

# Some results on hypersymplectic structures

by

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### **Author's Declaration**

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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## Abstract

A conjecture of Simon Donaldson is that on a compact 4-manifold  $X^4$  one can flow from a hypersymplectic structure to a hyperkähler structure while remaining in the same cohomology class. To this end the hypersymplectic flow was introduced by Fine–Yao. In this thesis the notion of a positive triple on  $X^4$  is used to define a hypersymplectic and hyperkähler structure. Given a closed positive triple one can define either a closed  $G_2$  structure or a coclosed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$ . The coclosed  $G_2$  structure is evolved under the  $G_2$  Laplacian coflow. The coflow descends to a flow of the positive triple on  $X^4$ , which is again the Fine–Yao hypersymplectic flow. In the second part of this thesis we let  $X^4 = \mathbb{R}^4 \setminus \{0\}$  with a particular cohomogeneity one action. A hypersymplectic structure built from data invariant under this action is introduced. The Riemann and Ricci curvature tensors are computed and we verify in a particular case that this hypersymplectic structure can be transformed to a hyperkähler structure. The notion of a soliton for the hypersymplectic flow in this particular case is introduced and it is found that steady solitons give rise to hypersymplectic structures that can be transformed to hyperkähler structures. Some other soliton solutions are also discussed.

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## Dedication

To my parents, Octavian and Mihaiela, my brother, Michael, and my cat, Mittens, who treated the papers on my desk as a midnight snack. And to all the younger versions of me who never saw herself as a mathematician. We did it.

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# Chapter 1

## Introduction

The notion of a hyperkähler manifold first arose in Berger’s classification of possible holonomy groups in his paper [Ber55], they are known in the literature to be a class of manifolds with special holonomy. Hyperkähler manifolds are a particular class of  $4n$ -dimensional manifolds whose metrics are Ricci-flat, see [Joy03]. At the time of Berger’s classification, non-flat examples of hyperkähler manifolds were not known. The first global examples of non-flat hyperkähler metrics were the Euguchi–Hanson metrics on  $T^*\mathbb{C}\mathbb{P}^1$  given in [EH79]. In [Yau78] Yau proved the Calabi conjecture by solving the Monge–Ampère equation; as a consequence, Yau showed that a  $K3$  surface admits a Ricci-flat Kähler metric whose holonomy is  $\mathrm{Sp}(1)$ , making it a hyperkähler manifold. It was then in [Cal79] that Calabi extended the work of Yau and Euguchi–Hanson by finding explicit hyperkähler metrics on  $T^*\mathbb{C}\mathbb{P}^n$  for  $n \geq 2$ .

Restricting to dimension 4, it was Donaldson who posed the idea of loosening the conditions of a hyperkähler structure to one where the triple of Kähler forms was only symplectic. This is what we call in this thesis a hypersymplectic structure. In [Don06], Donaldson conjectured that up to isotopy the only example of a hypersymplectic manifold is a hyperkähler manifold. More precisely, he conjectured that there should exist a one parameter family of diffeomorphisms connected to the identity that take a hypersymplectic structure to a hyperkähler structure while remaining in the same cohomology class. In a particular setting, Fine and Yao have done work to prove this conjecture. In [FY18] the authors define a closed  $G_2$  structure on  $M^7 = \mathbb{T}^3 \times X^4$  where  $X^4$  is compact and hypersymplectic. They evolve the  $G_2$  structure under the  $G_2$  Laplacian flow to create a new geometric flow on  $X^4$  that could deform the hypersymplectic structure to a hyperkähler structure. This flow is known as the hypersymplectic flow. In this paper they are able to show long time existence of the flow provided the torsion tensor of the  $G_2$  structure stays bounded as we deform

the hypersymplectic structures. In [FHY25] the authors prove Donaldson's conjecture explicitly on  $\mathbb{T}^3 \times \mathbb{T}^4$ , under the assumption that the hypersymplectic structure on  $\mathbb{T}^4$  is  $\mathbb{T}^3$ -invariant for some  $\mathbb{T}^3 \subset \mathbb{T}^4$ .

This thesis comprises of a few results involving hypersymplectic structures and the hypersymplectic flow. In the first part of this thesis we construct a coclosed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$  where  $X^4$  is compact and hypersymplectic. Following analogously the ideas of Fine and Yao, we show this  $G_2$  structure is invariant under  $\mathbb{T}^3$ . We evolve this  $G_2$  structure under the  $G_2$  Laplacian coflow and find that its solutions remain  $\mathbb{T}^3$  invariant as long as the flow exists. Furthermore, we find that for this particular ansatz the Laplacian coflow descends to a flow of the hypersymplectic structure on  $X^4$ , which is the same flow that Fine-Yao discovered known as the hypersymplectic flow.

The main content of this thesis concerns solitons of the hypersymplectic flow. Solitons are a triple  $(\lambda, X, \underline{\omega})$  of a constant  $\lambda$ , a vector field  $X$ , and an initial geometric structure  $\underline{\omega}$  that give rise to self similar solutions of a geometric flow. Self similar solutions of a geometric flow are solutions which evolve by scalings and diffeomorphisms. There is a one-to-one correspondence between solitons and self similar solutions for geometric flows. This is shown explicitly for solitons of the hypersymplectic flow in section 4.1. Solitons, or, equivalently, self similar solutions, can be categorized into three types: shrinking, steady or expanding, depending on the sign of the constant  $\lambda$ . There has been a lot of work done on solitons of the Ricci flow and flows of  $G_2$  structures such as the Laplacian flow and coflow, and the isometric flow, see [KMT12], [IK24], [Ive94], [DW11], [DHW11]. This is due to the interesting properties of Ricci solitons, for example when the soliton vector field vanishes Ricci solitons correspond to Einstein metrics, see [CK04]. Solitons of the Ricci flow are regarded as generalized fixed points of the Ricci flow and often arise as limits of dilations around singularities of the flow, see [CK04]. These properties about Ricci flow solitons are what incentivize researchers to look for solitons of other geometric flows. However, even though a lot of work has been done regarding solitons of other geometric flows the same relation between solitons and singularities is not yet confirmed but is suspected.

In chapter 1 we begin by introducing some background needed in order to explain the results in the later chapters. In section 1.2 we explain the necessary geometric structures to understand the problems. We introduce a positive triple of 2-forms on a 4-manifold, which we use to define a hypersymplectic structure and show how this gives rise to a Riemannian metric. In section 1.3 we explain the notion of a cohomogeneity one manifold and the particular cohomogeneity one action we use in this thesis. In section 1.4 we discuss the  $G_2$  structure defined by Fine and Yao in [FY18]. We also show how the hypersymplectic

flow descends from the  $G_2$  Laplacian flow and thus inherits the properties of short time existence and uniqueness. In this section we also explain a bit about the Laplacian coflow. In chapter 2 we explain how given a hypersymplectic triple we can either define a closed  $G_2$  structure or a coclosed  $G_2$  structure. We construct a coclosed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$  where  $X^4$  is compact. We find that this  $G_2$  structure is invariant under  $\mathbb{T}^3$ . We flow this under the  $G_2$  Laplacian coflow and find that it descends to the same flow of hypersymplectic structures found by Fine and Yao in [FY18]. In chapter 3 we introduce a particular cohomogeneity one hypersymplectic structure on  $\mathbb{R}^4 \setminus \{0\}$ . We compute the metric and both the Riemann and Ricci curvature tensors. In section 3.6 we verify a particular occurrence of when the hypersymplectic structure can be transformed to a hyperkähler structure. In chapter 4 we introduce the notion of a soliton for the hypersymplectic flow. In section 4.1 we derive the general soliton formula for the hypersymplectic flow. This formula gives rise to a triple of equations coming from each of the symplectic forms in the triple. In sections 4.2 and 4.3 we analyze the three soliton equations and solve for the soliton vector field coefficients. This leads us to two differential equations for which we know there exist solutions. Under some assumptions we can write down explicit solutions but in general the equations cannot be solved explicitly. In section 4.4 we discuss some particular solutions of the soliton equations which can be written down explicitly. We find that steady solitons correspond to a hypersymplectic triple that can be transformed to a hyperkähler triple. In chapter 5, we discuss future ideas along this line of research.

## 1.1 Notations and conventions

In this section we give a brief explanations of the notation and conventions that are used regularly throughout this thesis.

- The Einstein convention of a double repeated index is used to mean the sum over the possible values of that index. For example, for a function  $P_i$  and a 2-form  $\omega_i$ ,

$$P_i \omega_i = \sum_{i=1}^3 P_i \omega_i.$$

- A multi-index in an exponent is used to mean the wedge product. For example, given three 1-form  $e^i, e^j, e^k$ ,

$$e^{ijk} = e^i \wedge e^j \wedge e^k.$$

- Let  $dt^j$  be 1-forms on  $\mathbb{T}^3$ . The notation  $\hat{dt}^j$  is used to express the 2-form on  $\mathbb{T}^3$  that omits  $dt^j$ . That is

$$\hat{dt}^j = dt^i \wedge dt^k$$

where  $i, j, k$  are cyclically permuted  $i \rightarrow j \rightarrow k$ .

It will be made apparent in this thesis when these notations and conventions are used.

## 1.2 Geometric structures

In this section we explain the different structures needed to understand Donaldson's conjecture and the results of this thesis. Namely  $SU(2)$  structures, hypersymplectic structures, and hyperkähler structures. For this section we let  $X^4$  be any oriented manifold of dimension four. We fix a volume form  $\mu$  on  $X^4$ . The first structure we introduce is called a positive triple, this is the scaffolding necessary for understanding our definitions of  $SU(2)$ , hypersymplectic, and hyperkähler structures.

**Definition 1.2.1.** *Let  $X^4$  be an oriented manifold of dimension four, let  $\mu$  be a non-vanishing top form on  $X^4$ . Let  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  be a triple of 2-forms on  $X^4$ . Let  $Q$  be the symmetric  $3 \times 3$  matrix valued function such that*

$$\omega_i \wedge \omega_j = 2Q_{ij}\mu. \tag{1.1}$$

*We say the triple  $\underline{\omega}$  is a positive triple if  $Q$  is positive definite at all points.*

**Remark 1.2.2.** *When  $Q$  is positive definite, the 2-forms  $\omega_i$  are non-degenerate as*

$$\omega_i^2 = 2Q_{ii}\mu \neq 0$$

*which is non-vanishing since  $Q_{ii} > 0$  by positive definiteness.*

We also note that the matrix  $Q$  depends on  $\underline{\omega}$  but also on  $\mu$ . One can see that as  $\mu$  changes,  $Q$  also changes. There is a particular choice of volume form  $\mu$  so that  $\det(Q) = 1$ , this detail is explained in lemma 1.2.4. This is the volume form that is chosen in [FY18] and the one we use in section 4 when we look for solitons of the hypersymplectic flow. For now we do not make that assumption, we simply state it so that the reader is aware. It will be made apparent later in the thesis when we do decide to use this particular form. Furthermore, the positive triple  $\underline{\omega}$  gives rise to a Riemannian metric  $g_{\underline{\omega}}$  defined by

$$g_{\underline{\omega}}(u, v)\mu_{\underline{\omega}} = \frac{1}{6}\epsilon_{ijk}(u \lrcorner \omega_i) \wedge (v \lrcorner \omega_j) \wedge \omega_k. \tag{1.2}$$

It is important to note that  $\mu_{\underline{\omega}}$  is the Riemannian volume form of  $g_{\underline{\omega}}$  and is different from our reference form  $\mu$  on  $X^4$ . However  $\mu_{\underline{\omega}}$  can be expressed in terms of  $Q$  and  $\mu$ , as we show in the following lemma. We want to make clear that  $Q$  scales depending on  $\mu$  but  $\mu_{\underline{\omega}}$  is always expressed in the same way as in lemma 1.2.4. Furthermore, it is not obvious that (1.2) gives a Riemannian metric, this is due to the positive definiteness of  $Q$ , We refer the reader to [FY18] and [FY19] for more details regarding this metric construction.

**Remark 1.2.3.** *Each  $\omega_i$  in the positive triple is self-dual with respect to the metric and orientation given in (1.2) [FY18].*

**Lemma 1.2.4.** *Let  $\underline{\omega}$  be a positive triple on  $X^4$ . Let  $\mu$  be the reference volume form for  $\underline{\omega}$  as in definition 1.2.1. Let  $g_{\underline{\omega}} \otimes \mu_{\underline{\omega}}$  be defined as in (1.2). Then  $\mu_{\underline{\omega}} = (\det(Q))^{\frac{1}{3}} \mu$ .*

*Proof.* To begin we emphasize that we are doing these computations at some fixed point in  $X^4$  as this result and proof is entirely linear algebra. Let  $\underline{\omega}$  be a positive triple. Let  $\mu$  and  $Q$  be such that  $2Q_{ij}\mu = \omega_i \wedge \omega_j$ . Since  $Q$  is a symmetric matrix, there exists an invertible  $3 \times 3$  matrix  $P$  such that if  $\theta_1, \theta_2, \theta_3$  are defined by

$$\theta_i = P_{il}\omega_l \quad \text{then} \quad \theta_i \wedge \theta_j = 2\delta_{ij}\mu.$$

We note that  $\underline{\theta} = (\theta_1, \theta_2, \theta_3)$  is still a positive triple whose matrix  $Q_{\underline{\theta}} = \mathbb{I}_{3 \times 3}$ . When we expand the expression for the wedge product above, we obtain

$$\begin{aligned} P_{ik}\omega_k \wedge P_{jl}\omega_l &= P_{ik}P_{jl}\omega_k \wedge \omega_l = P_{ik}P_{jl}2Q_{kl}\mu, \\ \implies P_{ik}Q_{kl}P_{jl} &= \delta_{ij}. \end{aligned}$$

This tells us that  $PQP^T = \mathbb{I}_{3 \times 3}$ . Let  $M = P^{-1}$ . By taking the determinant of the expression above, we have

$$\det(P) \det(Q) \det(P) = 1 \quad \text{and hence} \quad \det(Q) = (\det(M))^2.$$

Since we let  $M = P^{-1}$ , we can say that  $\omega_i = M_{il}\theta_l$ . Now let  $g_{\underline{\theta}}$  be the metric coming from the triple  $\underline{\theta}$  defined as in (1.2). One can show that there locally exists a coframe  $e^0, e^1, e^2, e^3$  that is orthonormal with respect to  $g_{\underline{\theta}}$  so that  $\theta_i = e^0 \wedge e^i + \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k$  and  $\mu = e^0 \wedge e^1 \wedge e^2 \wedge e^3$ . Let  $e_i$  be the dual frame to  $e^i$ . We use the formula for the metric (1.2) to say

$$\frac{1}{6}\epsilon_{ijk} (e_a \lrcorner \theta_i) \wedge (e_b \lrcorner \theta_j) \wedge \theta_k = \delta_{ab}\mu. \quad (1.3)$$

We know that given a basis,  $\mu_{\underline{\omega}} = \sqrt{\det(g_{\underline{\omega}})}\mu$  where  $g_{\underline{\omega}}$  is the metric determined by the triple  $\underline{\omega}$ . We wish to determine  $\sqrt{\det(g_{\underline{\omega}})}$ . To do this we know that  $g_{\underline{\omega}}$  can be expressed

as a  $4 \times 4$  matrix, thus if we let  $R$  be a  $4 \times 4$  matrix such that  $R_{ab}\mu = g_{\underline{\omega}}(e_a, e_b)\mu_{\underline{\omega}}$  and use lemma A.2.1 and (1.3) then

$$\begin{aligned}
R_{ab}\mu &= g_{\underline{\omega}}(e_a, e_b)\mu_{\underline{\omega}} = \frac{1}{6}\epsilon_{ijk}(e_a \lrcorner \omega_i) \wedge (e_b \lrcorner \omega_j) \wedge \omega_k \\
&= \frac{1}{6}\epsilon_{ijk}M_{ip}M_{jq}M_{kl}(e_a \lrcorner \theta_p) \wedge (e_b \lrcorner \theta_q) \wedge \theta_l \\
&= \frac{1}{6}\det(M)\epsilon_{pql}(e_a \lrcorner \theta_p) \wedge (e_b \lrcorner \theta_q) \wedge \theta_l \\
&= \det(M)\delta_{ab}\mu.
\end{aligned}$$

The above computation determines that  $R = \det(M)\mathbb{I}_{4 \times 4}$ . We use this computation and the fact that  $\mu_{\underline{\omega}} = \sqrt{\det(g_{\underline{\omega}})}\mu$  to obtain

$$R = g_{\underline{\omega}}\sqrt{\det(g_{\underline{\omega}})} = \det(M)\mathbb{I}_{4 \times 4}. \quad (1.4)$$

We apply the determinant to (1.4) to obtain

$$\begin{aligned}
\det\left(g_{\underline{\omega}}\sqrt{\det(g_{\underline{\omega}})}\right) &= \det(\det(M)\mathbb{I}_{4 \times 4}) \\
\det(g_{\underline{\omega}})^3 &= \det(M)^4 \\
\det(g_{\underline{\omega}})^3 &= \det(Q)^2 \quad \text{since} \quad \det(Q) = \det(M)^2 \\
\sqrt{\det(g_{\underline{\omega}})} &= \det(Q)^{\frac{1}{3}}.
\end{aligned}$$

We substitute this last equation back into the formula for  $\mu_{\underline{\omega}}$  in terms of  $\mu$  and obtain our desired result  $\mu_{\underline{\omega}} = (\det(Q))^{\frac{1}{3}}\mu$ .  $\square$

This lemma tells us that when  $\det(Q) = 1$  the volume form  $\mu_{\underline{\omega}}$  is equal to the reference volume form  $\mu$ . In particular, it tells us how to choose  $\mu$  so that the determinant of  $Q$  is always 1. If we first construct  $\mu_{\underline{\omega}}$  using the formula for the metric (1.2) and choose  $\mu = \mu_{\underline{\omega}}$  this ensures that  $\det(Q) = 1$ . This is important in particular because as we flow our geometric structures in time  $\mu_{\underline{\omega}}$ ,  $\underline{\omega}$ , and  $Q$  will depend on time but the determinant of  $Q$  at any fixed point in time will always be 1.

The relation (1.1) and the Riemannian metric (1.2) give us three potential structures on  $X^4$ . The first happens when  $Q$  is the identity matrix. In this case the triple  $\underline{\omega}$  forms an  $SU(2)$  structure [FSE25]. The second is a hypersymplectic structure, this is the case where the forms  $\omega_i$  are symplectic, meaning they are closed, that is  $d\omega_i = 0$ . In this case  $Q$  need

not necessarily be the identity matrix [FY18]. The last is a hyperkähler structure. In this case the triple forms an  $SU(2)$  structure and a hypersymplectic structure, meaning the forms  $\underline{\omega}$  are symplectic (i.e. closed) and the matrix  $Q$  is the identity [FHY25], [Don17]. We give the full definition of hypersymplectic and hyperkähler below.

**Definition 1.2.5.** *Let  $(M, g)$  be a Riemannian manifold equipped with three complex structures  $I, J, K: TM \rightarrow TM$  such that*

$$I^2 = J^2 = K^2 = IJK = -1.$$

*If  $I, J, K$  are parallel then  $(M, g, I, J, K)$  is a hyperkähler manifold. Furthermore, we compute  $\omega_i(x, y) = -g(x, Jy)$  for the Kähler forms. There is one for each complex structure  $I, J, K$ . These 2-forms are closed and non-degenerate.*

**Remark 1.2.6.** *When a positive triple  $\underline{\omega}$  is hyperkähler, where the forms  $\omega_i$  are given as in the definition above, the matrix  $Q$  that comes from the positive triple is the identity. We show this below.*

To verify the remark made above, let  $M$  be a 4-dimensional hyperkähler manifold. Let  $e_0, e_1, e_2, e_3$  be orthonormal basis with respect to  $g$  as in definition 1.2.5, let  $e^i$  for  $i = 0, 1, 2, 3$  be its dual, and let  $\mu = e^{0123}$ . Let  $J_i$  be the three complex parallel structures on  $M$  so that  $\omega_i(x, y) = -g(x, J_i y)$  and  $J_i(e_0) = e_i$ , then we have the following relations:

$$\begin{aligned}\omega_i(e_0, e_i) &= -g(e_0, J_i(e_i)) = 1, \\ \omega_i(e_0, e_j) &= -g(e_0, J_i(e_j)) = -g(e_0, e_k) = 0, \\ \omega_i(e_j, e_i) &= -g(e_j, J_i(e_i)) = g(e_j, e_0) = 0, \\ \omega_i(e_j, e_k) &= -g(e_j, J_i(e_k)) = g(e_j, e_j) = 1.\end{aligned}$$

This allows us to say that in this basis  $\omega_i = e^0 \wedge e^i + e^j \wedge e^k$ . It is then easy to see that when the triple is written in this form the relation  $\omega_i \wedge \omega_j = 2\delta_{ij}\mu$  is satisfied. Since  $Q = \mathbb{I}_{3 \times 3}$  and  $\det(Q) = 1$  we have  $\mu = \mu_{\underline{\omega}}$ . Furthermore, one can show that  $g_{\underline{\omega}}$  is indeed the hyperkähler  $g$  from definition 1.2.5.

The holonomy of a hyperkähler structure is contained in  $Sp(n)$  and therefore are manifolds with special holonomy according to Berger's holonomy list; see [Dan99], [BB82], [Joy03]. One important property of hyperkähler metrics is that they are Ricci flat. Some examples of compact hyperkähler manifolds are  $T^4$  and K3 surfaces, see [Dan99].

**Remark 1.2.7.** *When the 2-forms  $\omega_i$  are all closed, the complex structures they define are integrable, see [Hit92], [Dan99]. Since the intrinsic torsion of an  $SU(2)$  structure vanishes if and only if each 2-form  $\omega_i$  is closed, see [FSE25], one can think of a hyperkähler structure in dimension 4 as a torsion free  $SU(2)$  structure.*

**Definition 1.2.8.** *Let  $X^4$  be a 4 dimensional oriented smooth manifold. A hypersymplectic structure on  $X^4$  is a positive triple of symplectic forms  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ .*

We can always take a hypersymplectic triple  $\underline{\omega}$  and diagonalize so that  $Q = \mathbb{I}_{3 \times 3}$ . This gives us a  $SU(2)$  structure at the cost of the closed-ness of the triple. When the matrix  $Q$  coming from the hypersymplectic triple is constant, we can apply a linear transformation to our triple  $\underline{\omega}$  so that it is now a triple  $\underline{\tilde{\omega}}$  for which  $Q$  is now the identity matrix, and  $\underline{\tilde{\omega}}$  is closed thus hyperkähler. Then  $\underline{\omega}$  induces the same metric as  $\underline{\tilde{\omega}}$  up to a constant factor. We show this explicitly now. Let  $\underline{\omega}$  be a hypersymplectic triple where  $Q$  given as in definition 1.2.8 is constant. Then there exists some constant matrix  $A$  that diagonalizes  $Q$ , meaning we can define another triple  $\underline{\tilde{\omega}}$  by  $\tilde{\omega}_i = A_{ik}\omega_k$  where  $\tilde{\omega}_i \wedge \tilde{\omega}_j = 2\delta_{ij}\mu$ . Since  $A$  is constant the triple  $\underline{\tilde{\omega}}$  is still closed since

$$d\tilde{\omega}_i = dA_{ik} \wedge \omega_k + A_{ik}d\omega_k = 0.$$

Furthermore, let the Riemannian metric as defined in (1.2) derived using the triple  $\underline{\tilde{\omega}}$  be  $g_{\underline{\tilde{\omega}}}$  and let the metric defined using  $\underline{\omega}$  be denoted by  $g_{\underline{\omega}}$ . Then the metrics are constant multiples of one another:

$$g_{\underline{\tilde{\omega}}} = \det(A)g_{\underline{\omega}}. \tag{1.5}$$

This computation can be easily seen from some of the computations done in the proof of lemma 1.2.4. Since  $A$  is a constant matrix,  $\det(A)$  is also constant. Therefore the metric  $g_{\underline{\omega}}$  is still Ricci-flat. We conclude that when  $Q$  is constant for a hypersymplectic triple we are able transform to a hyperkähler triple whose metric is equivalent up to a constant factor.

The last thing we should point out in this section is the two equivalent definitions of hypersymplectic. In [FY18], the authors offer two definitions of a hypersymplectic structure. The first is the definition we provided in 1.2.8. They also define a hypersymplectic structure as a triple of symplectic forms  $\underline{\omega}$  wherein any (non-zero) linear combination of the forms is again a symplectic form. We state it explicitly below and then we show that the definitions are indeed equivalent.

**Definition 1.2.9.** *Let  $X^4$  be an 4-dimensional manifold. Let  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  be a triple of symplectic forms. We say that  $\underline{\omega}$  is a hypersymplectic structure if for any  $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ ,  $a_1\omega_1 + a_2\omega_2 + a_3\omega_3$  is again a symplectic form.*

Let us show that definitions 1.2.8 and 1.2.9 are equivalent. Let  $\vec{a} = (a_1, a_2, a_3)$  be a non-zero vector in  $\mathbb{R}^3$  and let  $\omega_{\vec{a}} = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ . Let  $\mu$  be the reference volume form

such that  $\omega_i \wedge \omega_j = 2Q_{ij}\mu$ . We know  $Q$  is symmetric but we claim that  $Q$  being definite is equivalent to  $\omega_{\vec{a}}$  being non-degenerate. To see this, note that

$$\omega_{\vec{a}}^2 = (a_i\omega_i) \wedge (a_j\omega_j) = 2a_i a_j Q_{ij}\mu.$$

We observe that  $\omega_{\vec{a}}$  is non-degenerate for all non-zero  $\vec{a}$  if and only if  $Q(\vec{a}, \vec{a}) \neq 0$  for non-zero  $\vec{a}$ . We claim that given any non-zero  $\vec{a} = (a_1, a_2, a_3)$ ,  $\omega_{\vec{a}}$  all induce the same orientation. We can think of  $\omega_{\vec{a}}^2$  as

$$\omega_{\vec{a}}^2 = f(\vec{a})\mu$$

where  $f : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}$  is some non-vanishing smooth function. Since  $\mathbb{R}^3 \setminus \{0\}$  is connected and  $f$  is non-vanishing,  $f$  must be either positive everywhere or negative everywhere. Therefore any non-zero linear combination of  $\underline{\omega}$  induces the same orientation on the 4-manifold. Thus requiring  $Q$  to be positive definite gives a choice of orientation. Furthermore, it is easy to see that each  $\omega_i$  is closed if and only if  $\omega_{\vec{a}}$  is closed for all non-zero  $\vec{a}$ . This allows us to conclude that the two definitions of hypersymplectic are indeed equivalent.

Now we will introduce some background on  $G_2$  geometry. We refer the reader to the following sources for more details: [Kar25], [Kar20], [Joy00], [Sal17], [BS89]. Let  $\mathbb{O}$  be the octonions. They are an exceptional division algebra, and we can identify  $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ . Using octonionic multiplication we can define a cross product on  $\mathbb{R}^7$  in the following way:

$$a \times b = \text{Im}(ab) \quad a, b \in \text{Im}(\mathbb{O}). \quad (1.6)$$

This is a bilinear map on  $\mathbb{R}^7$  and we call it a vector cross product in the same way we do for the cross product on  $\mathbb{R}^3$  which arose as the imaginary projection of quaternionic multiplication on  $\mathbb{R}^4$ , as  $\mathbb{R}^3 \cong \text{Im}(\mathbb{H})$ . Furthermore, let  $\langle \cdot, \cdot \rangle$  be the standard Euclidean inner product on  $\mathbb{R}^7$  and let  $[a, b] = ab - ba$ ,  $a, b \in \text{Im}(\mathbb{O})$  be the commutator. Then we can define a 3-form on  $\mathbb{R}^7 \cong \text{Im}(\mathbb{O})$  by

$$\varphi(a, b, c) = \frac{1}{2} \langle [a, b], c \rangle = \langle a \times b, c \rangle \quad a, b, c \in \text{Im}(\mathbb{O}). \quad (1.7)$$

We use this to create a standard  $G_2$  structure on  $\mathbb{R}^7$ . Let  $g_0$  be the standard Euclidean metric on  $\mathbb{R}^7$  such that the standard dual basis  $e^1, \dots, e^7$  is orthonormal and the standard volume form is  $\mu_0 = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7$ . Then the metric  $g_0$  and the volume form  $\mu_0$  give rise to a Hodge star operator  $*_0$  and in turn a 4-form  $\psi_0 = *_0\varphi$  on  $\mathbb{R}^7$ . Using octonionic multiplication and (1.7) one can show that in this standard basis

$$\varphi_0 = e^{123} - e^{167} - e^{527} - e^{563} + e^{415} + e^{426} + e^{437} \quad (1.8)$$

where  $e^{ijk} = e^i \wedge e^j \wedge e^k$ . We can describe  $\psi_0$  in this basis as well but we omit that and refer the reader to [Kar20]. Note that with respect to the Euclidean metric  $g_0$  we know the following

$$\begin{aligned}\|\psi_0\|^2 &= \|\varphi_0\|^2 = 7, \\ \varphi_0 \wedge \psi_0 &= 7\mu_0.\end{aligned}$$

Therefore a  $G_2$  structure on  $\mathbb{R}^7$  is given by the package  $(\varphi_0, \psi_0, g_0, \mu_0, \times_0)$  where  $\times_0$  is the vector cross product we defined in (1.6). We note that  $\times_0$  and  $\psi_0$  are derived from  $g_0, \mu_0$  and  $\varphi_0$ . Furthermore the metric  $g_0$ , the volume form  $\mu_0$  and the 3-form  $\varphi_0$  are connected in the following way

$$g_0(a, b)\mu_0 = \frac{1}{6} (a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0. \quad (1.9)$$

Thus the 3-form  $\varphi_0$  can be used to extract the metric and the volume form. As a result the  $G_2$  structure on  $\mathbb{R}^7$  need only be given by  $\varphi_0$ . We can think of the group  $G_2$  as being the subgroup of  $\text{GL}(7, \mathbb{R})$  that preserves the 3-form  $\varphi_0$ . It follows that  $G_2$  is a simply connected, compact, Lie subgroup of  $\text{SO}(7)$ . Understanding the  $G_2$  structure on  $\mathbb{R}^7$  is imperative to the construction of general  $G_2$  manifolds.

Let  $M^7$  be a smooth orientable 7-manifold. We say  $M^7$  has a  $G_2$  structure given by a 3-form  $\varphi$  if at every point  $p$  there exists a linear isomorphism  $F: \mathbb{R}^7 \rightarrow T_p M$  such that the pullback of  $\varphi$  under  $F$  is the  $G_2$  structure  $\varphi_0$  on  $\mathbb{R}^7$  given in (1.8),  $F^*\varphi = \varphi_0 \in \bigwedge^3(\mathbb{R}^7)^*$  for  $\varphi \in \bigwedge^3(T_p^* M)$ . Given a  $G_2$  structure  $\varphi$ , we obtain a Riemannian metric  $g_\varphi$  on  $M^7$  in the following way

$$g_\varphi(a, b)\mu_\varphi = \frac{1}{6} (a \lrcorner \varphi) \wedge (b \lrcorner \varphi) \wedge \varphi, \quad (1.10)$$

from (1.10) one can obtain  $g_\varphi$  and  $\mu_\varphi$ . This in turn defines a Hodge star operator  $*_\varphi$  and the 4-form  $\psi = *_\varphi \varphi$  on  $M^7$ . We also point out that  $G_2$  manifolds are included in Berger's list of manifolds with special holonomy, that is manifolds whose holonomy is a subgroup of  $\text{SO}(n)$  [Ber55]. There is only one such kind of manifold for dimension 7 and they are  $G_2$  manifolds, we call them manifolds with exceptional holonomy since they are unique to dimension 7. We also want to mention that the  $G_2$  structure on  $M^7$  can be described using either the 3-form  $\varphi$  or the 4-form  $\psi$ , the difference being that when using the 4-form one needs to first choose an orientation in comparison to the 3-form which gives rise to an orientation. Using the 4-form we can still describe the Riemannian metric and the Hodge star, however currently there is no available literature explicitly stating how one might compute the metric given only the 4-form. A  $G_2$  structure is called closed if  $d\varphi = 0$  or co-closed if  $d\psi = 0$ .

Furthermore, since  $\varphi$  defines a Riemannian metric, it in turn gives a Levi-Cevita connection  $\nabla$ . We say that the  $G_2$  structure  $\varphi$  is torsion free if  $\varphi$  is parallel with respect to the Levi-Cevita connection,  $\nabla\varphi = 0$ . One can show that the torsion  $T$  of a  $G_2$  structure is an endomorphism of the tangent bundle characterized by

$$\nabla_X\varphi = T(X) \lrcorner \psi, \tag{1.11}$$

for more details on this definition we refer the reader to [Kar20]. This  $T$  is called the full torsion tensor of the  $G_2$  structure  $\varphi$ . We can easily see from (1.11) that  $\nabla\varphi = 0$  if and only if  $T = 0$ . It is a result of Fernandez and Gray in [FG82] that a  $G_2$  structure  $\varphi$  is torsion free if and only if it is closed and co-closed. Much work has been done on trying to find such manifolds [JK21], [Joy96], [Kov03], [CHNP15], and we refer the reader to the survey articles [Kov20], [Kar20] for more details and sources. Discussion of torsion free  $G_2$  structures is how the field of  $G_2$  geometric flows arose, by starting with some weaker assumptions than vanishing torsion (closed or co-closed) and seeing if one might be able to flow the geometric structure  $\varphi$  in time to a torsion free structure.

### 1.3 Cohomogeneity one manifolds

In this section we give a brief introduction to cohomogeneity one manifolds. Such manifolds play a special role in Riemannian geometry. They are used to create specific examples of interesting metrics. For more details see the following sources [DW98], [DW11] [AS21], [CS02], [GZ02], [Hoe10], [Zil09]. Much work has been done on studying cohomogeneity one manifolds with certain curvature restrictions. For example, any cohomogeneity one manifold admits a complete metric with nonnegative Ricci curvature, [GZ02]. Furthermore a compact cohomogeneity one manifold admits a metric with positive Ricci curvature if and only if its fundamental group is finite [GZ02]. In [Hoe10] a classification of compact, simply connected cohomogeneity one manifolds in dimensions 5, 6, and 7 is given. With some exceptions, all such manifolds admit metrics with nonnegative sectional curvature. Cohomogeneity one manifolds have also been used to create examples of manifolds with special holonomy. In [CS02] examples of cohomogeneity one  $G_2$  manifolds are studied, in particular the metrics are Einstein of positive sectional curvature.

**Definition 1.3.1.** *A cohomogeneity one manifold is a connected Riemannian manifold  $(M, g)$  with a smooth action  $\alpha : G \times M \rightarrow M$  of a compact, connected, Lie group  $G$  by isometries such that the generic orbits of the action have co-dimension one.*

Since  $G$  is connected the action of  $\alpha$  is proper [AS21], meaning the pre-image of every compact set is compact. Let  $\pi : M \rightarrow M/G$  be the map that takes  $M$  to the cohomogeneity one space  $M/G$ . Bérard-Bergery [BB82] says this is homeomorphic to  $S^1$  or an interval. This is decided by whether the space  $M/G$  is simply connected or not. One particularly obvious reason for using cohomogeneity one manifolds when studying geometric flows is that it significantly reduces the amount of computations. This is because the geometric structure on  $M$  is assumed to be invariant under the group  $G$  so it will only depend on a single variable. This in turn transforms many partial differential equations into ordinary differential equations by reducing the spacial dimensions in the computations to 1 dimension. In this thesis we work in dimension 4, and inducing a cohomogeneity one action reduces our computations to a single variable, thus the partial differential equations that arise from computing solitons of the hypersymplectic flow become ordinary differential equations. This makes cohomogeneity one manifolds good spaces on which to study certain geometric flows. For more references on geometric flows on manifolds with special holonomy see [DGK25], [LW17], [Bry06], [BX11] [CK04].

For the set up of this thesis we shall let  $M = \mathbb{R}^4$  and let  $G = \text{SO}(4)$ , where  $\text{SO}(4)$  acts on  $\mathbb{R}^4$  by rotations. The space  $M/G$  is homeomorphic to the ray  $[0, \infty)$ . For  $a \in (0, \infty)$ ,  $\pi^{-1}(a)$  is a principal orbit and  $\pi^{-1}(0)$  is a singular orbit. The principal or generic orbits for this action are  $S^3$ . Let  $X^4 = \mathbb{R}^4 \setminus \{0\} = (0, \infty) \times S^3$ . This space is not compact, and this is a key difference between the work done in this thesis and previous work done on the hypersymplectic flow. We say two points on  $X^4$  are equivalent if they lie on the same copy of  $S^3$ , meaning their distance from the origin in  $\mathbb{R}^4$  is the same. This allows us to reduce the dimension of our problem from 4 to 1. We note that we have removed the point at the origin, this is due to our choice of frame for  $T_p X^4$  that is described in chapter 3, as a result it is not immediate that any of the results of this thesis extend to the origin. In chapter 3 we show explicitly how we use the cohomogeneity one action to construct a hypersymplectic structure on  $X^4 = \mathbb{R}^4 \setminus \{0\}$ .

## 1.4 Geometric flows

In this section we introduce the hypersymplectic flow, a geometric flow defined in [FY18]. This is a flow that deforms hypersymplectic structures in time while keeping them in the same cohomology class. In the compact case, the fixed points of this flow are hyperkähler structures [FY19]. We explain how this flow descended from the  $G_2$ -Laplacian flow and as a result inherits properties of short time existence and uniqueness. For more details on the  $G_2$ -Laplacian flow we refer the reader to the following sources: [Bry06], [BX11],

[LW17], [Kar25]. We also introduce the Laplacian cflow, first defined in [KMT12]. This flows co-closed  $G_2$  structures in time and stops at torsion free  $G_2$  structures. Much work has been done on this flow, see also [Gri20] [Gri13]. On a compact manifold the  $G_2$  Laplacian flow has short time existence and uniqueness. When the torsion tensor of the  $G_2$  manifold is bounded the flow also has long time existence. It is not known if the Laplacian cflow has short time existence or uniqueness. However, in [Gri13] a modified Laplacian cflow was introduced which does have short time existence and uniqueness on a compact manifold. Furthermore, when the trace of the  $G_2$  torsion tensor vanishes and this condition is preserved by the flow, the cflow and modified cflow agree.

### 1.4.1 The hypersymplectic flow

Given a hypersymplectic triple  $\underline{\omega}$  using (1.2) the triple induces a metric  $g_{\underline{\omega}}$  and a volume form  $\mu_{\underline{\omega}}$ . The triple then gives a matrix  $Q$  defined using  $\mu_{\underline{\omega}}$  to ensure that  $\det(Q) = 1$ . Using this particular  $Q$ , the hypersymplectic flow was first defined in [FY18] as

$$\partial_t \underline{\omega} = d(Qd^*(Q^{-1}\underline{\omega})). \quad (1.12)$$

It arose in an attempt to solve a conjecture of Donaldson. In [Don06] Donaldson conjectured that up to isotopy on a compact manifold, the only example of a hypersymplectic structure is a hyperkähler structure. We state the conjecture below and refer the reader to [FY19] for more details on the conjecture and how it arose in the literature.

**Conjecture 1.4.1.** (*Donaldson's Conjecture [Don06]*) *Let  $M$  be a compact oriented 4-manifold. Let  $\underline{\omega}$  be a hypersymplectic structure on  $M$ . There exists a one-parameter family of diffeomorphisms  $F_t : M \rightarrow M$  connected to the identity, that take a hypersymplectic structure  $\underline{\omega}(0) = \underline{\omega}$  to a hyperkähler structure  $\underline{\omega}(1) = F_1^* \underline{\omega}$ , so that for  $0 \leq t \leq 1$ ,  $\underline{\omega}(t) = F_t^* \underline{\omega}$  remains in the same cohomology class.*

In order to attempt to tackle this problem, in [FY18] the authors started by working with a  $G_2$ -structure on  $M^7 = \mathbb{T}^3 \times X^4$  where  $X^4$  is a compact hypersymplectic manifold. Let  $t^i$  be the flat coordinates on  $\mathbb{T}^3$  and let  $\underline{\omega}$  be a hypersymplectic structure on  $X^4$ . Then define a 3-form  $\varphi$  on  $M$  by

$$\varphi = dt^{123} - dt^1 \wedge \omega_1 - dt^2 \wedge \omega_2 - dt^3 \wedge \omega_3. \quad (1.13)$$

This form gives a closed  $G_2$  structure on  $M$  that is invariant under  $\mathbb{T}^3$  [FY18]. We note that the  $G_2$  structure being closed is dependent on the forms  $\omega_i$  being closed. If the forms

were not closed a 3-form  $\varphi$  defined as in (1.13) would still give a  $G_2$  structure on  $M$  it just would not be closed [FY18]. Therefore we see that  $\varphi$  defines a closed  $G_2$ -structure if and only if  $d\omega_i$  is closed for each  $i = 1, 2, 3$ . This is interesting to point out as one might think to remove the closed condition effectively creating a flow of positive triples and attempting to see if that might flow to a hypersymplectic structure (that is flow to closed  $G_2$  structure). Using the particular  $\varphi$  from (1.13), [FY18] computed that the metric  $g_\varphi$ , given in (1.10), is a warped product of the metric on  $X^4$  given by the hypersymplectic structure and the Euclidean metric on  $\mathbb{T}^3$ . It is given as

$$g_\varphi = Q_{ij} dt^i \otimes dt^j + g_\omega, \quad (1.14)$$

where  $g_\omega$  is the metric defined as in (1.2) coming from the hypersymplectic triple  $\omega$ . We should point out that in this case the volume form for the positive triple is  $\mu_\omega$ , ensuring that  $\det(Q) = 1$ . Otherwise we would see a determinant of  $Q$  factor in the metric  $g_\varphi$ . One can see how the  $G_2$  structure on  $M$  and the hypersymplectic structure on  $X^4$  are inextricably linked together. The Riemannian metric  $g_\varphi$  gives rise to the Hodge star operator  $*_7$ . It is important at this point to mention that while the symplectic forms are self-dual with respect to the Hodge star  $*_4$  on  $X^4$  coming from the metric  $g_\omega$ , they are not self dual with respect to the Hodge star  $*_7$  on  $M$ . This is evident, as  $*_7\omega_i$  is a 5-form on  $M$ .

One of the most common geometric flows in  $G_2$  geometry is the  $G_2$ -Laplacian flow. This flow was first introduced in [Bry06] as a flow of closed  $G_2$ -structures on a fixed cohomology class  $[\varphi] = \{\varphi_0 + d\eta\}$  and is given by

$$\frac{\partial \varphi}{\partial t} = dd^{*\varphi_t} \varphi, \quad (1.15)$$

with initial conditions  $\varphi(0) = \varphi_0$  and  $d\varphi_0 = 0$ . For more details on the construction of this flow we refer the reader to [Bry06], [BX11] and [Kar25]. This is a flow of the  $G_2$  3-form whose fixed points are torsion free  $G_2$ -structures [Kar25], [LW17]. A torsion free  $G_2$ -structure occurs when the 3-form  $\varphi$  is closed and co-closed, that is  $d\varphi = d^*\varphi = 0$  [FG82]. In [BX11] it is shown that when the  $G_2$  manifold is compact, the flow (1.15) has short-time existence, uniqueness and remains in the same cohomology class, in particular this means the solutions of the flow remain closed for their entire existence. Much work has been done on this flow see, [LW17], [LW19b], [LW19a], [Che18], [BX11], [Bry06], [FR20]. In [FY18] the authors used the  $G_2$ -Laplacian flow to create a flow of a hypersymplectic triple. In this case the manifold  $M = \mathbb{T}^3 \times X^4$  is compact since  $X^4$  is compact and the  $G_2$  form  $\varphi$  (1.13) is closed since the positive triple on  $X^4$  is hypersymplectic. The authors then use the  $G_2$ -Laplacian flow on this particular  $\varphi$ . This descends to a geometric flow on just  $X^4$ , wherein the hypersymplectic structure evolves according to the equation (1.12).

Let  $Q^{ij}$  be the entries of  $Q^{-1}$  and we can write the hypersymplectic flow more explicitly on each form as

$$\frac{\partial \omega_i}{\partial t} = d(Q_{ik} d^*(Q^{kl} \omega_l)). \quad (1.16)$$

This is derived using the  $G_2$  Laplacian flow on the closed  $G_2$  form (1.13). Before we derive this flow we note that in [FY18] the authors show that any solution of (1.15) that begins with initial condition (1.13) remains  $\mathbb{T}^3$  invariant for as long as it exists. Thus any solution of the  $G_2$ -Laplacian flow of the form (1.13) remains of this form along the flow. We derive the hypersymplectic flow now. The Hodge star computations for the metric (1.14) can be found in (A.5):

$$\begin{aligned} \partial_t \varphi &= dd^{*7} \varphi = -d *_{7} d(\underline{\mu}_{\omega} - *_{7}(dt^i \wedge \omega_i)) \\ &= -d *_{7} d(\underline{\mu}_{\omega} - *_{3} dt^i \wedge *_{4} \omega_i) \\ &= -d *_{7} d\left(\underline{\mu}_{\omega} - \left(Q^{ij} \hat{d}t^j\right) \wedge \omega_i\right) \\ &= -d *_{7} \left(\hat{d}t^j \wedge d(Q^{ij} \omega_i)\right) \\ &= -d(Q_{jk} dt^k \wedge *_{4} d(Q^{ij} \omega_i)) \\ &= -d(dt^k \wedge Q_{jk} *_{4} d(Q^{ij} \omega_i)) \\ &= dt^k \wedge d(Q_{jk} *_{4} d(Q^{ij} \omega_i)) \\ &= -dt^k \wedge d(Q_{jk} d^{*4}(Q^{ij} \omega_i)) \end{aligned}$$

where  $\hat{d}t^j$  is the 2-form on  $\mathbb{T}^3$  that does not include  $dt^j$ , e.g.  $\hat{d}t^1 = dt^2 \wedge dt^3$ . We also used the fact that  $\omega_i$  is self dual with respect to the metric  $g_{\omega}$  that induces  $*_4$  and  $d^{*n} = (-1)^{nk+n+1} *_n d *_n$  on  $k$ -forms. Now we take the time derivative of the  $G_2$  form (1.15) and note that the angular coordinates  $t^i$  do not depend on time [FY18]. We obtain the following

$$\partial_t \varphi = -dt^k \wedge \partial_t \omega_k.$$

By equating the above two computations for  $\partial_t \varphi$  we obtain the hypersymplectic flow (1.16):

$$\begin{aligned} -dt^k \wedge \partial_t \omega_k &= -dt^k \wedge d(Q_{jk} d^{*4}(Q^{ij} \omega_i)) \\ \partial_t \omega_k &= d(Q_{jk} d^{*4}(Q^{ij} \omega_i)). \end{aligned}$$

We note that when  $Q$  is constant,  $\omega$  is a fixed point of the flow. In such cases there exists a hyperkähler structure whose metric is equivalent (up to some constant) to the hypersymplectic metric. The hypersymplectic flow (1.12) descended from the  $G_2$ -Laplacian flow therefore one can say that the flows are equivalent in this particular set up. Thus the

hypersymplectic flow inherits the properties of short-time existence and uniqueness from the  $G_2$ -Laplacian flow. Furthermore, if the initial  $G_2$  structure  $\varphi_0$  is closed then under the flow (1.15), the structure  $\varphi(t)$  remains closed and belongs to the same cohomology class as  $\varphi_0$  for all time  $t$  [BX11], [Kar25]. Thus the same can be said about the hypersymplectic flow. If  $\underline{\omega}_0$  is the initial hypersymplectic structure then as we flow using (1.12)  $\underline{\omega}(t)$  remains closed and cohomologous to  $\underline{\omega}_0$  for as long as it exists.

Thus the remaining problem in order to prove Donaldson's conjecture is to show that the hypersymplectic flow exists for all time and that in the limit, the structure converges to a hyperkähler structure. In [FY18] the crux of the paper is to show that under a technical assumption, solutions of the hypersymplectic flow (1.12) can be extended past some finite time. However they do not prove that the limit is hyperkähler. The assumption the authors make is that the scalar curvature of the  $G_2$  structure remains uniformly bounded along the flow. On a closed  $G_2$  manifold the norm of the torsion tensor is equal to the negative scalar curvature [LW17]. Thus the main result of [FY18] shows that if the norm of the torsion is bounded then the flow can be extended past a finite time. To achieve this, they use a proof by contradiction. They use a result by Lotay-Wei from [LW17] to do some parabolic rescaling and a generalization of Perelman's  $\kappa$ -non-collapsing theorem by Gao Chen to show the flow is uniformly non collapsed in finite time. This allows them to take the limit. In the limiting hypersymplectic structure  $\underline{\omega}_\infty$ , they show the matrix  $Q^\infty$  is constant and thus can be transformed to some hyperkähler structure. They then show that the limiting structure is asymptotically locally Euclidean (ALE) and non-compact, giving Euclidean volume growth in the limit. However, a result by Kronheimer classified all hyperkähler 4-manifolds with Euclidean volume growth, and in particular this means the limit would contain a 2-sphere which is holomorphic for one of its hyperkähler structures. This gives a contradiction. In [FY19], the authors prove that when  $X^4$  is compact the fixed points of the flow are hypersymplectic structures whose matrix  $Q$  is constant. In [FHY25] it is shown in the case of  $\mathbb{T}^3 \times \mathbb{T}^4$  where  $\underline{\omega}$  is a  $\mathbb{T}^3$  invariant hypersymplectic structure on  $\mathbb{T}^4$  for  $\mathbb{T}^3 \subset \mathbb{T}^4$ , the norm of the torsion of the  $G_2$  structure on  $\mathbb{T}^7$  is always bounded. Then as a result of their earlier paper [FY18], in this specific case, solutions of the hypersymplectic flow can be extended past some finite time. In [FHY25] the authors also manage to show that the limiting structure is hyperkähler, thus proving Donaldson's conjecture explicitly in this case. These papers are very analytical and complicated and we refer the reader to [FY18], [FY19] and [FHY25] for more details. Presently [FY18] and [FHY25] are the only known convergence results for the hypersymplectic flow.

## 1.4.2 The Laplacian coflow

One can define a  $G_2$ -structure and its corresponding Riemannian metric using a 4-form  $\psi$  (up to a choice of orientation). So instead of flowing the 3-form  $\varphi$  in time, one might choose to flow the 4-form in time, thus creating a different type of  $G_2$  geometric flow. Using this idea, the Laplacian coflow was first introduced in [KMT12]. This flow is given by

$$\frac{\partial \psi}{\partial t} = dd^* \psi \quad (1.17)$$

on the cohomology class  $[\psi] = \{\psi_0 + d\eta\}$  with initial conditions  $\psi(0) = \psi_0$  and  $d\psi_0 = 0$ . This flow shares some properties with the Laplacian flow. In particular the solutions stay co-closed and remain in the same cohomology class for their entire existence [Gri13], [Gri20]. However the main difference between the flows is that it is still not known if the Laplacian coflow has short time existence. In [BX11], the Laplacian flow starting from closed  $G_2$  structures is weakly parabolic in the direction of closed 3-forms and thus an application of DeTurk's trick transforms this flow to a strictly parabolic flow which then has short-time existence. However the same cannot be done for the Laplacian coflow. This flow is not weakly parabolic in the direction of closed 4-forms and thus cannot be shown to be equivalent to some parabolic flow [Gri20]. In the hopes of finding co-closed  $G_2$ -structures that can be deformed to torsion-free  $G_2$  structures, Grigorian introduced the modified Laplacian coflow in [Gri13]. This flow is given by

$$\frac{\partial \psi}{\partial t} = dd^* \psi + 2d((A - \text{Tr}T_\psi) *_\psi \psi) \quad (1.18)$$

where  $\text{Tr}T_\psi$  is the trace of the full torsion tensor  $T_\psi$  of the  $G_2$ -structure defined by  $\psi$  and  $A$  is a fixed constant. We note that we can pick  $A = 0$  and this is exactly what we do in chapter 2. It can be seen that when  $A = 0$  and  $\text{Tr}T_\psi = 0$  the coflow and modified coflow agree. In [Gri13], when starting from a co-closed  $G_2$ -structure the author shows that the modified coflow (2.7) is weakly parabolic in the direction of closed forms and hence an application of DeTurk's trick can be used to show that the flow has short time existence. The author also uses the Nash-Moser inverse function theorem to show that the modified coflow also has uniqueness. These are the same techniques used by Bryant-Xu for the Laplacian flow in [BX11]. We summarize this to say that the following initial value problem

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= dd^* \psi + 2d((A - \text{Tr}T_\psi) *_\psi \psi) \\ \psi(0) &= \psi_0 \\ d\psi_0 &= 0 \end{aligned} \quad (1.19)$$

on the space  $\{\psi_0 + d\eta\}$  has short-time existence and uniqueness. By construction any solution  $\psi(t)$  remains in the same cohomology class  $[\psi_0]$  for as long as it exists, and thus any solution stays co-closed for all time [Gri13].

# Chapter 2

## Laplacian coflow

In the first chapter we introduced a positive triple  $\underline{\omega}$ . In this chapter we begin by introducing a dual positive triple. Using a closed positive triple (a hypersymplectic structure) we can construct either a closed  $G_2$  structure or a coclosed  $G_2$  structure. Whichever one we choose to construct, the dual triple will show up in the other form. If we make the assumptions that the dual triple is also closed then we obtain a torsion free  $G_2$  structure. In [FY18], the authors used a hypersymplectic triple to define a 3-form  $\varphi$  that gave a closed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$ . We follow this analogy and construct a coclosed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$  by defining a 4-form using a hypersymplectic triple in a similar manner to Fine-Yao in [FY18] and the dual to this triple will show up in the construction of the 3-form. We find that this  $G_2$  structure is invariant under  $\mathbb{T}^3$ . Then we run the  $G_2$ -Laplacian coflow that is defined in [Gri20] on our 4-form. In our particular setting the Laplacian coflow and modified coflow agree. We confirm that solutions to this flow are also invariant under  $\mathbb{T}^3$ . We find that it descends to a flow on  $X^4$  of the dual hypersymplectic triple which is precisely the hypersymplectic flow defined in [FY18].

### 2.1 Dual positive triple

Given some positive triple  $\underline{\omega}$  as in definition 1.2.1 we can define a dual positive triple. If  $Q$  is the symmetric positive definite matrix that defines the positive triple  $\underline{\omega}$ , then its inverse matrix  $Q^{-1}$  is also symmetric and positive definite. We use this to define a new positive triple  $\underline{\sigma}$  called a dual positive triple.

**Lemma 2.1.1.** *Let  $X^4$  be an oriented manifold of dimension four, let  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  be a positive triple on  $X^4$ . Let  $Q$  be the matrix determined by  $\underline{\omega}$  and the volume form  $\mu_{\underline{\omega}}$  so*

that  $\det(Q) = 1$ , as in lemma 1.2.4. Then  $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  given by

$$\sigma_i = (Q^{-1})_{ik} \omega_k = Q^{ik} \omega_k \quad (2.1)$$

is another a positive triple on  $X^4$ .

*Proof.* We remark that since  $Q$  is symmetric and positive definite, so is  $Q^{-1}$ . Thus if  $\omega_i \wedge \omega_j = 2Q_{ij} \mu_{\underline{\omega}}$  then the triple  $\underline{\sigma}$  given as in (2.1) satisfies

$$\begin{aligned} \sigma_i \wedge \sigma_j &= (Q^{ik} \omega_k) \wedge (Q^{jl} \omega_l) = Q^{ik} Q^{jl} \omega_k \wedge \omega_l \\ &= 2Q^{ik} Q^{jl} Q_{kl} \mu_{\underline{\omega}} \\ &= 2Q^{ik} \delta_{jk} \mu_{\underline{\omega}} \\ &= 2Q^{ij} \mu_{\underline{\omega}}. \end{aligned} \quad (2.2)$$

Since  $Q^{-1}$  is positive definite this along side the computation (2.2) confirms that the triple  $\underline{\sigma}$  given as in (2.1) defines a positive triple on  $X^4$ .  $\square$

**Remark 2.1.2.** We call the positive triple (2.1) the dual positive triple to  $\underline{\omega}$ .

If we require the 2-forms  $\sigma_i$  to be closed, then the triple  $\sigma_i$  is a hypersymplectic triple. It is important to note that if the positive triple  $\underline{\omega}$  is closed it is not necessarily true that the dual triple  $\underline{\sigma}$  is also closed. Furthermore we note that the computation (2.2), alongside the way we have chosen  $\mu_{\underline{\omega}}$  so that  $\det(Q) = 1$  tells us that  $\mu_{\underline{\omega}} = \mu_{\underline{\sigma}}$ .

In [FY18] the authors use a hypersymplectic triple  $\underline{\omega}$  on  $X^4$  to define a  $G_2$  form  $\varphi$  on  $\mathbb{T}^3 \times X^4$  that is closed. This form induces the following metric

$$g_{\varphi} = Q_{ij} dt^i dt^j + g_{\underline{\omega}} \quad (2.3)$$

where  $g_{\underline{\omega}}$  is the metric on  $X^4$  as in (1.2),  $Q$  is the matrix from the triple  $\underline{\omega}$  and  $\mu_{\underline{\omega}}$  and  $t^i$  for  $i = 1, 2, 3$  are the coordinates on  $\mathbb{T}^3$ . We now show that if we use the dual positive triple, the metric we obtain via (1.2) is the same. Let  $\underline{\omega}$  be a positive triple and let  $\underline{\sigma}$  be the dual triple defined as in (2.1). Using the formula for the metric (1.2) we obtain

$$\begin{aligned} g_{\underline{\sigma}}(u, v) \mu_{\underline{\sigma}} &= \frac{1}{6} \epsilon_{ijk} (u \lrcorner \sigma_i) \wedge (v \lrcorner \sigma_j) \wedge \sigma_k \\ &= \frac{1}{6} \epsilon_{ijk} (u \lrcorner Q^{im} \omega_m) \wedge (v \lrcorner Q^{jn} \omega_n) \wedge Q^{kl} \omega_l \\ &= \frac{1}{6} \epsilon_{ijk} Q^{im} Q^{jn} Q^{kl} (u \lrcorner \omega_m) \wedge (v \lrcorner \omega_n) \wedge \omega_l \\ &= \frac{1}{6} \det(Q^{-1}) \epsilon_{mnl} (u \lrcorner \omega_m) \wedge (v \lrcorner \omega_n) \wedge \omega_l \\ &= g_{\underline{\omega}}(u, v) \mu_{\underline{\omega}} \end{aligned}$$

where the last line uses the fact that we defined  $Q$  using the volume form  $\mu_{\underline{\omega}}$  so that  $\det(Q) = 1$ . We will use a similar analogy to [FY18] but instead we define the  $G_2$  4-form  $\psi$  using the dual hypersymplectic structure. Let  $\underline{\omega}$  be a positive triple with matrix  $Q$  and let  $\underline{\sigma}$  be its dual positive triple. Suppose that  $d\underline{\sigma} = 0$ . Then we can define a closed 4-form  $\psi$  on  $\mathbb{T}^3 \times X^4$  as follows

$$\psi = \mu_{\underline{\sigma}} - dt^{12} \wedge \sigma_3 - dt^{31} \wedge \sigma_2 - dt^{23} \wedge \sigma_1. \quad (2.4)$$

We claim that this defines a co-closed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$  where  $g_\psi = g_\varphi$  where  $g_\varphi$  is as in (2.3). The way to see this is that using the Hodge star  $*_7$  coming from the metric  $g_\varphi$ , we compute  $*_7\psi$  and obtain the precise 3-form  $\varphi$  defined in [FY18] which induces the metric  $g_\varphi$ . Since every  $G_2$  4-form induces a unique metric we obtain that  $g_\varphi = g_\psi$ . To this end, the closed 4-form in (2.4) gives a co-closed  $G_2$  structure on  $X^4$ . Let us show that  $*_7\psi$  is indeed  $\varphi = dt^{123} - dt^i \wedge \omega_i$ . We compute

$$\begin{aligned} \varphi = *_7\psi &= dt^{123} - *_3 dt^{12} \wedge \sigma_3 - *_3 dt^{31} \wedge \sigma_2 - *_3 dt^{23} \wedge \sigma_1 \\ &= dt^{123} - (Q_{3k} dt^k) \wedge \sigma_3 - (Q_{2k} dt^k) \wedge \sigma_2 - (Q_{1k} dt^k) \wedge \sigma_1 \\ &= dt^{123} - (Q_{3k} dt^k) \wedge (Q^{3i} \omega_i) - (Q_{2k} dt^k) \wedge (Q^{2i} \omega_i) - (Q_{1k} dt^k) \wedge (Q^{1i} \omega_i) \\ &= dt^{123} - dt^i \wedge \omega_i. \end{aligned}$$

We reiterate that  $d\underline{\omega} = 0$  gives a closed  $G_2$  structure and  $d\underline{\sigma} = 0$  gives a co-closed  $G_2$  structure. However if both the positive triple  $\underline{\omega}$  and its dual positive triple  $\underline{\sigma}$  are closed (both are hypersymplectic) then the  $G_2$  structure they induce is both closed and co-closed and therefore torsion free.

## 2.2 The hypersymplectic flow from the Laplacian coflow

In the previous section we defined a co-closed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$  coming from a hypersymplectic structure in an analogous way to what was done [FY18]. In this section we compute the flow of the 4-form  $\psi$  from (2.4) under the modified Laplacian coflow from [Gri13]. To begin we note that in this section  $X^4$  is compact. We first show that the  $G_2$  structure given by  $\psi$  as in (2.4) and  $\varphi$  as in (1.13) the Laplacian coflow and modified coflow agree. Let us recall the forms

$$\begin{aligned} \varphi &= dt^{123} - dt^1 \wedge \omega_1 - dt^2 \wedge \omega_2 - dt^3 \wedge \omega_3, \\ \psi &= \mu_{\underline{\sigma}} - dt^{12} \wedge \sigma_3 - dt^{31} \wedge \sigma_2 - dt^{23} \wedge \sigma_1, \end{aligned} \quad (2.5)$$

where  $\omega$  is a positive triple with dual triple  $\underline{\sigma}$  with the further imposition that  $d\underline{\sigma} = 0$  giving a co-closed  $G_2$  structure as was defined in the previous section. The Laplacian coflow for coclosed  $G_2$  structures is given by

$$\frac{\partial \psi}{\partial t} = dd^{*\tau} \psi = d *_{\tau} d *_{\tau} \psi \quad (2.6)$$

where  $d^{*n} = (-1)^{nk+n+1} *_{n} d *_{n}$  on  $k$ -forms over a manifold of dimension  $n$ . Recall the modified Laplacian coflow for coclosed  $G_2$  structures is given by

$$\frac{\partial \psi}{\partial t} = dd^* \psi + 2d((A - \text{Tr}T_{\psi}) *_{\psi} \psi) \quad (2.7)$$

where  $A$  is a choice of constant and  $T$  is the full torsion tensor of the  $G_2$  structure. Before we can compute how the hypersymplectic triple  $\underline{\sigma}$  flows under this flow we first need to show that solutions to this flow remain  $\mathbb{T}^3$  invariant for all time.

**Theorem 2.2.1.** *Let  $\psi(t)$  be a solution of the modified Laplacian coflow (2.7) on  $\mathbb{T}^3 \times X^4$  with initial condition  $\psi_0$  given by*

$$\psi_0 = \mu_{\underline{\sigma}} - \frac{1}{2} \epsilon_{ijk} dt^{jk} \wedge \sigma_i \quad (2.8)$$

where  $\underline{\sigma}$  is hypersymplectic structure. Then  $\psi(t)$  is  $\mathbb{T}^3$  invariant for as long as it exists. Moreover,  $\psi(t)$  remains of the form (2.5) for as long as it exists.

*Proof.* Let  $\psi_0$  be as in (2.8). We want to show that if  $\psi_0$  is  $\mathbb{T}^3$  invariant then so is any solution of the modified Laplacian coflow with this initial condition. Let  $F$  be any diffeomorphism and given that  $\psi$  does not give an orientation we fix the standard orientation on  $\mathbb{T}^3 \times X^4$ . Let  $\psi_t$  be a solution for the modified coflow with initial condition  $\psi_0$ . Then if  $F$  is orientation preserving we fix the same orientation on  $\mathbb{T}^3 \times X^4$  for  $F^* \psi_t$  and if  $F$  is orientation reversing we choose the opposite orientation for  $F^* \psi_t$ , this ensures in the computation we below that the pullback of  $F$  commutes with the hodge star  $*_{\psi_t}$ , that is  $*_{F^* \psi_t} (F^* \alpha) = F^* (*_{\psi_t} \alpha)$  for a  $k$ -form  $\alpha$  on  $\mathbb{T}^3 \times X^4$ . Then we compute:

$$\begin{aligned} \frac{\partial}{\partial t} F^* \psi_t &= F^* \left( \frac{\partial}{\partial t} \psi_t \right) \\ &= F^* (dd^{*\psi_t} \psi_t + 2d((A - \text{Tr}T_{\psi_t}) *_{\psi_t} \psi_t)) \\ &= dd^{*F^* \psi_t} F^* \psi_t + 2d((A - \text{Tr}T_{F^* \psi_t}) *_{F^* \psi_t} F^* \psi_t). \end{aligned} \quad (2.9)$$

This means that  $\psi_t$  and  $F^* \psi_t$  both solve the modified coflow with the same initial condition  $\psi_0$  and therefore by uniqueness of solutions of the flow we obtain  $F^* \psi_t = \psi_t$  for all  $t$ . Thus

$\psi_t$  is  $\mathbb{T}^3$ -invariant for all time  $t$  for which it exists. Next we claim that any solution  $\psi(t)$  of the modified coflow has the form

$$\psi(t) = \mu_{\underline{\sigma}}(0) - \frac{1}{2}\epsilon_{ijk}dt^{jk} \wedge \sigma_i(t) \quad (2.10)$$

where  $\mu_{\underline{\sigma}}(0)$  is the volume form that depends on the initial coclosed  $G_2$  structure coming from the hypersymplectic triple  $\underline{\sigma}$ . To see this we let  $\psi(t)$  be a solution of the modified coflow with initial condition  $\psi_0$ . We can write  $\psi(t)$  as general as possible as

$$\psi(t) = \lambda(t)\mu_{\underline{\sigma}}(0) + dt^i \wedge A_i(t) + \frac{1}{2}\epsilon_{ijk}dt^{ij} \wedge B_k(t) + dt^{123} \wedge C(t) \quad (2.11)$$

where  $\lambda(t) \in C^\infty(X^4)$ ,  $A_i(t) \in \Omega^3(X^4)$ ,  $B_k(t) \in \Omega^2(X^4)$  for all  $i, k$  and  $C(t) \in \Omega^1(X^4)$ . Since we know that the modified coflow preserves the closedness of the 4-form we have that all of the forms in (2.11) are closed. This tells us that  $\lambda(t)$  is constant on  $X^4$  and thus only depends on  $t$ . Now let  $\theta : \mathbb{T}^3 \times X^4 \rightarrow \mathbb{T}^3 \times X^4$  be the map that takes  $(t, p) \mapsto (-t, p)$ . We also know from our earlier computation (2.9) that  $\theta^*\psi_t = \psi_t$ , since  $\theta^*\psi_0 = \psi_0$ . Then

$$\begin{aligned} \theta^*\psi_t &= \lambda(t)\mu_{\underline{\sigma}}(0) - dt^i \wedge A_i(t) + \frac{1}{2}\epsilon_{ijk}dt^{ij} \wedge B_k(t) - dt^{123} \wedge C(t) \\ &= \psi_t \end{aligned} \quad (2.12)$$

which tells that  $A_i(t) = 0$  for all  $i$  and  $C(t) = 0$ . Now we claim that  $\lambda(t)$  is constant. To see this we compute

$$\begin{aligned} \int_{X^4} \psi_t &= \int_{X^4} \lambda(t)\mu_{\underline{\sigma}}(0) + \int_{X^4} \frac{1}{2}\epsilon_{ijk}dt^{ij} \wedge B_k(t) \\ &= \int_{X^4} \lambda(t)\mu_{\underline{\sigma}}(0) + 0 \\ &= \lambda(t)\text{vol}_0(X^4). \end{aligned} \quad (2.13)$$

Now we take the time derivative of both sides of (2.13) and we recall that  $\psi_t$  is a solution of the modified coflow and thus  $\frac{\partial}{\partial t}\psi_t$  is an exact form so its integral will vanish. We obtain

$$0 = \frac{\partial}{\partial t} \left( \int_{X^4} \psi_t \right) = \frac{\partial}{\partial t} (\lambda(t)\text{vol}_0(X^4)) = \lambda'(t)\text{vol}_0(X^4). \quad (2.14)$$

Hence  $\lambda(t)$  is constant and therefore  $\lambda(t) = \lambda(0)$  for all  $t$ . This tells us that if we start with initial condition  $\psi_0$  then any solution  $\psi(t)$  of the modified coflow will remain of the form (2.10) for all time for which it exists. In particular this tells us that the only the  $\sigma_i$ 's will depend on time.  $\square$

Now we can return to the modified coflow and show that in our particular setting the coflow (2.6) and modified coflow (2.7) agree. In (2.7) we choose  $A = 0$ . The trace of the full torsion tensor given in [Kar08] is described using the  $G_2$  3-form as follows

$$\mathrm{Tr}(T) = \frac{1}{4} *_{7} (\varphi \wedge d\varphi). \quad (2.15)$$

Substituting the 3-form (2.5) into (2.15) we see that the trace of the torsion tensor vanishes. To see this we compute

$$\begin{aligned} \mathrm{Tr}(T) &= \frac{1}{4} *_{7} ((dt^{123} - dt^i \wedge \omega_i) \wedge (dt^j \wedge d\omega_j)) \\ &= \frac{1}{4} *_{7} (-dt^{ij} \wedge \omega_i \wedge d\omega_j) \\ &= 0 \end{aligned}$$

where the last equality comes from the fact that  $\omega_i \wedge d\omega_j$  is 5-form completely on  $X^4$  and therefore must vanish. As a result for this particular  $G_2$  structure the Laplacian coflow and modified coflow agree. Now we flow our coclosed  $G_2$  structure (2.5) under the Laplacian coflow and we find that it will descend to a flow of the hypersymplectic triple  $\underline{\sigma}$  on  $X^4$ . This flow is precisely the hypersymplectic flow found by Fine-Yao in [FY18].

**Theorem 2.2.2.** *Let  $\underline{\sigma}$  be a hypersymplectic structure on  $X^4$  and let  $\psi$  as in (2.5) be the associated coclosed  $G_2$  structure on  $\mathbb{T}^3 \times X^4$ . Then under the  $G_2$  Laplacian coflow this descends to a flow of the hypersymplectic triple  $\underline{\sigma}$  on  $X^4$  given by*

$$\partial_t \underline{\sigma} = d(Q_{\sigma} d^*(Q_{\sigma}^{-1} \underline{\sigma})) \quad (2.16)$$

or on each component

$$\frac{\partial \sigma_j}{\partial t} = d\left((Q_{\sigma})_{jk} d^*\left((Q_{\sigma})^{ki} \sigma_i\right)\right).$$

*Proof.* Let  $\underline{\sigma}$  be a hypersymplectic triple. Taking  $\psi$  as in (2.4), the flow of  $\psi$  is then

$$\frac{\partial \psi}{\partial t} = -dt^{12} \wedge \frac{\partial \sigma_3}{\partial t} - dt^{31} \wedge \frac{\partial \sigma_2}{\partial t} - dt^{23} \wedge \frac{\partial \sigma_1}{\partial t}. \quad (2.17)$$

Recall that in this setting the coflow and modified coflow agree so we expand the right hand side of (2.6) using the  $\psi$  we defined in (2.4). To make the following computation easier to read we let  $Q$  denote  $Q_{\underline{\omega}}$  the matrix associated with the positive triple  $\underline{\omega}$ . Recall

that  $\underline{\omega}$  is the dual positive triple to  $\underline{\sigma}$ , and that  $Q_{\underline{\omega}}^{-1} = Q_{\underline{\sigma}}$ . Therefore using the fact that the  $\sigma_i$ 's are self-dual, we have

$$\begin{aligned}
d *_{\mathbb{T}^3} d *_{\mathbb{T}^3} \psi &= d *_{\mathbb{T}^3} d\varphi = d *_{\mathbb{T}^3} d(dt^{123} - Q_{ik} dt^k \wedge \sigma_i) \\
&= d *_{\mathbb{T}^3} (dt^k \wedge d(Q_{ik} \sigma_i)) \\
&= d(*_3 dt^k \wedge *_4 d(Q_{ik} \sigma_i)) \\
&= d(Q^{kj} \hat{dt}^j \wedge *_4 d(Q_{ik} \sigma_i)) \\
&= \hat{dt}^j \wedge d(Q^{kj} *_4 d *_4 (Q_{ik} \sigma_i)) \\
&= -\hat{dt}^j \wedge d(Q^{kj} d^*(Q_{ik} \sigma_i))
\end{aligned}$$

where  $\hat{dt}^j$  denotes the 2-form on  $\mathbb{T}^3$  that omits  $dt^j$ , for example  $\hat{dt}^1 = dt^2 \wedge dt^3$ . We equate the two computations and obtain a flow of the hypersymplectic structure  $\underline{\sigma}$ . It is given by

$$\begin{aligned}
\frac{\partial \sigma_j}{\partial t} &= d(Q^{kj} d^*(Q_{ik} \sigma_i)) \\
&= d(Q_{\sigma_{kj}} d^*(Q_{\sigma}^{ik} \sigma_i)).
\end{aligned} \tag{2.18}$$

Since  $Q_{\underline{\omega}}^{-1} = Q_{\underline{\sigma}}$  this flow is precisely the hypersymplectic flow of Fine-Yao from [FY18].  $\square$

# Chapter 3

## A particular cohomogeneity one hypersymplectic structure

In this chapter we create scaffolding for the hypersymplectic structure we wish to examine. In section 3.1 we set up the hypersymplectic triple. In section 3.2 we compute the Riemannian metric that comes from the hypersymplectic triple. In sections 3.4 and 3.5 we compute the Riemann and Ricci curvature tensors of the hypersymplectic metric. Lastly in section 3.6 we verify that when the matrix  $Q$  coming from this particular hypersymplectic structure is constant the hypersymplectic structure does indeed induce a Ricci-flat metric.

### 3.1 A cohomogeneity one hypersymplectic structure

In this section we set up a cohomogeneity one hypersymplectic structure. We make a very particular choice of three symplectic forms  $\underline{\omega}$  that form a positive triple as in definition 1.2.8. We assume a cohomogeneity one setup on  $\mathbb{R}^4 \setminus \{0\}$ , the action is that of  $\text{SO}(4)$  on  $\mathbb{R}^4 \setminus \{0\}$  by rotations as we have described in section 1.3. This allows us to think of  $\mathbb{R}^4 \setminus \{0\}$  as  $(0, \infty) \times S^3$ . To begin we create the setup for our hypersymplectic structure. Let  $\frac{\partial}{\partial x_i}$  for  $i = 0, 1, 2, 3$  be the standard global coordinate frame for  $T\mathbb{R}^4$ . Furthermore, we endow  $\mathbb{R}^4$  with three complex structures. The first is given by

$$\begin{aligned} z_1^1 &:= x_0 + ix_1, \\ z_2^1 &:= x_2 + ix_3, \end{aligned} \tag{3.1}$$

with associated endomorphism  $J_1$  given by

$$\begin{aligned} J_1 \left( \frac{\partial}{\partial x_0} \right) &= \frac{\partial}{\partial x_1}, & J_1 \left( \frac{\partial}{\partial x_1} \right) &= -\frac{\partial}{\partial x_0}, \\ J_1 \left( \frac{\partial}{\partial x_2} \right) &= \frac{\partial}{\partial x_3}, & J_1 \left( \frac{\partial}{\partial x_3} \right) &= -\frac{\partial}{\partial x_2}. \end{aligned} \tag{3.2}$$

The other complex coordinates  $z_1^i$  and  $z_2^i$  and endomorphisms  $J_i$  for  $i = 2, 3$  are obtained by cyclically permuting  $1 \rightarrow 2 \rightarrow 3$  in (3.1) and (3.2). In this setup  $\partial_i$  and  $\bar{\partial}_i$  are differential operators that are associated to the complex operator  $J_i$ . The operators have the form

$$\begin{aligned} \partial_i &= \frac{\partial}{\partial z_1^i} \otimes dz_1^i + \frac{\partial}{\partial z_2^i} \otimes dz_2^i, \\ \bar{\partial}_i &= \frac{\partial}{\partial \bar{z}_1^i} \otimes d\bar{z}_1^i + \frac{\partial}{\partial \bar{z}_2^i} \otimes d\bar{z}_2^i. \end{aligned} \tag{3.3}$$

Note that there is no sum over  $i$  in (3.3). In terms of our real coordinates  $x_i$  on  $\mathbb{R}^4$  the operators are

$$\begin{aligned} \partial_1 &= \frac{1}{2} \left( \frac{\partial}{\partial x_0} - i \frac{\partial}{\partial x_1} \right) \otimes (dx_0 + idx_1) + \frac{1}{2} \left( \frac{\partial}{\partial x_2} - i \frac{\partial}{\partial x_3} \right) \otimes (dx_2 + idx_3), \\ \bar{\partial}_1 &= \frac{1}{2} \left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} \right) \otimes (dx_0 - idx_1) + \frac{1}{2} \left( \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \right) \otimes (dx_2 - idx_3). \end{aligned} \tag{3.4}$$

The other operators  $\partial_i$  and  $\bar{\partial}_i$  for  $i = 2, 3$  are obtained by cyclically permuting  $1 \rightarrow 2 \rightarrow 3$  in (3.4). Let  $r$  be the radial coordinate on  $\mathbb{R}^4 \setminus \{0\}$  so that  $r$  is defined by

$$r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

We use the standard cohomogeneity one action of  $\text{SO}(4)$  on  $\mathbb{R}^4$  and use the global frame  $E_i$  for  $T(\mathbb{R}^4 \setminus \{0\})$  defined as follows;

$$\begin{aligned} E_0 &= \frac{\partial}{\partial r} = \frac{1}{r} \left( x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right), \\ E_1 &= J_1 \left( \frac{\partial}{\partial r} \right) = \frac{1}{r} \left( -x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} \right), \\ E_2 &= J_2 \left( \frac{\partial}{\partial r} \right) = \frac{1}{r} \left( -x_2 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} \right), \\ E_3 &= J_3 \left( \frac{\partial}{\partial r} \right) = \frac{1}{r} \left( -x_3 \frac{\partial}{\partial x_0} - x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3} \right). \end{aligned} \tag{3.5}$$

Thus the dual frame  $E^i$  is

$$\begin{aligned}
E^0 &= dr = \frac{1}{r} (x_0 dx_0 + x_1 dx_1 + x_2 dx_2 + x_3 dx_3), \\
E^1 &= \frac{1}{r} (-x_1 dx_0 + x_0 dx_1 - x_3 dx_2 + x_2 dx_3), \\
E^2 &= \frac{1}{r} (-x_2 dx_0 + x_3 dx_1 + x_0 dx_2 - x_1 dx_3), \\
E^3 &= \frac{1}{r} (-x_3 dx_0 - x_2 dx_1 + x_1 dx_2 + x_0 dx_3).
\end{aligned} \tag{3.6}$$

We note that only the vector field  $E_0$  and consequently its dual element  $E^0$  are  $\text{SO}(4)$  invariant. The restrictions of  $E^1, E^2, E^3$  to  $r = 1$  form a global frame for  $T^*S^3$ . With respect to the Euclidean metric on  $\mathbb{R}^4$  the frame  $E^i$  is orthonormal. Then we can define six 2-forms on  $\mathbb{R}^4 \setminus \{0\}$  by

$$\begin{aligned}
\omega_i^+ &= E^0 \wedge E^i + *_3 E^i = E^0 \wedge E^i + \frac{1}{2} \epsilon_{ipq} E^p \wedge E^q, \\
\omega_i^- &= E^0 \wedge E^i - *_3 E^i = E^0 \wedge E^i - \frac{1}{2} \epsilon_{ipq} E^p \wedge E^q.
\end{aligned} \tag{3.7}$$

The forms  $\omega_i^+$  and  $\omega_i^-$  are respectively self-dual and anti-self-dual, with respect to the Euclidean metric on  $\mathbb{R}^4 \setminus \{0\}$ . We note that using the formula for the metric (1.2) the forms  $\omega_i^+$  give us the Euclidean metric with standard orientation. The Hodge star  $*_3$  is the Hodge star with respect to the round metric on  $S^3$ . We also note the following relations pertaining to the forms (3.7):

$$\begin{aligned}
\omega_i^+ \wedge \omega_j^+ &= 2\delta_{ij}\mu, \\
\omega_i^- \wedge \omega_j^- &= -2\delta_{ij}\mu, \\
\omega_i^+ \wedge \omega_j^- &= 0.
\end{aligned}$$

In terms of the coordinate frame  $dx_i$  on  $\mathbb{R}^4$  the forms  $\omega_i^\pm$  can be written as follows:

$$\begin{aligned}
\omega_1^+ &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3, \\
\omega_1^- &= \frac{1}{r^2} \left[ (x_0^2 + x_1^2 - x_2^2 - x_3^2)(dx_{01} - dx_{23}) \right. \\
&\quad \left. + 2(x_0 x_2 + x_1 x_3)(dx_{03} - dx_{12}) + 2(x_1 x_2 - x_0 x_3)(dx_{02} - dx_{31}) \right],
\end{aligned} \tag{3.8}$$

where  $dx_{ij}$  denotes  $dx_i \wedge dx_j$ . We obtain the remaining forms by cyclically permuting  $1 \rightarrow 2 \rightarrow 3$ . Now we want to construct a triple of symplectic forms  $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$  that

will give us a cohomogeneity one hypersymplectic structure on  $\mathbb{R}^4 \setminus \{0\}$ . We note that we are making a very particular choice for this structure and this is by no means the only way to construct a hypersymplectic structure. To begin our construction we let  $h_i(r)$  be functions of just the radial coordinate, and define the hypersymplectic structures  $\omega_k$  as

$$\omega_1 = \frac{i}{2} \partial_1 \bar{\partial}_1 (r^2), \quad \omega_2 = \frac{i}{2} \partial_2 \bar{\partial}_2 (h_2(r)), \quad \omega_3 = \frac{i}{2} \partial_3 \bar{\partial}_3 (h_3(r)). \quad (3.9)$$

We note that we have specifically chosen the first function  $h_1 = r^2$ . We did this to make our computations more tractable. When we choose the functions  $h_k = r^2$  we obtain the standard Euclidean structure and in the end we want to look for solutions close to this structure. Furthermore in section 4.3 the ODEs we obtain have invariant solutions and only involve the radial coordinate  $r$ , thus we have freedom to re-parametrize our functions  $h_k(r)$ .

The forms (3.9) are closed by construction. Let  $\omega_k$  be one of the forms in (3.9). Then

$$d\omega_k = (\partial_k + \bar{\partial}_k) \left( \frac{i}{2} \partial_k \bar{\partial}_k (h_k(r)) \right) = \frac{i}{2} (\partial_k^2 \bar{\partial}_k (h_k(r)) - \partial_k \bar{\partial}_k^2 (h_k(r))) = 0.$$

Above we used the fact that  $\partial_k^2 = \bar{\partial}_k^2 = 0$  and  $\partial_k \bar{\partial}_k = -\bar{\partial}_k \partial_k$ . For a hypersymplectic structure, we further require the forms (3.9) to be non-degenerate. Up to a choice of orientation, this is equivalent to the matrix  $Q$  from (1.1) being positive definite. Thus each  $\omega_k$  is a symplectic form, and the triple  $(\omega_1, \omega_2, \omega_3)$  forms a hypersymplectic structure. We now expand these forms and write them in terms of our frame  $E^i$ . We first need to do some basic calculations:

$$\begin{aligned} r^2 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ \implies \frac{\partial r}{\partial x_i} &= \frac{x_i}{r} \\ \implies \frac{\partial}{\partial x_i} h_j(r) &= \frac{\partial h_j}{\partial r} \cdot \frac{\partial r}{\partial x_i} = h'_j(r) \frac{x_i}{r}. \end{aligned}$$

If we write  $r$  in terms of the complex coordinates, we can write the form  $\partial_k r \wedge \bar{\partial}_k r$  which we need later. We do this in the first complex coordinate system and omit the other two, as the calculations are identical:

$$\begin{aligned} r^2 &= z_1^1 \bar{z}_1^1 + z_2^1 \bar{z}_2^1 \\ \implies 2r \cdot \partial_1 r &= \bar{z}_1^1 dz_1^1 + \bar{z}_2^1 dz_2^1, \\ 2r \cdot \bar{\partial}_1 r &= z_1^1 dz_1^{\bar{1}} + z_2^1 dz_2^{\bar{1}} \\ \implies 4r^2 \partial_1 r \wedge \bar{\partial}_1 r &= |z_1^1|^2 dz_1^1 \wedge dz_1^{\bar{1}} + |z_2^1|^2 dz_2^1 \wedge dz_2^{\bar{1}} + \bar{z}_1^1 z_2^1 dz_1^1 \wedge dz_2^{\bar{1}} + \bar{z}_2^1 z_1^1 dz_2^1 \wedge dz_1^{\bar{1}}. \end{aligned} \quad (3.10)$$

Analogous formulas hold for  $J_2$  and  $J_3$ . Thus, we can conclude

$$4r^2 \partial_k r \wedge \partial_k r = |z_1^k|^2 dz_1^k \wedge dz_1^{\bar{k}} + |z_2^k|^2 dz_2^k \wedge dz_2^{\bar{k}} + z_1^{\bar{k}} z_2^k dz_1^k \wedge dz_2^{\bar{k}} + z_2^{\bar{k}} z_1^k dz_2^k \wedge dz_1^{\bar{k}}. \quad (3.11)$$

Since the forms  $\omega_k^\pm$  form a basis for the space of 2-forms, we want to write our hypersymplectic forms in this basis. To this end we do the following computation:

$$\begin{aligned} \frac{i}{2} 4r^2 \partial_2 r \wedge \bar{\partial}_2 r &= \frac{i}{2} (x_0^2 + x_2^2) (-2i dx_0 \wedge dx_2) + \frac{i}{2} (x_3^2 + x_1^2) (-2i dx_3 \wedge dx_1) \\ &\quad + \frac{i}{2} (x_0 - ix_2)(x_3 + ix_1)(dx_0 + idx_2) \wedge (dx_3 - idx_1) \\ &\quad + \frac{i}{2} (x_3 - ix_1)(x_0 + ix_2)(dx_3 + idx_1) \wedge (dx_0 - idx_2) \\ &= \frac{1}{2} (x_0^2 + x_2^2) (dx_{02} + dx_{31} + dx_{02} - dx_{31}) \\ &\quad + \frac{1}{2} (x_3^2 + x_1^2) (dx_{02} + dx_{31} - dx_{02} + dx_{31}) \\ &\quad + \frac{i}{2} (2i(x_0 x_3 + x_1 x_2)(dx_{23} - dx_{01}) + 2i(x_0 x_1 - x_2 x_3)(dx_{03} + dx_{21})) \\ &= \frac{1}{2} r^2 \omega_2^+ + \frac{1}{2} (x_0^2 + x_2^2 - x_3^2 - x_1^2) (dx_{02} - dx_{31}) \\ &\quad + (x_0 x_3 + x_1 x_2) (dx_{01} - dx_{23}) + (x_2 x_3 - x_0 x_1) (dx_{03} - dx_{12}) \\ &= \frac{r^2}{2} \omega_2^+ + \frac{r^2}{2} \omega_2^-. \end{aligned} \quad (3.12)$$

Furthermore we note the following calculation:

$$\begin{aligned} dz_1^1 \wedge dz_1^{\bar{1}} + dz_2^1 \wedge dz_2^{\bar{1}} &= (dx_0 + idx_1) \wedge (dx_0 - idx_1) + (dx_2 + idx_3) \wedge (dx_2 - idx_3) \\ &= -2i dx_0 \wedge dx_1 - 2i dx_2 \wedge dx_3 \\ &= -2i (dx_0 \wedge dx_1 + dx_2 \wedge dx_3) \\ &= -2i \omega_1^+. \end{aligned}$$

We can repeat this calculation for each complex structure by cyclically permuting  $1 \rightarrow 2 \rightarrow 3$  and we obtain

$$\omega_k^+ = \frac{i}{2} \left( dz_1^k \wedge dz_1^{\bar{k}} + dz_2^k \wedge dz_2^{\bar{k}} \right). \quad (3.13)$$

Now we are ready to compute the hypersymplectic triple  $\omega_k = \frac{i}{2} \partial_k \bar{\partial}_k (h_k(r))$  where  $h_1 = r^2$  and  $h_2, h_3$  are arbitrary functions of  $r$ . Note that  $\omega_1 = E^{01} + E^{23}$  so we will do this

calculation for  $\omega_2$  and omit the  $\omega_3$  since the computation is identical and can be derived by simply permuting  $1 \rightarrow 2 \rightarrow 3$ .

$$\begin{aligned}
\bar{\partial}_2(h_2) &= \frac{1}{2} \left( \frac{\partial}{\partial x_0}(h_2) + i \frac{\partial}{\partial x_2}(h_2) \right) \otimes (dx_0 - idx_2) \\
&\quad + \frac{1}{2} \left( \frac{\partial}{\partial x_3}(h_2) + i \frac{\partial}{\partial x_1}(h_2) \right) \otimes (dx_3 - idx_1) \\
&= \frac{h'_2}{2r} (z_1^2 dz_1^{\bar{2}} + z_2^2 dz_2^{\bar{2}}), \\
\implies \partial_2 \bar{\partial}_2(h_2) &= \frac{h'_2}{2r} (dz_1^2 \wedge dz_1^{\bar{2}} + dz_2^2 \wedge dz_2^{\bar{2}}) + \partial_2 \left( \frac{h'_2}{2r} \right) (z_1^2 dz_1^{\bar{2}} + z_2^2 dz_2^{\bar{2}}), \\
\implies \partial_2 \bar{\partial}_2(h_2) &= \frac{h'_2}{2r} (dz_1^2 \wedge dz_1^{\bar{2}} + dz_2^2 \wedge dz_2^{\bar{2}}) \\
&\quad + \left( \frac{h''_2}{4r^2} - \frac{h'_2}{4r^3} \right) (\bar{z}_1^2 dz_1^2 + \bar{z}_2^2 dz_2^2) \wedge (z_1^2 dz_1^{\bar{2}} + z_2^2 dz_2^{\bar{2}}).
\end{aligned}$$

Now we multiply both sides by  $\frac{i}{2}$  and use (3.11) to obtain

$$\begin{aligned}
\frac{i}{2} \partial_2 \bar{\partial}_2(h_2) &= \frac{h'_2}{2r} \omega_2^+ + \frac{i}{2} \left( \frac{h''_2}{4r^2} - \frac{h'_2}{4r^3} \right) (|z_1^2|^2 dz_1^2 \wedge dz_1^{\bar{2}} + |z_2^2|^2 dz_2^2 \wedge dz_2^{\bar{2}} \\
&\quad + \bar{z}_1^2 z_2^2 dz_1^2 \wedge dz_2^{\bar{2}} + \bar{z}_2^2 z_1^2 dz_2^2 \wedge dz_1^{\bar{2}}) \\
&= \frac{h'_2}{2r} \omega_2^+ + \left( \frac{h''_2}{4r^2} - \frac{h'_2}{4r^3} \right) \left( \frac{i}{2} 4r^2 \partial_2 r \wedge \bar{\partial}_2 r \right).
\end{aligned}$$

Next we substitute the computation we did above in (3.11) to be able to write the hyper-symplectic forms  $\omega_k$  in terms of the forms  $\{\omega_k^\pm\}$ . Thus  $\omega_k$  has the following form in terms of the basis  $\{\omega_k^\pm\}$ :

$$\begin{aligned}
\omega_1 &= \omega_1^+, \\
\omega_2 &= \frac{h'_2}{2r} \omega_2^+ + \left( \frac{h''_2}{4r^2} - \frac{h'_2}{4r^3} \right) \left( \frac{r^2}{2} \omega_2^+ + \frac{r^2}{2} \omega_2^- \right), \\
\omega_3 &= \frac{h'_3}{2r} \omega_3^+ + \left( \frac{h''_3}{4r^2} - \frac{h'_3}{4r^3} \right) \left( \frac{r^2}{2} \omega_3^+ + \frac{r^2}{2} \omega_3^- \right).
\end{aligned} \tag{3.14}$$

For convenience, we write these in the following form and we note that there is no sum over  $k$ :

$$\omega_k = \left( A_k + \frac{r^2}{2} B_k \right) \omega_k^+ + B_k \frac{r^2}{2} \omega_k^- \tag{3.15}$$

where

$$A_k = \frac{h'_k}{2r} \quad \text{and} \quad B_k = \frac{h''_k}{4r^2} - \frac{h'_k}{4r^3} = \frac{A'_k}{2r} \quad (3.16)$$

for  $k = 2, 3$ . We wish to use a different notation for the forms  $\omega_k$  as it will be useful in sections 3.2, 3.4, 3.5 where we compute the metric and curvatures for this more general situation. Thus we write the symplectic forms in the following way and note that there is a sum over  $p$ :

$$\omega_i = T_{ip}\omega_p^+ + S_{ip}\omega_p^- \quad (3.17)$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_2 + \frac{r^2}{2}B_2 & 0 \\ 0 & 0 & A_3 + \frac{r^2}{2}B_3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{r^2}{2}B_2 & 0 \\ 0 & 0 & \frac{r^2}{2}B_3 \end{bmatrix}. \quad (3.18)$$

Now we recall that the hypersymplectic structure determines a matrix  $Q$  such that

$$\omega_i \wedge \omega_j = 2Q_{ij}\mu$$

where  $\mu = E^{0123}$  is our choice of volume form on  $\mathbb{R}^4 \setminus \{0\}$  and  $Q$  is a  $3 \times 3$  symmetric positive definite matrix. It is important to emphasize that for this particular  $Q$ ,  $\det(Q) \neq 1$  since we have not chosen  $\mu = \mu_\omega$  as we have in parts of the introduction chapter (see lemma 1.2.4). At the end of this section we will rescale our matrix  $Q$  in order for it to have  $\det(Q) = 1$  and choose  $\mu = \mu_\omega$ . Moving forward, we note in the following computation for  $Q$  that  $T_i = T_{ii}$  is the diagonal entry for the matrix  $T$ , similarly for  $S_i$ . Furthermore, there is no sum in the following computation:

$$\begin{aligned} 2Q_{ij}\mu &= \omega_i \wedge \omega_j \\ &= (T_i\omega_i^+ + S_i\omega_i^-) \wedge (T_j\omega_j^+ + S_j\omega_j^-) \\ &= T_iT_j (E^{0i} + *_3E^i) \wedge (E^{0j} + *_3E^j) + T_iS_j (E^{0i} + *_3E^i) \wedge (E^{0j} - *_3E^j) \\ &\quad + S_iT_j (E^{0i} - *_3E^i) \wedge (E^{0j} + *_3E^j) + S_iS_j (E^{0i} - *_3E^i) \wedge (E^{0j} - *_3E^j) \\ &= 2 \left[ (T_iT_j - S_iS_j)\delta_{ij} \right] \mu. \end{aligned}$$

Therefore we obtain

$$Q_{ij} = (T_iT_j - S_iS_j)\delta_{ij} \quad (3.19)$$

where there is no sum over  $i$  and  $j$  and thus  $Q$  is the following diagonal matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_2(A_2 + r^2B_2) & 0 \\ 0 & 0 & A_3(A_3 + r^2B_3) \end{bmatrix}. \quad (3.20)$$

Note that the condition on  $h_2$  and  $h_3$  that guarantees that  $\omega_2$  and  $\omega_3$  are non-degenerate is that the matrix  $Q$  is positive definite. This is equivalent to the conditions:

$$A_i (A_i + r^2 B_i) > 0 \quad \text{or} \quad A_i (2A_i + rA'_i) > 0 \quad (3.21)$$

for  $i = 2, 3$ , where  $B_i$  was replaced with the relation from (3.16).

In [FY18] the reference volume form is taken to be  $\mu_{\underline{\omega}}$ , this is to ensure that  $\det(Q) = 1$ . This is so that the metric defined by (1.14) does not have a factor of the determinant in it. The hypersymplectic flow in terms of  $Q$  depends on a choice of  $\mu$  and thus in order to follow analogously the work of Fine-Yao we must ensure that  $\det(Q) = 1$  implying that  $\mu = \mu_{\underline{\omega}}$ . In our specific situation, our reference volume form is taken to be  $\mu = E^{0123}$ . In order to ensure that  $\det(Q) = 1$ , we need to divide by a scaling factor of  $\det(Q)^{\frac{1}{3}}$ , this is due to the relation between the reference volume form of the positive triple and  $\mu_{\underline{\omega}}$  explained in lemma 1.2.4. Therefore our new matrix  $Q$  and the one we use in chapter 4 when looking for solitons of the hypersymplectic flow is given below:

$$Q = \begin{bmatrix} (A_2 A_3 (A_2 + r^2 B_2) (A_3 + r^2 B_3))^{-\frac{1}{3}} & 0 & 0 \\ 0 & \frac{(A_2 (A_2 + r^2 B_2))^{\frac{2}{3}}}{(A_3 (A_3 + r^2 B_3))^{\frac{1}{3}}} & 0 \\ 0 & 0 & \frac{(A_3 (A_3 + r^2 B_3))^{\frac{2}{3}}}{(A_2 (A_2 + r^2 B_2))^{\frac{1}{3}}} \end{bmatrix}.$$

After substituting  $B_i$  by the relation from (3.16) we obtain the following for the matrix  $Q$ :

$$Q = \begin{bmatrix} \frac{4^{\frac{1}{3}}}{(A_2 A_3 (2A_2 + rA'_2) (2A_3 + rA'_3))^{\frac{1}{3}}} & 0 & 0 \\ 0 & \frac{(A_2 (2A_2 + rA'_2))^{\frac{2}{3}}}{(2A_3 (2A_3 + rA'_3))^{\frac{1}{3}}} & 0 \\ 0 & 0 & \frac{(A_3 (2A_3 + rA'_3))^{\frac{2}{3}}}{(2A_2 (2A_2 + rA'_2))^{\frac{1}{3}}} \end{bmatrix}. \quad (3.22)$$

We note that in order for this new matrix  $Q$  to be positive definite, the requirements are still the same as the old before we divided by the determinant factor.

## 3.2 Metric calculations

In this section we compute the hypersymplectic metric that comes from a hypersymplectic triple. We point out that the computations in this section, that are summarized

in table 3.1, are done using as general of a structure as possible, meaning the frame we use in our computations,  $E^i$ , is just an orthonormal frame for  $T^*\mathbb{R}^4$  and not the specific frame we have chosen in (3.6). Similarly the matrices  $S$  and  $T$  in that table are arbitrary  $3 \times 3$  matrices. At the end of this section we substitute specific matrices  $S$  and  $T$ . In the following sections we use these metric computations and the frame (3.6) to compute the Christoffel symbols and eventually the curvature tensors. For now we wish to make the computation for the metric as general as possible. To begin we recall the formula for the metric derived from a positive triple that was given earlier in this thesis:

$$g_{\underline{\omega}}(X, Y)\mu_{\underline{\omega}} = \frac{1}{6}\epsilon_{ijk} (X \lrcorner \omega_i) \wedge (Y \lrcorner \omega_j) \wedge \omega_k.$$

In order to simplify the computation for the metric, we write the hypersymplectic forms  $\omega_k$  using the basis  $\omega_k^\pm$  given by (3.7). We do the computation using these pieces and put them all back together. First we need to set up some identities. Recall the standard self-dual and anti-dual forms are

$$\omega_i^\pm = E^0 \wedge E^i \pm *_3 E^i = E^0 \wedge E^i \pm \frac{1}{2}\epsilon_{ipq} E^p \wedge E^q$$

and that  $E^i$  for  $i = 0, 1, 2, 3$  is orthonormal with respect to the Euclidean metric on  $\mathbb{R}^4$ . We note that we are abusing notation and writing  $*_3 E^i$  as an abbreviation for  $\frac{1}{2}\epsilon_{ipq} E^p \wedge E^q$ . Then we have the following identities for any  $\omega_i^\pm$  where  $i \in \{1, 2, 3\}$ :

$$E_0 \lrcorner \omega_i^\pm = E^i \tag{3.23}$$

and for  $E_a$  where  $a \in \{1, 2, 3\}$  we have

$$E_a \lrcorner \omega_i^\pm = E_a \lrcorner (E^0 \wedge E^i \pm *_3 E^i) = -\delta_{ai} E^0 \pm \sum_p \epsilon_{iap} E^p. \tag{3.24}$$

We also need the following identities

$$\begin{aligned} \epsilon_{ijk}\epsilon_{abk} &= \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}, \\ \epsilon_{ijk}\epsilon_{ajk} &= 2\delta_{ia}. \end{aligned} \tag{3.25}$$

Now we are ready to compute the pieces of the metric.

**Lemma 3.2.1.** *Let  $E^0, E^1, E^2, E^3$  be an orthonormal frame for  $T^*\mathbb{R}^4$ , let  $\omega_i^\pm = E^{0i} + *_3 E^i$  and let  $\mu = E^0 \wedge E^1 \wedge E^2 \wedge E^3$ . Then*

$$(E_0 \lrcorner \omega_i^\pm) \wedge (E_0 \lrcorner \omega_j^\pm) \wedge \omega_k^\pm = \epsilon_{ijk}\mu,$$

$$\begin{aligned}
(E_a \lrcorner \omega_i^\pm) \wedge (E_b \lrcorner \omega_j^\pm) \wedge \omega_k^\pm &= \left[ (\pm)(\pm)\delta_{ai}\epsilon_{bjk} + (\pm)(\pm)\delta_{bj}\epsilon_{iak} \right. \\
&\quad \left. + (\pm)(\pm)\delta_{ak}\epsilon_{ijb} + (\pm)(\pm)\delta_{ik}\epsilon_{bja} \right] \mu, \\
(E_0 \lrcorner \omega_i^\pm) \wedge (E_a \lrcorner \omega_j^\pm) \wedge \omega_k^\pm &= \left[ (\pm)\delta_{aj}\delta_{ik} + (\pm)\delta_{jk}\delta_{ai} - (\pm)\delta_{ij}\delta_{ak} \right] \mu,
\end{aligned}$$

where  $a, b \in \{1, 2, 3\}$ .

*Proof.* Let us start with the first identity since it is simpler. Using (3.23) we have

$$\begin{aligned}
(E_0 \lrcorner \omega_i^\pm) \wedge (E_0 \lrcorner \omega_j^\pm) \wedge \omega_k^\pm &= E^i \wedge E^j \wedge (E^0 \wedge E^k \pm *_3 E^k) \\
&= E^0 \wedge E^i \wedge E^j \wedge E^k \\
&= \epsilon_{ijk} \mu.
\end{aligned}$$

Next we let  $a, b \in \{1, 2, 3\}$  and using (3.24) we obtain

$$\begin{aligned}
(E_a \lrcorner \omega_i^\pm) \wedge (E_b \lrcorner \omega_j^\pm) \wedge \omega_k^\pm &= (-\delta_{ai}E^0 \pm \epsilon_{iap}E^p) \wedge (-\delta_{bj}E^0 \pm \epsilon_{jbq}E^q) \wedge (E^0 \wedge E^k \pm *_3 E^k) \\
&= -(\pm)(\pm)\delta_{ai}\epsilon_{jbq}E^0 \wedge E^q \wedge *_3 E^k \\
&\quad - (\pm)(\pm)\delta_{bj}\epsilon_{iap}E^p \wedge E^0 \wedge *_3 E^k \\
&\quad + (\pm)(\pm)\epsilon_{iap}\epsilon_{jbq}E^p \wedge E^q \wedge E^0 \wedge E^k \\
&\quad + (\pm)(\pm)(\pm)\underbrace{\epsilon_{iap}\epsilon_{jbq}E^p \wedge E^q \wedge *_3 E^k}_{=0} \\
&= -(\pm)(\pm)\delta_{ai}\delta_{qk}\epsilon_{jbq}\mu + (\pm)(\pm)\epsilon_{iap}\delta_{bj}\delta_{pk}\mu \\
&\quad + (\pm)(\pm)\epsilon_{iap}\epsilon_{jbq}\epsilon_{pqk}\mu \\
&= \left[ (\pm)(\pm)\delta_{ai}\epsilon_{bjk} + (\pm)(\pm)\delta_{bj}\epsilon_{iak} \right. \\
&\quad \left. + (\pm)(\pm)\epsilon_{jbq}(\delta_{iq}\delta_{ak} - \delta_{ik}\delta_{aq}) \right] \mu \\
&= \left[ (\pm)(\pm)\delta_{ai}\epsilon_{bjk} + (\pm)(\pm)\delta_{bj}\epsilon_{iak} + (\pm)(\pm)\delta_{ak}\epsilon_{ijb} \right. \\
&\quad \left. + (\pm)(\pm)\delta_{ik}\epsilon_{bja} \right] \mu.
\end{aligned}$$

Finally using both (3.23) and (3.24) we have,

$$\begin{aligned}
(E_0 \lrcorner \omega_i^\pm) \wedge (E_a \lrcorner \omega_j^\pm) \wedge \omega_k^\pm &= E^i \wedge (-\delta_{aj} E^0 \pm \epsilon_{jap} E^p) \wedge (E^0 \wedge E^k \pm *_3 E^k) \\
&= (\pm) \delta_{aj} E^0 \wedge E^i \wedge *_3 E^k + (\pm) \epsilon_{jap} E^0 \wedge E^i \wedge E^p \wedge E^k \\
&\quad + (\pm)(\pm) \underbrace{\epsilon_{jap} E^i \wedge E^p \wedge *_3 E^k}_{=0} \\
&= \left[ (\pm) \delta_{aj} \delta_{ik} - (\pm) \epsilon_{jap} \epsilon_{ikp} \right] \mu \\
&= \left[ (\pm) \delta_{aj} \delta_{ik} + (\pm) \delta_{jk} \delta_{ai} - (\pm) \delta_{ij} \delta_{ak} \right] \mu
\end{aligned}$$

which gives our desired result.  $\square$

Now we are ready to compute the hypersymplectic metric. We need two identities from lemma A.2.1 that are proved in the appendix A.2. We use these identities quite often. Furthermore we recall that  $\omega_i^\pm$  for  $i = 1, 2, 3$  are a basis for  $\bigwedge^2(\mathbb{R}^4)$  thus we can write our hypersymplectic forms  $\omega_i$  using  $\omega_i^\pm$  as follows:

$$\omega_i = T_{ip} \omega_p^+ + S_{ip} \omega_p^-.$$

We input this into the formula (1.2) for the hypersymplectic metric.

$$\begin{aligned}
g_{\underline{\omega}}(E_a, E_b) \mu_{\underline{\omega}} &= \frac{1}{6} \epsilon_{ijk} \left[ (T_{ip} E_a \lrcorner \omega_p^+ + S_{ip} E_a \lrcorner \omega_p^-) \wedge (T_{jv} E_b \lrcorner \omega_v^+ + S_{jv} E_b \lrcorner \omega_v^-) \right. \\
&\quad \left. \wedge (T_{kt} \omega_t^+ + S_{kt} \omega_t^-) \right] \\
&= \frac{1}{6} \epsilon_{ijk} \left[ T_{ip} T_{jv} T_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^+ \right. \\
&\quad + T_{ip} T_{jv} S_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^- \\
&\quad + T_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+ \\
&\quad + S_{ip} T_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^+ \\
&\quad + T_{ip} S_{jv} S_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^- \\
&\quad + S_{ip} T_{jv} S_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^- \\
&\quad + S_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+ \\
&\quad \left. + S_{ip} S_{jv} S_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^- \right] \tag{3.26}
\end{aligned}$$

As an aside to simplify our computation we notice the six terms with mixed functions of  $S$  and  $T$  can be combined into two terms. Let us show this for the terms with two factors

of  $S$  and a factor of  $T$  and the other three terms will follow similarly. To begin we take the three terms and swap indices  $i \leftrightarrow k$  and  $p \leftrightarrow t$  in the first term, and indices  $k \leftrightarrow j$  and  $v \leftrightarrow t$  in the second term. Following this we use the fact that  $\omega_t^+ \wedge \omega_v^- = 0$  to obtain the identity  $(E_b \lrcorner \omega_t^+) \wedge \omega_v^- = -(E_b \lrcorner \omega_v^-) \wedge \omega_t^+$ . Using these we are able to combine all three terms together, and we obtain

$$\begin{aligned}
& \epsilon_{ijk} T_{ip} S_{jv} S_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^- + \epsilon_{ijk} S_{ip} T_{jv} S_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^- \\
& \quad + \epsilon_{ijk} S_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+ \\
& = \epsilon_{kji} T_{kt} S_{jv} S_{ip} (E_a \lrcorner \omega_t^+) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_p^- + \epsilon_{ikj} S_{ip} T_{kt} S_{jv} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_t^+) \wedge \omega_v^- \\
& \quad + \epsilon_{ijk} S_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+ \\
& = 3\epsilon_{ijk} S_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+.
\end{aligned}$$

We return to our computation (3.26) and use the computation above to combine the six mixed terms into two terms to obtain

$$\begin{aligned}
g_{\underline{\omega}}(E_a, E_b)\mu_{\underline{\omega}} &= \frac{1}{6}\epsilon_{ijk} \left[ T_{ip} T_{jv} T_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^+ \right. \\
& \quad + 3T_{ip} T_{jv} S_{kt} (E_a \lrcorner \omega_p^+) \wedge (E_b \lrcorner \omega_v^+) \wedge \omega_t^- \\
& \quad + 3S_{ip} S_{jv} T_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^+ \\
& \quad \left. + S_{ip} S_{jv} S_{kt} (E_a \lrcorner \omega_p^-) \wedge (E_b \lrcorner \omega_v^-) \wedge \omega_t^- \right].
\end{aligned} \tag{3.27}$$

For the remaining computations in this section let  $\mu = E^{0123}$ . Now we assume that  $a, b \in \{1, 2, 3\}$  and we use the result from lemma 3.2.1 to obtain the following:

$$\begin{aligned}
g_{\underline{\omega}}(E_a, E_b)\mu_{\underline{\omega}} &= \frac{1}{6}\epsilon_{ijk} \left[ T_{ip} T_{jv} T_{kt} (\delta_{ap}\epsilon_{bvt} + \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \right. \\
& \quad + 3T_{ip} T_{jv} S_{kt} (-\delta_{ap}\epsilon_{bvt} - \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \\
& \quad + 3S_{ip} S_{jv} T_{kt} (-\delta_{ap}\epsilon_{bvt} - \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \\
& \quad \left. + S_{ip} S_{jv} S_{kt} (\delta_{ap}\epsilon_{bvt} + \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \right] \mu.
\end{aligned}$$

We take the sum over  $i, j, k$  and factor in  $\epsilon_{ijk}$  which allows us to use the identities from lemma A.2.1. To make our computations simpler we introduce the following variables, let

$$f = \det(T) + \det(S), \tag{3.28}$$

$$h = \text{adj}(T)S + \text{adj}(S)T. \tag{3.29}$$

We obtain

$$g_{\underline{\omega}}(E_a, E_b)\mu_{\underline{\omega}} = \frac{1}{6} \left[ f\epsilon_{pvt} (\delta_{ap}\epsilon_{bvt} + \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \right. \\ \left. + 3h_{mt}\epsilon_{mpv} (-\delta_{ap}\epsilon_{bvt} - \delta_{bv}\epsilon_{pat} + \delta_{at}\epsilon_{pvb} + \delta_{pt}\epsilon_{bva}) \right] \mu.$$

Then using identities (3.25) we obtain

$$g_{\underline{\omega}}(E_a, E_b)\mu_{\underline{\omega}} = \left[ \frac{1}{6} f (2\delta_{ap}\delta_{pb} + 2\delta_{bv}\delta_{av} + 2\delta_{at}\delta_{bt}) \right. \\ \left. + \frac{1}{2} h_{mt} (\epsilon_{mav}\epsilon_{btv} + 2\delta_{at}\delta_{bm} + \epsilon_{atp}\epsilon_{mbp} - \epsilon_{mtv}\epsilon_{bav}) \right] \mu \\ = \left[ f\delta_{ab} + \frac{1}{2} h_{mt} (-2\delta_{mt}\delta_{ab} + 2\delta_{at}\delta_{mb} + 2\delta_{ma}\delta_{tb}) \right] \mu \\ = \left[ f\delta_{ab} - \text{Tr}(h)\delta_{ab} + h_{ba} + h_{ab} \right] \mu.$$

Now we can substitute back the expressions for  $f$  from (3.28) and  $h$  from (3.29) and we obtain the following:

$$g_{\underline{\omega}}(E_a, E_b)\mu_{\underline{\omega}} = \left[ (\det(T) + \det(S))\delta_{ab} - \text{Tr}(\text{adj}(T)S + \text{adj}(S)T)\delta_{ab} \right. \\ \left. + (\text{adj}(T)S + \text{adj}(S)T)_{ba} + (\text{adj}(T)S + \text{adj}(S)T)_{ab} \right] \mu.$$

Next consider the case where  $E_a, E_b = E_0$ . We simply take  $a = b = 0$  in (3.27) and with some relabeling of indices we obtain

$$g_{\underline{\omega}}(E_0, E_0)\mu_{\underline{\omega}} = \frac{1}{6} \epsilon_{ijk} \left[ T_{ip}T_{jv}T_{kt}\epsilon_{pvt} + 3T_{ip}T_{jv}S_{kt}\epsilon_{pvt} + 3S_{ip}S_{jv}T_{kt}\epsilon_{pvt} + S_{ip}S_{jv}S_{kt}\epsilon_{pvt} \right] \mu.$$

Now we use the identities from lemma A.2.1 to obtain

$$g_{\underline{\omega}}(E_0, E_0)\mu_{\underline{\omega}} = \frac{1}{6} \epsilon_{ijk} \left[ f\epsilon_{ijk} + 3h_{mt}\epsilon_{tpv}\epsilon_{mpv} \right] \mu. \\ = \left[ f + \text{Tr}(h) \right] \mu \\ = \left[ \det(T) + \det(S) + \text{Tr}(\text{adj}(T)S + \text{adj}(S)T) \right] \mu.$$

The last case we consider is when  $E_a = E_0$  and  $E_b$  is one of  $E_1, E_2$ , or  $E_3$ . Again we use

(3.27) to obtain

$$\begin{aligned}
g_{\underline{\omega}}(E_0, E_b)\mu_{\underline{\omega}} &= \frac{1}{6}\epsilon_{ijk} \left[ T_{ip}T_{jv}T_{kt} (\delta_{bv}\delta_{pt} + \delta_{vt}\delta_{bp} - \delta_{pv}\delta_{bt}) \right. \\
&\quad + 3T_{ip}T_{jv}S_{kt} (-\delta_{pt}\delta_{bv} + \delta_{vt}\delta_{bp} - \delta_{pv}\delta_{bt}) \\
&\quad + 3S_{ip}S_{jv}T_{kt} (\delta_{pt}\delta_{bv} - \delta_{vt}\delta_{bp} + \delta_{pv}\delta_{bt}) \\
&\quad \left. + S_{ip}S_{jv}S_{kt} (-\delta_{pt}\delta_{bv} - \delta_{vt}\delta_{bp} + \delta_{pv}\delta_{bt}) \right] \mu.
\end{aligned}$$

We factor in the  $\epsilon_{ijk}$ , use the identities from lemma A.2.1, and, for simplicity we let

$$\tilde{f} = \det(T) - \det(S), \quad (3.30)$$

$$\tilde{h} = \text{adj}(T)S - \text{adj}(S)T. \quad (3.31)$$

We then obtain

$$\begin{aligned}
g_{\underline{\omega}}(E_0, E_b)\mu_{\underline{\omega}} &= \frac{1}{6} \left[ \tilde{f}\epsilon_{pvt} (\delta_{bv}\delta_{pt} + \delta_{vt}\delta_{bp} - \delta_{pv}\delta_{bt}) + 3\tilde{h}_{mt}\epsilon_{mpv} (-\delta_{pt}\delta_{bv} + \delta_{vt}\delta_{bp} - \delta_{pv}\delta_{bt}) \right] \mu \\
&= \left[ 0 - \frac{1}{2}\tilde{h}_{mp}\epsilon_{mpb} + \frac{1}{2}\tilde{h}_{mv}\epsilon_{mbv} - 0 \right] \mu \\
&= \left[ -\tilde{h}_{mp}\epsilon_{mpb} \right] \mu.
\end{aligned}$$

Now we substitute back  $\tilde{h}$  from (3.31) and obtain the following:

$$g_{\underline{\omega}}(E_0, E_b)\mu_{\underline{\omega}} = \left[ (\text{adj}(S)T - \text{adj}(T)S)_{mp} \epsilon_{mpb} \right] \mu. \quad (3.32)$$

We notice that when  $S$  and  $T$  are diagonal matrices, the expression in (3.32) vanishes since the matrix  $\text{adj}(S)T - \text{adj}(T)S$  is diagonal and thus symmetric. Before we substitute in our particular matrices  $S$  and  $T$  we want to summarize the results of our metric coefficients  $(g_{\underline{\omega}})_{ab}$  in a table. However it is important that on one side of the metric computations we have  $\mu_{\underline{\omega}}$  and on the other we have  $\mu$  which we have chosen to be  $\mu = E^{0123}$ . In order to find the exact metric coefficients for  $g_{\underline{\omega}}$  we must divide by  $\sqrt{\det(g_{\underline{\omega}})}$ . Recall that the coefficients in the metric  $g_{\underline{\omega}}$  can be seen as a  $4 \times 4$  matrix. Let  $N$  be a  $4 \times 4$  matrix such that

$$g_{\underline{\omega}}\mu_{\underline{\omega}} = N\mu.$$

We have

$$(g_{\underline{\omega}})_{ab} \sqrt{\det(g_{\underline{\omega}})}\mu = (g_{\underline{\omega}})_{ab} \mu_{\underline{\omega}} = N_{ab}\mu.$$

Thus,

$$(g_{\omega})_{ab} = \frac{1}{\sqrt{\det(g_{\omega})}} N_{ab}.$$

To be able to find  $\sqrt{\det(g_{\omega})}$  we compute

$$\begin{aligned} \det(N) &= \det\left(g_{\omega}\sqrt{\det(g_{\omega})}\right) \\ &= \left(\sqrt{\det(g_{\omega})}\right)^4 \det(g_{\omega}) \\ &= \det(g_{\omega})^3. \end{aligned}$$

Thus,

$$\sqrt{\det(g_{\omega})} = (\det(N))^{\frac{1}{6}}.$$

We summarize these results in table 3.1.

$(g_{\omega})_{00}$	$\frac{1}{(\det(N))^{1/6}} (\det(T) + \det(S) + \text{Tr}(\text{adj}(T)S + \text{adj}(S)T))$
$(g_{\omega})_{a,b}$ $a, b \in \{1, 2, 3\}$	$\frac{1}{(\det(N))^{1/6}} (\det(T) + \det(S)) \delta_{ab} - \text{Tr}(\text{adj}(T)S + \text{adj}(S)T) \delta_{ab}$ $+ (\text{adj}(T)S + \text{adj}(S)T)_{ba} + (\text{adj}(T)S + \text{adj}(S)T)_{ab}$
$(g_{\omega})_{0,b}$ $b \in \{1, 2, 3\}$	$\frac{1}{(\det(N))^{1/6}} (\text{adj}(S)T - \text{adj}(T)S)_{mp} \epsilon_{mpb}$

Table 3.1: Metric coefficients

Now we can substitute in the values for  $S$  and  $T$  and evaluate the metric coefficients. First we recall the matrices  $S$  and  $T$  and compute their adjoints

$$\begin{aligned} T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_2 + \frac{r^2}{2} B_2 & 0 \\ 0 & 0 & A_3 + \frac{r^2}{2} B_3 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{r^2}{2} B_2 & 0 \\ 0 & 0 & \frac{r^2}{2} B_3 \end{bmatrix}, \\ \text{Adj}(T) &= \begin{bmatrix} \frac{(B_2 r^2 + 2A_2)(B_3 r^2 + 2A_3)}{4} & 0 & 0 \\ 0 & A_3 + B_3 \frac{r^2}{2} & 0 \\ 0 & 0 & A_2 + B_2 \frac{r^2}{2} \end{bmatrix}, \quad \text{Adj}(S) = \begin{bmatrix} \frac{r^4}{4} B_2 B_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\text{Adj}(S)T + \text{Adj}(T)S = \begin{bmatrix} \frac{r^4}{4}B_2B_3 & 0 & 0 \\ 0 & \frac{r^2}{2}B_2 \left( A_3 + B_3 \frac{r^2}{2} \right) & 0 \\ 0 & 0 & \frac{r^2}{2}B_3 \left( A_2 + B_2 \frac{r^2}{2} \right) \end{bmatrix},$$

$$\det(S) = 0,$$

$$\det(T) = \frac{(2A_2 + r^2B_2)(2A_3 + r^2B_3)}{4},$$

$$\text{Tr}(\text{Adj}(S)T + \text{Adj}(T)S) = \frac{3}{4}r^4B_2B_3 + \frac{r^2}{2}(A_3B_2 + A_2B_3).$$

Now we can substitute these into our expressions for  $g_{\underline{\omega}}\mu_{\underline{\omega}}$  and we obtain

$$\begin{aligned} g_{\underline{\omega}}(E_0, E_0)\mu_{\underline{\omega}} &= (A_2 + r^2B_2)(A_3 + r^2B_3)\mu, \\ g_{\underline{\omega}}(E_1, E_1)\mu_{\underline{\omega}} &= A_2A_3\mu, \\ g_{\underline{\omega}}(E_2, E_2)\mu_{\underline{\omega}} &= A_3(A_2 + r^2B_2)\mu, \\ g_{\underline{\omega}}(E_3, E_3)\mu_{\underline{\omega}} &= A_2(A_3 + r^2B_3)\mu. \end{aligned}$$

All other  $g_{\underline{\omega}}(E_a, E_b)$  are 0. In order to find the exact coefficients  $(g_{\underline{\omega}})_{ab}$  we need to divide by  $\sqrt{\det(g_{\underline{\omega}})}$ . As we have explained earlier, we can compute this to be

$$\sqrt{\det(g_{\underline{\omega}})} = \left( (A_2 + r^2B_2)^2 A_2^2 A_3^2 (A_3 + r^2B_3)^2 \right)^{\frac{1}{6}} = (A_2A_3(A_2 + r^2B_2)(A_3 + r^2B_3))^{\frac{1}{3}}$$

then the metric coefficients are

$$\begin{aligned} g_{\underline{\omega}}(E_0, E_0) &= \left( \frac{(A_2 + r^2B_2)^2 (A_3 + r^2B_3)^2}{A_2A_3} \right)^{\frac{1}{3}}, \\ g_{\underline{\omega}}(E_1, E_1) &= \left( \frac{A_2^2 A_3^2}{(A_2 + r^2B_2)(A_3 + r^2B_3)} \right)^{\frac{1}{3}}, \\ g_{\underline{\omega}}(E_2, E_2) &= \left( \frac{A_3^2 (A_2 + r^2B_2)^2}{A_2(A_3 + r^2B_3)} \right)^{\frac{1}{3}}, \\ g_{\underline{\omega}}(E_3, E_3) &= \left( \frac{A_2^2 (A_3 + r^2B_3)^2}{A_3(A_2 + r^2B_2)} \right)^{\frac{1}{3}}. \end{aligned}$$

Let

$$\begin{aligned}
f_0(r) &:= \left( \frac{(A_2 + r^2 B_2)^2 (A_3 + r^2 B_3)^2}{A_2 A_3} \right)^{\frac{1}{6}} = \left( \frac{(r A'_2 + 2A_2)^2 (r A'_3 + 2A_3)^2}{16 A_2 A_3} \right)^{\frac{1}{6}}, \\
f_1(r) &:= \left( \frac{A_2^2 A_3^2}{(A_2 + r^2 B_2)(A_3 + r^2 B_3)} \right)^{\frac{1}{6}} = \left( \frac{4 A_2^2 A_3^2}{(r A'_2 + 2A_2)(r A'_3 + 2A_3)} \right)^{\frac{1}{6}}, \\
f_2(r) &:= \left( \frac{A_3^2 (A_2 + r^2 B_2)^2}{A_2 (A_3 + r^2 B_3)} \right)^{\frac{1}{6}} = \left( \frac{A_3^2 (r A'_2 + 2A_2)^2}{2 A_2 (r A'_3 + 2A_3)} \right)^{\frac{1}{6}}, \\
f_3(r) &:= \left( \frac{A_2^2 (A_3 + r^2 B_3)^2}{A_3 (A_2 + r^2 B_2)} \right)^{\frac{1}{6}} = \left( \frac{A_2^2 (r A'_3 + 2A_3)^2}{2 A_3 (r A'_2 + 2A_2)} \right)^{\frac{1}{6}}.
\end{aligned} \tag{3.33}$$

The metric  $g_\omega$  can therefore be written as

$$g_\omega = f_0^2 (E^0)^2 + f_1^2 (E^1)^2 + f_2^2 (E^2)^2 + f_3^2 (E^3)^2.$$

The frame  $E^0, E^1, E^2, E^3$  is orthogonal with respect to  $g_\omega$ , but not orthonormal. Let

$$\tilde{E}^i = f_i E^i \quad \text{and} \quad \tilde{E}_i = \frac{1}{f_i} E_i. \tag{3.34}$$

We can write

$$g_\omega = (\tilde{E}^0)^2 + (\tilde{E}^1)^2 + (\tilde{E}^2)^2 + (\tilde{E}^3)^2.$$

Later we will need to use this orthonormal frame in order to simplify our computations for the soliton equation and thus writing our hypersymplectic structure in terms of this

orthonormalized basis will be useful. Using (3.15), we get:

$$\begin{aligned}
\omega_1 &= E^{01} + E^{23} \\
&= \frac{1}{f_0 f_1} \tilde{E}^{01} + \frac{1}{f_2 f_3} \tilde{E}^{23} \\
&= \left( \frac{4}{A_2 A_3 (r A'_2 + 2A_2)(r A'_3 + 2A_3)} \right)^{\frac{1}{6}} (\tilde{E}^{01} + \tilde{E}^{23}), \\
\omega_2 &= \left( A_2 + \frac{r^2}{2} B_2 \right) (E^{02} + E^{31}) + \frac{r^2}{2} B_2 (E^{02} - E^{31}) \\
&= (A_2 + r^2 B_2) E^{02} + A_2 E^{31} \\
&= (A_2 + r^2 B_2) \frac{1}{f_0 f_2} \tilde{E}^{02} + A_2 \frac{1}{f_1 f_3} \tilde{E}^{31} \\
&= \left( \frac{A_2^2 (r A'_2 + 2A_2)^2}{2A_3 (r A'_3 + 2A_3)} \right)^{\frac{1}{6}} (\tilde{E}^{02} + \tilde{E}^{31}), \\
\omega_3 &= (A_3 + r^2 B_3) \frac{1}{f_0 f_3} \tilde{E}^{03} + A_3 \frac{1}{f_1 f_2} \tilde{E}^{12} \\
&= \left( \frac{A_3^2 (r A'_3 + 2A_3)^2}{2A_2 (r A'_2 + 2A_2)} \right)^{\frac{1}{6}} (\tilde{E}^{03} + \tilde{E}^{12}).
\end{aligned} \tag{3.35}$$

### 3.3 The Christoffel symbols

In this section we compute the Christoffel symbols that come from  $g_{\underline{\omega}}$  with respect to the frame  $\tilde{E}_i$  from (3.34). Recall the form of the metric:

$$g_{\underline{\omega}} = (\tilde{E}^0)^2 + (\tilde{E}^1)^2 + (\tilde{E}^2)^2 + (\tilde{E}^3)^2.$$

In this section we will not make any assumptions about the structure of the  $\omega_k$ , meaning we will leave the functions  $A_k, B_k$  as functions of  $r$ . To compute the Christoffel symbols we will use the Koszul Formula

$$\begin{aligned}
2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) \\
&\quad - g([X, Z], Y) - g([Y, Z], X).
\end{aligned}$$

Since our frame is orthogonal we obtain

$$2g(\nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_k) = g([\tilde{E}_i, \tilde{E}_j], \tilde{E}_k) - g([\tilde{E}_i, \tilde{E}_k], \tilde{E}_j) - g([\tilde{E}_j, \tilde{E}_k], \tilde{E}_i).$$

For simplicity, we write  $g$  to mean  $g_\omega$ , the metric that comes from the triple  $\omega$ . The computations for the Lie brackets of the  $\tilde{E}_i$  is done in appendix A.1. We also notice the Christoffel symbols are skew symmetric. Indeed, since

$$0 = \tilde{E}_i \left( g(\tilde{E}_j, \tilde{E}_k) \right) = g \left( \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_k \right) + g \left( \tilde{E}_j, \nabla_{\tilde{E}_i} \tilde{E}_k \right),$$

we have 
$$\Gamma_{ij}^k = g \left( \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_k \right) = -g \left( \tilde{E}_j, \nabla_{\tilde{E}_i} \tilde{E}_k \right) = -\Gamma_{ik}^j.$$

Note that the derivative of  $r$  in the direction of  $E_i$  for  $i \neq 0$  is 0, that is  $E_i(r) = 0$ .

In the following computations there are sums over  $m$  from 1 to 3 but not  $i, j, k$ .

Case 1:  $\Gamma_{ij}^k$  where  $i, j, k \in \{1, 2, 3\}$ . Using the computations in section A.1 we compute

$$\begin{aligned} 2g \left( \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_k \right) &= g \left( \frac{-2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \tilde{E}_m, \tilde{E}_k \right) - g \left( \frac{-2}{r} \epsilon_{ikm} (f_i f_k)^{-1} f_m \tilde{E}_m, \tilde{E}_j \right) \\ &\quad - g \left( \frac{-2}{r} \epsilon_{jkm} (f_j f_k)^{-1} f_m \tilde{E}_m, \tilde{E}_i \right) \\ &= \frac{2}{r} \left( -\epsilon_{ijk} (f_i f_j)^{-1} f_k + \epsilon_{ikj} (f_i f_k)^{-1} f_j + \epsilon_{jki} (f_j f_k)^{-1} f_i \right) \\ &= \frac{2}{r} \epsilon_{ijk} \left( f_i (f_j f_k)^{-1} - f_j (f_k f_i)^{-1} - f_k (f_i f_j)^{-1} \right). \end{aligned}$$

Let

$$H_{ijk} := f_i (f_j f_k)^{-1} - f_j (f_k f_i)^{-1} - f_k (f_i f_j)^{-1}. \quad (3.36)$$

We note that  $H_{ijk} = H_{ikj}$ . We thus have

$$\Gamma_{ij}^k = \frac{1}{r} \epsilon_{ijk} H_{ijk}.$$

Case 2:  $\Gamma_{0j}^k$  where  $j, k \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} 2g \left( \nabla_{\tilde{E}_0} \tilde{E}_j, \tilde{E}_k \right) &= g \left( [\tilde{E}_0, \tilde{E}_j], \tilde{E}_k \right) - g \left( [\tilde{E}_0, \tilde{E}_k], \tilde{E}_j \right) - g \left( [\tilde{E}_j, \tilde{E}_k], \tilde{E}_0 \right) \\ &= \left( f_j f_0^{-1} (f_j^{-1})' - \frac{f_0^{-1}}{r} \right) \delta_{jk} - \left( f_k f_0^{-1} (f_k^{-1})' - \frac{f_0^{-1}}{r} \right) \delta_{jk} \\ &= \delta_{jk} f_0^{-1} \left( f_j (f_j^{-1})' - f_k (f_k^{-1})' \right). \end{aligned}$$

When  $j = k$ , the elements in parentheses cancel out, and otherwise  $\delta_{jk} = 0$ . Thus

$$\Gamma_{0j}^k = 0$$

Case 3:  $\Gamma_{i0}^k, \Gamma_{ik}^0$  where  $i, k \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} 2g\left(\nabla_{\tilde{E}_i} \tilde{E}_0, \tilde{E}_k\right) &= g\left([\tilde{E}_i, \tilde{E}_0], \tilde{E}_k\right) - g\left([\tilde{E}_i, \tilde{E}_k], \tilde{E}_0\right) - g\left([\tilde{E}_0, \tilde{E}_k], \tilde{E}_i\right) \\ &= \delta_{ki} f_0^{-1} \left(\frac{1}{r} - f_i(f_i^{-1})'\right) - \delta_{ki} f_0^{-1} \left(f_k(f_k^{-1})' - \frac{1}{r}\right) \\ &= \delta_{ki} f_0^{-1} \left(\frac{2}{r} - f_i(f_i^{-1})' - f_k(f_k^{-1})'\right). \end{aligned}$$

Thus we have

$$\begin{aligned} \Gamma_{i0}^k &= \delta_{ki} \frac{f_0^{-1}}{2} \left(\frac{2}{r} - f_i(f_i^{-1})' - f_k(f_k^{-1})'\right), \\ \Gamma_{ik}^0 &= -\delta_{ki} \frac{f_0^{-1}}{2} \left(\frac{2}{r} - f_i(f_i^{-1})' - f_k(f_k^{-1})'\right). \end{aligned}$$

Case 4:  $\Gamma_{i0}^0$  where  $i \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} 2g\left(\nabla_{\tilde{E}_i} \tilde{E}_0, \tilde{E}_0\right) &= g\left([\tilde{E}_i, \tilde{E}_0], \tilde{E}_0\right) - g\left([\tilde{E}_i, \tilde{E}_0], \tilde{E}_0\right) - g\left([\tilde{E}_0, \tilde{E}_0], \tilde{E}_i\right) \\ &= 0. \end{aligned}$$

Thus

$$\Gamma_{i0}^0 = 0.$$

Case 5:  $\Gamma_{0j}^0, \Gamma_{00}^j$  where  $j \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} 2g\left(\nabla_{\tilde{E}_0} \tilde{E}_j, \tilde{E}_0\right) &= g\left([\tilde{E}_0, \tilde{E}_j], \tilde{E}_0\right) - g\left([\tilde{E}_0, \tilde{E}_0], \tilde{E}_j\right) - g\left([\tilde{E}_j, \tilde{E}_0], \tilde{E}_0\right) \\ &= 0. \end{aligned}$$

Thus

$$\Gamma_{0j}^0 = \Gamma_{00}^j = 0.$$

Case 6:  $\Gamma_{00}^0$

$$\begin{aligned} 2g\left(\nabla_{\tilde{E}_0}\tilde{E}_0, \tilde{E}_0\right) &= g\left([\tilde{E}_0, \tilde{E}_0], \tilde{E}_0\right) - g\left([\tilde{E}_0, \tilde{E}_0], \tilde{E}_0\right) - g\left([\tilde{E}_0, \tilde{E}_0], \tilde{E}_0\right) \\ &= 0. \end{aligned}$$

Thus

$$\Gamma_{00}^0 = 0.$$

We summarize these results in table 3.2 where  $i, j, k \in \{1, 2, 3\}$  and  $f_i$  are the metric coefficients (3.33). All other Christoffel symbols can be found by skew-symmetry.

$\Gamma_{ij}^k$	$\frac{2}{r}\epsilon_{ijk}(f_i(f_j f_k)^{-1} - f_j(f_k f_i)^{-1} - f_k(f_i f_j)^{-1})$
$\Gamma_{i0}^k$	$\delta_{ki}\frac{f_0^{-1}}{2}\left(\frac{2}{r} - f_i(f_i^{-1})' - f_k(f_k^{-1})'\right)$
$\Gamma_{0j}^k$	0
$\Gamma_{0j}^0$	0

Table 3.2: Christoffel symbols

### 3.4 The Riemann curvature tensor

In this section we compute the Riemann curvature tensor. Again we note that we do not place any assumptions on the functions  $A_k, B_k$ , other than that they are arbitrary functions of the radial coordinate  $r$ . We use the Christoffel symbols from the previous section in these calculations. By the symmetries of the Riemann curvature tensor we need only consider three cases:  $R_{ijpq}, R_{0ip0}, R_{0ipq}$  where  $i, j, p, q \in \{1, 2, 3\}$ . For simplicity, we write  $g$  to mean  $g_\omega$  the metric that comes from the triple  $\omega$ . Lastly, the usual Einstein summation breaks in this section. It is to be assumed that there is no sum unless explicitly specified.

Case 1:  $R_{ijpq}$  where  $i, j, p, q \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} R_{ijpq} &= g\left(\nabla_{\tilde{E}_i}\left(\nabla_{\tilde{E}_j}\tilde{E}_p\right) - \nabla_{\tilde{E}_j}\left(\nabla_{\tilde{E}_i}\tilde{E}_p\right) - \nabla_{[\tilde{E}_i, \tilde{E}_j]}\tilde{E}_p, \tilde{E}_q\right) \\ &= g\left(\nabla_{\tilde{E}_i}\left(\Gamma_{jp}^0\tilde{E}_0 + \Gamma_{jp}^a\tilde{E}_a\right) - \nabla_{\tilde{E}_j}\left(\Gamma_{ip}^0\tilde{E}_0 + \Gamma_{ip}^a\tilde{E}_a\right) + \frac{2}{r}\epsilon_{ijm}(f_i f_j)^{-1}f_m\nabla_{\tilde{E}_m}\tilde{E}_p, \tilde{E}_q\right). \end{aligned}$$

where there is a sum over  $a$  and  $m$  but not over  $i, j, p, q$ . Expanding the second covariant derivative we obtain

$$\begin{aligned}
R_{ijpq} &= g\left(\underbrace{\tilde{E}_i(\Gamma_{jp}^0) \cdot \tilde{E}_0}_{0} + \Gamma_{jp}^0 \nabla_{\tilde{E}_i} \tilde{E}_0 + \underbrace{\tilde{E}_i(\Gamma_{jp}^a) \cdot \tilde{E}_a}_{0} + \Gamma_{jp}^a \nabla_{\tilde{E}_i} \tilde{E}_a, \tilde{E}_q\right) \\
&\quad - g\left(\underbrace{\tilde{E}_j(\Gamma_{ip}^0) \cdot \tilde{E}_0}_{0} + \Gamma_{ip}^0 \nabla_{\tilde{E}_j} \tilde{E}_0 + \underbrace{\tilde{E}_j(\Gamma_{ip}^a) \cdot \tilde{E}_a}_{0} + \Gamma_{ip}^a \nabla_{\tilde{E}_j} \tilde{E}_a, \tilde{E}_q\right) \\
&\quad + \frac{2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m g\left(\Gamma_{mp}^0 \tilde{E}_0 + \Gamma_{mp}^a \tilde{E}_a, \tilde{E}_q\right).
\end{aligned}$$

The frame  $\tilde{E}_i$  is orthonormal so many of the terms above vanish. Furthermore, the Christoffel symbols are functions of only  $r$  and thus vanish when differentiated in the direction of  $\tilde{E}_i$  for  $i \neq 0$ . Then the above becomes

$$\begin{aligned}
R_{ijpq} &= \Gamma_{jp}^0 g\left(\underbrace{\Gamma_{i0}^0 \tilde{E}_0}_{0} + \Gamma_{i0}^a \tilde{E}_a, \tilde{E}_q\right) + \Gamma_{jp}^a g\left(\Gamma_{ia}^0 \tilde{E}_0 + \Gamma_{ia}^b \tilde{E}_b, \tilde{E}_q\right) \\
&\quad - \Gamma_{ip}^0 g\left(\underbrace{\Gamma_{j0}^0 \tilde{E}_0}_{0} + \Gamma_{j0}^a \tilde{E}_a, \tilde{E}_q\right) - \Gamma_{ip}^a g\left(\Gamma_{ja}^0 \tilde{E}_0 + \Gamma_{ja}^b \tilde{E}_b, \tilde{E}_q\right) + \frac{2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \Gamma_{mp}^a \delta_{aq} \\
&= \Gamma_{jp}^0 \Gamma_{i0}^a \delta_{aq} + \Gamma_{jp}^a \Gamma_{ia}^b \delta_{bq} - \Gamma_{ip}^0 \Gamma_{j0}^a \delta_{aq} - \Gamma_{ip}^a \Gamma_{ja}^b \delta_{bq} + \frac{2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \Gamma_{mp}^a \delta_{aq}
\end{aligned}$$

where there is a sum over  $a, b, m$  from 1 to 3 but not over  $i, j, p, q$ . Once we apply the Kronecker delta function and input the functions for the Christoffel symbols we obtain

$$\begin{aligned}
R_{ijpq} &= \Gamma_{jp}^0 \Gamma_{i0}^q + \Gamma_{jp}^a \Gamma_{ia}^q - \Gamma_{ip}^0 \Gamma_{j0}^q - \Gamma_{ip}^a \Gamma_{ja}^q + \frac{2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \Gamma_{mp}^q \\
&= \left( \frac{f_0^{-1}}{2} \right)^2 \delta_{jp} \delta_{iq} \left( f_p (f_p^{-1})' + f_j (f_j^{-1})' - \frac{2}{r} \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_q (f_q^{-1})' \right) \\
&\quad + \left( \frac{1}{r} \right)^2 \epsilon_{jpa} H_{jpa} \epsilon_{iaq} H_{iaq} \\
&\quad + \left( \frac{f_0^{-1}}{2} \right)^2 \delta_{ip} \delta_{jq} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_q (f_q^{-1})' \right) \\
&\quad - \left( \frac{1}{r} \right)^2 \epsilon_{ipa} H_{ipa} \epsilon_{jaq} H_{jaq} + \frac{2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \frac{1}{r} \epsilon_{mpq} H_{mpq} \\
&= \left( \frac{f_0^{-1}}{2} \right)^2 \delta_{jp} \delta_{iq} \left( f_p (f_p^{-1})' + f_j (f_j^{-1})' - \frac{2}{r} \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_q (f_q^{-1})' \right) \\
&\quad + \left( \frac{f_0^{-1}}{2} \right)^2 \delta_{ip} \delta_{jq} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_q (f_q^{-1})' \right) \\
&\quad - \frac{1}{r^2} H_{iqa} H_{jpa} \epsilon_{iqa} \epsilon_{jpa} + \frac{1}{r^2} H_{ipa} H_{jq a} \epsilon_{ipa} \epsilon_{jq a} + \frac{2}{r^2} (f_i f_j)^{-1} f_m \epsilon_{ijm} \epsilon_{pqm} H_{mpq}
\end{aligned}$$

where there is a sum over  $a$  and  $m$  from 1 to 3 but no sum over  $i, j, k$ . Thus we obtain

$$\begin{aligned}
R_{ijpq} &= \left( \frac{f_0^{-1}}{2} \right)^2 \left[ \delta_{ip} \delta_{jq} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_q (f_q^{-1})' \right) \right. \\
&\quad \left. - \delta_{iq} \delta_{jp} \left( \frac{2}{r} - f_p (f_p^{-1})' - f_j (f_j^{-1})' \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_q (f_q^{-1})' \right) \right] \\
&\quad + \left( \frac{1}{r^2} \right) (H_{ipa} H_{jq a} \epsilon_{ipa} \epsilon_{jq a} - H_{iqa} H_{jpa} \epsilon_{iqa} \epsilon_{jpa}) + \frac{2}{r^2} (f_i f_j)^{-1} f_m \epsilon_{ijm} \epsilon_{pqm} H_{mpq}.
\end{aligned}$$

Case 2:  $R_{0ip0}$  where  $i, p \in \{1, 2, 3\}$ . We compute

$$\begin{aligned}
R_{0ip0} &= g \left( \nabla_{\tilde{E}_0} \left( \nabla_{\tilde{E}_i} \tilde{E}_p \right) - \nabla_{\tilde{E}_i} \left( \nabla_{\tilde{E}_0} \tilde{E}_p \right) - \nabla_{[\tilde{E}_0, \tilde{E}_i]} \tilde{E}_p, \tilde{E}_0 \right) \\
&= g \left( \nabla_{\tilde{E}_0} \left( \Gamma_{ip}^0 \tilde{E}_0 + \Gamma_{ip}^a \tilde{E}_a \right) - \nabla_{\tilde{E}_0} \left( \underbrace{\Gamma_{0p}^0}_0 \tilde{E}_0 + \underbrace{\Gamma_{0p}^a}_0 \tilde{E}_a \right), \tilde{E}^0 \right) \\
&\quad - g \left( \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \left( \Gamma_{ip}^0 \tilde{E}_0 + \Gamma_{ip}^a \tilde{E}_a \right), \tilde{E}_0 \right)
\end{aligned}$$

where there is a sum over  $a$  from 1 to 3 but no sum over  $i$  or  $p$ . We remind ourselves that the frame  $\tilde{E}^i$  is orthonormal and thus we obtain the following where there is a sum over  $a, b$  but not over  $i$  or  $p$ :

$$\begin{aligned}
R_{0ip0} &= g \left( \tilde{E}_0 (\Gamma_{ip}^0) \cdot \tilde{E}_0 + \Gamma_{ip}^0 \underbrace{\nabla_{\tilde{E}_0} \tilde{E}_0}_0 + \tilde{E}_0 (\Gamma_{ip}^a) \cdot \tilde{E}_a + \Gamma_{ip}^a \nabla_{\tilde{E}_0} \tilde{E}_a, \tilde{E}_0 \right) \\
&\quad - \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \Gamma_{ip}^0 \\
&= \tilde{E}_0 (\Gamma_{ip}^0) + \Gamma_{ip}^a g \left( \underbrace{\Gamma_{0a}^0}_0 \tilde{E}_0 + \Gamma_{0a}^b \tilde{E}_b, \tilde{E}_0 \right) - \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \Gamma_{ip}^0 \\
&= \tilde{E}_0 (\Gamma_{ip}^0) - \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \Gamma_{ip}^0 \\
&= f_0^{-1} \frac{d}{dr} \left( -\delta_{ip} \frac{f_0^{-1}}{2} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \right) \\
&\quad + \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \left( \delta_{ip} \frac{f_0^{-1}}{2} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \right).
\end{aligned}$$

Now we can split this piece into two cases:

- If  $i \neq p \implies R_{0ip0} = 0$ .
- If  $i = p$  then

$$R_{0ii0} = f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_i (f_i^{-1})' - \frac{1}{r} \right)^2.$$

Case 3:  $R_{0ipq}$  where  $i, p, q \in \{1, 2, 3\}$ . The Riemann curvature in this case is

$$\begin{aligned} R_{0ipq} &= g \left( \nabla_{\tilde{E}_0} \left( \nabla_{\tilde{E}_i} \tilde{E}_p \right) - \nabla_{\tilde{E}_i} \left( \nabla_{\tilde{E}_0} \tilde{E}_p \right) - \nabla_{[\tilde{E}_0, \tilde{E}_i]} \tilde{E}_p, \tilde{E}_q \right) \\ &= g \left( \nabla_{\tilde{E}_0} \left( \Gamma_{ip}^0 \tilde{E}_0 + \Gamma_{ip}^a \tilde{E}_a \right) - \nabla_{\tilde{E}_0} \left( \underbrace{\Gamma_{0p}^0}_0 \tilde{E}_0 + \underbrace{\Gamma_{0p}^a}_0 \tilde{E}_a \right) \right. \\ &\quad \left. - f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \left( \Gamma_{ip}^0 \tilde{E}_0 + \Gamma_{ip}^a \tilde{E}_a \right), \tilde{E}_q \right) \end{aligned}$$

where there is a sum over  $a$  from 1 to 3 but no sum over  $i, p$ , or  $q$ . Now we input the Christoffel symbols and use the fact that the frame is orthonormal to obtain

$$\begin{aligned} R_{0ipq} &= g \left( \tilde{E}_0 \left( \Gamma_{ip}^0 \right) \tilde{E}_0 + \Gamma_{ip}^0 \underbrace{\nabla_{\tilde{E}_0} \tilde{E}_0}_0 + \tilde{E}_0 \left( \Gamma_{ip}^a \right) \tilde{E}_a + \Gamma_{ip}^a \nabla_{\tilde{E}_0} \tilde{E}_a, \tilde{E}_q \right) \\ &\quad - f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \Gamma_{ip}^q \\ &= \tilde{E}_0 \left( \Gamma_{ip}^q \right) + \sum_{a,b=1}^3 \Gamma_{ip}^a g \left( \underbrace{\Gamma_{0a}^0}_0 \tilde{E}_0 + \underbrace{\Gamma_{0a}^b}_0 \tilde{E}_b, \tilde{E}_q \right) - f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \Gamma_{ip}^q \\ &= \epsilon_{ipq} f_0^{-1} \left( \frac{d}{dr} \left( \frac{1}{r} H_{ipq} \right) - \frac{1}{r} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) H_{ipq} \right) \end{aligned}$$

where there is no sum over  $i, p$ , or  $q$ . We summarize these results in table 3.3 where there is a sum over  $a$  and  $m$  from 1 to 3 but no sum over  $i, j, p$ , or  $q$ . The function  $H_{ijk}$  is given in (3.36) and the functions  $f_i$  are the metric coefficients which can be found in (3.33).

$R_{ijpq}$	$\begin{aligned} &\left( \frac{f_0^{-1}}{2} \right)^2 \left[ \delta_{ip} \delta_{jq} \left( \frac{2}{r} - f_i (f_i^{-1})' - f_p (f_p^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_q (f_q^{-1})' \right) \right. \\ &\quad \left. - \delta_{iq} \delta_{jp} \left( \frac{2}{r} - f_p (f_p^{-1})' - f_j (f_j^{-1})' \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_q (f_q^{-1})' \right) \right] \\ &\quad + \left( \frac{1}{r^2} \right) \left( H_{ipa} H_{jqa} \epsilon_{ipa} \epsilon_{jqa} - H_{iqa} H_{jpa} \epsilon_{iqa} \epsilon_{jpa} \right) + \frac{2}{r^2} (f_i f_j)^{-1} f_m \epsilon_{ijm} \epsilon_{pqm} H_{mpq} \end{aligned}$
$R_{0ip0}$	$\delta_{ip} \left[ f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_i (f_i^{-1})' - \frac{1}{r} \right)^2 \right]$
$R_{0ipq}$	$\epsilon_{ipq} f_0^{-1} \left( \frac{d}{dr} \left( \frac{1}{r} H_{ipq} \right) - \frac{1}{r} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) H_{ipq} \right)$

Table 3.3: Riemann curvature components

### 3.5 The Ricci curvature tensor

In this section we compute the general Ricci curvature tensor with respect to the metric  $g_{HS}$  that we computed in section 3.2. We use  $R_{jk}$  to mean the components of the Ricci curvature tensor and  $R_{ijpq}$  to mean the components of the Riemann curvature tensor. Recall that our chosen frame  $\tilde{E}_i$  is orthonormal, thus the Ricci curvature tensor coefficients simplify greatly from  $R_{jk} = g^{ip}R_{jipk}$  to  $R_{jk} = R_{jikk}$ . Lastly, similar to the previous subsection the Einstein summation convention breaks, therefore we will indicate a summation with the usual notation.

Case 1:  $R_{00}$ . We compute

$$\begin{aligned} R_{00} &= R_{0000} + \sum_{i=1}^3 R_{0i i 0} \\ &= 0 + \sum_{i=1}^3 \left( f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_i (f_i^{-1})' - \frac{1}{r} \right)^2 \right) \\ &= - \sum_{i=1}^3 \left( f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_i))' + \frac{1}{r} \right) \right) + (f_0^{-1})^2 \left( (\log(f_i))' + \frac{1}{r} \right)^2 \right) \end{aligned}$$

Case 2:  $R_{0k}$  for  $k \in \{1, 2, 3\}$ . We compute

$$\begin{aligned} R_{0k} &= R_{000k} + \sum_{i=1}^3 R_{0i i k} \\ &= 0 + \sum_{i=1}^3 \left( \epsilon_{iik} f_0^{-1} \left( \frac{d}{dr} \left( \frac{1}{r} H_{iik} \right) - \frac{1}{r} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) H_{iik} \right) \right) \\ &= 0. \end{aligned}$$

Case 3:  $R_{jk}$  for  $j, k \in \{1, 2, 3\}$  when  $j \neq k$ . We compute

$$\begin{aligned}
R_{jk} &= R_{j00k} + \sum_{i=1}^3 R_{jiiik} \\
&= 0 + \sum_{i=1}^3 \left[ \left( \frac{f_0^{-1}}{2} \right)^2 \left[ \underbrace{\delta_{ji}\delta_{ik}}_0 \left( \frac{2}{r} - f_j(f_j^{-1})' - f_i(f_i^{-1})' \right) \left( \frac{2}{r} - f_i(f_i^{-1})' - f_k(f_k^{-1})' \right) \right. \right. \\
&\quad \left. \left. - \underbrace{\delta_{jk}\delta_{ii}}_0 \left( \frac{2}{r} - f_i(f_i^{-1})' - f_i(f_i^{-1})' \right) \left( \frac{2}{r} - f_j(f_j^{-1})' - f_k(f_k^{-1})' \right) \right] \right. \\
&\quad \left. + \left( \frac{1}{r^2} \right) \left( H_{jia}H_{ika}\epsilon_{jia}\epsilon_{ika} - H_{jka}H_{ia}\epsilon_{jka}\underbrace{\epsilon_{ia}}_0 \right) \right. \\
&\quad \left. + \frac{2}{r^2} (f_j f_i)^{-1} \left[ f_1 H_{1ik}\epsilon_{ji1}\epsilon_{ik1} + f_2 H_{2ik}\epsilon_{ji2}\epsilon_{ik2} + f_3 H_{3ik}\epsilon_{ji3}\epsilon_{ik3} \right] \right]. \\
&= \sum_{i=1}^3 \left[ \frac{1}{r^2} H_{jia}H_{ika}\epsilon_{jia}\epsilon_{ika} + \frac{2}{r^2} (f_i f_j)^{-1} f_a H_{aik}\epsilon_{jia}\epsilon_{ika} \right] \\
&= 0.
\end{aligned}$$

In the last step above, since  $j \neq k$  one of  $\epsilon_{jia}$  or  $\epsilon_{ika}$  will be zero since there are only three choices for the indices, therefore this coefficient will vanish.

Case 4:  $R_{jk}$  for  $j, k \in \{1, 2, 3\}$  when  $j = k$ . We compute

$$\begin{aligned}
R_{jj} &= R_{j00j} + \sum_{i=1}^3 R_{jii} \\
&= f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_j (f_j^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_j (f_j^{-1})' - \frac{1}{r} \right)^2 \\
&\quad + \left( \frac{f_0^{-1}}{2} \right)^2 \sum_{i=1}^3 \left[ \delta_{ij} \delta_{ij} \left( \frac{2}{r} - f_j (f_j^{-1})' - f_i (f_i^{-1})' \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_j (f_j^{-1})' \right) \right. \\
&\quad \quad \left. - \left( \frac{2}{r} - f_i (f_i^{-1})' - f_i (f_i^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_j (f_j^{-1})' \right) \right] \\
&\quad + \left( \frac{1}{r^2} \right) \sum_{i,a=1}^3 (H_{jia} H_{ija} \epsilon_{jia} \epsilon_{ija} - H_{jja} H_{iia} \epsilon_{jja} \epsilon_{iia}) \\
&\quad + \frac{2}{r^2} \sum_{i=1}^3 (f_i f_j)^{-1} \left[ f_1 H_{1ij} \epsilon_{ij1} \epsilon_{ji1} + f_2 H_{2ij} \epsilon_{ij2} \epsilon_{ji2} + f_3 H_{3ij} \epsilon_{ij3} \epsilon_{ji3} \right] \\
&= f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_j (f_j^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_j (f_j^{-1})' - \frac{1}{r} \right)^2 \\
&\quad + \left( \frac{f_0^{-1}}{2} \right)^2 \sum_{i=1}^3 \left[ \delta_{ij} \left( \frac{2}{r} - f_j (f_j^{-1})' - f_i (f_i^{-1})' \right) \left( \frac{2}{r} - f_i (f_i^{-1})' - f_j (f_j^{-1})' \right) \right. \\
&\quad \quad \left. - \left( \frac{2}{r} - f_i (f_i^{-1})' - f_i (f_i^{-1})' \right) \left( \frac{2}{r} - f_j (f_j^{-1})' - f_j (f_j^{-1})' \right) \right] \\
&\quad - \sum_{i,a=1}^3 \left[ \frac{1}{r^2} (H_{jia} H_{ija} (\epsilon_{ija})^2) + \frac{2}{r^2} (f_i f_j)^{-1} (f_a H_{aij} (\epsilon_{ija})^2) \right] \\
&= f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_j (f_j^{-1})' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( f_j (f_j^{-1})' - \frac{1}{r} \right)^2 \\
&\quad + \left( \frac{f_0^{-1}}{2} \right)^2 \left[ \left( \frac{2}{r} - 2f_j (f_j^{-1})' \right)^2 - \left( \frac{2}{r} - 2f_j (f_j^{-1})' \right) \sum_{i=1}^3 \left( \frac{2}{r} - 2f_i (f_i^{-1})' \right) \right] \\
&\quad - \sum_{i,a=1}^3 \left[ \frac{1}{r^2} (H_{jia} H_{ija} (\epsilon_{ija})^2) + \frac{2}{r^2} (f_i f_j)^{-1} (f_a H_{aij} (\epsilon_{ija})^2) \right].
\end{aligned}$$

We can simplify the above by noticing for a function  $f$  that  $f (f^{-1})' = -(\log(f))'$ . There-

fore, we obtain the following expression:

$$\begin{aligned}
R_{jj} &= f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( -(\log(f_j))' - \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( -(\log(f_j))' - \frac{1}{r} \right)^2 \\
&\quad + (f_0^{-1})^2 \left[ \left( \frac{1}{r} + (\log(f_j))' \right)^2 - \left( \frac{1}{r} + (\log(f_j))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \right] \\
&\quad - \sum_{i,a=1}^3 \left[ \frac{1}{r^2} (H_{jia} H_{ija} (\epsilon_{ija})^2) + \frac{2}{r^2} (f_i f_j)^{-1} (f_a H_{aij} (\epsilon_{ija})^2) \right] \\
&= -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_j))' + \frac{1}{r} \right) \right) - (f_0^{-1})^2 \left( \frac{1}{r} + (\log(f_j))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\
&\quad - \sum_{i,a=1}^3 \left[ \frac{1}{r^2} (H_{jia} H_{ija} (\epsilon_{ija})^2) + \frac{2}{r^2} (f_i f_j)^{-1} (f_a H_{aij} (\epsilon_{ija})^2) \right].
\end{aligned}$$

We summarize these results in table 3.4 where the function  $H_{ija}$  can be found in (3.36) and the functions  $f_i$  are the metric coefficients found in (3.33):

$R_{00}$	$-\sum_{i=1}^3 \left[ f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_i))' + \frac{1}{r} \right) \right) + (f_0^{-1})^2 \left( (\log(f_i))' + \frac{1}{r} \right)^2 \right]$
$R_{jk}$ for $j \neq k$	0
$R_{jj}$	$ \begin{aligned} & -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_j))' + \frac{1}{r} \right) \right) \\ & - (f_0^{-1})^2 \left( \frac{1}{r} + (\log(f_j))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\ & - \sum_{i,a=1}^3 \left[ \frac{1}{r^2} (H_{jia} H_{ija} (\epsilon_{ija})^2) + \frac{2}{r^2} (f_i f_j)^{-1} (f_a H_{aij} (\epsilon_{ija})^2) \right] \end{aligned} $

Table 3.4: Ricci curvature components

## 3.6 A hyperkähler structure

It was shown in the introduction using (1.5) that when the matrix  $Q$  that defines the hypersymplectic structure on  $X^4$  is constant there exists a hyperkähler structure on  $X^4$

inducing the same metric as the hypersymplectic structure up to some constant factor, thus the metric is Ricci flat. In this section, we verify explicitly for our particular set up that when  $Q$  as defined in (3.22) is constant, our metric is Ricci flat but not necessarily flat. Recall  $Q$  from (3.22). We claim that when  $Q$  is constant, there are positive real constants  $c_2, c_3$  for which

$$A_2(2A_2 + rA'_2) = c_2 \quad \text{and} \quad A_3(2A_3 + rA'_3) = c_3. \quad (3.37)$$

To begin let  $\alpha_i$  be positive real constants so that  $Q_{ii} = \alpha_i$ . Using (3.22) we obtain

$$\frac{(A_2(2A_2 + rA'_2))^2}{2A_3(2A_3 + rA'_3)} = \alpha_2^3,$$

and thus

$$A_3(2A_3 + rA'_3) = \frac{1}{2\alpha_2^3} (A_2(2A_2 + rA'_2))^2. \quad (3.38)$$

Similarly,

$$A_2(2A_2 + rA'_2) = \frac{1}{2\alpha_3^3} (A_3(2A_3 + rA'_3))^2. \quad (3.39)$$

Lastly,

$$\begin{aligned} \frac{4}{\alpha_1^3} &= A_2(2A_2 + rA'_2) A_3(2A_3 + rA'_3) \\ &= A_2(2A_2 + rA'_2) \frac{1}{2\alpha_2^3} (A_2(2A_2 + rA'_2))^2 \end{aligned}$$

and thus

$$(A_2(2A_2 + rA'_2))^3 = \frac{8\alpha_2^3}{\alpha_1^3}.$$

In particular,  $c_2 = 2\alpha_2/\alpha_1$  and identity (3.37) for  $A_2$  holds. We can repeat the same computation only now substituting (3.39) in and we obtain the identity in (3.37) for  $A_3$ . We will need these identities quite often in this section.

To begin we use the identities in (3.37) and substitute them back into the equations for the metric coefficients  $f_i$  for  $i \in \{0, 1, 2, 3\}$ . These functions can be found in (3.33) and we obtain the following

$$f_0 = \frac{(c_2 c_3)^{\frac{1}{3}}}{4^{\frac{1}{3}} \sqrt{A_2 A_3}}, \quad f_1 = \frac{2^{\frac{1}{3}} \sqrt{A_2 A_3}}{(c_2 c_3)^{\frac{1}{6}}}, \quad f_2 = \sqrt{\frac{A_3}{A_2}} \left( \frac{c_2^2}{2c_3} \right)^{\frac{1}{6}}, \quad f_3 = \sqrt{\frac{A_2}{A_3}} \left( \frac{c_3^2}{2c_2} \right)^{\frac{1}{6}}. \quad (3.40)$$

We show that with these coefficients the metric is Ricci-flat. We do this computation component by component, we first start with  $R_{00}$

Case 1:  $= R_{00}$

$$\begin{aligned}
R_{00} &= R_{i00i} \\
&= \sum_{i=1}^3 f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( f_i (f_i^{-1})' - \frac{1}{r} \right) \right) - \sum_{i=1}^3 (f_0^{-1})^2 \left( f_i (f_i^{-1})' - \frac{1}{r} \right)^2 \\
&= - \sum_{i=1}^3 f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_i))' + \frac{1}{r} \right) \right) - \sum_{i=1}^3 (f_0^{-1})^2 \left( (\log(f_i))' + \frac{1}{r} \right)^2.
\end{aligned}$$

When we substitute in the functions  $f_i$  with the constants in them, we obtain

$$\begin{aligned}
R_{00} &= \frac{-2^{\frac{1}{3}}}{r (c_2 c_3)^{\frac{2}{3}}} (r A_3 A_2'' + r A_2 A_3'' + A_2' (2r A_3' + 3A_3) + 3A_2 A_3') \\
&\quad + \frac{2^{\frac{1}{3}}}{r A_2 (c_2 c_3)^{\frac{2}{3}}} \left( r A_2 A_3 A_2'' - r A_2^2 A_3'' - r A_3 (A_2')^2 + A_2 A_2' (r A_3' + A_3) - 3A_2^2 A_3' \right) \\
&\quad + \frac{2^{\frac{1}{3}}}{r A_3 (c_2 c_3)^{\frac{2}{3}}} \left( r A_3 A_2 A_3'' - r A_3^2 A_2'' - r A_2 (A_3')^2 + A_3 A_3' (r A_2' + A_2) - 3A_3^2 A_2' \right) \\
&= \frac{-2^{\frac{1}{3}}}{r (c_2 c_3)^{\frac{2}{3}} A_2 A_3} \left( r A_3^2 A_2 A_2'' + r A_2^2 A_3 A_3'' + r A_3^2 (A_2')^2 + r A_2^2 (A_3')^2 \right. \\
&\quad \left. + 5A_3^2 A_2 A_2' + 5A_2^2 A_3 A_3' \right).
\end{aligned}$$

Since  $Q$  is a constant matrix, we will use the expressions (3.37). First we differentiate both expressions in (3.37) to obtain

$$5A_i A_i' = -r (A_i')^2 - r A_i A_i'' \quad (3.41)$$

for  $i = 2, 3$ . We then substitute this into the expression we have above for  $R_{00}$  and we obtain the following:

$$\begin{aligned}
R_{00} &= \frac{-2^{\frac{1}{3}}}{r (c_2 c_3)^{\frac{2}{3}} A_2 A_3} \left[ r A_3^2 A_2 A_2'' + r A_2^2 A_3 A_3'' + r A_3^2 (A_2')^2 + r A_2^2 (A_3')^2 \right. \\
&\quad \left. + A_3^2 \left( -r (A_2')^2 - r A_2 A_2'' \right) + A_2^2 \left( -r (A_3')^2 - r A_3 A_3'' \right) \right] \\
&= 0.
\end{aligned}$$

Case 1: =  $R_{11}$

$$\begin{aligned}
R_{11} &= R_{1001} + R_{1221} + R_{1331} \\
&= -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_1))' + \frac{1}{r} \right) \right) \\
&\quad - \left( \frac{1}{f_0} \right)^2 \left( \frac{1}{r} + (\log(f_1))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\
&\quad - \frac{1}{r^2} \sum_{i,a=1}^3 H_{1ia} H_{i1a} (\epsilon_{1ia})^2 - \frac{2}{r^2} \sum_{i,a=1}^2 \frac{f_a}{f_1 f_i} H_{ai1} (\epsilon_{1ia})^2 \\
&= -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_1))' + \frac{1}{r} \right) \right) \\
&\quad - \left( \frac{1}{f_0} \right)^2 \left( \frac{1}{r} + (\log(f_1))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\
&\quad - \left( \frac{1}{r^2} \right) (H_{123} H_{213} (\epsilon_{123})^2 + H_{132} H_{312} (\epsilon_{132})^2) \\
&\quad - \left( \frac{2}{r^2} \right) \left( \frac{f_3}{f_1 f_2} H_{321} (\epsilon_{123})^2 + \frac{f_2}{f_1 f_3} H_{231} (\epsilon_{132})^2 \right).
\end{aligned}$$

Recall from (3.36),  $H_{ijk} = \frac{f_i}{f_j f_k} - \frac{f_j}{f_i f_k} - \frac{f_k}{f_i f_j}$  and  $H_{ijk} = H_{ikj}$ . In the following step we input the functions  $H_{ijk}$  and the equations for  $f_i$  from (3.40). We obtain

$$\begin{aligned}
R_{11} &= \frac{-2^{\frac{1}{3}}}{r (c_2 c_3)^{\frac{2}{3}}} (r A_3 A_2'' + r A_2 A_3'' + A_2' (2r A_3' + 3A_3) + 3A_2 A_3') \\
&\quad - \frac{16^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}}} (r A_2 A_3' + r A_3 A_2' + 2A_2 A_3) \\
&\quad + \frac{2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} (4A_3^4 A_2^4 - c_2^2 A_3^4 + 2c_2 c_3 A_3^2 A_2^2 - c_3^2 A_2^4) \\
&= \frac{-2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( r^2 A_3^4 A_2^3 A_2'' + r^2 A_2^4 A_3^3 A_3'' + 2r^2 A_3^3 A_2^3 A_2' A_3' + 5r A_3^4 A_2^3 A_2' \right. \\
&\quad \left. + 5r A_2^4 A_3^3 A_3' + (c_2 A_3^2 - c_3 A_2^2)^2 \right).
\end{aligned}$$

We replace all occurrences of  $c_2$  and  $c_3$  in the numerator with the formulas in (3.37) to

obtain

$$R_{11} = \frac{-2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( r^2 A_3^4 A_2^3 A_2'' + r^2 A_2^4 A_3^3 A_3'' + 5r A_3^4 A_2^3 A_2' + 5r A_2^4 A_3^3 A_3' \right. \\ \left. + r^2 A_3^4 A_2^2 (A_2')^2 + r^2 A_2^4 A_3^2 (A_3')^2 \right).$$

Now we rearrange the equation we had in (3.41) to obtain

$$r (A_i')^2 = -5A_i A_i' - r A_i A_i'', \quad (3.42)$$

and we substitute this into our expression for  $R_{11}$  to obtain

$$R_{11} = \frac{-2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left[ r^2 A_3^4 A_2^3 A_2'' + r^2 A_2^4 A_3^3 A_3'' + 5r A_3^4 A_2^3 A_2' + 5r A_2^4 A_3^3 A_3' \right. \\ \left. - r A_3^4 A_2^2 (5A_2 A_2' + r A_2 A_2'') - r A_2^4 A_3^2 (5A_3 A_3' + r A_3 A_3'') \right] \\ = 0.$$

The computations for the remaining cases  $R_{22}$  and  $R_{33}$  are fairly similar to the case for  $R_{11}$  so we will not add as much detail.

Case 3:  $R_{22}$

$$R_{22} = R_{2002} + R_{2112} + R_{2332} \\ = -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_2))' + \frac{1}{r} \right) \right) \\ - \left( \frac{1}{f_0} \right)^2 \left( \frac{1}{r} + (\log(f_2))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\ - \frac{1}{r^2} (H_{123} H_{213} (\epsilon_{123})^2 + H_{231} H_{321} (\epsilon_{231})^2) \\ - \frac{2}{r^2} \left( \frac{f_3}{f_1 f_2} H_{321} (\epsilon_{123})^2 + \frac{f_1}{f_2 f_3} H_{132} (\epsilon_{231})^2 \right).$$

We substitute the expression for  $H_{ijk}$  and the functions  $f_i$  from (3.40).

$$R_{22} = \frac{2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( r^2 A_3^4 A_2^3 A_2'' - r^2 A_2^4 A_3^3 A_3'' + r A_3^4 A_2^3 A_2' - 5r A_2^4 A_3^3 A_3' - 8A_3^4 A_2^4 \right. \\ \left. + 8c_3 A_2^4 A_3^2 - 4c_3^2 A_2^4 + 4c_2^2 A_3^4 \right).$$

We substitute the expressions we have for  $c_i$  from (3.37) and and we obtain

$$R_{22} = \frac{2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( r^2 A_3^4 A_2^3 A_2'' - r^2 A_2^4 A_3^3 A_3'' + 5r A_3^4 A_2^3 A_2' - 5r A_2^4 A_3^3 A_3' \right. \\ \left. - r^2 A_2^4 A_3^2 (A_3')^2 + r^2 A_3^4 A_2^2 (A_2')^2 \right).$$

Next we do the same thing we did for the  $R_{11}$  case and substitute the expression for  $r(A_i)^2$  from (3.42). After this substitution all the terms cancel and we are left with  $R_{22} = 0$ .

Case 4:  $R_{33}$

$$R_{33} = R_{3003} + R_{3113} + R_{3223} \\ = -f_0^{-1} \frac{d}{dr} \left( f_0^{-1} \left( (\log(f_3))' + \frac{1}{r} \right) \right) \\ - \left( \frac{1}{f_0} \right)^2 \left( \frac{1}{r} + (\log(f_3))' \right) \left( \frac{3}{r} + (\log(f_1 f_2 f_3))' \right) \\ - \frac{1}{r^2} (H_{132} H_{312} + H_{231} H_{321}) - \frac{2}{r^2} \left( \frac{f_2}{f_1 f_3} H_{231} + \frac{f_1}{f_2 f_3} H_{132} \right).$$

We substitute the expression for  $H_{ijk}$  and the functions  $f_i$  from (3.40).

$$R_{33} = \frac{2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( -r^2 A_3^4 A_2^3 A_2'' + r^2 A_2^4 A_3^3 A_3'' - 5r A_3^4 A_2^3 A_2' + r A_2^4 A_3^3 A_3' \right. \\ \left. - 8A_3^4 A_2^4 + 8c_2 A_3^4 A_2^2 - 4c_2^2 A_3^4 + 4c_3^2 A_2^4 \right).$$

We substitute the expressions we have for  $c_i$  from (3.37) and obtain

$$R_{33} = \frac{2^{\frac{1}{3}}}{r^2 (c_2 c_3)^{\frac{2}{3}} (A_2 A_3)^3} \left( -r^2 A_3^4 A_2^3 A_2'' + r^2 A_2^4 A_3^3 A_3'' - 5r A_3^4 A_2^3 A_2' + 5r A_2^4 A_3^3 A_3' \right. \\ \left. - r^2 A_3^4 A_2^2 (A_2')^2 + r^2 A_2^4 A_3^2 (A_3')^2 \right).$$

Next we do the same thing we did for the  $R_{11}$  and  $R_{22}$  case and substitute the expression for  $r(A_i)^2$  from (3.42). After this substitution all the terms cancel and we are left with  $R_{33} = 0$ .

Now we show that the structure is not necessarily flat, meaning there is a component

of the Riemann curvature tensor that does not vanish. Using the computation we did in section 3.4 and the metric coefficients (3.40), we compute some of the Riemann curvature tensors to be:

$$R_{1001} = \frac{-2^{\frac{1}{3}}}{r(c_2c_3)^{\frac{2}{3}}} (rA_3A_2'' + rA_2A_3'' + A_2'(2rA_3' + 3A_3) + 3A_2A_3'). \quad (3.43)$$

Let us examine  $R_{1001}$  more closely, note that there are of course specific choices we can make for  $A_i$  so that this does vanish. In particular we note that if  $A_i$  are constant then this gives  $R_{1001} = 0$ . We want to look for solutions that do not vanish. If we solve equations (3.37) then we obtain

$$A_i = \pm \frac{\sqrt{2r^4c_i + 4b_i}}{2r^2}$$

where  $b_i$  are the constants of integration. Then we can substitute this solution into (3.43) to obtain

$$R_{1001} = \frac{16^{\frac{1}{3}} (2b_2c_2c_3^2r^{12} + 2b_3c_2^2c_3r^{12} + 24b_2b_3c_2c_3r^8 + 24b_2^2b_3c_3r^4 + 24b_2b_3^2c_2r^4 + 32b_2^2b_3^2)}{r^6(c_2c_3)^{2/3}((r^4c_2 + 2b_2)(r^4c_3 + 2b_3))^{3/2}}. \quad (3.44)$$

We know that  $c_i > 0$ , but we can see in the above that if both  $b_i = 0$  then  $A_i$  is constant and all of the metric coefficients (3.33) are the same and thus  $g_{\omega}$  is constant multiple of the Euclidean metric. If at least one of  $b_i$  is non-zero then the expression (3.44) is non-zero. Taking both  $b_i$ 's to be positive assures that  $A_i$  is defined on all of  $\mathbb{R}^4 \setminus \{0\}$ . Thus choosing  $b_i > 0$  gives us a class of hyperkähler structures on  $\mathbb{R}^4 \setminus \{0\}$ . Lastly we point out that this class of hyperkähler metrics looks very similar to a class of metrics on the cone over  $\mathbb{RP}^3$ , that is  $(0, \infty) \times \mathbb{RP}^3$ , given in the paper [FLS17]. If we pulled back our metrics from  $\mathbb{R}^4 \setminus \{0\}$  to  $(0, \infty) \times \mathbb{RP}^3$  it is likely that the metrics in [FLS17] are a subset of the class of hyperkähler metrics we have found.

# Chapter 4

## The soliton equation

In this chapter we discuss some solitons of the hypersymplectic flow. In [4.1](#) we introduce a special class of solutions to the hypersymplectic flow called self similar solutions. Self similar solutions are solutions that evolve by a scaling factor and a diffeomorphism. They are essentially equivalent to solitons of the flow. We will show how solitons give rise to self similar solutions and vice versa. In this same section we compute the general soliton equation. This gives rise to three different equations (one for each symplectic form). In [section 4.2](#) we compute explicitly the soliton equations for each  $\omega_i$ . This tells that all but one of the vector field coefficients for the soliton triple vanishes. In [section 4.3](#) we equate three different expressions for our vector field coefficient and find two differential equations whose solutions would give us the soliton vector field. In [section 4.4](#) we find explicit solitons for the hypersymplectic flow when the matrix  $Q$  from the hypersymplectic structure is constant. We also see that when the constant  $\lambda$  in the soliton triple vanishes, the matrix  $Q$  must be constant. In this case we can explicitly solve for the functions  $A_k$  that comprise the metric. Since  $Q$  is constant the hypersymplectic triple can be transformed into a hyperkähler triple whose metric differs from the hypersymplectic metric by a constant. Thus concluding in this particular setting, steady solitons give rise to hypersymplectic structures whose metrics are a constant multiple of a hyperkähler metric.

## 4.1 General soliton equation

In this section we let  $N$  be a smooth manifold of dimension 4. The hypersymplectic flow is given as a triple of partial differential equations defined as

$$\frac{d}{dt}\underline{\omega}(t) = d(Qd^*(Q^{-1}\underline{\omega})). \quad (4.1)$$

Let  $Q^{ij}$  denote the components of  $Q^{-1}$ . We have the following equation for each component  $\omega_i$  of  $\underline{\omega}$ :

$$\frac{d}{dt}\omega_i(t) = d(Q_{ik}d^*(Q^{kl}\omega_l)).$$

We begin by introducing the notion of a self similar solution, this definition is given in an analogous way to self similar solutions for other geometric flows such as the  $G_2$ - Laplacian flow and the isometric flow [FR20],[DGK19].

**Definition 4.1.1.** *Let  $N$  be a smooth manifold of dimensions 4. Let  $\underline{\omega}(t)$   $t \in [0, T)$  be a solution of the hypersymplectic flow (4.1). We say that  $\underline{\omega}(t)$  is self similar if there exists a positive function  $a(t)$  with  $a(0) = 1$ , a hypersymplectic structure  $\underline{\omega}(0)$  on  $N$  and a family of diffeomorphisms  $F_t : N \rightarrow N$  with  $F_0 = Id_N$  such that*

$$\underline{\omega}(t) = a(t)^2 F_t^* \underline{\omega}(0). \quad (4.2)$$

**Remark 4.1.2.** *The exponent on  $a(t)$  is not necessary but is there to avoid rational factors later in the computations.*

**Definition 4.1.3.** *A soliton of the hypersymplectic flow (4.1) is a triple  $(\underline{\omega}(0), \lambda, X)$  where  $\underline{\omega}(0)$  is a hypersymplectic structure on  $N$ ,  $X$  is vector field on  $N$ , and  $\lambda$  is a constant satisfying*

$$\lambda \underline{\omega}(0) + \mathcal{L}_X \underline{\omega}(0) = d(Qd^*(Q^{-1}\underline{\omega}(0))). \quad (4.3)$$

We show that self similar solutions give rise to solitons and solitons give rise to self similar solutions. Indeed the formula (4.3) was constructed by taking the time derivative of a self similar solution. Let us show that. Let  $\underline{\omega}(t)$  be some self similar solution as in definition 4.1.1. That is  $\underline{\omega}(t) = (a(t))^2 F_t^* \underline{\omega}(0)$  where  $a(0) = 1$  and  $F_t \in \text{Diff}(M)$  with  $F_0 = Id_M$  is a solution of the hypersymplectic flow. First we take the time derivative of  $\underline{\omega}(t)$ . Let  $X$  be the infinitesimal generator of  $F_t$ . We obtain

$$\frac{\partial \underline{\omega}(t)}{\partial t} = 2a(t)a'(t)F_t^*(\underline{\omega}(0)) + (a(t))^2 F_t^*(\mathcal{L}_X \underline{\omega}(0)).$$

Thus,

$$\left. \frac{\partial \underline{\omega}(t)}{\partial t} \right|_{t=0} = 2a'(0)\underline{\omega}(0) + \mathcal{L}_X \underline{\omega}(0).$$

Let  $\lambda = 2a'(0)$ . Since  $\underline{\omega}(0)$  is a solution of the hypersymplectic flow then  $(X, \lambda, \underline{\omega}(0))$  satisfies

$$\lambda \underline{\omega}(0) + \mathcal{L}_X \underline{\omega}(0) = \left. \frac{\partial \underline{\omega}(t)}{\partial t} \right|_{t=0} = d(Qd^*(Q^{-1}\underline{\omega}(0)))$$

making  $(X, \lambda, \underline{\omega}(0))$  a soliton. Now suppose we have a soliton, a triple  $(X, \lambda, \underline{\omega}(0))$  that satisfies equation (4.3). Define  $\underline{\omega}(t) = e^{\lambda t} F_t^*(\underline{\omega}(0))$  where  $F_t$  is the flow of the vector field  $X$ , that is  $\left. \frac{\partial F_t}{\partial t} \right|_p = X(F_t(p))$ . We want to show that  $\underline{\omega}(t)$  is indeed a solution to the hypersymplectic flow. We use the fact that  $F_t^*$  commutes with  $d$  and (4.3), to obtain

$$\begin{aligned} \frac{\partial \underline{\omega}(t)}{\partial t} &= e^{\lambda t} F_t^*(\lambda \underline{\omega}(0)) + e^{\lambda t} F_t^*(\mathcal{L}_X \underline{\omega}(0)) \\ &= e^{\lambda t} F_t^*(\lambda \underline{\omega}(0) + \mathcal{L}_X \underline{\omega}(0)) \\ &= e^{\lambda t} F_t^*(d(Qd^*(Q^{-1}\underline{\omega}(0)))) \\ &= d(F_t^*(Q)d^{*F_t^*\underline{\omega}}(F_t^*(Q^{-1})e^{\lambda t} F_t^*\underline{\omega}(0))) \\ &= d(F_t^*(Q)d^{*F_t^*\underline{\omega}}(F_t^*(Q^{-1})\underline{\omega}(t))). \end{aligned}$$

Above  $F_t^*(Q)$  is the matrix that corresponds to the positive triple  $\underline{\omega}(t) = e^{\lambda t} F_t^*(\underline{\omega}(0))$  and  $*_{F_t^*\underline{\omega}}$  is the Hodge star induced by the pullback triple  $F_t^*\underline{\omega}$ . Thus we can conclude that  $\underline{\omega}(t) = e^{\lambda t} F_t^*(\underline{\omega}(0))$  is a solution to the hypersymplectic flow arising from the soliton triple. So we see how solitons give rise to solutions of the flow.

Now we expand the soliton equation (4.3) so that we are able to say something about its potential solutions. When we expand the soliton equation for each component  $\omega_i$  we get

$$\lambda \omega_i(0) + \mathcal{L}_X \omega_i(0) = d(Q_{ik}d^*(Q^{kl}\omega_l)). \quad (4.4)$$

Recall from (3.20) that  $Q$  is a diagonal matrix. Thus in this case the equation becomes

$$\lambda \omega_i + \mathcal{L}_X \omega_i = d(Q_{ii}d^*(Q^{ii}\omega_i)) \quad (4.5)$$

where  $\omega_i$  is the  $i$ th component of  $\underline{\omega}(0)$ . Note there is no sum in the formula above. We can further expand the left hand side of this equation. Since we know that the entries of the matrix  $Q$  are just functions of  $r$ , we can also use the Hodge star  $*_{\underline{\omega}}$  that comes from

the metric  $g_{\underline{\omega}}$  in the orthonormal frame  $\tilde{E}^i$ . We also use the fact that  $d^* = (-1)^{nk+n+1}$  on  $k$ -forms, and  $n = 4$  is the dimension of our manifold. Thus the previous equation becomes

$$\begin{aligned}
\lambda\omega_i + \mathcal{L}_X\omega_i &= -d(Q_{ii} *_{\underline{\omega}} d *_{\underline{\omega}} (Q^{ii}\omega_i)) \\
&= -d(Q_{ii} *_{\underline{\omega}} d(Q^{ii}\omega_i)) \\
&= -d(Q_{ii} *_{\underline{\omega}} (d(Q^{ii}) \wedge \omega_i)) \\
&= -d(Q_{ii} *_{\underline{\omega}} ((Q^{ii})' E^0 \wedge \omega_i)) \\
&= -d\left(Q_{ii}(Q^{ii})' \frac{1}{f_0} *_{\underline{\omega}} (\tilde{E}^0 \wedge \omega_i)\right)
\end{aligned}$$

where we used the fact that  $\omega_i$  is closed and self dual with respect to the Hodge star  $*_{\underline{\omega}}$  that comes from the metric  $g_{\underline{\omega}}$ . If we further expand the exterior derivative in the last equation above we obtain

$$\begin{aligned}
&\lambda\omega_i + \mathcal{L}_X\omega_i \\
&= -\left(\left(Q_{ii}(Q^{ii})' \frac{1}{f_0}\right)' \frac{1}{f_0} \tilde{E}^0 \wedge *_{\underline{\omega}} (\tilde{E}^0 \wedge \omega_i) + Q_{ii}(Q^{ii})' \frac{1}{f_0} d(*_{\underline{\omega}} (\tilde{E}^0 \wedge \omega_i))\right). \quad (4.6)
\end{aligned}$$

We note that each symplectic form  $\omega_k$  yields a soliton equation. The symplectic forms  $\omega_k$  are given by

$$\omega_1 = P_1 (\tilde{E}^{01} + \tilde{E}^{23}), \quad \omega_2 = P_2 (\tilde{E}^{02} + \tilde{E}^{31}), \quad \omega_3 = P_3 (\tilde{E}^{03} + \tilde{E}^{12}) \quad (4.7)$$

where  $P_i$  are functions of  $r$  that can be found in (3.35). We state them again below:

$$\begin{aligned}
P_1 &= \frac{1}{f_0 f_1} = \frac{1}{f_2 f_3} = \left(\frac{4}{A_2 A_3 (r A_2' + 2A_2)(r A_3' + 2A_3)}\right)^{\frac{1}{6}}, \\
P_2 &= \frac{A_2}{f_1 f_3} = \frac{2A_2 + r A_2'}{2f_0 f_2} = \left(\frac{A_2^2 (r A_2' + 2A_2)^2}{2A_3 (r A_3' + 2A_3)}\right)^{\frac{1}{6}}, \\
P_3 &= \frac{A_3}{f_1 f_2} = \frac{2A_3 + r A_3'}{2f_0 f_3} = \left(\frac{A_3^2 (r A_3' + 2A_3)^2}{2A_2 (r A_2' + 2A_2)}\right)^{\frac{1}{6}}.
\end{aligned}$$

We will compute the left hand side of (4.6). We note that the derivation of  $d\tilde{E}^i$  is found in section A.1. First we note the following computations for each  $\omega_i$ . Since

$$*_{\underline{\omega}} (\tilde{E}^0 \wedge \omega_i) = -P_i \tilde{E}^i,$$

we have

$$\begin{aligned} d\left(*_{\omega}\left(\tilde{E}^0 \wedge \omega_i\right)\right) &= -d\left(P_i\right)\tilde{E}^i - P_i d\tilde{E}^i \\ &= -\frac{1}{f_0} P_i' \tilde{E}^{0i} - P_i d\tilde{E}^i. \end{aligned} \quad (4.8)$$

## 4.2 Soliton equations for $\omega_1$ , $\omega_2$ , and $\omega_3$

In this section we compute the specific soliton equations (4.5) for  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . First we recall the matrix  $Q$  for this hypersymplectic structure:

$$Q = \begin{bmatrix} \frac{1}{\left(A_2 A_3 (2A_2 + rA_2') (2A_3 + rA_3')\right)^{\frac{1}{3}}} & 0 & 0 \\ 0 & \frac{\left(A_2 (2A_2 + rA_2')\right)^{\frac{2}{3}}}{\left(2A_3 (2A_3 + rA_3')\right)^{\frac{1}{3}}} & 0 \\ 0 & 0 & \frac{\left(A_3 (2A_3 + rA_3')\right)^{\frac{2}{3}}}{\left(2A_2 (2A_2 + rA_2')\right)^{\frac{1}{3}}} \end{bmatrix}.$$

In the previous section we expanded the soliton equations as (4.6). We start with the right hand side of (4.6) for  $\omega_1$  and we use the identity for  $d\tilde{E}^i$  that is found in A.1 and (4.8) to obtain

$$\begin{aligned} d\left(Q_{11} d^* \left(Q^{11} \omega_1\right)\right) &= \left(Q_{11} \left(Q^{11}\right)' \frac{1}{f_0}\right)' \frac{P_1}{f_0} \tilde{E}^{01} + \left(Q_{11} \left(Q^{11}\right)' \frac{1}{f_0}\right) \left(\frac{P_1'}{f_0} \tilde{E}^{01} + P_1 d\tilde{E}^1\right) \\ &= \left[\left(Q_{11} \left(Q^{11}\right)' \frac{1}{f_0}\right)' \frac{P_1}{f_0} + Q_{11} \left(Q^{11}\right)' \frac{1}{f_0} \left(\frac{P_1'}{f_0} + \frac{P_1 (f_1)'}{f_0 f_1} + \frac{P_1}{r f_0}\right)\right] \tilde{E}^{01} \\ &\quad + \left[\frac{2Q_{11} \left(Q^{11}\right)' P_1 f_1}{r f_0 f_2 f_3}\right] \tilde{E}^{23}. \end{aligned} \quad (4.9)$$

For the left hand side of (4.6) we assume that the vector field  $X$  for the soliton equation is  $X = F^i \tilde{E}_i$  where  $F^i$  are functions that depend only on  $r$ . The left hand side of (4.6) is then

$$\lambda \omega_1 + \mathcal{L}_X \omega_1 = \lambda P_1 \left(\tilde{E}^{01} + \tilde{E}^{23}\right) + d\left(F^i \tilde{E}_i \lrcorner P_1 \left(\tilde{E}^{01} + \tilde{E}^{23}\right)\right)$$

where we used the fact that  $d\omega_1 = 0$ . We recall the metric coefficients from (3.33) and the formulas for  $d\tilde{E}^i$  from section A.1. Then the above becomes

$$\begin{aligned}
\lambda\omega_1 + \mathcal{L}_X\omega_1 &= \lambda P_1 \tilde{E}^{01} + \lambda P_1 \tilde{E}^{23} + \frac{(P_1 F^0)'}{f_0} \tilde{E}^{01} + \frac{(P_1 F^2)'}{f_0} \tilde{E}^{03} - \frac{(P_1 F^3)'}{f_0} \tilde{E}^{02} \\
&\quad + P_1 F^0 d\tilde{E}^1 + P_1 F^2 d\tilde{E}^3 - P_1 F^3 d\tilde{E}^2 \\
&= \lambda P_1 \tilde{E}^{01} + \lambda P_1 \tilde{E}^{23} + \frac{(P_1 F^0)'}{f_0} \tilde{E}^{01} + \frac{(P_1 F^2)'}{f_0} \tilde{E}^{03} - \frac{(P_1 F^3)'}{f_0} \tilde{E}^{02} \\
&\quad + P_1 F^0 \left( \left[ \frac{(f_1)'}{f_1 f_0} + \frac{1}{r f_0} \right] \tilde{E}^{01} + \left[ \frac{2f_1}{r f_2 f_3} \right] \tilde{E}^{23} \right) \\
&\quad + P_1 F^2 \left( \left[ \frac{(f_3)'}{f_3 f_0} + \frac{1}{r f_0} \right] \tilde{E}^{03} + \left[ \frac{2f_3}{r f_1 f_2} \right] \tilde{E}^{12} \right) \\
&\quad - P_1 F^3 \left( \left[ \frac{(f_2)'}{f_2 f_0} + \frac{1}{r f_0} \right] \tilde{E}^{02} + \left[ \frac{2f_2}{r f_1 f_3} \right] \tilde{E}^{31} \right) \\
&= \left[ \lambda P_1 + \frac{(P_1 F^0)'}{f_0} + \frac{P_1 F^0 (f_1)'}{f_1 f_0} + \frac{P_1 F^0}{r f_0} \right] \tilde{E}^{01} \\
&\quad - \left[ \frac{(P_1 F^3)'}{f_0} + \frac{P_1 F^3 (f_2)'}{f_2 f_0} + \frac{P_1 F^3}{r f_0} \right] \tilde{E}^{02} \\
&\quad + \left[ \frac{(P_1 F^2)'}{f_0} + \frac{P_1 F^2 (f_3)'}{f_3 f_0} + \frac{P_1 F^2}{r f_0} \right] \tilde{E}^{03} + \left[ \lambda P_1 + \frac{2P_1 F^0 f_1}{r f_2 f_3} \right] \tilde{E}^{23} \\
&\quad - \left[ \frac{2P_1 F^3 f_2}{r f_1 f_3} \right] \tilde{E}^{31} + \left[ \frac{2P_1 F^2 f_3}{r f_1 f_2} \right] \tilde{E}^{12}.
\end{aligned} \tag{4.10}$$

We equate the computations we did above, using the fact that  $P_i \neq 0$ ,  $f_i \neq 0$  for all  $i$  we obtain the following information for the vector field  $X = F^i \tilde{E}^i$ :

$$F^2 = F^3 = 0, \tag{4.11}$$

$$\begin{aligned}
F^0 &= Q_{11} (Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2f_1}, \\
(F^0)' &= \left( Q_{11} (Q^{11})' \frac{1}{f_0} \right)' - \frac{\lambda f_2 f_3}{2f_1} - \frac{\lambda r}{2} \left( \frac{f_2 f_3}{f_1} \right)',
\end{aligned} \tag{4.12}$$

$$\begin{aligned} \left( Q_{11} (Q^{11})' \frac{1}{f_0} \right)' \frac{P_1}{f_0} + Q_{11} (Q^{11})' \frac{1}{f_0} \left( \frac{P_1'}{f_0} + \frac{P_1 (f_1)'}{f_0 f_1} + \frac{P_1}{r f_0} \right) &= \lambda P_1 + \frac{P_1' F^0}{f_0} + \frac{P_1 (F^0)'}{f_0} \\ &+ \frac{P_1 F^0 (f_1)'}{f_1 f_0} + \frac{P_1 F^0}{r f_0}. \end{aligned} \quad (4.13)$$

We substitute (4.12) into (4.13) and allow  $\lambda$  to be a parameter. We obtain

$$\begin{aligned} &\left( Q_{11} (Q^{11})' \frac{1}{f_0} \right)' \frac{P_1}{f_0} + Q_{11} (Q^{11})' \frac{1}{f_0} \left( \frac{(P_1)'}{f_0} + \frac{P_1 (f_1)'}{f_1 f_0} + \frac{P_1}{r f_0} \right) \\ &= \lambda P_1 + \frac{P_1}{f_0} \left[ \left( Q_{11} (Q^{11})' \frac{1}{f_0} \right)' - \frac{\lambda f_2 f_3}{2 f_1} - \frac{\lambda r}{2} \left( \frac{f_2 f_3}{f_1} \right)' \right] + \frac{(P_1)'}{f_0} \left[ Q_{11} (Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2 f_1} \right] \\ &+ \frac{P_1 (f_1)'}{f_1 f_0} \left[ Q_{11} (Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2 f_1} \right] + \frac{P_1}{r f_0} \left[ Q_{11} (Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2 f_1} \right]. \end{aligned}$$

Many of the terms cancel each other out and we are left with the equation

$$0 = \lambda \left( P_1 - \frac{r P_1}{2 f_0} \left( \frac{f_2 f_3}{f_1} \right)' - \frac{r (P_1)' f_2 f_3}{2 f_0 f_1} - \frac{r P_1 (f_1)' f_2 f_3}{2 f_1^2 f_0} - \frac{P_1 f_2 f_3}{f_0 f_1} \right). \quad (4.14)$$

Using the fact that  $P_1 = \frac{1}{f_1 f_0} = \frac{1}{f_2 f_3}$  we can significantly simplify the above into the following

$$\begin{aligned} 0 &= \lambda \left( P_1 - \frac{P_1 r}{2 f_0} (f_0)' - \frac{(P_1)' r}{2} - (f_1)' \frac{P_1 r}{2 f_1} - P_1 \right) \\ &= \lambda \left( \frac{r P_1 (f_0)'}{2 f_0} + (P_1)' \frac{r}{2} + \frac{r P_1 (f_1)'}{2 f_1} \right). \end{aligned} \quad (4.15)$$

We can replace  $P_1 = \frac{1}{f_0 f_1}$  and its derivative

$$(P_1)' = -\frac{(f_0)'}{f_0^2 f_1} - \frac{(f_1)'}{f_0 f_1^2}$$

in (4.15) to obtain an equation that is always automatically satisfied

$$0 = \lambda \left( -\frac{r (f_0)'}{2 f_0^2 f_1} - \frac{r (f_1)'}{2 f_0 f_1^2} + \frac{r (f_0)'}{2 f_0^2 f_1} + \frac{r (f_1)'}{2 f_0 f_1^2} \right).$$

Thus the soliton equation for  $\omega_1$  gives us the following

$$F^0 = Q_{11} (Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2 f_1}, \quad F^2 = F^3 = 0. \quad (4.16)$$

In order to obtain the soliton equations for  $\omega_2$  and  $\omega_3$  we can repeat this same process as we did for  $\omega_1$ . Alternatively one can cyclically permute  $1 \rightarrow 2 \rightarrow 3$  in the equations above to obtain the remaining soliton equations. Doing this we obtain from  $\omega_2$  the following new piece of information:

$$F^0 = Q_{22}(Q^{22})' \frac{1}{f_0} - \frac{\lambda r f_1 f_3}{2f_2}, \quad F^1 = 0. \quad (4.17)$$

We cyclically permute again and obtain from  $\omega_3$  the only non-trivial new information is:

$$F^0 = Q_{33}(Q^{33})' \frac{1}{f_0} - \frac{\lambda r f_1 f_2}{2f_3}. \quad (4.18)$$

Therefore  $X$  is radial.

### 4.3 Finding $F^0$

In the previous section we computed the soliton equations that came from each  $\omega_i$ . In the end we were left with three different expressions for  $F^0$ . In this section we will set them all equal to one another and ideally find the constraints  $\lambda$  and  $A_2, A_3$  that will satisfy our equations for  $F^0$ . We see in (4.16), (4.17), (4.18) that the different expressions of  $F^0$  are:

$$F^0 = Q_{11}(Q^{11})' \frac{1}{f_0} - \frac{\lambda r f_2 f_3}{2f_1} = Q_{22}(Q^{22})' \frac{1}{f_0} - \frac{\lambda r f_1 f_3}{2f_2} = Q_{33}(Q^{33})' \frac{1}{f_0} - \frac{\lambda r f_1 f_2}{2f_3}.$$

We would like to be able to determine possible  $A_2, A_3, \lambda$  that could solve the equations above. We first create two equations from these expressions for  $F^0$ . The equations are created by taking the  $F^0$  that came from  $\omega_2$  and  $\omega_3$  and subtracting them from the  $F^0$  that came from  $\omega_1$ . The result is the following two equations

$$\begin{aligned} 0 &= \frac{1}{f_0} \left( Q_{11} (Q^{11})' - Q_{22} (Q^{22})' \right) + \frac{\lambda r}{2} \left( \frac{f_1 f_3}{f_2} - \frac{f_2 f_3}{f_1} \right), \\ 0 &= \frac{1}{f_0} \left( Q_{11} (Q^{11})' - Q_{33} (Q^{33})' \right) + \frac{\lambda r}{2} \left( \frac{f_1 f_2}{f_3} - \frac{f_2 f_3}{f_1} \right). \end{aligned} \quad (4.19)$$

We can replace the metric coefficients  $f_i$  and the entries of the matrix  $Q$  using the identities (3.33) and (3.22) and doing some simplifications we obtain the following two equations

$$\begin{aligned} 0 &= \lambda r^2 A_2' (A_2 A_3 (r A_2' + 2A_2) (r A_3' + 2A_3))^{\frac{2}{3}} - 4 \cdot 2^{\frac{1}{3}} A_3 (r A_2 A_2'' + r (A_2')^2 + 5A_2 A_2'), \\ 0 &= \lambda r^2 A_3' (A_2 A_3 (r A_2' + 2A_2) (r A_3' + 2A_3))^{\frac{2}{3}} - 4 \cdot 2^{\frac{1}{3}} A_2 (r A_3 A_3'' + r (A_3')^2 + 5A_3 A_3'). \end{aligned} \quad (4.20)$$

Since these are first order differential equations we know that there exist unique solutions to these equations given an initial condition at  $r = r_0$  at least in some open neighborhood of  $r_0$ . One would then need to study if the solution extends to  $r = 0$  and  $r = \infty$ . However these equations are not linear and difficult to solve. The author searched for a compact invariant set for solutions to these differential equations. Namely looking for a function  $G(r, A_2, A_3, A'_2, A'_3)$  which is decreasing along the solutions of (4.20) and whose level sets are compact. As a result any solution of (4.20) would stay in these compact level sets for all time. The obvious functions  $G$  to try would be polynomials whose terms only have even exponents. Among some of the other functions  $G$  we tried were  $\text{Tr}(Q)$ ,  $\sigma_2(Q)$  (the second symmetric polynomial of  $Q$ ), and the sum of the squared entries of  $Q$  where  $Q$  is given by (3.22) or (3.20). However, after some attempts these computations are either not feasible or the level sets of the function  $G$  are not compact.

## 4.4 Some soliton solutions when $Q$ is constant

In the previous sections we computed the soliton equations. Recall for our vector field  $X = F^i \tilde{E}_i$  we were only left with the function  $F^0$  being non-zero. In the previous section we set the three different expressions for  $F^0$  equal to one another and obtained two equations involving  $A_2$ ,  $A_3$ , and  $\lambda$ . In this section we will examine two cases of solitons, one being the case when  $\lambda = 0$ . This case forces  $Q$  to be a constant matrix and thus we obtain explicit solutions inducing a Ricci flat metric. The second case we will examine is when  $\lambda \neq 0$ . For this case, in general it is not clear how to proceed. However, there do exist solutions when  $Q$  is a constant matrix. In particular, when  $\lambda \neq 0$  and  $Q$  is constant, this forces the metric  $g_{\omega}$  to be Euclidean. However in general we know there exist solutions to the soliton equations for  $\lambda \neq 0$  and  $Q$  not being constant, however we just cannot write them down explicitly and furthermore we do not know if they are even defined for all  $r > 0$ .

We begin with the case of  $\lambda = 0$ . Recall that for each  $\omega_i$  we had a different equation for  $F^0$ . If we substitute  $\lambda = 0$  into our equations for  $F_0$  (4.16), (4.17) and (4.18) we obtain

$$F^0 = Q_{11} (Q^{11})' \frac{1}{f_0} = Q_{22} (Q^{22})' \frac{1}{f_0} = Q_{33} (Q^{33})' \frac{1}{f_0}. \quad (4.21)$$

Furthermore, out of three equations for  $F^0$  we created two equations (4.20). We note that substituting  $\lambda = 0$  into (4.20) and solving (4.21) is equivalent, as we would repeat the same

steps. If we substitute  $\lambda = 0$  into these equations we obtain the following equations

$$\begin{aligned} 0 &= -4 \cdot 2^{\frac{1}{3}} A_3 (r A_2 A_2'' + r(A_2')^2 + 5A_2 A_2'), \\ 0 &= -4 \cdot 2^{\frac{1}{3}} A_2 (r A_3 A_3'' + r(A_3')^3 + 5A_3 A_3'). \end{aligned}$$

We can further simplify the above by noticing that the second factor can be written as the derivative of  $A_i (rA_i' + 2A_i)$  and thus we obtain

$$\begin{aligned} 0 &= A_3 (A_2 (rA_2' + 2A_2))', \\ 0 &= A_2 (A_3 (rA_3' + 2A_3))'. \end{aligned}$$

Since  $A_i \neq 0$  since that would contradict the hypothesis that  $Q$  is positive definite, we obtain

$$\begin{aligned} (A_i (rA_i' + 2A_i))' &= 0 \\ \implies A_i (rA_i' + 2A_i) &= c_i \end{aligned} \tag{4.22}$$

where  $c_i$  is some positive constant ensuring that  $Q$  is still positive definite. If we substitute (4.22) into the matrix  $Q$  we computed in (3.22) we obtain a constant matrix:

$$Q = \begin{bmatrix} \frac{4^{1/3}}{(c_2 c_3)^{1/3}} & 0 & 0 \\ 0 & \frac{c_2^{2/3}}{(2c_3)^{1/3}} & 0 \\ 0 & 0 & \frac{c_3^{2/3}}{(2c_2)^{1/3}} \end{bmatrix}.$$

As we have shown in section 3.6, when  $Q$  is constant the hypersymplectic structure defines a non-flat Ricci flat metric structure and thus can be transformed to a hyperkähler structure. In this case, we see that  $F^0 = 0$  and thus the vector field  $X = 0$ . We can return to (4.22) and solve them explicitly:

$$A_i = \pm \frac{\sqrt{2r^4 c_i + \alpha_i}}{2r^2} \tag{4.23}$$

where  $c_i > 0$  and  $\alpha_i$  are the constants of integration. If we substitute (4.22) into our formula for the hypersymplectic triple  $\underline{\omega}$  (3.35) then the triple becomes:

$$\begin{aligned}\omega_1 &= \left(\frac{4}{c_2 c_3}\right)^{\frac{1}{6}} \left(\tilde{E}^{01} + \tilde{E}^{23}\right) \\ \omega_2 &= \left(\frac{c_2^2}{2c_3}\right)^{\frac{1}{6}} \left(\tilde{E}^{02} + \tilde{E}^{31}\right) \\ \omega_3 &= \left(\frac{c_3^2}{2c_2}\right)^{\frac{1}{6}} \left(\tilde{E}^{03} + \tilde{E}^{12}\right).\end{aligned}\tag{4.24}$$

Therefore our soliton triple is  $\lambda = 0$ ,  $X = 0$  and  $\underline{\omega}$  as in (4.24). To see the Ricci curvature tensor we direct the reader to section 3.6 as we have shown in that section that when  $Q$  is constant in this particular set up the Ricci curvature vanishes. We have also shown in that same section that the Riemann curvature tensor does not necessarily vanish, thus giving us a class of non-flat Ricci flat structures.

The second case we consider is when  $\lambda \neq 0$ . As previously mentioned in general we do not have explicit solutions to the soliton equations for this case. However there exists some explicit solutions when we assume  $Q$  to be constant. We will show that this implies that the functions  $A_i$  are constant and our metric  $g_{HS}$  is then Euclidean. If we assume  $\lambda \neq 0$  and  $Q$  to be constant and we set equal the three equations for  $F^0$  (4.16), (4.17) and (4.18) we obtain

$$F^0 = -\frac{\lambda r f_2 f_3}{2f_1} = -\frac{\lambda r f_1 f_3}{2f_2} = -\frac{\lambda r f_1 f_2}{2f_3}.\tag{4.25}$$

From (4.25) and from our computations in (A.2) we obtain the following

$$f_0 = \frac{f_2 f_3}{f_1} = \frac{f_1 f_3}{f_2} = \frac{f_1 f_2}{f_3} \implies f_0 = f_1 = f_2 = f_3.$$

We equate the functions  $f_1 = f_2$  and  $f_1 = f_3$  found in (3.33) and we obtain the following

$$\begin{aligned}f_1 &= f_2 \\ \implies \frac{4A_2^2 A_3^2}{(rA_2' + 2A_2)(rA_3' + 2A_3)} &= \frac{A_3^2 (rA_2' + 2A_2)^2}{2A_2 (rA_3' + 2A_3)} \\ 8A_2^3 &= (rA_2' + 2A_2)^3.\end{aligned}$$

We can repeat this exact computation with  $f_1 = f_3$  and we obtain the following relations

$$2A_i = rA_i' + 2A_i\tag{4.26}$$

for  $i = 2$  and  $3$ . We also assumed the entries of  $Q$  to be constant, if  $c_i \in \mathbb{R}$  where  $c_i > 0$ , we set the entries  $Q_{ii}$  from (3.22) to be equal to  $c_i$  and we substitute the relation from (4.26). Setting  $Q_{22}$  equal to some positive constant gives

$$\begin{aligned} \frac{A_2^2 (rA_2' + 2A_2)}{A_3 (rA_3' + 2A_3)} &= c_2 \\ \implies A_2^2 (rA_2' + 2A_2)^2 &= c_2 A_3 (rA_3' + 2A_3). \end{aligned} \quad (4.27)$$

Setting  $Q_{33}$  equal to some positive constant gives

$$A_3^2 (rA_3' + 2A_3)^2 = c_3 A_2 (rA_2' + 2A_2). \quad (4.28)$$

Lastly we set  $Q_{11}$  equal to some positive constant and use (4.27) as well as (4.26) to obtain the following

$$\begin{aligned} A_2 A_3 (rA_2' + 2A_2) (rA_3' + 2A_3) &= c_1 \\ \implies A_2^3 (rA_2' + 2A_2)^3 &= c_1 c_2 \\ \implies 8A_2^6 &= c_1 c_2 \\ \implies A_2 &= c \end{aligned}$$

for some positive  $c \in \mathbb{R}$ . We can repeat this for  $A_3$  to indicate that our functions  $A_i$  are constant. When the functions  $A_i$  are constant and using identity (4.26) our metric coefficients (3.33) are given by  $f_i = f = (A_2 A_3)^{\frac{1}{6}}$  for  $i = 0, 1, 2, 3$ . When we substitute in the constants  $\alpha_i$  for  $A_i$  our metric becomes

$$g_{\underline{\omega}} = f (E_0^2 + E_1^2 + E_2^2 + E_3^2) = (\alpha_2 \alpha_3)^{\frac{1}{6}} (E_0^2 + E_1^2 + E_2^2 + E_3^2) \quad (4.29)$$

which is a scalar multiple of the Euclidean metric. In this case our  $F^0$  becomes

$$F^0 = -\frac{\lambda r}{2} f = -\frac{\lambda (\alpha_2 \alpha_3)^{\frac{1}{6}}}{2} r. \quad (4.30)$$

Thus when  $\lambda \neq 0$  and  $Q$  is constant, our hypersymplectic structure  $\underline{\omega}$  is Euclidean and our vector field is

$$X = -\left(\frac{\lambda (\alpha_2 \alpha_3)^{\frac{1}{6}}}{2} r\right) \tilde{E}_0 = -\left(\frac{\lambda (\alpha_2 \alpha_3)^{\frac{1}{6}}}{2} r\right) \frac{1}{f} E_0 = -\left(\frac{\lambda r}{2}\right) \frac{d}{dr}. \quad (4.31)$$

We can now evaluate the self similar solution that corresponds to this soliton. The flow of the vector field of  $X$  in (4.31) is  $F_t(x) = x e^{\frac{\lambda}{2} t}$  for  $x \in \mathbb{R} \setminus \{0\}$ . We show that the self

similar solution  $\underline{\omega}(t) = e^{\lambda t} F_t^* \underline{\omega}(0)$  for  $\underline{\omega}(0)$  given in (4.7) is the standard hyperkähler triple. We show this for  $\omega_1(0) = P_1 \left( \tilde{E}^{01} + \tilde{E}^{23} \right)$ . We first compute that

$$\begin{aligned} F_t^*(E^0) &= F_t^*(dr) = d(re^{-\frac{\lambda}{2}t}) = e^{-\frac{\lambda}{2}t} dr = e^{-\frac{\lambda}{2}t} E^0, \\ F_t^*(E^i) &= e^{-\frac{\lambda}{2}t} E^i. \end{aligned} \tag{4.32}$$

Then using (4.32) we obtain

$$\begin{aligned} F_t^*(E^0 \wedge E^1) &= (e^{-\frac{\lambda}{2}t} E^0) \wedge (e^{-\frac{\lambda}{2}t} E^1) = e^{-\lambda t} (E^0 \wedge E^1), \\ F_t^*(E^2 \wedge E^3) &= (e^{-\frac{\lambda}{2}t} E^2) \wedge (e^{-\frac{\lambda}{2}t} E^3) = e^{-\lambda t} (E^2 \wedge E^3). \end{aligned} \tag{4.33}$$

We therefore obtain

$$\begin{aligned} F_t^* \omega_1(0) &= e^{-\lambda t} P_1 \left( re^{-\frac{\lambda}{2}t} \right) \left[ f_0 \left( re^{-\frac{\lambda}{2}t} \right) f_1 \left( re^{-\frac{\lambda}{2}t} \right) E^0 \wedge E^1 \right. \\ &\quad \left. + f_2 \left( re^{-\frac{\lambda}{2}t} \right) f_3 \left( re^{-\frac{\lambda}{2}t} \right) E^2 \wedge E^3 \right]. \end{aligned}$$

Using (A.2), the self-similar solution that corresponds to this soliton is given by

$$\begin{aligned} \omega_1(t) &= P_1 \left( re^{-\frac{\lambda}{2}t} \right) \left[ f_0 \left( re^{-\frac{\lambda}{2}t} \right) f_1 \left( re^{-\frac{\lambda}{2}t} \right) E^0 \wedge E^1 + f_2 \left( re^{-\frac{\lambda}{2}t} \right) f_3 \left( re^{-\frac{\lambda}{2}t} \right) E^2 \wedge E^3 \right] \\ &= E^{01} + E^{23}. \end{aligned}$$

The remaining 2-forms can be found by cyclically permuting  $1 \rightarrow 2 \rightarrow 3$ . We conclude that the self similar solution that corresponds the soliton  $(\lambda, X, \underline{\omega}(0))$  where  $X$  is given by (4.31) and  $\underline{\omega}(0)$  is given in (4.7) is the hyperkähler triple

$$\omega_i(t) = E^{0i} + \frac{1}{2} \epsilon_{ijk} E^{jk}.$$

# Chapter 5

## Future ideas

There are many avenues one can take from here. One interesting avenue that was explored during my doctorate was removing the closedness assumption on the positive triple. Recall that the closedness condition was not required to create a  $G_2$  structure on  $\mathbb{T}^3 \times X^4$ . For the  $G_2$  form

$$\varphi = dt^{123} - dt^i \wedge \omega_i$$

we could loosen the conditions of the triple  $\underline{\omega}$  so that they need not be closed. This would still give a  $G_2$  structure it would just not be closed. From here one could apply the  $G_2$ -isometric flow given in [DGK19] to  $\varphi$  and see how this flows in time. The isometric flow is given by

$$\partial_t \varphi = \text{Div} T \lrcorner \psi$$

where  $\text{Div} T$  is the divergence of the torsion tensor of the  $G_2$  structure. Its fixed points include closed  $G_2$  structures. With that in mind, one might be able to use the isometric flow to ideally flow in time to a closed  $G_2$  structure.

Another interesting avenue to take would be to look for other cohomogeneity one solitons of the hypersymplectic flow. We only explored the action of  $\text{SO}(4)$  on  $\mathbb{R}^4$ , however one might just take an arbitrary cohomogeneity one action on other 4-manifolds and see how solitons in this case might look.

Lastly in the paper [FHY25] the space  $X^4$  was taken to be  $\mathbb{T}^4$  invariant under the action of  $\mathbb{T}^3 \subset \mathbb{T}^4$ . Another avenue to explore would be to look for solitons of the hypersymplectic flow with this ansatz.

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# APPENDICES

# Appendix A

## A.1 Lie bracket calculations

In this section we include the Lie bracket calculations of the frames  $E_i$  and  $\tilde{E}_i$ . Recall the frame  $E_0, E_1, E_2, E_3$  from (3.5). The frame  $, E_1, E_2, E_3$  forms a global frame for the tangent bundle for  $S^3 = \text{SU}(2) = \text{Sp}(1)$ . Let  $r$  be the radial coordinate on  $\mathbb{R}^4 \setminus \{0\}$  so that  $r$  is defined by

$$r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

We let  $i, j \in \{1, 2, 3\}$  and thus the Lie Bracket of the frame (3.6) has the following form

$$\begin{aligned} [E_0, E_i] &= \frac{-1}{r} E_i, \\ [E_i, E_j] &= \epsilon_{ijm} \frac{-2}{r} E_m. \end{aligned}$$

Then the Lie brackets of the orthonormal frame vector fields (3.34) are

$$\begin{aligned} [\tilde{E}_0, \tilde{E}_i] &= [f_0^{-1} E_0, f_i^{-1} E_i] \\ &= f_0^{-1} E_0 (f_i^{-1})' E_i - f_i^{-1} E_i (f_0^{-1})' E_0 + (f_0 f_i)^{-1} [E_0, E_i] \\ &= f_0^{-1} (f_i^{-1})' f_i f_i^{-1} E_i + (f_0 f_i)^{-1} \left( \frac{-1}{r} E_i \right) \\ &= f_0^{-1} (f_i^{-1})' f_i \tilde{E}_i - \frac{f_0^{-1}}{r} \tilde{E}_i \\ &= \left( f_0^{-1} f_i (f_i^{-1})' - \frac{f_0^{-1}}{r} \right) \tilde{E}_i, \end{aligned}$$

$$\begin{aligned}
[\tilde{E}_i, \tilde{E}_j] &= [f_i^{-1}E_i, f_j^{-1}E_j] \\
&= f_i^{-1}E_i(f_j^{-1})E_j - f_j^{-1}E_j(f_i^{-1})E_i + (f_i f_j)^{-1}[E_i, E_j] \\
&= (f_i f_j)^{-1} \left( \frac{-2\epsilon_{ijm}}{r} E_m \right) \\
&= \frac{-2}{r} \epsilon_{ijm} (f_i f_j)^{-1} f_m \tilde{E}_m.
\end{aligned}$$

Now using the formula

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

for 1-forms  $\alpha$  we can find the exterior derivative of each element in the frame. Note the following exterior derivatives that come from the  $S^3$  structure

$$dE^1 = \frac{1}{r}E^{01} + \frac{2}{r}E^{23}, \quad dE^2 = \frac{1}{r}E^{02} + \frac{2}{r}E^{31}, \quad dE^3 = \frac{1}{r}E^{03} + \frac{2}{r}E^{12},$$

and  $dE^0 = 0$  since  $E^0 = dr$ . Then we have the following for the orthonormalized frame:

$$d\tilde{E}^0 = d(f_0 E^0) = 0,$$

$$\begin{aligned}
d\tilde{E}^1 &= d(f_1 E^1) = (f_1)' E^{01} + f_1 dE^1 \\
&= (f_1)' E^{01} + f_1 \left( \frac{1}{r} E^{01} + \frac{2}{r} E^{23} \right) \\
&= \left[ \frac{(f_1)'}{f_0 f_1} + \frac{1}{r f_0} \right] \tilde{E}^{01} + \left[ \frac{2f_1}{r f_2 f_3} \right] \tilde{E}^{23},
\end{aligned}$$

$$d\tilde{E}^2 = \left[ \frac{(f_2)'}{f_0 f_2} + \frac{1}{r f_0} \right] \tilde{E}^{02} + \left[ \frac{2f_2}{r f_1 f_3} \right] \tilde{E}^{31},$$

$$d\tilde{E}^3 = \left[ \frac{(f_3)'}{f_0 f_3} + \frac{1}{r f_0} \right] \tilde{E}^{03} + \left[ \frac{2f_3}{r f_1 f_2} \right] \tilde{E}^{12}.$$

## A.2 Useful identities

In this section of the appendix we derive some useful identities that we need for our computations. This first identity is primarily used in the computation for the metric in section 3.2 but also shows up in other computations in the thesis.

**Lemma A.2.1.** *Let  $T$  and  $S$  be two  $3 \times 3$  matrices. Let  $\det(S)$  be the determinant of  $S$ , and let  $\text{adj}(S)$  be the adjugate of  $S$ . Then*

$$\begin{aligned} \sum_{i,j,k} \epsilon_{ijk} S_{ip} S_{jq} S_{kl} &= (\det(S)) \epsilon_{pql}, \\ \sum_{i,j,k} \epsilon_{ijk} S_{ip} S_{jq} T_{kl} &= \sum_m \left[ \text{adj}(S)T \right]_{ml} \epsilon_{mpq}. \end{aligned} \tag{A.1}$$

*Proof.* The first identity comes from the definition of the determinant. The identity  $\epsilon_{ijk} S_{ip} S_{jq} S_{kl}$  is skew in  $p, q$ , and  $l$  thus we can write the expression as being equal to  $\lambda \epsilon_{pql}$  for some scalar  $\lambda$ . That is,

$$\sum_{i,j,k} \epsilon_{ijk} S_{ip} S_{jq} S_{kl} = \lambda \epsilon_{pql}.$$

If we take the above and contract it with  $\epsilon_{pql}$  we obtain

$$6 \det(S) = \sum_{p,q,l} \sum_{i,j,k} \epsilon_{pql} \epsilon_{ijk} S_{ip} S_{jq} S_{kl} = \sum_{p,q,l} \lambda \epsilon_{pql} \epsilon_{pql} = 6\lambda.$$

We substitute  $\lambda = \det(S)$  and we obtain the desired result.

Let us prove the second identity in (A.1). We first take the right hand side that just has the pieces from  $S$  in it and sum over  $i, j$  but not  $k$ . We know this is then skew in  $p, q$  and there is no symmetry with  $l$ , so there exist  $X_{mk}$  such that

$$\sum_{i,j} \epsilon_{ijk} S_{ip} S_{jq} = \sum_m X_{mk} \epsilon_{mpq}$$

and we need to determine what the coefficients  $X_{mk}$  are. We take the expression above and contract it with  $\epsilon_{pql}$  to obtain the following:

$$\begin{aligned} \sum_{p,q} \epsilon_{pql} \left( \sum_{i,j} \epsilon_{ijk} S_{ip} S_{jq} \right) &= \sum_{p,q} \left( \sum_m X_{mk} \epsilon_{mpq} \right) \epsilon_{pql} \\ \sum_{p,q,i,j} \epsilon_{pql} \epsilon_{ijk} S_{ip} S_{jq} &= \sum_m X_{mk} 2\delta_{ml} \\ &= 2X_{lk}. \end{aligned}$$

Therefore we can say that  $X_{lk} = \frac{1}{2} \sum_{p,q,i,j} \epsilon_{pql} \epsilon_{ijk} S_{ip} S_{jq}$ . We would like to determine what this expression for  $X_{lk}$  is so we take this expression and we multiply both sides by  $S_{km}$  and use the first identity:

$$\begin{aligned} \sum_k X_{lk} S_{km} &= \frac{1}{2} \sum_{p,q,i,j,k} \epsilon_{pql} \epsilon_{ijk} S_{ip} S_{jq} S_{km} \\ &= \frac{1}{2} \sum_{p,q} \det(S) \epsilon_{pqm} \epsilon_{pql} \\ &= \det(S) \delta_{lm}. \end{aligned}$$

We recall that  $\text{adj}(S)$  is the unique matrix such that  $\text{adj}(S)S = \det(S)I$  thus we can conclude that

$$X_{lk} = \text{adj}(S)_{lk}.$$

Now we can put everything together to obtain

$$\sum_{i,j,k} \epsilon_{ijk} S_{ip} S_{jq} T_{kl} = \sum_{m,k} X_{mk} \epsilon_{mpq} T_{kl} = \sum_{m,k} \text{adj}(S)_{mk} T_{kl} \epsilon_{mpq} = \sum_m \left[ \text{adj}(S)T \right]_{ml} \epsilon_{mpq}.$$

Thus we have obtained our desired identity.  $\square$

Next we derive some identities that come from the coefficients of the symplectic forms  $\omega_i$  that can be found in (3.35). Recall that  $\omega_i = P_i \left( \tilde{E}^{0i} + \frac{1}{2} \epsilon_{iab} \tilde{E}^{ab} \right)$  and from the computations in (3.35) we have the following identities.

$$P_1 = \frac{1}{f_0 f_1} = \frac{1}{f_2 f_3} \implies f_0 f_1 = f_2 f_3 \tag{A.2}$$

$$P_2 = \frac{A_2 + r^2 B_2}{f_0 f_2} = \frac{A_2}{f_1 f_3} \implies \frac{f_1 f_3}{f_0 f_2} = \frac{A_2}{A_2 + r^2 B_2} = \frac{2A_2}{2A_2 + rA'_2} \tag{A.3}$$

$$P_3 = \frac{A_3 + r^2 B_3}{f_0 f_3} = \frac{A_3}{f_1 f_2} \implies \frac{f_1 f_2}{f_0 f_3} = \frac{2A_3}{2A_3 + rA'_3} \tag{A.4}$$

### A.3 Hodge star identities and computations

**Lemma A.3.1.** *Let  $(N^3, g_3)$  and  $(X^4, g_4)$  be two Riemannian manifolds. Let  $M^7 = N^3 \times X^4$  be a Riemannian manifold where at each point the metric splits as  $g_7 = g_3 \oplus g_4$ . Let  $\alpha \in \Omega^k(N^3)$  and  $\beta \in \Omega^l(X^4)$ . Then*

$$*_7(\alpha \wedge \beta) = (-1)^{l(k+1)}(*_3\alpha) \wedge (*_4\beta).$$

*Proof.* Let  $\text{vol}_7$ ,  $\text{vol}_3$ , and  $\text{vol}_4$  be the volume forms on  $M^7$ ,  $N^3$ , and  $X^4$  respectively thus  $\text{vol}_7 = \text{vol}_3 \wedge \text{vol}_4$ . Then by definition of  $*_7$  we have

$$\begin{aligned} \alpha \wedge \beta \wedge *_7(\alpha \wedge \beta) &= |\alpha \wedge \beta|_{g_7}^2 \text{vol}_7 \\ &= |\alpha|_{g_3}^2 |\beta|_{g_4}^2 \text{vol}_3 \wedge \text{vol}_4 \\ &= \alpha \wedge *_3\alpha \wedge \beta \wedge *_4\beta \\ &= (-1)^{l(3-k)} \alpha \wedge \beta \wedge *_3\alpha \wedge *_4\beta \\ &= (-1)^{l(k+1)} \alpha \wedge \beta \wedge *_3\alpha \wedge *_4\beta \end{aligned}$$

Therefore we can conclude that  $*_7(\alpha \wedge \beta) = (-1)^{l(k+1)}(*_3\alpha) \wedge (*_4\beta)$ .  $\square$

**Lemma A.3.2.** *Let  $X^4$  is a 4-manifold with a positive triple  $\underline{\omega}$ . Let  $M^7 = \mathbb{T}^3 \times X^4$ . Let  $Q$  be the matrix determined by the positive triple and the reference volume form chosen so that  $\det(Q) = 1$ , as in definition 1.2.1. Let  $t^i$  be the flat coordinates on  $\mathbb{T}^3$ . Let  $\varphi = dt^{123} - dt^i \wedge \omega_i$  be  $G_2$  structure on  $M^7$  with induced metric  $g_7 = Q_{ij}dt^i dt^j + g_{\underline{\omega}}$ . Then*

$$\begin{aligned} *_3 dt^1 &= Q^{13} dt^{12} + Q^{11} dt^{23} + Q^{21} dt^{31}, \\ *_3 dt^2 &= Q^{23} dt^{12} + Q^{21} dt^{23} + Q^{22} dt^{31}, \\ *_3 dt^3 &= Q^{33} dt^{12} + Q^{31} dt^{23} + Q^{32} dt^{31}, \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} *_3 dt^{31} &= Q_{12} dt^1 + Q_{22} dt^2 + Q_{32} dt^3, \\ *_3 dt^{12} &= Q_{31} dt^1 + Q_{32} dt^2 + Q_{33} dt^3, \\ *_3 dt^{23} &= Q_{11} dt^1 + Q_{12} dt^2 + Q_{13} dt^3. \end{aligned} \tag{A.6}$$

*Proof.* Let us start with proving (A.5). To begin we know from lemma A.3.1 that  $*_7 dt^i = *_3 dt^i \wedge \text{vol}_4$ . Let  $\alpha \in \Omega^1(M^7)$ , so

$$\begin{aligned} \alpha \wedge *_7 dt^i &= \alpha \wedge *_3 dt^i \wedge \text{vol}_4 \\ \implies g_7(\alpha, dt^i) \text{vol}_3 \wedge \text{vol}_4 &= \alpha \wedge *_3 dt^i \wedge \text{vol}_4 \\ \implies g_7(\alpha, dt^i) dt^{123} &= \alpha \wedge *_3 dt^i \end{aligned}$$

which tells us that  $*_3 dt^i$  can be computed using the metric  $g_7$ . Furthermore since  $g_7$  has no mixed pieces we need only consider  $\alpha \in \Omega^1(\mathbb{T}^3)$ . Note that a similar argument can be made for  $*_3 dt^{ij}$  and thus in that case we need only consider  $\beta \in \Omega^2(\mathbb{T}^3)$ . Returning to  $*_3 dt^i$ , we only show the details for  $*_3 dt^1$  as the remaining computations are done similarly. Let the entries of  $Q^{-1}$  be denoted by  $Q^{ij}$ . Let  $*_3 dt^1 = Adt^{12} + Bdt^{23} + Cdt^{31}$  where  $A, B, C$  are smooth functions on  $M^7$ . We compute the following

$$\begin{aligned}
dt^1 \wedge *_3 dt^1 &= g_7(dt^1, dt^1) dt^{123} \\
Bdt^{123} &= Q^{11} dt^{123} \\
\implies B &= Q^{11}, \\
dt^2 \wedge *_3 dt^1 &= g_7(dt^2, dt^1) dt^{123} \\
Cdt^{123} &= Q^{21} dt^{123} \\
\implies C &= Q^{21}, \\
dt^3 \wedge *_3 dt^1 &= g_7(dt^3, dt^1) dt^{123} \\
\implies A &= Q^{31}.
\end{aligned}$$

Combining these together we obtain that  $*_3 dt^1 = Q^{13} dt^{12} + Q^{11} dt^{23} + Q^{21} dt^{31}$ . We can repeat this same process for  $*_3 dt^2$  and  $*_3 dt^3$  and obtain the other desired identities in (A.5). Now we compute the identities (A.6), and by similar logic to the case of 1-forms, the computation of  $*_3 dt^{ij}$  can be done using only  $g_7$  and we need only consider 2-forms on  $\mathbb{T}^3$ . Recall that  $\det(Q) = \det(Q^{-1}) = 1$  and therefore we can use the following formula for  $Q = \frac{\text{adj}(Q^{-1})}{\det(Q^{-1})} = \text{adj}(Q^{-1})$ . Using this identity and the fact that  $Q$  is symmetric we can compute  $*_3 dt^{ij}$ . We will only do this computation for  $*_3 dt^{31}$  as the remaining computations are done similarly. Let  $*_3 dt^{31} = Adt^{12} + Bdt^{23} + Cdt^{31}$  then we obtain the following:

$$\begin{aligned}
dt^{12} \wedge *_3 dt^{31} &= g_7(dt^{12}, dt^{31}) dt^{123} \\
Cdt^{123} &= -(Q^{23}Q^{11} - Q^{13}Q^{21}) dt^{123} \\
\implies C &= Q_{32}, \\
dt^{31} \wedge *_3 dt^{31} &= g_7(dt^{31}, dt^{31}) dt^{123} \\
Bdt^{123} &= (Q^{33}Q^{11} - (Q^{13})^2) dt^{123} \\
\implies B &= Q_{22}, \\
dt^{23} \wedge *_3 dt^{31} &= g_7(dt^{23}, dt^{31}) dt^{123} \\
Adt^{123} &= -(Q^{33}Q^{21} - Q^{23}Q^{31}) dt^{123} \\
\implies A &= Q_{12}.
\end{aligned}$$

Combining these together we obtain the desired result  $*_3 dt^{31} = Q_{12} dt^{12} + Q_{22} dt^{23} + Q_{32} dt^{31}$ .  $\square$