n = 0$. But now, since A is Noetherian we can write $\mathfrak{m} = (a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in A$ and furthermore, since $A/\mathfrak{m} \cong k$, we have $A \cong k \oplus \mathfrak{m}$ as a k-vector space. Since $\mathfrak{m}^n = 0$ we can conclude that A is a quotient of $k[x_1, \cdots, x_n]$ by some power of the irrelevant ideal $(x_1, \cdots, x_n)$, as required. $\square$

Definition 3.2.6. We denote by Art_k the category of all local Artinian k-algebras $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A$ isomorphic to k. Morphisms in Art_k will be pointed morphisms in the sense that they will be algebra homomorphisms $f : A \to B$ such that $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Right. So we now know precisely what an infinitesimal deformation of a k-algebra is. What about an infinitesimal deformation of a compact complex manifold?

We know from [26] that for any complex manifold X of (complex) dimension n and any point $p \in X$ we have

$$\mathcal{O}_{X,p} \cong \mathbb{C}\{z_1, \cdots, z_n\}$$

where $\mathcal{O}_{X,p}$ is a algebra of germs of holomorphic functions at p and $\mathbb{C}\{z_1, \cdots, z_n\}$ is the algebra of convergent power series.

Proposition 3.2.7. *[26] The algebra $\mathbb{C}\{z_1, \cdots, z_n\}$ is Noetherian.*

Notice how this is in stark contrast to the case of the ring $C_p^\infty(M)$ of germs of smooth functions. Now, for the purposes of finding appropriate parameter spaces for infinitesimal deformations of compact complex manifolds, it seems reasonable to consider all $\mathbb{C}$-algebras of the form

$$\mathbb{C}\{z_1, \cdots, z_n\}/I$$

of Krull dimension zero. However, since these algebras are all local rings as well, it follows that such algebras all Lie in $\mathrm{Art}_\mathbb{C}$. Furthermore, since $\mathbb{C}\{z_1, \cdots, z_n\}$ contains the polynomial algebra $\mathbb{C}[z_1, \cdots, z_n]$ it follows that this category of infinitesimal base spaces is precisely $\mathrm{Art}_\mathbb{C}$! So, intuitively a deformation of a compact complex manifold X should be a sequence of $\mathbb{C}$-algebra homomorphisms

$$A \to \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$$

where the last homomorphism is given by composition with a holomorphic map $X \to Y$ and $A \in \mathrm{Art}_\mathbb{C}$ with

$$\mathcal{O}_Y(Y) \otimes_A \mathbb{C} \cong \mathcal{O}_X(X)$$

via our natural map. Unfortunately a problem remains. We need an algebraic analogue of a "proper submersion". In algebraic geometry, one often talks about *flat families* [22] as an analogue of this. For us, we will do something slightly different. The key thing to remember is that, by Ehressman's theorem, the fibers of a deformation

$$X \to Y \to B$$

of compact complex manifolds sufficently close to the distinguished point $b \in B$ are all diffeomorphic, and so it is only the complex structure that varies under deformation.

Now, recall that a $\mathrm{GL}(\mathbb{C}^n)$-structure on a manifold X can be encoded in a tensor

$$J \in \Gamma(X, T_X^* \otimes T_X).$$

This makes T_X into a sheaf of complex vector spaces and so its complexification $T_{X,\mathbb{C}}$ decomposes into the eigenspaces of the action of J as

$$T_{X,\mathbb{C}} \cong T^{1,0} \oplus T^{0,1}$$

with $T^{1,0}$ being the $+i$-eigenspace and $T^{0,1}$ the $-i$-eigenspace. After dualizing and taking exterior powers we obtain decompositions:

$$\Lambda^k T^*_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q} T^*_X$$

and the complexification of the exterior derivative d on functions

$$d : C^\infty_{X,\mathbb{C}} \to T^*_{X,\mathbb{C}}$$

decomposes as

$$d = \partial + \overline{\partial}$$

where

$$\partial : C^\infty_{X,\mathbb{C}} \to \Lambda^{1,0} T^*_X \text{ and } \overline{\partial} : C^\infty_{X,\mathbb{C}} \to \Lambda^{0,1} T^*_X.$$

One can the prove the following proposition.

Proposition 3.2.8. *[26] Recall that the Nijenhuis tensor $N_J \in \Omega^2(X, TX)$ of the almost complex structure J is given by*

$$N_J(X,Y) := [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY].$$

If $N_J = 0$ then for all (p,q) the exterior derivative takes values in the subbundle

$$d : \Lambda^{p,q} T^*_X \to \Lambda^{p+1,q} T^*_X \oplus \Lambda^{p,q+1} T^*_X$$

thus $\overline{\partial} : \Lambda^{p,q} T^*_X \to \Lambda^{p,q+1} T^*_X$ is defined for all p, q. Furthermore, again in the case $N_J = 0$, we have $\overline{\partial}^2 = 0$.

The point of this is that one can encode the complex structure on a complex manifold very succinctly using J. Furthermore, one can prove that on a complex manifold, a function f is holomorphic if and only if $\overline{\partial} f = 0$. So, if we want to encode the notion that $A \to \mathcal{O}_Y(Y)$ "is" a proper holomorphic submersion in some way, it is convenient to make use of J and $\overline{\partial}$. Indeed, the following theorem describes how the complex structure is encoded by J.

Theorem 3.2.9. Newlander-Nirenberg*[24]*
An almost complex structure J arises from an atlas of holomorphic charts if and only if $N_J = 0$.

We are now in a position to describe what an infinitesimal deformation of a compact complex manifold is. First, given a complex manifold X we denote by $\mathcal{T}_X$ the sheaf of holomorphic vector fields on X. The module

$$\Omega^{0,*}(X, \mathcal{T}_X) := \Gamma(X, \Lambda^{0,*} T^*_X \otimes \mathcal{T}_X)$$

comes equipped with the natural structure of a differential graded Lie algebra where the graded Lie bracket is given in local coordinates by

$$\left[f d\overline{z}^I \otimes \frac{\partial}{\partial z^i}, g d\overline{z}^J \otimes \frac{\partial}{\partial z^j} \right] := d\overline{z}^I \wedge dz^J \left(f \frac{\partial g}{\partial z^i} \frac{\partial}{\partial z^j} - g \frac{\partial f}{\partial j^j} \frac{\partial}{\partial z^i} \right)$$

and the differential $\overline{\partial}$ given in local coordinates by

$$\overline{\partial} \left(\alpha \otimes \frac{\partial}{\partial z^i} \right) := (\overline{\partial} \alpha) \otimes \frac{\partial}{\partial z^i}.$$

The essential reason these are well-defined and, furthermore, make $\Omega^{0,*}(X, \mathcal{T}_X)$ into a differential graded Lie algebra is that sections of $\mathcal{T}_X$ are holomorphic and $\overline{\partial} f = 0$ for holomorphic functions f. A proof of all of this can be found in [46].

Definition 3.2.10. An **infinitesimal deformation** of a compact complex manifold over $A \in \mathrm{Art}_\mathbb{C}$ is an element

$$\alpha \in \Omega^{0,1}(X, \mathcal{T}_X) \otimes \mathfrak{m}_A$$

such that $[\bar{\partial}+\alpha, \bar{\partial}+\alpha] = 0$ where $[-,-]$ is the graded Lie bracket on derivations of the algebra $\Omega^{0,1}(X, \mathcal{T}_X)$ and α acts as a derivation via

$$\left(\beta \otimes \frac{\partial}{\partial z^i}\right)(f d\bar{z}^I) := \beta \wedge \frac{\partial f}{\partial z^i} d\bar{z}^I.$$

Finally we arrive at our main result for the deformation theory of compact complex manifolds. It is not a result guaranteeing the existence of a universal deformation unfortunately, but it instead guarantees the existence of a **semiuniversal deformation**, the definition of which can be found in [46].

Theorem 3.2.11. *[46] Let X be a compact complex manifold and write F_X for the functor which assigns to an object of $\mathrm{Art}_{\mathbb{C}}$ the set of infinitesimal deformations of X over that object up to isomorphism. If for every $n > 2$ the map*

$$F_X(\mathbb{C}[z]/(z^n)) \to F_X(\mathbb{C}[z]/(z^2))$$

obtained by applying F_X to the quotient map, is surjective then X admits a semiuniversal deformation over the base $H^1(X, \mathcal{T}_X) \ni 0$.

There are two important remarks to be made here. First, notice that not only did we glean information about actual deformations from infinitesimal ones, but we actually only needed to look at deformations over the bases $\mathbb{C}[z]/(z^n)$. The second remark is with regards to the condition $[\bar{\partial}+\alpha, \bar{\partial}+\alpha] = 0$ from before. One can show [26, 46] that the inner derivation $[\bar{\partial}, -]$ on the collection of derivations corresponds to the operator $\bar{\partial}$ on $\Omega^{0,*}(X, \mathcal{T}_X)$. In this way, using that $[-,-]$ is a graded Lie bracket, we can compute:

$$[\bar{\partial}+\alpha, \bar{\partial}+\alpha] = 2[\bar{\partial}, \alpha] + [\alpha, \alpha] = 2\bar{\partial}\alpha + [\alpha, \alpha]$$

and so our condition $[\bar{\partial}+\alpha, \bar{\partial}+\alpha] = 0$ becomes equivalent to the **Maurer-Cartan equations**:

$$\bar{\partial}\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

As we're about to see, these equations come up in the study of deformations of algebras as well.

For now, let's try to develop a better understanding of what morphisms in Art_k look like. The following definition and proposition will help with this.

Definition 3.2.12. A **small extension** in Art_k is a short exact sequence of k-vector spaces

$$0 \to I \to A \to B \to 0$$

where $A \to B$ is a morphism in Art_k, I is a not necessarily unital k-algebra and $I \to A$ is multiplicative; such that $\mathfrak{m}_A$ annihilates the image of I in A. Often it is the morphism $A \to B$ itself which is refered to as a small extension or we'll say that A is a small extension of B.

The prototypical example of a small extension is

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A^2 \to k \to 0$$

for $A \in \mathrm{Art}_k$. In fact, any object $A \in \mathrm{Art}_k$ can be shown to be obtainable by taking a finite number of iterated small extensions of k. Indeed, if n is the smallest natural number such that $\mathfrak{m}_A^{n+1} = 0$ then the following is a sequence of small extensions:

$$\begin{gathered}
0 \to\to \mathfrak{m}_A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A^2 \to k \to 0 \\
0 \to \mathfrak{m}_A^2/\mathfrak{m}_A^3 \to A/\mathfrak{m}_A^3 \to A/\mathfrak{m}_A^2 \to 0 \\
\vdots \\
0 \to \mathfrak{m}_A^{n-1}/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^{n-1} \to 0 \\
0 \to \mathfrak{m}_A^n \to A \to A/\mathfrak{m}_A^n \to 0.
\end{gathered}$$

It will be useful to us to prove the following more general result.

Proposition 3.2.13. *[46] Any surjective morphism in* Art_k *factors into a composition of a finite number of small extensions.*

Now, let's try to develop an explicit description of what infinitesimal deformations of an algebra look like. For starters, suppose we were given a k-algebra R whose deformations we want to study as well as a deformation

$$A \to S \to R$$

over the base $A := k[t]/(t^{n+1})$. This is a reasonable starting point considering the importance of such deformations in the context of complex geometry. It then follows from the fact that $S \otimes_A k \cong R$ as k-algebras that:

$$R \otimes_k A \cong (S \otimes_A k) \otimes_k A \cong S \text{ as } A\text{-modules.}$$

Therefore the A-module underlying S is just $R \otimes_k A$ which, in our present scenario, is given explicitly by

$$R \otimes_k A \cong R[t]/(t^{n+1}).$$

So we are really only varying the multiplication map by the parameter t. So, if

$$m : S \otimes_k S \to S$$

denotes the multiplication on S then we can expand it out in its components via the A-module isomorphism $S \cong R[t]/(t^{n+1})$ to get

$$m(r,s) = m_0(r,s) + m_1(r,s)t + \cdots + m_n(r,s)t^n$$

for all $r, s \in R$ where each m_i is a k-bilinear map

$$m_i : R \otimes_k R \to R.$$

Comparing coefficients of powers of t we see that a multiplication m on $R \otimes_k A \cong R[t]/(t^{n+1})$ given by a sequence $m_0, \cdots, m_n : R \otimes_k R \to R$ defines a deformation of R if and only if m_0 is the original multiplication on R, each m_i is commutative, and for each $0 \leq \ell \leq n$ we have

$$\sum_{\substack{i+j=\ell \\ i,j \geq 0}} m_j(m_i(a,b),c) = \sum_{\substack{i+j=\ell \\ i,j > 0}} m_j(a, m_i(b,c))$$

for all $a, b, c \in R$ (this is associativity). Since m_0 is the original multiplication we will denote it by concatenation. We can then re-arrange the above expression as:

$$m_\ell(ab,c) + m_\ell(a,b)c - m_\ell(a,bc) - am_\ell(b,c) = \sum_{\substack{i+j=\ell \\ i,j > 0}} \left(m_j(a, m_i(b,c)) - m_j(m_i(a,b),c) \right).$$

The claim is now that the above equations can be expressed in terms of the Maurer-Cartan equations. Indeed, the way in which deformations of complex manifolds were written using the Maurer-Cartan equations was by encoding the complex structure in a variant of the algebra of differential forms.

We know that for a C^∞-ring A, the cohomology of the normalized complex $N^\bullet(S^1 \otimes A)$ provides a good substitute for the algebra of differential forms. So, let's consider

$$N_\bullet(S^1 \otimes A) := N^{-\bullet}(S^1 \otimes A)$$

for a k-algebra A. This is known as the **Hochshild homology complex** of A. The following construction from [41] yields a complex which is naturally quasi-isomorphic to this one (and is what is classically called the Hochshild homology complex).

First, we define operators (here tensor products are over k)

$$
\begin{aligned}
\partial_i : A^{\otimes n+1} &\to A^{\otimes n} \\
(a_0, \cdots, a_n) &\mapsto \begin{cases} (a_0a_1, a_2, \cdots, a_n) & \text{if } i = 0 \\ (a_0, \cdots, a_ia_{i+1}, \cdots, a_n) & \text{if } 1 \leq i < n \\ (a_na_0, a_1, \cdots, a_{n-1}) & \text{if } i = n. \end{cases}
\end{aligned}
$$

We then set

$$C_n(A) := A^{\otimes n+1}$$

with differential

$$
\begin{aligned}
b : C_n(A) &\to C_{n-1}(A) \\
b &:= \sum_{i=0}^{n} (-1)^i \partial_i.
\end{aligned}
$$

If one recalls our explicit presentation of S^1 as a simplicial set then the following proposition becomes immediate.

Proposition 3.2.14. *[41] There is a natural quasi-isomorphism of complexes*

$$(C_\bullet(A), b) \to N_\bullet(S^1 \otimes A).$$

The homology of this complex is called the ***Hochshild homology*** *of A and denoted by $HH_\bullet(A)$.*

For the purposes of deformation theory it is not the Hochshild homology, but instead the Hochshild *cohomology* which is relevant. It's worth noting that there are two different objects which are often called the Hochshild cohomology of an algebra A and they are typicall denoted by [41]:

$$H^\bullet(A, A) \text{ and } H^\bullet(A, A^*).$$

The version we will work with here is $H^\bullet(A, A)$.

We begin by denoting

$$C^n(A, A) := \mathrm{Hom}_k(A^{\otimes n}, A)$$

where again the tensor products are over k. We then define a differential

$$
\begin{aligned}
b : C^n(A, A) &\to C^{n+1}(A, A) \\
b(f)(a_1, \cdots, a_{n+1}) &:= a_1 f(a_2, \cdots, a_{n+1}) \\
&\quad + \sum_{0<i<n+1} (-1)^i f(a_1, \cdots, a_ia_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} f(a_1, \cdots, a_n)a_{n+1}.
\end{aligned}
$$

Up to a sign (which varies depending on n) this differential can be identified with pre-composition by the Hochshild homology differential. Also, having just seen some explicit descriptions of deformations of an algebra, the above expressions should look familiar.

If we are to have any hope of obtaining some sort of Maurer-Cartan description of deformations, we should first obtain a differential graded Lie algebra. In the case of complex manifolds, if we counted holomorphic vector fields as having degree one then in reality it wasn't $\Omega^{0,*}(X, \mathcal{T}_X)$ which was a differential graded Lie algebra, but instead it was $\Omega^{0,*}(X, \mathcal{T}_X)[1]$. The same thing happens here.

Proposition 3.2.15. *[41] $C^\bullet(A, A)[1]$ is naturally a differential graded Lie algebra.*

Proof. We won't actually provide the proof here, but will instead simply state the definition of the graded Lie bracket. The differential is, as expected, given by b.

Suppose we were given $f \in C^k(A, A)$ and $g \in C^\ell(A, A)$. We define the bracket

$$[f, g] \in C^{k+\ell-1}(A, A)$$

by first defining the follow "composition" rule:

$$(f \bar{\circ} g)(a_1, \cdots, a_{k+\ell-1}) := \sum_{i=1}^{k} (-1)^{(k-1)(\ell-1)} f(a_1, \cdots, a_{i-1}, g(a_i, \cdots, a_{i+\ell-1}), a_{i+\ell}, \cdots, a_{k+\ell-1}).$$

Recalling that f is thought of as having degree $k-1$ while g is thought of as having degree $\ell - 1$, we then set:

$$[f, g] := f \bar{\circ} g - (-1)^{(k-1)(\ell-1)} g \bar{\circ} f.$$

One then simply computes to show that $(C^\bullet(A, A)[1], b, [-, -])$ is a differential graded Lie algebra. $\square$

We now return to deformations of an algebra R over $A := k[t]/(t^{n+1})$. The map

$$m_2 : A \otimes_k A \to A$$

defines an element $m_2 \in C^1(A, A)[1]^1 = C^2(A, A)$ and, via our explicit formula for the differential b, we have that our associativity equations for m_2 become

$$bm_2 = 0.$$

For our other equations, it becomes convenient to consider the differential graded Lie algebra

$$C^\bullet(A, A)[1] \otimes \mathfrak{m}_A$$

where again $A = k[t]/(t^{n+1})$. The graded Lie bracket here is defined using the graded Lie bracket on $C^\bullet(A, A)[1]$ together with the multiplication on $\mathfrak{m}_A$, while the differential b is extended $k[t]/(t^{n+1})$-linearly. Recalling that we had set

$$m := m_0 + m_1 t + \cdots + m_n t^n$$

we have now obtained the following result.

Proposition 3.2.16. *Assuming k has characteristic zero, m defines an associative multiplication on $A[t]/(t^{n+1})$ if and only if*

$$0 = bm + \frac{1}{2}[m, m]$$

in $C^\bullet(A, A)[1] \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $k[t]/(t^{n+1})$.

Furthermore, it is shown in [46, 47] that the obstructions to extending a deformation over $A \in \mathrm{Art}_k$ to a deformation over a small extension B of A in Art_k also live in the Hochshild cohomology. In fact, in these two papers partial criteria for representability of the full (non-infinitesimal) deformation functor associated to A are given.

Let's now link up $C^\bullet(A, A)[1]$ with our simplicial algebra $S^1 \otimes A$. Denoting $H^\bullet(A, A)[1]$ for the graded Lie algebra given by the cohomology of $C^\bullet(A, A)[1]$ it follows from the universal coefficient theorem (k is a field) that

$$H^\bullet(A, A) \cong \mathrm{Hom}_A(HH_\bullet(A), A).$$

This will be useful when defining a version of this differential graded Lie algebra for C^∞-rings.

There is one more important remark to make here: while the above product is associative it need not be commutative in general. To obtain a Maurer-Cartan description of commutative deformations one needs to look at *Harisson cohomology* [23].

As our final topic of this section we turn to C^∞-rings. By our above discussion, the graded Lie algebra which should intuitively control the deformation theory of a C^∞-ring A is given by

$$\mathrm{Hom}_A(H_*(N_\bullet(S^1 \otimes A)), A).$$

Indeed, we have the following proposition. As with the case of the computation of $H^*(N^\bullet(S^1 \otimes A))$, I believe this proposition is well-known but I could not find an explicit proof in the literature.

Proposition 3.2.17. *Consider the C^∞-ring $C^\infty(M)$ for M a smooth manifold. Then*

$$\mathrm{Hom}_{C^\infty(M)}(H_*(N_\bullet(S^1 \otimes C^\infty(M))), C^\infty(M)) \cong \Gamma(M, \Lambda^* T_M)$$

is the module of poly-vector-fields. Furthermore, $\Gamma(M, \Lambda^* T_M)[1]$ is naturally a differential graded Lie algebra.

Proof. We've already seen that $H^*(N^\bullet(S^1 \otimes C^\infty(M))) \cong \Omega^{-*}(M)$ and so the homological version is $\Omega^*(M)$. Dualizing this indeed yields the algebra of poly-vector-fields

$$\mathfrak{X}^*(M) := \Gamma(M, \Lambda^* T_M).$$

The graded Lie bracket on $\mathfrak{X}^*(M)[1]$ is called the **Schouten-Nijenhuis bracket** or **Schouten bracket** and is defined on vector fields $X_1, \cdots, X_k, Y_1, \cdots, Y_\ell$ by

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_\ell] := \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_\ell$$

where the "hat" denotes omission. If f is a function on M then we set

$$[f, X_1 \wedge \cdots \wedge X_k] := -\iota_{df}(X_1 \wedge \cdots \wedge X_k).$$

For the details of the proof that this is indeed a graded Lie bracket, see [52]. $\square$

As our final example of a deformation theory we mention Kontsevich's *Deformation Quantization of Poisson Manifolds* [36]. The paper [36] studied deformations of *star-products* on the algebra $C^\infty(M)$ where $(M, \{-,-\})$ is a Poisson manifold. If $\hbar$ denotes the formal deformation parameter, then he wanted the star product on $C^\infty(M)[\![\hbar]\!]$ (possibly after applying some "gauge transformation") to be given by

$$f \star g = fg + \frac{\hbar}{2}\{f, g\} \ \text{ modulo } \hbar^2.$$

As we might have expected by this point, it was $\mathfrak{X}^*(M)[\![\hbar]\!]$ which controlled the deformation theory. In fact, it is well-known [52] that if $\{-,-\}$ is a Poisson bracket on a manifold M, then $\{-,-\}$ defines an element $\pi \in \mathfrak{X}^2(M)$ via the equation

$$\{f, g\} = \pi(df \wedge dg) \ \text{ for all } f, g \in C^\infty(M).$$

Furthermore, one has $[\pi, \pi]_{SN} = 0$ where we write $[-,-]_{SN}$ to denote the Schouten bracket. Conversely, an element $\pi \in \mathfrak{X}^2(M)$ defines a Poisson bracket via the above formula if and only if $[\pi, \pi]_{SN} = 0$ [52].

The point of all of this is the following. We are trying to explain the "coincidence" that the Frölicher-Nijenhuis bracket detects first order formal integrability of G_2, $\mathrm{Spin}(7)$ and $\mathrm{GL}(\mathbb{C}^n)$-structures while it also detects the curvature of connections. We have seen that there is a similarity between the jet short exact sequences and small extensions in Art_k. In particular, the fibers of the jet sequences are precisely the small extensions of the form $A/\mathfrak{m}_A^{r+1}$ where A is a formal power series ring.

We've seen that, for the purposes of deformation theory, much is encoded in the algebras $k[t]/(t^{n+1})$. As such, it is reasonable to expect that much would be encoded in the jet sequences despite the fact that their fibers on modelled on very particular Weil algebras.

Finally we've seen the Schouten bracket encode an "integrability condition" of sorts for Poisson tensors. We know that there is a relationship between Poisson tensors and symplectic forms (every symplectic form gives rise to a Poisson bracket) and furthermore, we'll see in section 3.4 that the Schouten bracket actually *subsumes* both the Frölicher-Nijenhuis bracket and the Nijenhuis-Richardson bracket. But first, we should reformulate the integrability problem for G-structures in terms of some homological algebra.

3.3 Formal Integrability and the Spencer Complex

In this section, we'll see how to formulate the formal integrability problem for a G-structure in terms of the Spencer complex. To do this we'll need a slightly more general notion of a jet than that used in the previous chapter. [51] and [21] are the main references for this section.

Definition 3.3.1. Let M, N be smooth manifolds, $x \in M$, $y \in N$. Then we define the space of r**-jets** of smooth functions $M \to N$ with **source** x and **target** y to be the quotient of the set of smooth functions f from an open neighbourhood of x in M to an open neighbourhood of y in N such that $f(x) = y$ by the relation

$$f \sim_r g \text{ if and only if for all } \varphi \in C^\infty_{N,y} \text{ we have } \varphi \circ f - \varphi \circ g = 0 \in C^\infty_{N,p}/\mathfrak{m}^{r+1}_{N,p}.$$

We denote this set of equivalence classes by $J^r_x(M,N)_y$. Similarly, we set

$$\begin{aligned} J^r_x(M,N) &:= \bigsqcup_{y \in N} J^r_x(M,N)_y \\ J^r(M,N)_y &:= \bigsqcup_{x \in M} J^r_x(M,N)_y \\ J^r(M,N) &:= \bigsqcup_{(x,y) \in M \times N} J^r_x(M,N)_y. \end{aligned}$$

Given $f : M \to N$ with $f(x) = y$ we'll write the image of f in $J^r(M,N)$ as $j^r f(x)$.

Proposition 3.3.2. *Composition of jets*

$$\begin{aligned} J^r_x(M,N)_y \times J^r_y(N,P)_z &\to J^r_x(M,P)_z \\ (j^r f(x), j^r g(y)) &\mapsto j^r(g \circ f)(x) \end{aligned}$$

is well-defined.

Proof. Suppose we had jets $j^r f_1(x) = j^r f_2(x)$ and $j^r g_1(y) = j^r g_2(y)$ with $f_1(x) = y$, $g_1(y) = z$. Arbitrarily select a smooth function $\varphi \in C^\infty_{N,z}$. Then $\varphi \circ g_1, \varphi \circ g_2 \in C^\infty_{M,y}$ and since $j^r f_1(x) = j^r f_2(x)$ it then follows that

$$(\varphi \circ g_1) \circ f_1 - (\varphi \circ g_2) \circ f_2 \in C^\infty_{M,x}/\mathfrak{m}^{r+1}_{M,x}$$

and so $j^r(g_1 \circ f_1)(x) = j^r(g_2 \circ f_2)(x)$, as required. □

Proposition 3.3.3. *The sets $J^r(M,N)_y$, $J^r_x(M,N)$ and $J^r(M,N)$ all come naturally equipped with the structure of smooth manifolds such that the source, target and source-target maps into M, N and $M \times N$ respectively are surjective submersions making these not only into locally trivial fibrations over $M, N, M \times N$ respectively, but actually fibre bundles.*

Proof. First we consider $J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0$. By Taylor's theorem, elements of $J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0$ correspond to m-tuples

$$\left(\sum_{1 \leq |\alpha| \leq r} a^1_\alpha x^\alpha, \cdots, \sum_{1 \leq |\alpha| \leq r} a^m_\alpha x^\alpha \right)$$

where $x^1, \cdots, x^n$ are the standard coordinates on $\mathbb{R}^n$ and $a^k_\alpha \in \mathbb{R}$. Hence $J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0$ is a finite dimensional vector space, thus a manifold. Now, for $J^r(\mathbb{R}^n, \mathbb{R}^m)$ we can write

$$\begin{aligned} J^r(\mathbb{R}^n, \mathbb{R}^m) &\cong \mathbb{R}^n \times J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0 \times \mathbb{R}^m \\ j^r f(x) &\leftrightarrow (x, (j^r_{f(x)} \ell_{-f(x)}(f(x))) \circ j^r f(x) \circ (j^r_0 \ell_x(x)), f(x)) \end{aligned}$$

where ℓ_x denotes the map given by addition by x. In this way we also get a smooth manifold structure on $J^r(\mathbb{R}^n, \mathbb{R}^m)$ and furthermore this makes it into a fibre bundle over $\mathbb{R}^n \times \mathbb{R}^m$. To replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by general manifolds we then use the above to construct local charts centered at each point, making use of the fact that for $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ open neighbourhoods of the origin we have

$$J^r_0(U, V)_0 \cong J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0.$$

Furthermore, built into this construction is the fact that the source and target maps make these spaces into fibre bundles and the standard fibre of all of these bundles is $J^r_0(\mathbb{R}^n, \mathbb{R}^m)_0$. $\square$

Proposition 3.3.4. *The composition maps for jets are actually smooth maps.*

Proof. This is because they are given by composing the relevant tuples of polynomials and then forgetting all of the terms of order greater than r. $\square$

Something worth noticing is that for any smooth manifold M, $J^r(M,M)$ is naturally a category whose objects are the points of M and whose morphisms $x \to y$ for $x, y \in M$ are the elements of $J^r_x(M,M)_y$. As this category is actually a fibered manifold over $M \times M$ we see that the subcategory of $J^r(M,M)$ with the same objects but only allowing those morphisms which are isomorphisms is an open submanifold of $J^r(M,M)$ and is in fact a *Lie groupoid*. This is important if one wishes to reformulate this theory in terms of Lie groupoids.

Let's now show that this definition of jets is in agreeance with our previous one.

Proposition 3.3.5. *Let $\pi : E \to M$ be a vector bundle. Then the underlying smooth manifold of the vector bundle $J^r E_M$ is naturally isomorphic to the submanifold of $J^r(M,E)$ given by*

$$J^r(E) := \{ j^r s(x) \in J^r(M,E) \ : \ (j^r \pi(s(x))) \circ (j^r s(x)) = j^r \mathrm{id}_M(x) \}$$

and the projection $J^r(E) \to M$ is given by the source map.

Now, let's begin to formulate our formal integrability problem for G-structures. Recall that we denote by F the frame bundle of a manifold M and that a G-structure on M for $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ a Lie subgroup is a principal G-subbundle $P \subseteq F$ such that the G-equivariance of the inclusion is in terms of the inclusion $G \subseteq \mathrm{GL}(\mathbb{R}^n)$.

Definition 3.3.6. Let P be a G-structure on an n-dimensional manifold M. We define P^r to be the subset of $J^{r+1}_0(\mathbb{R}^n, M)$ consisting of all $r+1$-jets $j^{r+1} f(0)$ such that $f_{*,0} : T_0\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \to T_{f(0)}M$ is invertible and there exists $g \in G$ such that

1. $f_{*,0}(0, g) \in P \subseteq F$, and

2. there is a neighbourhood U of $(0, g)$ in $\mathbb{R}^n \times \mathrm{GL}(\mathbb{R}^n)$ such that $f_*|_U$ is a diffeomorphism onto its image and the submanifolds $f_*(U \cap (\mathbb{R}^n \times G)), P \subseteq F$ have r'th order contact at $f_{*,0}(0, g)$.

We've already seen that the above notion is indeed well-defined. One important remark is that the superscript r on P^r does **not** refer to the order of the jets in P^r, which are $(r+1)$'st order, but instead refers to the **order of contact** of the submanifolds of F.

Proposition 3.3.7. *For any G-structure P and any $r \geq 0$, P^r is a smooth submanifold of $J_0^r(\mathbb{R}^n, M)$ and the target map $P^r \to M$ is a (not-necessarily surjective) submersion.*

As we're about to see, $P^r \to M$ is in fact a principal bundle itself. The structure group of this bundle is, in some sense, a *prolongation* of the Lie group G.

Definition 3.3.8. Let $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ be a Lie subgroup and let P be the standard G-structure on $\mathbb{R}^n$. Write $\pi : P^r \to \mathbb{R}^n$ for the projection and denote

$$G^r := \pi^{-1}(\{0\}),$$

equipped with its smooth structure as a submanifold of P^r.

Proposition 3.3.9. *G^r is a Lie subgroup of the group of invertible jets in $J_0^{r+1}(\mathbb{R}^n, \mathbb{R}^n)_0$.*

Proof. Since $\pi : P^r \to \mathbb{R}^n$ is a surjective submersion we already know that G^r is a smooth manifold and in fact a smooth submanifold of P^r. Since P^r is in turn a smooth submanifold of the manifold of invertible jets in $J_0^{r+1}(\mathbb{R}^n, \mathbb{R}^n)$ and π is the target map it folows that G^r is a smooth submanifold of the Lie group of invertible jets in $J_0^{r+1}(\mathbb{R}^n, \mathbb{R}^n)_0$. Since the group operation in G^r is precisely the composition of jets it follows that it is indeed a Lie subgroup. $\square$

Corollary 3.3.10. *For any G-structure P on a smooth n-dimensional manifold M and any $r \geq 0$, P^r is a principal G^r-subbundle of F^r, which is itself a principal bundle with structure group given by the group of invertible jets in $J_0^r(\mathbb{R}^n, \mathbb{R}^n)_0$.*

There is one subtlety in the constructions outlined above: throughout we implicitky assumed that the fibres of P^r over any point in M were non-empty. A priori, we can only assume zero'th order formal integrability of any given G-structure P, i.e. the fact that P^0 has non-empty fibres.

The question now is the following: if we were given that P^r had non-empty fibres over every point of M, what are the obstructions to extending the elements of these fibres to elements of the fibres of P^{r+1} over M? In view of the previous chapter, we therefore want to understand the potential surjectivity of the following morphisms of principal bundles in terms of the prolongations of some differential equation.

Proposition 3.3.11. *For each $r \geq 0$ the projection $G^{r+1} \to G^r$ is a morphism of Lie groups and through this the projection $P^{r+1} \to P^r$ becomes a morphism of principal bundles.*

Proof. All that needs to be checked here is that composition of jets is compatible with the projection, i.e. that

$$\pi^{r+1,r}(j^{r+1}_{g(x)} f \circ j^{r+1}_x g) = j^r_{g(x)} f \circ j^r_x g.$$

However, as we mentioned before, composition of jets is given in local coordinates by composition of polynomials follows by a truncation to the appropriate degree. Hence the further truncation of the degree $r+1$ polynomials by $\pi^{r+1,r}$ is precisely the same as just truncating to degree r from the start and so $\pi^{r+1,r} : G^{r+1} \to G^r$ is indeed a morphism of Lie groups (it is smooth since truncation of polynomials is smooth). $\square$

Now, the bundle P^r is a principal subbundle of F^r which itself lives in $J_0^r(\mathbb{R}^n, M)$. If we want to use the language of the previous chapter then we'll need to replace P^r with a subbundle of $J^r E_M$ for some vector bundle E_M. Since we're linearizing, in some sense, it is probably prudent of us to understand the Lie algebra of G^r.

Proposition 3.3.12. *Let G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$, $r \geq 0$, and write $\mathfrak{g} = \mathrm{Lie}(G)$. For $\ell \geq 1$ we'll denote*

$$\mathfrak{g}^{(\ell)} := (\mathbb{R}^n \otimes \mathrm{Sym}^{\ell+1}(\mathbb{R}^n)^*) \cap (\mathfrak{g} \otimes \mathrm{Sym}^{\ell}(\mathbb{R}^n)^*).$$

*Also, write $\mathfrak{g}^{(0)} := \mathfrak{g}$. Then, as a vector space (**not** as a Lie algebra) we have:*

$$\mathrm{Lie}(G^r) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(r)}.$$

Proof. Choose an arbitrary smooth curve $\gamma : (-\epsilon, \epsilon) \to G^r$ with $\gamma(0) = j_0^{r+1}\,\mathrm{id}$. Pulling back the coordinates on $J_0^{r+1}(\mathbb{R}^n, \mathbb{R}^n)_0$ to G^r we can write γ as:

$$\gamma(t) = \left(\sum_{1 \leq |\alpha| \leq r+1} a_\alpha^1(t) x^\alpha, \cdots, \sum_{1 \leq |\alpha| \leq r+1} a_\alpha^n(t) x^\alpha \right).$$

In order for this to lie in G^r we need

$$\frac{\partial \gamma^i}{\partial x^j} = a_j^i(t) + \sum_{2 \leq |\alpha \leq r+1} a_\alpha^i(t) \frac{\partial}{\partial x^j} x^\alpha$$

to lie in G to r'th order. Differentiating this with respect to t and evaluating at $t = 0$ then yields

$$\left(\frac{d}{dt}\Big|_{t=0} a_j^i(t) \right)_{i,j} \in \mathfrak{g} = \mathfrak{g}^{(0)} \text{ and}$$
$$\sum_i \sum_{|\alpha| = \ell+1} \left(\frac{d}{dt}\Big|_{t=0} a_\alpha^i(t) \right) \frac{\partial}{\partial x^j} x^\alpha \in (\mathbb{R}^n \otimes \mathrm{Sym}^{\ell+1}(\mathbb{R}^n)^*) \cap (\mathfrak{g} \otimes \mathrm{Sym}^{\ell}(\mathbb{R}^n)^*)$$

thus, as a vector space, we do indeed have

$$\mathrm{Lie}(G^r) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(r)}.$$

$\square$

We now ask: how is the Lie bracket on $\mathrm{Lie}(G^r)$ described when this is written as the above direct sum of vector spaces? For this, we notice that the infinite direct sum of vector spaces

$$\bigoplus_{k \geq 0} ((\mathbb{R}^n \otimes \mathrm{Sym}^{k+1}(\mathbb{R}^n)^*) \cap (\mathrm{End}_{\mathbb{R}}(\mathbb{R}^n) \otimes \mathrm{Sym}^k(\mathbb{R}^n)^*) = \bigoplus_{k \geq 0} \mathfrak{gl}(\mathbb{R}^n)^{(k)}$$

has a natural Lie bracket defined as follows. For $v \in V$ we write ι_v for the degree -1 derivation on $\mathrm{Sym}^*((\mathbb{R}^n)^*)$ extending $v : (\mathbb{R}^n)^* \to \mathbb{R}$. For $v_1 \otimes s_1 \in \mathbb{R}^n \otimes \mathrm{Sym}^{k+1}((\mathbb{R}^n)^*)$ and $v_2 \otimes s_2 \in \mathbb{R}^n \otimes \mathrm{Sym}^{\ell+1}((\mathbb{R}^n)^*)$ we can then define

$$[v_1 \otimes s_1, v_2 \otimes s_2] := v_2 \otimes ((\iota_{v_1} s_2) \circ s_1) - v_1 \otimes ((\iota_{v_2} s_1) \circ s_2) \in \mathbb{R}^n \otimes \mathrm{Sym}^{k+\ell+1}((\mathbb{R}^n)^*).$$

This descends to a Lie bracket on the direct sum of vector spaces:

$$\mathfrak{g}^{(\infty)} := \bigoplus_{k \geq 0} \mathfrak{g}^{(k)}.$$

Note: this does **not** make this into a graded Lie algebra since the Lie bracket is not graded anticommutative (nor does it satisfy the graded version of the Jacobi identity). This is just a usual ungraded Lie algebra. However, this Lie bracket does satisfy $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}] \subseteq \mathfrak{g}^{(k+\ell)}$ and therefore it endows our direct sum with the structure of a *filtered Lie algebra*.

Proposition 3.3.13. *The Lie bracket on* $\mathrm{Lie}(G^r)$ *is the truncation of the Lie bracket on*

$$\bigoplus_{k \geq 0} \mathfrak{g}^{(k)}$$

defined above to the finite sum

$$\mathrm{Lie}(G^r) = \mathfrak{g}^{(0)} \oplus \cdots \oplus \mathfrak{g}^{(k)}.$$

As was mentioned in Guillemin's paper [21], this gives us an alternate description of the Lie group G^r as the semidirect product of G with the Lie group associated to the nilpotent Lie algebra

$$\left(\bigoplus_{k \geq 1} \mathfrak{g}^{(k)} \right) \Big/ \left(\bigoplus_{k \geq r+1} \mathfrak{g}^{(k)} \right)$$

whose underlying set is the same as the Lie algebra and whose group multiplication is defined using the Baker-Campbell-Hausdorff formula.

Now, let's see how to get the Spencer complex involved. Our goal is to express the non-emptiness of the fibres of P^1 in terms of a first order equation and the rest of the P^r's as prolongations of this equation. Since we're working on the frame bundle F this means we'll need to look at TF and pushing forward along the projection $\pi : F \to M$ yields a morphism of vector bundles $\pi_* : TF \to \pi^* TM$ whose kernel is a vector bundle called the *vertical bundle* of the fibre bundle F:

$$0 \to VF \to TF \to \pi^* TM \to 0.$$

The fiber of the vertical bundle $VF \to F$ at a point $f \in F$ can be shown to be precisely the fiber over f of the tangent bundle

$$T(F_{\pi(f)}) \to F_{\pi(f)} = \pi^{-1}(\pi(f)).$$

Something worth keeping in mind is that $\pi^* TM$ has rank n as a bundle over F, while F is $n + n^2$-dimensional and so TF has rank $n + n^2$ over F. By the exactness of the above sequence it then follows that the rank of the vertical bundle VF is n^2. We have seen another sequence of vector bundles, this time on M, with these ranks. Namely the first jet bundle sequence of the tangent bundle:

$$0 \to T^*M \otimes TM \to J^1(TM) \to TM \to 0.$$

Indeed $T^*M \otimes TM$ has rank $n \cdot n = n^2$ and TM has rank n so by exactness $J^1(TM)$ has rank $n + n^2$ (all over M). Moreover, we have a free and proper action of $\mathrm{GL}(\mathbb{R}^n)$ on F with

$$F / \mathrm{GL}(\mathbb{R}^n) \cong M$$

via the projection $\pi : F \to M$. Since the induced action of $\mathrm{GL}(\mathbb{R}^n)$ on TF preserves the subbundle VF and acts trivially on $\pi^* TM$ (this is because the action on F preserves the fibres of π) we can try to quotient our exact sequence of bundles over F by $\mathrm{GL}(\mathbb{R}^n)$ to see if we obtain our jet sequence from this. If so, then we might obtain a method of phrasing the non-emptyness of the fibres of P^1 in terms of a first order equation on TM, i.e. a subbundle of $J^1(TM)$.

Lemma 3.3.14. *[51]*
The vertical bundle VF is naturally a trivial bundle

$$VF \cong F \times \mathfrak{gl}(\mathbb{R}^n).$$

Proposition 3.3.15. *[51]*
Quotienting by the action of $\mathrm{GL}(\mathbb{R}^n)$ *on* VF *yields a vector bundle over* $M \cong F/\,\mathrm{GL}(\mathbb{R}^n)$ *given by*

$$VF/\,\mathrm{GL}(\mathbb{R}^n) \cong F \times_{\mathrm{GL}(\mathbb{R}^n)} \mathfrak{gl}(\mathbb{R}^n) \cong T^*M \otimes TM.$$

Proposition 3.3.16. *[51]*
$\mathrm{GL}(\mathbb{R}^n)$ *acts trivially on* π^*TM *and the quotient recovers the tangent bundle on* M*:*

$$\pi^*TM/\,\mathrm{GL}(\mathbb{R}^n) \cong TM.$$

So, all that remains is to understand the quotient $TF/\,\mathrm{GL}(\mathbb{R}^n)$. At a first glance, this is more difficult, however we can actually obtain our desired result from the previous two propositions.

Proposition 3.3.17. $TF/\,\mathrm{GL}(\mathbb{R}^n)$ *is naturally a vector bundle over* M *and the morphism* $\pi_* : TF \to \pi^*TM$ *descends to a quotient map* $TF/\,\mathrm{GL}(\mathbb{R}^n) \to TM$ *whose kernel is naturally isomorphic to* $T^*M \otimes TM$*. Hence the map* $TF/\,\mathrm{GL}(\mathbb{R}^n) \to TM$ *factors through to a morphism of vector bundles* $TF/\,\mathrm{GL}(\mathbb{R}^n) \to J^1(TM)$ *fitting into the following commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & VF/\,\mathrm{GL}(\mathbb{R}^n) \arrow[r] \arrow[d, "\cong"] & TF/\,\mathrm{GL}(\mathbb{R}^n) \arrow[r] \arrow[d] & \pi^*TM/\,\mathrm{GL}(\mathbb{R}^n) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & T^*M \otimes TM \arrow[r] & J^1(TM) \arrow[r] & TM \arrow[r] & 0
\end{tikzcd}$$

Therefore, by the five lemma, the above map yields a natural isomorphism $TF/\,\mathrm{GL}(\mathbb{R}^n) \cong J^1(TM)$*.*

So, we can now perform the following construction. Given a G-structure P on M we have $TP \subseteq TF$ is a vector subbundle over F since $P \subseteq F$ is a principal subbundle. Unfortunately, this subbundle need not be $\mathrm{GL}(\mathbb{R}^n)$-invariant. However, we can notice the following.

Proposition 3.3.18. *[51]*
Let M *be a smooth manifold equipped with a free and proper action of a Lie group* G *by diffeomorphisms. If* $\pi : M \to M/G$ *denotes the quotient map then the sheaf of sections of the vector bundle* TM/G *over* M/G *is naturally identifying with the push-forward*

$$\pi_* T_M^G$$

of the sheaf of sections of TM *which are fixed by the action of* G*.*

So, returning to our G-structure $P \subseteq F$ we get a subsheaf

$$R^{1,P} := \pi_* T_P^{\mathrm{GL}(\mathbb{R}^n)} \ \text{ of } \ \pi_* T_F^{\mathrm{GL}(\mathbb{R}^n)} \cong J^1 T_M.$$

Our claim is now that this is precisely the differential equation we were looking for.

Proposition 3.3.19. *[39]*
Let P *be a* G*-structure on* M*. Then the subsheaf* $R^{1,P}$ *of* $J^1 T_M$ *is in fact a vector subbundle.*

Proposition 3.3.20. *[39]*
Let P *be a* G*-structure on* M *and* $g^{1,P}$ *be the kernel of the restriction of the morphism* $J^1 T_M \to T_M$ *to a morphism* $R^{1,P} \to T_M$*. Then, if* $g^{r+1,P}$ *denotes the* $(r+1)$*'st prolongation of* $g^{1,P}$ *we obtain a natural isomorphism of vector bundles over* M*:*

$$g^{r+1,P} \cong P^r \times_{G^r} (\mathbb{R}^n \oplus \mathrm{Lie}(G^r))$$

where $G \subseteq \mathrm{GL}(\mathbb{R}^n)$*.*

This gives us a nice explicit description of our Spencer complex:

$$(\Lambda^* T_M^* \otimes g^{\infty,P}, \delta)$$

as arising from the Lie algebra

$$V \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \cdots$$

with differential given by the restriction of the differential

$$\delta : \Lambda^p(\mathbb{R}^n)^* \otimes \mathrm{Sym}^{q+1}(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \Lambda^{p+1}(\mathbb{R}^n)^* \otimes \mathrm{Sym}^q(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

arising from skew-symmetrizing the $(p+1)$'st factor in the tensor product with the first p factors.

Let's now see how the obstructions to an atlas of r'th order G-structure-preserving charts arising from an atlas of $(r+1)$'st order G-structure-preserving charts naturally live in the Spencer cohomology. The idea comes from the following. Recall how $\mathrm{O}(n)$-structures were always 1'st order formally integrable due to the torsion-freeness of the Levi-Civita connection. This is actually a more general phenomenon since the torsion of a connection lives as a section of the vector bundle associated to

$$\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

where two different affine G-connections have difference given by a section of the bundle associated to

$$(\mathbb{R}^n)^* \otimes \mathfrak{g} \subseteq (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

when $G \subseteq \mathrm{GL}(\mathbb{R}^n)$. Thus the obstruction to the existence of a torsion-free affine G-connection is a section of the vector bundle associated to the cohomology of the middle term of

$$(\mathbb{R}^n)^* \otimes \mathfrak{g}^{(0)} \xrightarrow{\delta} \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to 0.$$

As we'll see, the obstructions we're looking for in the case of G-structures can be measured using the existence of torsion-free connections on certain fibre bundles associated to the P^r. With this in mind, we consider the following constructions.

If we wanted to construct a connection on the principal bundle P_G^r then we would need a one-form on P_G^r taking values in the Lie algebra of G^r. Since one-forms are sections of the cotangent bundle they naturally act on derivatives of things and so if we want a naturally associated connection then an intuitive idea would be to consider some point $j_0^{r+1} f \in P_G^r$ over which we'll want to define our form on the fibre and to look at its image $j_0^r f \in P_G^{r-1}$.

Since these jets are invertible by definition and invertibility is an open condition it follows that there is an open neighbourhood

$$j_0^r f \in U \subseteq P_G^{r-1}$$

on which the following map is well-defined

$$\begin{aligned} U &\mapsto \mathrm{inv}\, J_0^r(\mathbb{R}^n, \mathbb{R}^n) \\ \varphi &\mapsto (j_0^r f)^{-1} \circ \varphi. \end{aligned}$$

Furthermore, this map takes $j_0^r f$ to $j_0^r\,\mathrm{id}$. Now, the derivative of this map only depends on $j_0^{r+1} f \in P_G^r$ and defines a map

$$T_{j_0^r f} P_G^{r-1} \to \mathbb{R}^n \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{r-1}.$$

Pre-composing this with the push-forward

$$T_{j_0^{r+1} f} P_G^r \to T_{j_0^r f} P_G^{r-1}$$

and doing this for each $j_0^{r+1}f$ (at least for each one that exists) gives a smooth 1-form

$$\Gamma_G^r \in \Omega^1(P_G^r, \mathbb{R}^n \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(r-1)})$$

which we will think of as being similar to some sort of connection. Indeed, suppose we considered the principal bundle

$$Q_G^r := P_G^{r-1} \times_{G^{r-1}} (\mathbb{R}^n \rtimes G^{r-1}).$$

Then the Lie algebra of the structure group of Q_G^r is

$$V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(r-1)}$$

and one can show that Guillemin's construction above also defines Γ_G^r as a principal connection on Q_G^r, in the sense of a Lie algebra valued 1-form satisfying certain compatibility conditions with the action of the structure group of Q_G^r.

The point is that the "torsion" of Γ_G^r will end up corresponding to an r'th order version of the intrinsic torsion of the G-structure P_G. Indeed, let's expand out Γ_G^r in terms of its components

$$\Gamma_G^r = \omega_G + \Omega_G^0 + \cdots + \Omega_G^{r-1}$$

where $\omega_G \in \Omega^1(P_G^r, \mathbb{R}^n)$ and

$$\Omega_G^i \in \Omega^1(P_G^r, \mathfrak{g}^{(i)}) \text{ for all } i \geq 0.$$

Now, arbitrarily select $\varphi \in P_G^r$ and let's work on the fibre of this bundle on P_G^r over this point. Here, as mentioned by Guillemin, we can pick a horizontal subspace

$$H_\varphi = H \subseteq T_\varphi P_G^r$$

so that

$$\Omega^i|_H = 0 \text{ for all } i \geq 0.$$

Just as with the definition of the intrinsic torsion, this subspace need not be unique but by passing to an appropriate cohomology theory later our construction will be independent of our choice of H. Using D_H to denote the covariant derivative arising from H we obtain a linear map

$$\begin{aligned} \Pi_H : \Lambda^2 T_\varphi P_G^r &\to V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(r-1)} \\ X \wedge Y &\mapsto (D_H \Gamma_H^r)(X \wedge Y). \end{aligned}$$

Guillemin then proves the following where we begin to see the Maurer-Cartan equations appearing again.

Proposition 3.3.21. *[21]*
For $i \neq k-1$ the homogeneous component $\Pi_H^{(i)}$ of Π_H (taking values in $\mathfrak{g}^{(i)}$) vanishes.

Now, there is a natural map

$$\Lambda^2 \mathbb{R}^n \to \Lambda^2 T_\varphi P_G^r$$

given by using the chart determined by φ to identify $\Lambda^2 TM$ at the corresponding point with $\Lambda^2 \mathbb{R}^n$ and then taking the horizontal lift. In this way, $\Pi_H^{(r-1)}$ becomes naturally an element of

$$\Pi_H^{(r-1)} \in \Lambda^2 (\mathbb{R}^n)^* \otimes \mathfrak{g}^{(r-1)}.$$

The real vector space above is indeed something we already know. It is a fibre of the formal Spencer complex associated to the differential equation $R^{1,P}$ discussed earlier. This is the cohomology theory in which we will obtain a class independent of our choice of H. Indeed, Guillemin now proves the following.

Proposition 3.3.22. *[21]*
The element $\Pi_H^{(r-1)}$ is δ-closed and therefore defines an element of the fibre over φ of the $(2, r)$ formal Spencer cohomology:

$$[\Pi_H^{(r-1)}] \in H_S^{(2,r)}(P_G, g^{1,P})_\varphi.$$

Furthermore, this cohomology class is independent of our choice of H and the collection of such classes as we range over all φ varies smoothly, yielding a class

$$c^r \in H_S^{(2,r)}(P_G, g^{1,P})$$

*which we call the r'th **structure tensor** of P_G.*

Theorem 3.3.23. Guillemin [21]
Let M be a smooth manifold, $r \geq 0$ and P a G-structure on M such that the target map $P^r \to M$ is surjective. If the r'th structure tensor c^r vanishes at $x \in M$ then x is in the image of $P^{r+1} \to M$.

Corollary 3.3.24. *[21]*
If the target map $P_G^r \to M$ is surjective then the target map $P_G^{r+1} \to M$ is surjective if and only if $c^r = 0$ identically.

The point of all of this formalism is the following. It was mentioned in [21] and proven by Cartan that if $\mathfrak{g}$ is the Lie algebra of a Lie subgroup $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ which preserves some symmetric positive definite bilinear form on $\mathbb{R}^n$ then $\mathfrak{g}^{(k)} = 0$ for all $k \geq 1$. This happens somewhat generically since, for example, any compact Lie subgroup $G \subseteq \mathrm{GL}(\mathbb{R}^n)$ preserves such a form. Meaning that in general there are only two obstructions to full integrability, those coming from the cohomology of the following two complexes:

$$0 \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{g} \to \Lambda^3(\mathbb{R}^n)^* \otimes \mathbb{R}^n \text{ and}$$
$$(\mathbb{R}^n)^* \otimes \mathfrak{g} \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to 0.$$

The obstruction corresponding to the second of the above two complexes is precisely the torsion of any G-compatible connection. The obstruction corresponding to the first of the above complexes, which appears only once we know that the other obstruction vanishes, is the curvature of the (unique since $\mathfrak{g}^{(1)} = 0$) torsion free G-connection.

This was used, together with the Borel-Weil-Bott theorem, in [50] in order to classify the possible holonomy groups of torsion-free affine connections. In the author's opinion, this also serves to demonstrate why manifolds with special holonomy are are great interest to those studying the integrability problem for G-structures.

3.4 The Frölicher-Nijenhuis and Schouten Brackets

The main reference for this section is [51].

Having expressed our integrability problem in the language of Spencer complexes, we now return to the "coincidence" mentioned in the beginning of this thesis. If we are to have any hope of explaining the relationship between the Frölicher-Nijenhuis bracket and the integrability of G-structures, we should probably also define this bracket. We will also introduce the Nijenhuis-Richardson and Schouten (or Schouten-Nijenhuis) brackets since this is a natural place to do so and we will need them later.

From this point forward, we will abbreviate the Frölicher-Nijenhuis, Nijenhuis-Richardson and Schouten-Nijenhuis brackets respectively as the **FN,NR and SN brackets**. The FN and NR brackets appear naturally when one is studying graded derivations on the algebra of differential forms.

Definition 3.4.1. A (degree k) **derivation** on the $\mathbb{Z}$-graded-commutative algebra $\Omega^*(M)$ of differential forms is a $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to \Omega^*(M)[k]$$

satisfying the (graded) Leibniz rule

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{k \deg(\alpha)} \alpha \wedge D\beta$$

for homogeneous forms α, β. Writing $\mathrm{Der}^k(\Omega^*(M))$ for the k-vector space of degree k derivations we obtain a graded Lie algebra whose underlying graded vector space is

$$\mathrm{Der}(\Omega^*(M)) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Der}^k(\Omega^*(M))[-k]$$

together with graded Lie bracket given by the graded commutator

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{\deg(D_1)\deg(D_2)} D_2 \circ D_1.$$

The point of the FN and NR brackets is that the graded Lie algebra $\mathrm{Der}(\Omega^*(M))$ actually breaks up into a direct sum of graded vector spaces, each of which is a Lie subalgebra in a natural way (however, the direct sum here is not a direct sum of graded Lie algebras). This direct sum splits up the derivations into those which are purely algebraic (like interior product ι_X for a vector field X), and those which actually differentiate the differential forms (like the exterior derivative d or Lie derivative $\mathcal{L}_X$). Also, something probably worth mentioning here is that in the exact same way as above, the sheaf

$$\mathcal{D}\mathrm{er}(\Omega_M^*) = \bigoplus_{k \in \mathbb{Z}} \mathcal{D}\mathrm{er}^k(\Omega_M^*)[-k]$$

is a sheaf of graded Lie algebras.

Now, let's characterize which elements of $\mathrm{Der}(\Omega^*(M))$ are merely algebraic derivations. Fix $D \in \mathrm{Der}^k(\Omega^*(M))$ and suppose that D was algebraic, i.e. for all smooth functions $f \in C^\infty(M)$ and all forms $\alpha \in \Omega^*(M)$ we had

$$D(f\alpha) = fD\alpha.$$

Since D is a derivation we have $D1 = 0$ and so $Df = 0$ for all smooth functions $f \in C^\infty(M) = \Omega^0(M)$. On the other hand, as a map $\Omega^1(M) \to \Omega^{1+k}(M)$, D is a morphism of $C^\infty(M)$-modules and therefore determines and a uniquely specified global section

$$K \in \Gamma(M, \mathcal{H}\mathrm{om}_{C_M^\infty}(T_M^*, \Lambda^{k+1} T_M^*)) \cong \Omega^{k+1}(M, TM).$$

But D is completely determined by its action on 1-forms via the Leibniz rule and hence this D is completely determined by K and we write

$$\iota_K := D.$$

The reason for this notation is that, due to the next proposition, in the case $D \in \mathrm{Der}^{-1}(\Omega^*(M))$ then K is a vector field and D will actually be given by interior product ι_X.

Proposition 3.4.2. *Let $K \in \Omega^{k+1}(M, TM)$ and $\alpha \in \Omega^\ell(M)$. Then*

$$(\iota_K \alpha)(X_1, \cdots, X_{k+\ell}) = \frac{1}{(k+1)!(\ell-1)!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(K(X_{\sigma(1)}, \cdots, X_{\sigma(k+1)}), X_{\sigma(k+2)}, \cdots, X_{\sigma(k+\ell)}).$$

Using our newly defined morphism of sheaves of graded vector spaces

$$\iota : (\Lambda^* T_M^* \otimes T_M)[1] \to \mathcal{D}\mathrm{er}(\Omega_M^*)$$

we can define a graded Lie bracket on $\Omega^*(M, TM)[1]$ making ι into a monomorphism of graded Lie algebras.

Proposition 3.4.3. *If D_1, D_2 are algebraic derivations in $\mathrm{Der}^k(\Omega^*(M))$ and $\mathrm{Der}^\ell(\Omega^*(M))$ respectively then so is $[D_1, D_2] \in \mathrm{Der}^{k+\ell}(\Omega^*(M))$ and so we can define the **Nijenhuis-Richardson bracket** (or NR-bracket) $[-,-]_{NR}$ on $\Omega^*(M, TM)[1]$ via*

$$[\iota_K, \iota_L] = \iota_{[K,L]_{NR}}.$$

Proof. Let $D_1 \in \mathrm{Der}^k(\Omega^*(M))$ and $D_2 \in \mathrm{Der}^\ell(\Omega^*(M))$ be two algebraic derivations and arbitrarily select $\alpha \in \Omega^p(M)$ and $f \in C^\infty(M)$. We can then compute

$$[D_1, D_2](f\alpha) = D_1 D_2(f\alpha) - (-1)^{k\ell} D_2 D_1(f\alpha) = f D_1 D_2 \alpha - (-1)^{k\ell} D_2 D_1 \alpha = f[D_1, D_2]\alpha$$

so indeed the bracket remains algebraic. $\square$

The next step is to handle the non-algebraic derivations such as d and the Lie derivatives. Its worth noticing that given a vector field X on M, the Lie derivative $\mathcal{L}_X$ can be built up from the algebraic derivation ι_X and the exterior derivative d via *Cartan's formula*:

$$\mathcal{L}_X = d\iota_X + \iota_X d = [d, \iota_X] = [\iota_X, d].$$

Analogous to this, we can try constructing new non-algebraic derivations by definining, for $K \in \Omega^k(M, TM)$, the derivation

$$\mathcal{L}_K := [d, \iota_K] \in \mathrm{Der}^k(\Omega^*(M)).$$

One important question here is to what extent the above actually deserves the moniker and notation of a Lie derivative? As we will now see, these are in some sense all of the non-algebraic derivations.

Lemma 3.4.4. *The derivations $D \in \mathrm{Der}^k(\Omega^*(M))$ which are of the form $\mathcal{L}_K$ for some $K \in \Omega^k(M, TM)$ are precisely those for which $[d, D] = 0$. We call such derivations **Lie derivations**.*

Proof. First notice that given any $K \in \Omega^k(M, TM)$ we have

$$0 = [d, [d, \iota_K]] + [[d, d], \iota_K] + (-1)^{k-1}[d, [\iota_K, d]] = 2[d, [d, \iota_K]]$$

and so indeed $\mathcal{L}_K$ commutes with d. For the converse we arbitrarily select $D \in \mathrm{Der}^k(\Omega^*(M))$ such that $[d, D] = 0$ as well as k arbitrary vector fields $X_1, \cdots, X_k$ on M. Then we obtain a derivation

$$\begin{aligned} C^\infty(M) &\to C^\infty(M) \\ f &\mapsto (Df)(X_1 \wedge \cdots \wedge X_k). \end{aligned}$$

Thus there is a unique vector field $Y \in \Gamma(M, T_M)$ such that

$$(Df)(X_1 \wedge \cdots \wedge X_k) = Yf$$

for all $f \in C^\infty(M)$. This vector field depends multilinearly on $X_1, \cdots, X_k$, is alternating and is $C^\infty(M)$-linear in $X_1, \cdots, X_k$. Thus the assignment

$$X_1 \wedge_{C^\infty(M)} \cdots \wedge_{C^\infty(M)} X_k \mapsto Y$$

defines a unique vector-valued k-form

$$K \in \Omega^k(M, TM).$$

By construction, for $f \in C^\infty(M)$ we have

$$Df = \iota_K df = \mathcal{L}_K f.$$

The claim is now that $D = \mathcal{L}_K$. Indeed, given a ℓ-form α and $f \in C^\infty(M)$ we can compute

$$\begin{aligned}(D - \mathcal{L}_K)(f\alpha) &= (Df) \wedge \alpha + fD\alpha - (\mathcal{L}_K f) \wedge \alpha - f\mathcal{L}_K\alpha \\ &= f(D - \mathcal{L}_K)\alpha\end{aligned}$$

and so $D - \mathcal{L}_K$ is algebraic. All that remains then is to show that if an algebraic derivation commutes with d then it is necessarily zero. i.e. we need to prove that the map $K \mapsto \mathcal{L}_K$ is injective. This is done in [51]. □

Proposition 3.4.5. *Every $D \in \mathrm{Der}^k(\Omega^*(M))$ admits a unique decomposition as*

$$D = \iota_K + \mathcal{L}_L$$

for some $K \in \Omega^{k+1}(M, TM)$ and $L \in \Omega^\ell(M, TM)$.

Now just as we did with the algebraic derivations we can define a graded Lie bracket on $\Omega^*(M, TM)$ using our monomorphism of sheaves of graded vector spaces

$$\mathcal{L} : \Lambda^* T_M^* \otimes T_M \to \mathcal{D}\mathrm{er}(\Omega_M^*)$$

making this into a monomorphism of sheaves of graded Lie algebras.

Proposition 3.4.6. *If D_1, D_2 are two (homogeneous) Lie derivations then so is $[D_1, D_2]$ and so we can define the* ***Frölicher-Nijenhuis bracket*** *(or FN-bracket) $[-,-]_{FN}$ on $\Omega^*(M, TM)$ via*

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]_{FN}}.$$

So we now have that the sheaf of graded Lie algebras $\mathcal{D}\mathrm{er}(\Omega_M^*)$ decomposes as a direct sum of sheaves of graded vector spaces:

$$\mathcal{L}(\Lambda^* T_M^* \otimes T_M) \oplus \iota((\Lambda^* T_M^* \otimes T_M)[1])$$

where both of these subsheaves are in fact sub graded Lie algebras. This is not a semidirect product of graded Lie algebras (the precise product structure it has was investigated by Michor). For us, the important point regarding the above decomposition is that the Lie derivations can be thought of as those vector fields on the derived loop space which are invariant under the S^1-action.

But what then, geometrically, is this identification of S^1-invariant vector fields on the derived loop space with vector-valued differential forms? Let's first think about this intuitively. We have an action of the diffeomophism group $\mathrm{Diff}(M)$ on the loop space $C^\infty(S^1, M)$ given by post-composition. If we could interpret this action on the derived loop space $\mathcal{L}M$ and differentiate it this would give a Lie algebra homomorphism

$$\mathcal{L} : \Gamma(M, T_M) \to \mathrm{Der}^*(\Omega^*(M))$$

assuming the action is invariant under the Hochschild differential. Now, suppose we were given a point $z \in S^1$ interpreted as a map $S^1 \to S^1$ by multiplication, a loop $\gamma \in C^\infty(S^1, M)$ and a diffeomorphism $f \in \mathrm{Diff}(M)$. Using a dot to denote our actions we have for all $w \in S^1$:

$$(f \cdot (z \cdot \gamma))(w) = f((z \cdot \gamma)(w)) = f(\gamma(zw)) = (f \circ \gamma)(zw) = (z \cdot (f \cdot \gamma))(w)$$

and so the actions commute. Thus our representation of $\Gamma(M, T_M)$ is S^1-equivariant and therefore lifts to a morphism

$$\mathcal{L} : \Omega^*(M, TM) = \Omega^*(M) \otimes_{C^\infty(M)} \Gamma(M, T_M) \to \mathrm{Der}(\Omega^*(M)).$$

Supposing we made this rigorous, this should intuitively be the same "$\mathcal{L}$" we used to define the Frölicher-Nijenhuis bracket. This makes sense since for a vector-valued form K we showed that $[\mathcal{L}_K, d] = 0$ and so the vector field $\mathcal{L}_K$ on the derived loop space is S^1-invariant. One of my future projects is to make this precise.

As a final remark, we state the link between the Schouten and Frölicher-Nijenhuis brackets. The author was made aware of this in [19] but the original proof is in [18]. It is a standard fact from differential geometry [52] that the cotangent bundle T^*M of any smooth manifold M comes equipped with a natural symplectic form

$$\omega \in \Gamma(T^*M, \Lambda^2 T^*_{T^*M}).$$

Given a vector field $X \in \mathfrak{X}(M)$ we obtain a function

$$\iota_X : T^*M \to \mathbb{R}$$

given by interior product. Denote by $H(X) \in \mathfrak{X}(T^*M)$ its Hamiltonian vector field (via ω). If we let q^i, p^j denote the canonical coordinates on T^*M associated to ω we can also vertically lift 1-forms on M to vertical vector fields

$$\alpha_i(q)dq^i \mapsto \alpha_i(q)\frac{\partial}{\partial p^i} \text{ in local coordinates}$$

and this lift extends over the wedge product to a map

$$V : \Omega^*(M) \to \Omega^*(T^*M).$$

Given a pure vector-valued form $\alpha \otimes X$ on M we then define its **Hamiltonian lift** to be

$$H(\alpha \otimes X) := H(X) \wedge V(\alpha) - \iota_X V(d\alpha).$$

The following can then be proven.

Proposition 3.4.7. *[18] The Hamiltonian lift is injective and identifies the Frölicher-Nijenhuis bracket on M with the Schouten bracket on T^*M.*

Something else worth noticing is that since T^*M has a canonical symplectic form, it is also orientable and has a canonical volume form vol $\in \Omega^{2n}(T^*M)$. Contraction with vol gives us isomorphisms:

$$\iota_{\mathrm{vol}} : \Lambda^k T_{T^*M} \xrightarrow{\cong} \Lambda^{2n-k} T^*_{T^*M}$$

giving rise to a degree -1 differential

$$b := \iota_{\mathrm{vol}}^{-1} \circ d \circ \iota_{\mathrm{vol}} : \Lambda^k T_{T^*M} \to \Lambda^{k-1} T_{T^*M}.$$

We can now ask the following question: how does the differential graded Lie algebra

$$(\mathfrak{X}^*(T^*M)[1], b, [-,-]_{SN})$$

relate to, for example, the Spencer complex describing the integrability problem for G-structures? Can the integrability of G-structures always be phrased in terms of the Schouten bracket and the differential b? What if G is given as a stabilizer subgroup of an element of some representation of $\mathrm{GL}(\mathbb{R}^n)$? All of this is for the future.

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Appendix A

Category Theory Review

Here we recall some basic category theory [69]. This is mostly meant as a reference for the reader so they don't have to go look up these definitions and results.

First, recall that monomorphisms are morphisms f for which

$$f \circ g = f \circ h \implies g = h$$

and epimorphisms are morphisms f for which

$$g \circ f = h \circ f \implies g = h.$$

All categories are assumed to be locally small (i.e. hom sets are sets) so that Yoneda's lemma holds.

Lemma A.0.1. Yoneda's Lemma
Let $\mathcal{C}$ be a category and write $\mathrm{PSh}(\mathcal{C})$ for the category of functors $\mathcal{C}^{op} \to \mathrm{Set}$ with natural transformations as morphisms. Then the functor

$$\begin{aligned} \mathcal{C} &\to \mathrm{PSh}(\mathcal{C}) \\ C &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, C) \end{aligned}$$

*is **fully faithful**.*

Recall that a **full functor** is one that induces a surjection on the hom-sets and a **faithful functor** is one that induces an injection on the hom-sets. A **full subcategory** of a category is a subcategory for which the inclusion is full (for subcategories the inclusion is always assumed to be faithful).

Let $\mathcal{I}$ be a small category (i.e. the collection of objects is a set). Then a **cone** over a diagram $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ is an object $C \in \mathcal{C}$ together with morphisms from C to every object in $F(\mathcal{I})$ such that all relevant diagrams commute. A **cocone** under $F : \mathcal{I} \to \mathcal{C}$ is an object $C \in \mathcal{C}$ together with morphisms into C from each object in $F(\mathcal{I})$, again making all relevant diagrams commute. Morphisms between cones and cocones are morphisms between the underlying objects making all diagrams commute.

Definition A.0.2. Let $\mathcal{I}$ be a small category and $F : \mathcal{I} \to \mathcal{C}$ a diagram, a.k.a. a functor. A **limit** of F, written

$$\varprojlim_{I \in \mathcal{I}} F(I)$$

is a terminal object in the category of cones over F. i.e. it is a cone over F such that every other cone over F admits a unique morphism into the limit. Dually, a **colimit** of F, written

$$\varinjlim_{I \in \mathcal{I}} F(I)$$

is an initial object in the category of cocones under F. i.e. it is a cocone under F such that every other cocone admits a unique morphism to the cocone from the colimit.

One should think of limits as infimums and colimits as supremums. The names "limit" and "colimit" are unfortunate since they are poor analogies for limits in analysis despite that being the motivation for the name. Really one should think of them as infimums and supremums.

Example A.0.3. If $\mathcal{C}$ is a category determined by a poset $(X, \leq)$ (i.e. the objects of $\mathcal{C}$ are the elements of X and one has a unique morphism $a \to b$ if and only if $a \leq b$) and $\mathcal{I}$ is a small category with no non-identity morphisms then for any functor $F : \mathcal{I} \to \mathcal{C}$ we have

$$\varprojlim_{I \in \mathcal{I}} F(I) \cong \inf F(\mathcal{I})$$

and

$$\varinjlim_{I \in \mathcal{I}} F(I) \cong \sup F(\mathcal{I}).$$

Example A.0.4. Products are limits over diagrams in which the modelling category $\mathcal{I}$ has no non-identity morphisms. Coproducts are colimits over such diagrams. Initial objects are colimits over the empty diagram and terminal objects are limits over the empty diagram.

Example A.0.5. Let $f : M \to N$ be a morphism of smooth manifolds and $E \to N$ a vector bundle. Then the pullback f^*E is a limit over the diagram

$$\begin{tikzcd} & E \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

in the category Man of all smooth manifolds with smooth maps as morphisms. Limits over diagrams of the above shape in general categories are called **pullbacks**. Dually, limits over diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

are called **pushouts**.

For the next example we make the following definition.

Definition A.0.6. A **locally ringed space** is a topological space X together with a sheaf $\mathcal{O}_X$ of commutative unital rings such that for each $p \in X$ the **stalk**

$$\mathcal{O}_{X,p} := \varinjlim_{p \in U} \mathcal{O}_X(U)$$

is a local ring. The above colimit is taken over the diagram consisting of all open neighbourhoods of p, which is a diagram in the poset category of all open subsets of X with $\subseteq$ as the partial order. A morphism of locally ringed spaces is a pair

$$(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

where $f : X \to Y$ is a continuous map and $f^\#$ is a morphism of sheaves

$$f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

(here $f_*\mathcal{O}_X$ is the sheaf $(f_*\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U)))$ such that for each $p \in X$ the naturally induced composition

$$\mathcal{O}_{Y,f(p)} \xrightarrow{f_p^\#} \varinjlim_{p\in f^{-1}(U)} \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_{X,p}$$

has the property that the preimage of the maximal ideal $\mathfrak{m}_{X,p} \trianglelefteq \mathcal{O}_{X,p}$ is equal to the maximal ideal $\mathfrak{m}_{Y,f(p)} \trianglelefteq \mathcal{O}_{Y,p}$. Such ring homomorphisms are called morphisms of local rings.

Example A.0.7. Let M be a smooth manifold and C^∞_M the sheaf of smooth functions on M. Then for each $p \in M$ the stalk $C^\infty_{M,p}$ is the ring of germs of smooth functions at p. This is a local ring with unique maximal ideal

$$\mathfrak{m}_{M,p} := \ker(\mathrm{ev}_p)$$

where $\mathrm{ev}_p : C^\infty_{M,p} \to \mathbb{R}$ is evaluation at p and thus smooth manifolds M yield locally ringed spaces (M, C^∞_M).

Example A.0.8. Limits of diagrams of the form

$$A \rightrightarrows B$$

are called **equalizers** and colimits of diagrams of the above form are called **coequalizers**. A **zero** object in a category $\mathcal{C}$ is an object which is both initial and terminal. If $\mathcal{C}$ has a zero object $0 \in \mathcal{C}$ then for each pair of objects $A, B \in \mathcal{C}$ there is a unique morphism of the form

$$A \to 0 \to B$$

called the zero map. Equalizers for which one of the morphisms is the zero map are called **kernels** and coequalizers for which one of the morphisms is the zero map are called **cokernels**. The kernel of a cokernel is called the **image** and the cokernel of a kernel is called the **coimage**. There is always a natural morphism from the coimage to the image supposing they exist and if this morphism ends up being an isomorphism we can interpret this as saying that "the first isomorphism theorem holds".

Definition A.0.9. We call a category $\mathcal{C}$ **complete** if and only if it contains the limit of every diagram $\mathcal{I} \to \mathcal{C}$ for which $\mathcal{I}$ is small. Dually we call $\mathcal{C}$ **cocomplete** if and only if it contains the colimit of every small diagram in $\mathcal{C}$.

Definition A.0.10. An **abelian category** is an additive category $\mathcal{A}$ (i.e. the hom-sets are abelian groups, composition is bilinear, and $\mathcal{A}$ has both a zero object and all finite products) satisfying the following axioms:

1. every morphism has a kernel and cokernel;
2. monomorphisms are kernels of their cokernels;
3. epimorphisms are cokernels of their kernels.

A functor between additive categories is called **additive** if and only if it induces group homomorphisms on the hom-abelian groups.

Definition A.0.11. Let $\mathcal{C}, \mathcal{D}$ be two categories and suppose we had functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We say that F is a **left adjoint** of G, G is a **right adjoint** of F and call the pair (F, G) an **adjunction** if and only if we have a natural isomorphism of functors

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(-)).$$

The motivation for this terminology is that the easy way to remember the definition of an adjunction is to pretend $\mathrm{Hom}(-,-)$ was an inner product.

Proposition A.0.12. *Left adjoints take colimits to colimits and right adjoints take limits to limits.*