Topics in G_2 geometry and geometric flows

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Chapter 4 is based on a paper by the author [Dwi19] which appeared in the Journal of Geometry and Physics. Chapter 5 is based on a preprint by the author [Dwi18]. Chapter 6 is based on the contributions of the author (all the results of §6.3) to a joint work with Panagiotis Gianniotis and Spiro Karigiannis [DGK19] which is to appear in the Journal of Geometric Analysis.

Abstract

We study three different problems in this thesis, all related to G_2 structures and geometric flows. In the first problem we study hypersurfaces in a nearly G_2 manifold. We define various quantities associated to such a hypersurface using the G_2 structure of the ambient manifold and establish several relations between them. In particular, we give a necessary and sufficient condition for a hypersurface with an almost complex structure induced from the G_2 structure of the ambient manifold to be nearly Kähler. Then using the nearly G_2 structure on the round sphere S^7, we prove that for a compact minimal hypersurface M^6 of constant scalar curvature in S^7 with the shape operator A satisfying $|A|^2 > 6$, there exists an eigenvalue $\lambda > 12$ of the Laplace operator on M such that $|A|^2 = \lambda - 6$, thus giving the next discrete value of $|A|^2$ greater than 0 and 6 in terms of the spectrum of the Laplace operator on M. The latter is related to a question of Chern on the values of the scalar curvature of compact minimal hypersurfaces in S^n of constant scalar curvature.

The second problem is related to the study of solitons and almost solitons of the Ricci-Bourguignon flow. We prove some characterization results for compact Ricci-Bourguignon solitons. Taking motivation from Ricci almost solitons, we then introduce the notion of Ricci-Bourguignon almost solitons and prove some results about them which generalize previous results for Ricci almost solitons. We also derive integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. Finally, using the integral formula we show that a compact gradient Ricci-Bourguignon almost soliton is isometric to an Euclidean sphere if it has constant scalar curvature or if its associated vector field is conformal.

In the third problem we study a flow of G_2 structures that all induce the same Riemannian metric. This isometric flow is the negative gradient flow of an energy functional. We prove Shi-type estimates for the torsion tensor T along the flow. We show that at a finite-time singularity the torsion must blow up, so the flow will exist as long as the torsion remains bounded. We prove a Cheeger–Gromov type compactness theorem for the flow. One possible motivation for studying this isometric flow of G_2-structures is that it can be coupled with "Ricci flow" of G_2 structures, which is a flow of G_2 structures that induces *precisely* the Ricci flow on metrics, in contrast to the Laplacian flow which induces Ricci flow plus lower order terms involving the torsion. In effect, one may hope to first flow the 3-form in a way that improves the metric, and then flow the 3-form in a way that preserves the metric but still decreases the torsion. More generally, the isometric flow is a particular geometric flow of G_2-structures distinct from the Laplacian flow, and both fit into a broader class of geometric flows of G_2-structures with good analytic properties. In the final section, we summarize the rest of the results on the isometric flow which include

an Uhlenbeck type trick and the definition of a scale-invariant quantity Θ for any solution of the flow and the proof that it is almost monotonic along the flow. We also introduce an entropy functional and prove that low entropy initial data lead to solutions of the flow that exist for all time and converge smoothly to a G_2 structure with divergence-free torsion. We study the singular set of the flow. Finally, we prove that if the singularity is of Type-I then a sequence of blow-ups of a solution admits a subsequence that converges to a shrinking soliton for the flow.

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मम्मी, डैडी, बाबा -दादी और नाना -नानीजी को समर्पित जिनके स्नेह और आशीर्वाद के बिना मैं अपने जीवन की परिकल्पना भी नहीं कर सकता ।

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Chapter 1

Introduction

The two unifying themes in this thesis are: special algebraic structures on manifolds, specifically a G_2 structure on a seven dimensional manifold, and geometric flows on manifolds.

Let (M^n, g) be a Riemannian manifold. An algebraic structure on M corresponds to a smooth section of some tensor bundle of M, satisfying some natural algebraic condition at each point $p \in M$. For example, an *orientation* on M is a nowhere vanishing smooth section $\mu \in \Lambda^n(T^*M)$. An orientation compatible with the metric g is called an $\mathrm{SO}(n)$ structure on M which is equivalent to a reduction of the structure group of the frame bundle of M from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{SO}(n)$. Another example is an *almost Hermitian* structure on M which is a smooth section J of the tensor bundle $TM \otimes T^*M$ such that $J^2 = -I$ and $g(JX, JY) = g(X, Y)$ for all $X, Y \in \Gamma(TM)$. It is easy to check that such a structure exists if and only if $n = 2m$ is even. An almost Hermitian structure on M is a reduction of the structure group of the frame bundle from $\mathrm{GL}(2m, \mathbb{R})$ to $\mathrm{U}(m)$.

A G_2 structure on M, the subject matter of this thesis, is a special algebraic structure which we can only consider in dimension $n = 7$. Topologically, a G_2 structure on M is a reduction of the structure group of the frame bundle of M from $\mathrm{SO}(7)$ to the Lie group G_2. From the point of view of differential geometry, a G_2 structure on M corresponds to a special kind of 3-form φ on M. Such a pair (M, φ) is a manifold with a G_2 structure. In addition, if the 3-form is parallel with respect to the Levi-Civita connection ∇ of g, i.e., $\nabla\varphi = 0$ then (M, φ) is called a G_2 *manifold*.

One motivation for studying manifolds with a G_2 structure is the following. The holonomy group of a Riemannian manifold is an invariant of the metric, defined as the group generated by parallel transport around closed loops. In [Ber55], Berger classified the possible holonomy groups of simply-connected, irreducible and non-symmetric Riemannian manifolds. Berger's classification is given in the following table.

Dimension	Holonomy group	Remarks
n	SO(n)	Generic Riemannian manifold
2m	U(m)	Kähler
2m	SU(m)	Calabi-Yau
4q	Sp(q)	Hyper-Kähler
4q	$\mathrm{Sp(q)} \cdot \mathrm{Sp(1)}$	Quaternionic-Kähler
7	G_2	G_2-holonomy
8	Spin(7)	Spin(7)-holonomy

Manifolds with holonomy SU(m), Sp(q), $\mathrm{Sp(q)}\cdot\mathrm{Sp(1)}$, G_2 or Spin(7) are called manifolds with special holonomy. With the exception of manifolds with holonomy $\mathrm{Sp(q)} \cdot \mathrm{Sp(1)}$, all other special holonomy manifolds are *Ricci-flat*. In fact, all currently known examples of irreducible and non-symmetric, *compact* Ricci-flat manifolds have special holonomy. In particular, G_2 manifolds are always Ricci-flat.

Another motivation is that G_2 manifolds have *parallel spinors* which make them very useful in physics, especially in string theory. M-theory is an 11-dimensional theory which is "compactified" on a 7-dimensional manifold (4 dimensions for space-time). For "super-symmetric" reasons, the 7-dimensional manifolds must admit a non-zero parallel spinor and hence must be a G_2 manifold. In this respect, G_2 manifolds are analogous to Calabi–Yau 3-folds which are the "compactified" spaces in the 10-dimensional string theory.

In Chapter 2 we discuss preliminaries on G_2 geometry. We start with the definition of a G_2 structure on a seven dimensional Riemannian manifold and discuss the decomposition of the space of differential forms on such a manifold. We describe the *torsion* of a G_2 structure and give an explicit description of the four torsion tensors along with their expressions in local coordinates. This also allows us to define manifolds with *nearly* G_2 structures which are the subject matter of Chapter 4. We then see the relationship between the torsion of a G_2 structure and the curvature of the induced metric.

The second theme of the thesis is geometric flows on a manifold. Geometric flows have been a very powerful tool to study geometric structures. The Ricci flow introduced by Hamilton paved the way for the solution of the Poincaré conjecture by Perelman. Similarly, other geometric flows like the Kähler-Ricci flow, the harmonic map heat flow, the Yang-Mills flow and others have been indispensable tools for a variety of geometric problems. In Chapter 3, we discuss some preliminaries on geometric flows. We start by describing the basic idea of a flow on a manifold. We review the notion of parabolic partial differential equations and discuss short-time existence and uniqueness for a solution of such an equation. We also discuss maximum principles for scalar and tensor equations.

Chapter 2 and Chapter 3 contains an introduction to the basic ideas involved in the thesis.

The new results start from Chapter 4 where we are concerned with minimal hypersurfaces in nearly G_2 manifolds. A G_2 structure on a manifold $\overline{M}^7$ induces a Hermitian structure on any oriented hypersurface M. A G_2 structure φ is called a nearly G_2 structure if

$$d\varphi = \lambda \star \varphi,$$

where λ is a non-zero constant and $\star$ is the Hodge-star operator (see §4.1 for relevant definitions). A Hermitian structure on a manifold M with an almost complex structure J is called a nearly Kähler structure if

$$(\nabla_X J)X = 0, \quad \text{for all} \quad X \in \Gamma(TM).$$

Nearly Kähler 6-manifolds and nearly G_2 manifolds, apart from being interesting in their own right, are also important as the Riemannian cones over them has holonomy contained in G_2 and Spin(7) respectively, both of which appear on Berger's list. The first main result in Chapter 4 is the following. The reader is urged to see the chapter for relevant definitions.

Theorem 4.1.2. Let M be an oriented hypersurface of a nearly G_2 manifold $(\overline{M}, \varphi)$. Then (M, g, ξ) is a nearly Kähler structure if and only if M is totally umbilic, i.e., for all $X \in \Gamma(TM)$

$$AX = \alpha X, \tag{1.0.1}$$

where A is the shape operator of M in $\overline{M}$ and $\alpha \in C^\infty(M)$.

Since there are very few known examples of nearly Kähler 6-manifolds, Theorem 4.1.2 can potentially be useful in finding some more examples.

The next main result in that chapter concerns the following question of Chern (cf. [Yau82, pg.693])

Question 1.0.1. [Chern] *Consider the set of all compact minimal hypersurfaces in S^{n+1} with constant scalar curvature. Think of the scalar curvature as a function on this set. Is the image of the scalar curvature function a discrete set of positive numbers ?*

Since for any minimal hypersurface M^n with scalar curvature R in S^{n+1}, $R = n(n-1) - |A|^2$ (cf. (4.2.7) in §4.2), the above question asks whether the set of $|A|^2$ for such hypersurfaces M is a discrete set. The round unit sphere S^7 has a nearly G_2 structure, so a natural question is whether we can say anything about the values of $|A|^2$ for compact minimal hypersurfaces with constant scalar curvature in S^7 by using the nearly G_2 structure on it. The second main result in the chapter concerns this.

Theorem 4.1.5. Let M^6 be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^7. If the shape operator A of M satisfies $|A|^2 > 6$, then there exists an eigenvalue $\lambda > 12$ of the Laplace operator on M such that $|A|^2 = \lambda - 6$.

Chapter 3 is concerned with the study of solitons and almost solitons of the Ricci-Bourguignon flow which is a flow of Riemannian metrics on an n-dimensional Riemannian manifold.

A family of metrics $g(t)$ on an n-dimensional Riemannian manifold (M^n, g) is said to evolve by the *Ricci-Bourguignon* flow (RB flow for short) if $g(t)$ satisfies the following evolution equation

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} - \rho R g),$$

where Ric is the Ricci tensor of the metric, R is the scalar curvature and $\rho \in \mathbb{R}$ is a constant. We notice that the Ricci flow is a special case of the RB flow for $\rho = 0$. Moreover, from [LW17, eqn. (3.6)] we see that up to the leading order term, the metric along the Laplacian flow of closed G_2 structures evolves by RB flow for $\rho = \dfrac{2}{3}$. This was the original motivation for the author to study the RB flow. We prove various rigidity results for the solitons and almost solitons of the Ricci-Bourguignon flow. The first main result of this chapter is the following

Theorem 5.1.3. Let $(M^n, g, X, \lambda, \rho)$ be a RB soliton with $n \geq 3$ and suppose that the soliton vector field X is a conformal vector field.

1. If M is compact then X is a Killing vector field and hence $(M^n, g, X, \lambda, \rho)$ is a trivial RB soliton.

2. If M is non-compact, complete and gradient RB soliton then either X is a Killing vector field or $(M^n, g, X, \lambda, \rho)$ is isometric to the Euclidean space.

We characterize compact almost soliton (see Definition 5.1.1) of the Ricci-Bourguignon flow in the following

Theorem 5.1.5. Let $(M^n, g, X, \lambda, \rho)$ be a compact RB almost soliton with $n \geq 3$. If X is a nontrivial conformal vector field then M^n is isometric to an Euclidean sphere.

We also obtain an integral formula for compact gradient RB almost soliton. Precisely, we prove the following

Theorem 5.1.9. Let $(M^n, g, \nabla f, \lambda, \rho)$ be a compact gradient RB almost soliton. Then

$$\int_M \Big|\nabla^2 f - \frac{\Delta f}{n} g\Big|^2 \operatorname{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) \operatorname{vol}, \tag{1.0.2}$$

$$\int_M \Big|\mathrm{Ric} - \frac{R}{n} g\Big|^2 \operatorname{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) \operatorname{vol}. \tag{1.0.3}$$

The preceding theorem allows us to give some conditions when a compact gradient RB soliton is isometric to an Euclidean sphere. More precisely, we prove the following

Corollary 5.1.10. A nontrivial compact gradient RB almost soliton $(M^n, g, \nabla f, \lambda, \rho)$, $n \geq 3$ is isometric to an Euclidean sphere if any one of the following holds

1. M^n has constant scalar curvature.
2. $\int_M g(\nabla R, \nabla f) \operatorname{vol} \leq 0$.
3. M^n is a homogenous manifold.

In Chapter 6 we introduce and study a geometric flow of G_2 structures all of which induce the same Riemannian metric (see (6.2.8) for the precise expression). The *isometric flow* of G_2 structures is the negative gradient flow of a natural energy functional on the space of isometric G_2 structures, namely the the square of the L^2 norm of the torsion of the G_2 structure.

One possible motivation for studying this isometric flow of G_2-structures is that it can be coupled with "Ricci flow" of G_2-structures, which is a flow of G_2-structures that induces *precisely* the Ricci flow on metrics. In effect, one may hope to first flow the 3-form in a way that improves the metric, and then flow the 3-form in a way that preserves the metric but still decreases the torsion. More generally, the isometric flow is a particular geometric flow of G_2-structures distinct from the Laplacian flow, and both fit into a broader class of geometric flows of G_2-structures with good analytic properties. A detailed study of a general class of flows that includes both the Laplacian flow and the isometric flow is undertaken in [DGK]. After some initial discussion about the flow we prove the first main result of the chapter.

Theorem 6.2.12. Let (M^7, φ_0) be a compact manifold with G_2-structure. Then the isometric flow (6.2.8) has a unique solution for a short time $t \in [0, \varepsilon)$.

Afterwards, we discuss the solitons for the flow. We then proceed to prove Shi-type estimates for the torsion T of the the G_2 structure along the flow. More precisely, we prove the following theorem.

Theorem 6.3.3. Suppose that $K > 0$ is a constant and $\varphi(t)$ is a solution to the isometric flow on a closed manifold M^7 with $t \in [0, \frac{1}{K^2}]$. For all $m \in \mathbb{N}$, there exists a constant C_m *depending only on* (M, g) such that if

$$\sup_M |T(x,t)| \leq K \text{ and } |\nabla^j \mathrm{Rm}| \leq B_j K^{2+j} \quad \text{for all } j \geq 0 \text{ on } M^7 \times [0, \tfrac{1}{K^2}],$$

then for all $t \in [0, \frac{1}{K^2}]$ we have

$$|\nabla^m T| \leq C_m t^{-\frac{m}{2}} K.$$

We also prove the local version of the above theorem. Using the Shi-type estimates we give a characterization of the singular time or the maximal existence time for any solution of the isometric flow in terms of the norm of the torsion tensor. Roughly speaking, we prove that a solution of the isometric flow exists as long as the torsion T remains bounded. We then proceed to prove a Cheeger–Gromov type compactness theorem for the solutions of the flow. We finish the chapter (and the thesis) by briefly summarizing other results we obtained on the isometric flow of G_2 structures.

1.1 Notations and conventions

Throughout the thesis, unless stated otherwise, we use the symbol $*$ to denote various contractions between tensors whose precise form is not important, and thus we instead use $\star$ for the Hodge star operator. When estimating the norms of various tensors, the symbol C will be used to denote some positive constant, which may change from line to line in the derivation of an estimate.

We very frequently use Young's inequality

$$ab \leq \tfrac{1}{2\varepsilon} a^2 + \tfrac{\varepsilon}{2} b^2,$$

for any $\varepsilon, a, b > 0$.

Throughout the thesis, we do computations with respect to a local orthonormal frame, so all indices are subscripts (unless stated otherwise) and any repeated indices are summed over all values from 1 to 7. The symbol Δ always denotes the *analyst's Laplacian* $\Delta = \nabla_k \nabla_k$ which is the negative of the *rough Laplacian* $\nabla^* \nabla$.

Our convention for labelling the Riemann curvature tensor is

$$R_{ijkm}\frac{\partial}{\partial x^m} = (\nabla_i\nabla_j - \nabla_j\nabla_i)\frac{\partial}{\partial x^k},$$

in terms of coordinate vector fields. With this convention, the Ricci tensor is $R_{jk} = R_{ljkl}$, and the Ricci identity is

$$\nabla_k\nabla_i X_l - \nabla_i\nabla_k X_l = -R_{kilm}X_m. \tag{1.1.1}$$

Schematically, the Ricci identity implies that

$$\nabla^m\Delta S - \Delta\nabla^m S = \sum_{i=0}^{m}\nabla^{m-i}S * \nabla^i\mathrm{Rm}, \tag{1.1.2}$$

for any tensor S, where as mentioned above, $*$ denotes the contraction between tensors. Here Rm denotes the Riemann curvature tensor. Thus for instance, for a $(0,2)$-tensor α the Ricci identity is

$$\nabla_i\nabla_j\alpha_{kl} - \nabla_j\nabla_i\alpha_{kl} = -R_{ijkm}\alpha_{ml} - R_{ijlm}\alpha_{km}.$$

We also have the Riemannian second Bianchi identity

$$\nabla_i R_{jkab} + \nabla_j R_{kiab} + \nabla_k R_{ijab} = 0,$$

which when contracted on i, a gives

$$\nabla_i R_{ibjk} = \nabla_k R_{jb} - \nabla_j R_{kb}. \tag{1.1.3}$$

Contracting (1.1.3) on b and j gives the contracted second Bianchi identity

$$\nabla_i R_{ik} = \frac{1}{2}\nabla_k R.$$

Chapter 2

Preliminaries on G_2 geometry

In this chapter, we review the notion of a G_2 structure on a 7-dimensional manifold. In §2.2, we will discuss the associated decomposition of the space of forms on such a manifold. We describe the *torsion* of a G_2 structure and give an explicit description of the four torsion tensors along with their expressions in local coordinates in §2.3. We then see the relationship between the torsion of a G_2 structures and the curvature of the induced metric in §2.4. We list several identities which are used in later chapters. The main references for this chapter are [Bry06], [Joy00], [Kar09] and [Kar19]. In particular, we use the sign convention used in [Kar09] (which is opposite to that used in [Bry06] and [Joy00]).

2.1 Manifolds with a G_2 structure

Let M^7 be a smooth manifold. A G_2 structure on M is a reduction of the structure group of the frame bundle from $\mathrm{GL}(7, \mathbb{R})$ to $G_2 \subset \mathrm{SO}(7)$. It is well-known that such a structure exists on M if and only if the manifold is orientable and spinnable (i.e., M can be given a spin structure), conditions which are respectively equivalent to the vanishing of the first and second Stiefel-Whitney classes. Thus, the existence of a G_2 structure on a manifold is purely a topological problem. From the point of view of differential geometry, a G_2 structure on M can also be equivalently defined by a 3-form φ on M that satisfies a certain pointwise algebraic non-degeneracy condition, which we will now describe *via* the notion of vector cross product.

Definition 2.1.1. Let (M^n, g) be a Riemannian manifold. An r-fold *vector cross product*

(VCP, for short) is an alternating r-linear smooth bundle map

$$B : \wedge^r TM \to TM \tag{2.1.1}$$

satisfying the following conditions

$$\begin{cases} g(B(v_1, ..., v_r), v_i) = 0, & 1 \le i \le r \\ \|B(v_1, ..., v_r)\|^2 = \|v_1 \wedge \cdots \wedge v_r\|^2, \end{cases}$$

for any $v_i \in TM$.

Such a cross product gives rise to a $(r+1)$-differential form ϕ defined as

$$\phi(v_1, v_2, ..., v_{r+1}) = g(B(v_1, ..., v_r), v_{r+1}).$$

The VCP is called *parallel/closed* if and only if the corresponding differential form is *parallel/closed*.

Cross products on real vector spaces were classified by Brown and Gray in [BG67] and global cross products on manifolds are discussed in Gray [Gra69]. The classification of VCPs on a real vector space V with a positive definite inner product g is as follows:

1. $r = 1$. Then a 1-fold VCP B on V is equivalent to an almost complex structure on V, i.e., $B^2 = -I$ on V. The associated VCP form is the Kähler form ω.

2. $r = n - 1$, where n is the dimension of V. An $(n-1)$-fold VCP B on V is the *Hodge star* operator $\star$ given by g on $\Lambda^{n-1}V$ and the VCP form of degree n is the volume form on V. Thus B is equivalent to an orientation.

3. $r = 2$. A 2-fold VCP B on $\mathbb{R}^7$ is a cross product defined as $B(u, v) = \mathrm{Im}(u \cdot v)$, for u, v in $\mathbb{R}^7 \cong \mathrm{Im}\mathbb{O}$, the set of imaginary octonions where $\mathbb{O}$ is an 8-dimensional nonassociative real normed division algebra. Here $\cdot$ is the octonionic multiplication. For coordinates $\{x_1, ..., x_7\}$ on $\mathrm{Im}\mathbb{O}$, the VCP form of degree 3 can be written as follows

$$\phi_0 = dx^{123} - dx^{167} - dx^{527} - dx^{563} - dx^{415} - dx^{426} - dx^{437}, \tag{2.1.2}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. Bryant [Bry87] showed that the group of linear transformations of $\mathbb{O}$ which preserves ϕ_0 also preserves g and B and is the exceptional Lie group G_2 which is also the automorphism group of $\mathbb{O}$.

4. $r = 3$. A 3-fold VCP B on $\mathbb{R}^8$ is a cross product defined as $B(u,v,w) = \frac{1}{2}(u(\bar{v}w) - w(\bar{v}u))$ for any u, v, w in $\mathbb{R}^8 \cong \mathbb{O}$. For coordinates $x_1, ..., x_8$ on $\mathbb{O}$, the VCP form of degree 4 can be written as

$$\begin{aligned}\Omega_0 = &-dx^{1234} - dx^{5678} - (dx^{21+dx^{34}})(dx^{65} + dx^{78}) - (dx^{31} + dx^{42})(dx^{75} + dx^{86}) \\ &- (dx^{41} + dx^{23})(dx^{85} + dx^{67}).\end{aligned}$$

Bryant [Bry87] showed that the group of linear transformations of $\mathbb{O}$ preserving Ω_0 also preserves g and B and is the group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

In particular, the cross product on a 7-dimensional vector space induced by the octonionic multiplication has the following properties. For all $u, v, w \in (V, g)$

$$g(u, B(v,w)) = g(B(u,v), w), \tag{2.1.3}$$
$$B(u, B(u,v)) = -g(u,u)v + g(u,v)u, \tag{2.1.4}$$
$$B(u, B(v,w)) + B(v, B(u,w)) = g(u,w)v + g(v,w)u - 2g(u,v)w. \tag{2.1.5}$$

Let $\{e_1, ..., e_7\}$ be the standard basis of $\mathbb{R}^7$ and $\{e^1, ..., e^7\}$ be the dual basis. The VCP form ϕ_0 induced by the vector cross product on $\mathbb{R}^7$ is described in (2.1.2). The group G_2 preserves ϕ_0 and it also preserves the metric and orientation for which $\{e_1, ..., e_7\}$ is an oriented orthonormal basis. If $\star_{\phi_0}$ denotes the Hodge star determined by the metric and the orientation, then G_2 preserves the 4-form

$$\psi_0 = \star_{\phi_0}\phi_0 = dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} - dx^{2637} - dx^{1537} - dx^{1526}, \tag{2.1.6}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$.

Let M be a 7-manifold. For $x \in M$, we denote by

$$\Lambda^3_+(M)_x = \{\varphi_x \in \Lambda^3 T^*_x M \mid \exists \text{ isomorphism } \rho : T_x M \to \mathbb{R}^7, \rho^*\phi_0 = \varphi_x\}.$$

The fibre bundle $\Lambda^3_+(M) = \sqcup_{x\in M}\Lambda^3_+(M)_x$ is an open subbundle of $\Lambda^3 T^*M$. A section φ of $\Lambda^3_+(M)$ is called a *positive* 3-form on M and the space of positive 3-forms on M is denoted by $\Omega^3_+(M)$. **Such a $\boldsymbol{\varphi}$ is also called a $\mathbf{G_2}$ structure on $\boldsymbol{M}$**. Thus, a G_2 structure on M is a positive 3-form φ.

To see that $\Lambda^3_+(M)$ is an open subbundle of $\Lambda^3 T^*M$, observe that at any point $x \in M$, the space of all G_2 structures $\Lambda^3_+(M)_x$ can be identified with the orbit of ϕ_0 in $\Lambda^3(\mathbb{R}^7)^*$ by the action of $\mathrm{GL}(7,\mathbb{R})$ quotiented by the stabilizer subgroup of ϕ_0 which is the Lie group G_2. Thus, $\dim\, \Lambda^3_+(M)_x = 49 - 14 = 35 = \dim\, \Lambda^3(T^*_x M)$. Hence $\Lambda^3_+(M)_x$ is an open subset of $\Lambda^3(T^*_x M)$. We emphasize that the fibre bundle $\Lambda^3_+(M)$ always exists for any 7-manifold but it doesn't always have a section. A section of $\Lambda^3_+(M)$ exists if and only if M is orientable and spinnable which is equivalent to the vanishing of the first and second Stiefel–Whitney classes respectively.

A G_2 structure induces a unique Riemannian metric and orientation in a nonlinear way. For a 3-form φ, we define

$$S_\varphi(u,v) = -\frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi, \tag{2.1.7}$$

for tangent vectors u, v on M, which is a $\Lambda^7(T^*M)$-valued bilinear form as 2-forms commute. The 3-form φ is a positive 3-form if and only if S_φ is the tensor product of a positive definite bilinear form and a nowhere vanishing 7-form which defines a unique metric g with volume form vol_g by

$$g(u,v)\,\mathrm{vol}_g = S_\varphi(u,v). \tag{2.1.8}$$

The metric and orientation determine the Hodge star operator $\star_\varphi$ and we define $\psi = \star_\varphi \varphi$, the Hodge dual of φ.

Remark 2.1.2. Even though we used a vector cross product and a metric to define G_2 structures on a manifold, one need only start with a positive 3-form and then obtain the metric using (2.1.8).

Remark 2.1.3. The nonlinear map $\varphi \to g$ is not one-to-one. In fact, given a metric g on M induced from a G_2 structure φ, at each point $p \in M$, the space of G_2 structures at p inducing g_p is diffeomorphic to $\mathbb{RP}^7$. Thus the G_2 structures inducing the same metric g correspond to sections of an $\mathbb{RP}^7$-bundle over M. See Chapter 6 for more details on *isometric* G_2 structures.

There are useful contraction identities involving the 3-form φ and the 4-form ψ of a G_2 structure, which we collect here. Their proofs can be found in [Kar09, Appendix A].

Contractions of φ with φ

$$\varphi_{ijk}\varphi_{abk} = g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab}, \tag{2.1.9}$$
$$\varphi_{ijk}\varphi_{ajk} = 6g_{ia}, \tag{2.1.10}$$
$$\varphi_{ijk}\varphi_{ijk} = 42. \tag{2.1.11}$$

Contractions of φ with ψ

$$\varphi_{ijk}\psi_{abck} = g_{ia}\varphi_{jbc} + g_{ib}\varphi_{ajc} + g_{ic}\varphi_{abj} - g_{ja}\varphi_{ibc} - g_{jb}\varphi_{aic} - g_{jc}\varphi_{abi}, \tag{2.1.12}$$
$$\varphi_{ijk}\psi_{abjk} = -4\varphi_{iab}, \tag{2.1.13}$$
$$\varphi_{ijk}\psi_{aijk} = 0. \tag{2.1.14}$$

Contractions of ψ with ψ

$$\psi_{ijkl}\psi_{abkl} = 4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{ijab}, \tag{2.1.15}$$
$$\psi_{ijkl}\psi_{ajkl} = 24g_{ia}, \tag{2.1.16}$$
$$\psi_{ijkl}\psi_{ijkl} = 168. \tag{2.1.17}$$

2.2 Decomposition of the space of forms

The existence of a G_2 structure φ on M (with no condition on $\nabla\varphi$) determines a decomposition of the spaces of differential forms on M into irreducible G_2 representations. This is analogous to the decomposition of complex-valued differential forms on an almost complex manifold into forms of type (p, q).

Since the Lie group G_2 stabilizes φ, all the tensors determined by φ will be invariant under G_2 and hence any subspaces of Ω^k defined using φ, ψ, g and $\star$ will be G_2 representations. The space Ω^k is irreducible if $k = 0, 1, 6, 7$. However, for $k = 2, 3, 4, 5$ we have a nontrivial decomposition.

The space of 2-forms $\Omega^2(M)$ and 3-forms $\Omega^3(M)$ decompose as

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M), \tag{2.2.1}$$
$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M). \tag{2.2.2}$$

where Ω^k_l has pointwise dimension l. More precisely, we have the following description of the space of forms:

$$\begin{aligned} \Omega^2_7(M) &= \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Omega^2(M) \mid \star(\varphi \wedge \beta) = -2\beta\}, \\ \Omega^2_{14}(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \psi = 0\} = \{\beta \in \Omega^2(M) \mid \star(\varphi \wedge \beta) = \beta\} \end{aligned} \tag{2.2.3}$$

and

$$\Omega^3_1(M) = \{f\varphi \mid f \in C^\infty(M)\}, \tag{2.2.4}$$

$$\Omega^3_7(M) = \{X \lrcorner \psi \mid X \in \Gamma(TM)\}, \tag{2.2.5}$$

$$\begin{aligned} \Omega^3_{27}(M) &= \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0 = \gamma \wedge \psi\} \\ &= \{h_{ij} dx^i \wedge (\frac{\partial}{\partial x^j} \lrcorner \varphi) \mid h_{ij} = h_{ji}, \mathrm{Tr}_g\, h = 0\} \end{aligned} \tag{2.2.6}$$

in local coordinates $\{x^1, ..., x^7\}$ on M. In (2.2.6), h is a symmetric 2-tensor. The decompositions of $\Omega^4(M) = \Omega^4_1(M) \oplus \Omega^4_7(M) \oplus \Omega^4_{27}(M)$ and $\Omega^5(M) = \Omega^5_7(M) \oplus \Omega^5_{14}(M)$ are obtained by taking the Hodge star of (2.2.1) and (2.2.2) respectively.

For doing explicit computations, it is convenient to write key quantities in local coordinates. In local coordinates $\{x^1, \ldots, x^7\}$ on M, we write a k-form α as

$$\alpha = \frac{1}{k!} \alpha_{i_i i_2 \cdots i_k} dx^{i_1} \wedge \cdots dx^{i_k},$$

where $\alpha_{i_i i_2 \cdots i_k}$ is totally skew-symmetric in its indices. In particular, φ and ψ are locally written as

$$\varphi = \frac{1}{6} \varphi_{ijk} dx^i \wedge dx^j \wedge dx^k \qquad \text{and} \qquad \psi = \frac{1}{24} \psi_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$

To describe Ω^2 locally, consider the G_2-invariant linear map $P : \Omega^2 \to \Omega^2$ given by $P\beta = \star(\varphi \wedge \beta)$. If we write $\beta = \frac{1}{2}\beta_{ij} dx^i \wedge dx^j$ and $P\beta = \frac{1}{2}(P\beta)_{ab} dx^a \wedge dx^b$, then one can show [Kar09, §2.2] that

$$(P\beta)_{ab} = \frac{1}{2} \psi_{abcd} \beta_{cd}. \tag{2.2.7}$$

If α is any 2-form then

$$\begin{aligned} \langle P\beta, \alpha \rangle &= \frac{1}{2} (P\beta)_{ab} \alpha_{ab} \\ &= \frac{1}{4} \psi_{abcd} \beta_{cd} \alpha_{ab} \\ &= \frac{1}{4} \psi_{cdab} \alpha_{ab} \beta_{cd} \\ &= \langle P\alpha, \beta \rangle. \end{aligned}$$

Thus, P is a self-adjoint map and hence is orthogonally diagonalizable with real eigenvalues. To find the eigenvalues, we compute

$$\begin{aligned}(P^2\beta)_{ab} &= \frac{1}{2}\psi_{abcd}(P\beta)_{cd} \\ &= \frac{1}{4}\psi_{abcd}\psi_{cdij}\beta_{ij} \\ &= \frac{1}{4}(4g_{ia}g_{jb} - 4g_{ib}g_{ja} - 2\psi_{abij})\beta_{ij} \\ &= 2\beta_{ab} - (P\beta)_{ab},\end{aligned}$$

where we have used the contraction identity (2.1.15) in the third equality. So $P^2 = 2I - P$ and hence the eigenvalues of P are -2 and $+1$, as described in (2.2.3). Notice that if $\beta \in \Omega^2_7$ and we let $\beta_{ij} = X_m\varphi_{mij}$ for some $X \in \Gamma(TM)$ then

$$(P\beta)_{ab} = \frac{1}{2}\psi_{abij}X_m\varphi_{mij} = \frac{1}{2}(-4X_m\varphi_{abm}) = -2\beta_{ab}$$

thus verifying that Ω^2_7 is the -2 eigenspace of P. The condition that $\Omega^2_{14} = (\Omega^2_7)^\perp$ gives that $\beta \in \Omega^2_{14}$ satisfies $X_m\varphi_{mij}\beta_{ij} = 0$ for all X. This is equivalent to $\beta_{ij}\varphi_{mij} = 0$. Thus, we can describe the decomposition (2.2.3) of Ω^2 in local coordinates as

$$\begin{aligned}\beta_{ij} \in \Omega^2_7 \quad &\iff \quad \beta_{ij} = X_m\varphi_{mij} \quad \iff \quad \frac{1}{2}\psi_{ijcd}\beta_{cd} = -2\beta_{ij}, \\ \beta_{ij} \in \Omega^2_{14} \quad &\iff \quad \beta_{ij}\varphi_{ijm} = 0 \quad \iff \quad \frac{1}{2}\psi_{ijcd}\beta_{cd} = \beta_{ij}.\end{aligned} \tag{2.2.8}$$

Moreover, using (2.1.10), for $\beta \in \Omega^2_7$

$$\beta_{ij} = X_m\varphi_{mij} \qquad \iff \qquad X_m = \frac{1}{6}\beta_{ij}\varphi_{ijm}. \tag{2.2.9}$$

Remark 2.2.1. Many authors prefer to use the opposite orientation than we do for the orientation induced by φ (for example in [Bry06] and [Joy00]). This changes the sign of $\star$ and the eigenvalues $(-2, +1)$ in (2.2.3) and (2.2.8) are replaced by $(+2, -1)$.

To understand the local coordinate description of the decomposition (2.2.5) and (2.2.6) of Ω^3 we consider the infinitesimal action of $(1,1)$ tensors $\Gamma(T^*M \otimes TM)$ on φ. Let $A = A_{il} \in \Gamma(T^*M \otimes TM)$. At each point $p \in M$, we have $e^{At} \in \mathrm{GL}(T_pM)$ and so we get

$$e^{At} \cdot \varphi = \frac{1}{6}\varphi_{ijk}(e^{At}dx^i) \wedge (e^{At}dx^j) \wedge (e^{At}dx^k). \tag{2.2.10}$$

We define $A \diamond \varphi \in \Omega^3$ by

$$A \diamond \varphi = \frac{d}{dt}\Big|_{t=0}(e^{At} \cdot \varphi).$$

From (2.2.10) we have

$$(A \diamond \varphi) = \frac{1}{6}(A_{il}\varphi_{ljk} + A_{jl}\varphi_{ilk} + A_{kl}\varphi_{ijl})dx^i \wedge dx^j \wedge dx^k$$

and hence

$$(A \diamond \varphi)_{ijk} = A_{il}\varphi_{ljk} + A_{jl}\varphi_{ilk} + A_{kl}\varphi_{ijl}. \tag{2.2.11}$$

If we write $\mathcal{S} = \Gamma(S^2(T^*M))$ for the space of smooth symmetric 2-tensors on M and $\mathcal{S}_0$ to denote those sections h of $\mathcal{S}$ that are traceless with respect to the metric g on M then $\mathcal{S} \cong \Omega^0 \oplus \mathcal{S}_0$ as for $h \in \mathcal{S}$ we have $h = \frac{(\operatorname{Tr} h)}{7}g + h_0$ and so we have the decomposition

$$\Gamma(T^*M \otimes T^*M) = \Omega^0 \oplus \mathcal{S}_0 \oplus \Omega^2,$$

where the splitting is pointwise orthogonal with respect to the metric on $T^*M \otimes T^*M$ induced by g. Using (2.2.1) we can further decompose this as

$$\Gamma(T^*M \otimes T^*M) = \Omega^0 \oplus \mathcal{S}_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}.$$

With respect to this splitting, we can write $A = \frac{(\operatorname{Tr} A)}{7}g + A_0 + A_7 + A_{14}$, where A_0 is a traceless symmetric tensor. Thus from (2.2.11), we have a *linear map* $A \mapsto A \diamond \varphi$ from $\Omega^0 \oplus \mathcal{S}_0 \oplus \Omega^2_7 \oplus \Omega^2_{14} \to \Omega^3$. We describe the decomposition (2.2.5) and (2.2.6) in the following

Proposition 2.2.2. *The kernel of $A \mapsto A \diamond \varphi$ is Ω^2_{14} and Ω^0, $\mathcal{S}_0$ and Ω^2_7 are mapped isomorphically onto Ω^3_1, Ω^3_{27} and Ω^3_7 respectively. Explicitly, if $A = \frac{(\operatorname{Tr} A)}{7}g + A_0 + A_7 + A_{14}$ then*

$$A \diamond \varphi = \underbrace{\frac{3}{7}(\operatorname{Tr} A)g}_{\Omega^3_1} + \underbrace{A_0 \diamond \varphi}_{\Omega^3_{27}} + \underbrace{X \lrcorner \psi}_{\Omega^3_7}$$

where $X_m = -\frac{1}{2}A_{ij}\varphi_{ijm}$.

Proof. See [Kar09, Section 2.2]. □

Remark 2.2.3. If h is a symmetric 2-tensor then $h \diamond \varphi$ is a constant multiple of $i_\varphi(h)$ where i_φ is the map defined in [Bry06].

Remark 2.2.4. Since G_2 preserves φ hence the kernel of $A \mapsto A \diamond \varphi$ is isomorphic to $\mathfrak{g}_2$ and at every point $p \in M$, $\Lambda^2_{14}(T^*_pM) \cong \mathfrak{g}_2$.

2.3 Torsion of a G_2 structure

Let (M, φ) be a manifold with a G_2 structure. Let ∇ be the Levi-Civita connection of the metric g induced by φ. Consider the tensor $\nabla\varphi \in \Gamma(T^*M \otimes \Lambda^3 T^*M)$.

Definition 2.3.1. The G_2 structure φ is *torsion-free* and M is a G_2 manifold if $\nabla\varphi = 0$.

We emphasize that $\nabla\varphi = 0$ is a nonlinear partial differential equation as ∇ is induced from g which itself is induced from φ in a nonlinear way. Thus, $\nabla\varphi$ is interpreted as the torsion of the G_2 structure. The fundamental observation about the torsion is the following lemma whose proof can be found in [Kar09, Lemma 2.24]

Lemma 2.3.2. *For a vector field X on M, $\nabla_X\varphi$ lies in the subspace Ω^3_7 of Ω^3. Thus $\nabla_X\varphi \in \Gamma(T^*M \otimes \Lambda^3_7(T^*M))$.*

Lemma 2.3.2 motivates the following

Definition 2.3.3. As $\nabla_X\varphi \in \Omega^3_7$, from (2.2.5) we can write

$$\nabla_X\varphi = T(X) \lrcorner \psi$$

for some vector field $T(X)$ on M. That is, there exists a tensor $T \in \Gamma(T^*M \otimes T^*M)$ such that

$$\nabla_i\varphi_{jkl} = T_{ip}\psi_{pjkl}. \tag{2.3.1}$$

T is called the *full torsion tensor* of φ.

Contracting (2.3.1) with ψ_{ajkl} and using (2.1.16) we get

$$T_{mn} = \frac{1}{24}\nabla_m\varphi_{abc}\psi_{nabc}. \tag{2.3.2}$$

Moreover, taking the covariant derivative of (2.1.9) and using (2.1.12) and (2.3.1) we get

$$\nabla_m\psi_{ijkl} = -T_{mi}\varphi_{jkl} + T_{mj}\varphi_{ikl} - T_{mk}\varphi_{ijl} + T_{ml}\varphi_{ijk}. \tag{2.3.3}$$

We see from (2.3.1) and (2.3.2) that $\nabla\varphi = 0 \iff T = 0$ and hence φ is torsion-free if and only if $T = 0$. Since $T \in \Gamma(T^*M \otimes T^*M) \cong \Omega^0 \oplus \mathcal{S}_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$, therefore we can decompose T further as

$$T = T_1 + T_{27} + T_7 + T_{14}, \tag{2.3.4}$$

where $T_1 = \frac{1}{7}(\operatorname{Tr} T)g$ and $T_{27} = T_0$ is the traceless symmetric part of T.

Note that $d\varphi$ and $d^*\varphi$ are linear in $\nabla\varphi$ and hence from (2.3.1) are linear in T. Since $d\varphi \in \Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$ and $d^*\varphi \in \Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$, by Schur's Lemma, the independent components of $d\varphi$ and $d^*\varphi$ must be equal to the 1, 7, 14 and 27 components of T as in (2.3.4), up to some constant factors. So if $d\varphi = 0$ and $d^*\varphi = 0$ then $T = 0$ and φ is torsion-free. If $T = 0$ then φ is parallel and since a parallel form is always closed and co-closed, $d\varphi = 0$ and $d^*\varphi = 0$. Thus we have proved the following theorem which was originally proved by Fernández and Gray by a different method.

Theorem 2.3.4 (Fernández–Gray [FG82]). *Let φ be a G_2 structure on M. Then φ is torsion-free if and only if $d\varphi = 0$ and $d^*\varphi = 0$.*

Remark 2.3.5. As $d^*\varphi = -\star d \ast \varphi = -\star d\psi$, Theorem 2.3.4 says that φ is torsion-free if and only if $d\varphi = 0$ and $d\psi = 0$.

Remark 2.3.6. A differential form α on (M, g) is *harmonic* if $\Delta_d\alpha = (dd^* + d^*d)\alpha = 0$. If M is compact then integration by parts yields that α harmonic $\iff d\alpha = 0$ and $d^*\alpha = 0$. Thus Theorem 2.3.4 says that for compact M, a G_2 structure φ is torsion-free if and only if it is harmonic with respect to its induced metric.

Since the torsion T of φ decomposes into four independent components as in (2.3.4), each component can be zero or nonzero. This gives $2^4 = 16$ distinct classes of G_2 structures. Some of the classes of G_2 structures are given in the following table.

T_1	T_{27}	T_7	T_{14}	G_2 structure	Name
0	0	0	0	$\nabla\varphi = 0$	torsion-free
0	0	0	$\star$	$d\varphi = 0$	closed G_2 structure
$\star$	$\star$	0	0	$d^*\varphi = 0$	co-closed G_2 structure
$\star$	0	0	0	$d\varphi = \lambda\psi$, $\lambda \neq 0$	nearly G_2 structure

Table 2.1

2.4 Relation between curvature and torsion for a G_2 structure

Let (M, φ) be a manifold with a G_2 structure and g_φ be the induced metric on M. Let Rm denote the Riemann curvature tensor of g_φ. There is an important relationship between Rm and ∇T called the "G_2 *Bianchi identity*" which was first proved by Karigiannis [Kar09, Theorem 4.2] using the diffeomorphism invariance of the torsion tensor T. Another proof using (2.3.2) and the Ricci identities was given by Lotay–Wei [LW17, Lemma 2.1]. We review the latter proof here for completeness.

Theorem 2.4.1. *On (M, φ), if T is the torsion and* Rm *is the Riemann curvature tensor with R_{abcd} denoting the $(4,0)$ Riemann curvature tensor in local coordinates, then the* G_2 *Bianchi identity is the following*

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ia} T_{jb} \varphi_{abk} + \frac{1}{2} R_{ijab} \varphi_{abk}. \tag{2.4.1}$$

Proof. Contracting (2.3.2) with ψ_{mabc}, using (2.1.16) and taking the covariant derivative, we get

$$\begin{aligned}
\nabla_i \nabla_j \varphi_{abc} &= \nabla_i T_{jk} \psi_{kabc} + T_{jk} \nabla_i \psi_{kabc} \\
&= \nabla_i T_{jk} \psi_{kabc} + T_{jk}(-T_{ik}\varphi_{abc} + T_{ia}\varphi_{kbc} - T_{ib}\varphi_{kac} + T_{ic}\varphi_{kab})
\end{aligned}$$

where we used (2.3.3). Interchanging the roles of i and j and noting that $T_{ik}T_{jk}$ is symmetric in i and j we have

$$\begin{aligned}
\nabla_i \nabla_j \varphi_{abc} - \nabla_j \nabla_i \varphi_{abc} = (\nabla_i T_{jk} - \nabla_j T_{ik})\psi_{kabc} &+ T_{jk}(T_{ia}\varphi_{kbc} - T_{ib}\varphi_{kac} + T_{ic}\varphi_{kab}) \\
&- T_{ik}(T_{ja}\varphi_{kbc} - T_{jb}\varphi_{kac} + T_{jc}\varphi_{kab}).
\end{aligned}$$

Using the Ricci identity (1.1.1) for the left hand side we get

$$\begin{aligned}
-R_{ijam}\varphi_{mbc} - R_{ijbm}\varphi_{amc} - R_{ijcm}\varphi_{abm} = &(\nabla_i T_{jk} - \nabla_j T_{ik})\psi_{kabc} \\
&+ T_{jk}(T_{ia}\varphi_{kbc} - T_{ib}\varphi_{kac} + T_{ic}\varphi_{kab}) \\
&- T_{ik}(T_{ja}\varphi_{kbc} - T_{jb}\varphi_{kac} + T_{jc}\varphi_{kab})
\end{aligned}$$

which on contracting with ψ_{sabc} on both sides and observing that the left hand side and each of the three terms on the right hand side are totally skew in i, j and k gives

$$-3R_{ijam}\varphi_{mbc}\psi_{sabc} = (\nabla_i T_{jk} - \nabla_j T_{ik})\psi_{kabc}\psi_{sabc} + 3T_{ia}T_{jk}\varphi_{kbc}\psi_{sabc} - 3T_{ik}T_{ja}\varphi_{kbc}\psi_{sabc}.$$

Using (2.1.13) and (2.1.16) we get

$$12R_{ijam}\varphi_{ams} = 24(\nabla_i T_{js} - \nabla_j T_{is}) + 12T_{ia}T_{jk}\varphi_{aks} - 12T_{ik}T_{ja}\varphi_{aks},$$

which on reindexing gives (2.4.1). □

Contracting (2.4.1) with φ_{mik} on both sides and using (2.1.9) gives

$$(\nabla_i T_{jk} - \nabla_j T_{ik})\varphi_{mik} = T_{im}T_{ji} - (\operatorname{tr} T)T_{jm} - T_{ia}T_{jb}\psi_{iamb} + R_{jm} - \frac{1}{2}R_{ijab}\psi_{abmi}.$$

Using the fact that $R_{ijab}\psi_{abmi} = 0$ (see [Kar09, Lemma 4.9]) we get the following

Theorem 2.4.2. *If φ is a G_2 structure on M with the associated metric g_φ then the Ricci curvature R_{jk} is given by*

$$R_{jk} = (\nabla_i T_{jm} - \nabla_j T_{im})\varphi_{imk} - T_{ji}T_{ik} + (\operatorname{tr} T)T_{jk} + T_{ia}T_{jb}\psi_{kiab}. \tag{2.4.2}$$

In particular, (2.4.2) shows that the metric of a torsion-free G_2 structure is Ricci-flat. Taking the trace of (2.4.2) we get the expression for the scalar curvature R.

Proposition 2.4.3. *The scalar curvature R of a metric g_φ induced by a G_2 structure φ is*

$$R = -2\operatorname{div}(\widehat{T_7}) + \frac{6}{7}(\operatorname{tr} T)^2 + 5|T_7|^2 - |T_{14}|^2 - |T_{27}|^2. \tag{2.4.3}$$

where $\widehat{(T_7)}_k = T_{ij}\varphi_{ijk}$ is the vector field corresponding to T_7 as described in (2.2.9).

Proof. We take the trace of (2.4.2) to get

$$R = (\nabla_i T_{jm} - \nabla_j T_{im})\varphi_{imj} - T_{ij}T_{ji} + (\operatorname{tr} T)^2 + T_{ia}T_{jb}\psi_{iajb}.$$

Decomposing T into its components as in (2.3.4) and using (2.2.8) we get

$$\begin{aligned}
R &= -\nabla_i(T_{jm}\varphi_{ijm}) + T_{jm}\nabla_i\varphi_{ijm} - \nabla_j(T_{im}\varphi_{imj}) + T_{im}\nabla_j\varphi_{jim} \\
&\quad - ((\frac{\operatorname{tr} T}{7})g_{ij} + (T_7)_{ij} + (T_{14})_{ij} + (T_{27})_{ij})((\frac{\operatorname{tr} T}{7})g_{ji} + (T_7)_{ji} + (T_{14})_{ji} + (T_{27})_{ji}) + (\operatorname{tr} T)^2 \\
&\quad + T_{ia}((\frac{\operatorname{tr} T}{7})g_{jb} + (T_7)_{jb} + (T_{14})_{jb} + (T_{27})_{jb})\psi_{iajb} \\
&= -\operatorname{div}(\widehat{T_7}) + T_{jm}T_{is}\psi_{sijm} - \operatorname{div}(\widehat{T_7}) + T_{im}T_{js}\psi_{sjim} - \frac{(\operatorname{tr} T)^2}{7} + |T_7|^2 + |T_{14}|^2 - |T_{27}|^2 \\
&\quad + (\operatorname{tr} T)^2 + T_{ia}(-4(T_7)ia + 2(T_{14})_{ia})
\end{aligned}$$

where we have used (2.2.9), (2.3.1), the fact that the decomposition in (2.3.4) is point wise orthogonal and the facts that T_1 and T_{27} are symmetric whereas T_7 and T_{14} are skew-symmetric. Decomposing the torsion terms again in the second, fourth and the last terms above and using (2.2.1) we get

$$R = -2\,\mathrm{div}(\widehat{T_7}) + \frac{6}{7}(\mathrm{tr}\, T)^2 + 5|T_7|^2 - |T_{14}|^2 - |T_{27}|^2$$

which completes the proof. $\square$

Chapter 3

Preliminaries on geometric flows

Geometric flows are a powerful tool in the study of geometric structures on a manifold. Loosely speaking, a geometric flow is a mechanism for "simplifying" a given geometric structure to a "canonical" or "special" one. The flow will be different depending on the context and some particular examples of flows are: Ricci flow of metrics where one starts with an arbitrary Riemannian metric and the flow "deforms" that towards a metric which is Ricci-flat; mean curvature flow of immersions, where the flow deforms a submanifold towards a minimal submanifold; Yang-Mills flow where the flow deforms a connection on a principal bundle over a manifold towards a connection with minimum L^2-norm of the curvature.

The existence of torsion-free G_2 structures on a manifold is a challenging problem and given the success of geometric flows in the study of other geometric structures, it is very natural to attempt to do the same for G_2 structures on manifolds. One hopes that a suitable flow of G_2 structures might help in establishing the existence of torsion-free G_2 structures.

In this chapter we discuss the basic idea and importance of a geometric flow along with the tools and techniques in the study of flow, which will be needed later. The idea of a geometric flow is simple: one starts with an arbitrary geometric structure on a manifold (for example a G_2 structure) and then let it evolve in time by a certain rule (we will see examples of such rules in later chapters). The hope is that the solution to the initial value problem exists for all times and converges to a "special" structure (which for the case of G_2 structures can be torsion-free G_2 structures, or G_2 structures with divergence free torsion which is the subject matter of Chapter 6). However, if the geometric structure is evolving then any other quantity associated to it will also evolve in time and their properties along the flow become very important in analyzing the long-term behavior of the solutions.

Given a family $\varphi(t)$ of G_2 structures, in §3.1 we derive the variation formulas of various quantities associated to a G_2 structure. They are used in Chapter 6 when we look at flows of G_2 structures. In §3.2 we review what parabolic partial differential equations are and discuss DeTurck's trick in the context of flows of G_2 structures. Finally in §3.3 we discuss maximum principles for scalar and tensor equations which are extremely useful in later chapters. The main references for this chapter are the books by Chow–Knopf [CK04] and Chow–Lu–Ni [CLN06] and the paper by Karigiannis [Kar09].

3.1 Variation formulas

In this section we derive variation formulas of quantities associated to a G_2 structure. Recall from §2.2 that using G_2-equivariant isomorphism, a 3-form on a manifold with a G_2 structure is given by a symmetric 2-tensor h and a vector field X on M. Thus, given a manifold M^7, the general flow of G_2 structures is given by

$$\frac{\partial}{\partial t}\varphi(t) = h(t) \diamond \varphi(t) + X(t) \lrcorner \psi(t) \tag{3.1.1}$$

where $h(t)$ is a 1-parameter family of symmetric 2-tensors and $X(t)$ is a 1-parameter family of vector fields on M and $\diamond$ is the operation introduced in (2.2.11). In the following subsections we derive the evolution of the quantities related to a G_2 structure and hence also the metric induced by the G_2 structure. Most of what follows is from [Kar09] and [CK04, Chapter 3].

3.1.1 Evolution of the metric and associated quantities

We start by stating the evolution of the induced metric $g(t)$ along (3.1.1).

Lemma 3.1.1. *The evolution of the metric g_{ij} under (3.1.1) is given by*

$$\frac{\partial}{\partial t} g_{ij} = 2h_{ij}. \tag{3.1.2}$$

Proof. The proof follows from finding the evolution of the tensor S_φ in (2.1.7) and the relation (2.1.8). The precise details can be found in [Kar09, Theorem 3.1]. □

Lemma 3.1.2. *The metric inverse g^{-1} evolves by*

$$\frac{\partial}{\partial t} g^{ij} = -2g^{ik}g^{jl}h_{kl}. \tag{3.1.3}$$

Proof. We have

$$\delta^i_l = g^{ik} g_{kl}$$

where δ is the Kronecker delta. Differentiating above with respect to t, we get

$$\begin{aligned} 0 &= \partial_t(g^{ik}) g_{kl} + g^{ik} \partial_t(g_{kl}) \\ &= \partial_t(g^{ik}) g_{kl} + g^{ik}(2h_{kl}) \end{aligned}$$

and hence

$$\partial_t(g^{ij}) = -2g^{ik} g^{jl} h_{kl}.$$

□

Lemma 3.1.3. *The variation of the Christoffel symbols* Γ^k_{ij} *is given by*

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}). \tag{3.1.4}$$

Proof. In local coordinates $\{x^i\}$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Hence

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma^k_{ij} &= \frac{1}{2}\Big(\frac{\partial}{\partial t} g^{kl}\Big) \cdot (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &\quad + \frac{1}{2} g^{kl}\Big(\partial_i\Big(\frac{\partial}{\partial t} g_{jl}\Big) + \partial_j\Big(\frac{\partial}{\partial t} g_{il}\Big) - \partial_l\Big(\frac{\partial}{\partial t} g_{ij}\Big)\Big). \end{aligned}$$

In normal coordinates at a point $p \in M$, we have $\Gamma^k_{ij}(p) = 0$ and hence $\partial_i A_{jk} = \nabla_i A_{jk}$ at p for any tensor A and $\partial_i g_{jk}(p) = 0$ for all $i,\ j,\ k$. So using (3.1.2) we get

$$\frac{\partial}{\partial t}\Gamma^k_{ij}(p) = \frac{1}{2} g^{kl}\Big(\nabla_i(2h_{jl}) + \nabla_j(2h_{il}) - \nabla_l(2h_{ij})\Big).$$

Since both sides of the preceding equation are components of a tensor, the result holds in any coordinates system and at any point, thus giving (3.1.4). □

We now derive the evolution of the curvature tensors along (3.1.1). If we denote the $(3,1)$ Riemann curvature tensor by R^l_{ijk} in local coordinates then we define the $(4,0)$ Riemann curvature tensor by

$$R_{ijkl} = g_{lm} R^m_{ijk}. \tag{3.1.5}$$

Lemma 3.1.4. *The evolution of the* $(3,1)$ *Riemann curvature tensor along* (3.1.1) *is given by*

$$\frac{\partial}{\partial t}R^l_{ijk} = g^{lp}\big(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\big) \quad (3.1.6)$$

Proof. We recall that in local coordinates $\{x^i\}$ we have the standard formula

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t}R^l_{ijk} = \partial_i\Big(\frac{\partial}{\partial t}\Gamma^l_{jk}\Big) - \partial_j\Big(\frac{\partial}{\partial t}\Gamma^l_{ik}\Big) + \Big(\frac{\partial}{\partial t}\Gamma^p_{jk}\Big)\cdot\Gamma^l_{ip} + \Gamma^p_{jk}\cdot\Big(\frac{\partial}{\partial t}\Gamma^l_{ip}\Big)\\ - \Big(\frac{\partial}{\partial t}\Gamma^p_{ik}\Big)\cdot\Gamma^l_{jp} - \Gamma^p_{ik}\cdot\Big(\frac{\partial}{\partial t}\Gamma^l_{jp}\Big).\end{aligned}$$

We again use normal coordinates at a point $p \in M$ to get

$$\frac{\partial}{\partial t}R^l_{ijk} = \nabla_i\Big(\frac{\partial}{\partial t}\Gamma^l_{jk}\Big) - \nabla_j\Big(\frac{\partial}{\partial t}\Gamma^l_{ik}\Big).$$

The proof now follows from using (3.1.4) in the above equation. □

Using (3.1.5) we get the following corollary of Lemma 3.1.4

Corollary 3.1.5. *Along* (3.1.1), *the* $(4,0)$ *Riemann curvature tensor evolves as*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = (\nabla_i\nabla_j h_{kl} + \nabla_i\nabla_k h_{jl} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_i h_{kl}\\ - \nabla_j\nabla_k h_{il} + \nabla_j\nabla_l h_{ik}) + 2R^m_{ijk}h_{lm}.\end{aligned} \quad (3.1.7)$$

Remark 3.1.6. Using the Ricci identity (1.1.1) on the first and fourth term, (3.1.7) can also be written as

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = (\nabla_i\nabla_k h_{jl} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il} + \nabla_j\nabla_l h_{ik})\\ - R^q_{ijk}h_{ql} - R^q_{ijl}h_{kq} + 2R^m_{ijk}h_{lm}.\end{aligned} \quad (3.1.8)$$

Since the Ricci tensor is given by

$$R_{jk} = g^{il}R_{ijkl}$$

contracting (3.1.7) on i and l and noting that there is an extra term due to the variation of g^{-1} we get

Lemma 3.1.7. *The evolution of the Ricci tensor along* (3.1.1) *is given by*

$$\frac{\partial}{\partial t}R_{jk} = \nabla_i \nabla_j h_{ik} + \nabla_i \nabla_k h_{ij} - \Delta h_{jk} - \nabla_j \nabla_k (\operatorname{tr} h). \tag{3.1.9}$$

Similarly, we have the evolution of the scalar curvature R.

Lemma 3.1.8. *The scalar curvature evolves by*

$$\frac{\partial}{\partial t}R = -2\Delta(\operatorname{tr} h) + \nabla_i \nabla_j h_{ij} - 2h_{jk}R_{jk}. \tag{3.1.10}$$

We complete this subsection by computing the evolution of the volume form vol and the total scalar curvature.

Lemma 3.1.9. *The volume form* vol *evolves by*

$$\frac{\partial}{\partial t}\operatorname{vol} = (\operatorname{tr} h)\operatorname{vol}. \tag{3.1.11}$$

Proof. In oriented local coordinates $\{x^i\}$

$$\operatorname{vol} = \sqrt{\det g}\; dx^1 \wedge \cdots \wedge dx^7.$$

Jacobi's formula for a matrix $A(t)$ in variable t states that

$$\partial_t(\det A(t)) = \det(A(t)) \operatorname{tr}[A^{-1}(t) \cdot \partial_t A(t)] \tag{3.1.12}$$

hence

$$\begin{aligned}\partial_t(\operatorname{vol}) &= \frac{1}{2\sqrt{\det g}}\det(g(t)) \operatorname{tr}[g^{-1}(t) \cdot \partial_t g(t)]\; dx^1 \wedge \cdots \wedge dx^7 \\ &= (\operatorname{tr} h)\operatorname{vol}.\end{aligned}$$

∎

Corollary 3.1.10. *The total scalar curvature* $\int_M R \operatorname{vol}$ *on a compact* M *evolves by*

$$\frac{\partial}{\partial t}\Big(\int_M R\operatorname{vol}\Big) = \int_M \Big(R \operatorname{tr} h - 2\langle h, \operatorname{Ric}\rangle\Big)\; \operatorname{vol}. \tag{3.1.13}$$

Proof. Using (3.1.10) and (3.1.11) we get

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_M R \,\mathrm{vol}\Big) &= \int_M (\partial_t R \,\mathrm{vol} + R\partial_t \,\mathrm{vol}) \\
&= \int_M (-2\Delta(\operatorname{tr} h) + \nabla_i\nabla_j h_{ij} - 2h_{jk}R_{jk} + R \operatorname{tr} h) \,\mathrm{vol} \\
&= \int_M (-2\Delta(\operatorname{tr} h) + \operatorname{div}(\operatorname{div} h) - 2\langle h, \mathrm{Ric}\rangle + R \operatorname{tr} h) \,\mathrm{vol}
\end{aligned}$$

Since M is compact, integral of the first two terms on the right hand side of the last equality is zero by the divergence theorem. Thus we get (3.1.13). □

3.1.2 Evolution of the torsion tensor

In this subsection we will derive the evolution of the torsion tensor under (3.1.1). Recall from (2.3.2) that the torsion tensor of a G_2 structure is given by

$$T_{ij} = \frac{1}{24}(\nabla_i \varphi_{abc})\psi_{jabc}.$$

Thus to find the evolution of the torsion we need to find the evolution of the dual 4-form $\psi = \star\varphi$ and $\nabla_i\varphi_{abc}$. We state the results here.

Lemma 3.1.11. *The evolution of the* 4*-form* ψ_{ijkl} *under* (3.1.1) *is given by*

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_{ijkl} &= h_{im}\psi_{mjkl} + h_{jm}\psi_{imkl} + h_{km}\psi_{ijml} + h_{lm}\psi_{ijkm} \\
&\quad - X_i\varphi_{jkl} + X_j\varphi_{ikl} - X_k\varphi_{ijl} + X_l\varphi_{ijk}.
\end{aligned} \tag{3.1.14}$$

Proof. The idea is to differentiate the contraction identity (2.1.9). The detailed proof can be found in [Kar09, Theorem 3.5]. □

Lemma 3.1.12. *The evolution of* $\nabla_l\varphi_{ijk}$ *under* (3.1.1) *is given by*

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_l\varphi_{ijk} &= h_{im}(\nabla_l\varphi_{mjk}) + h_{jm}(\nabla_l\varphi_{imk}) + h_{km}(\nabla_l\varphi_{ijm}) + X_m(\nabla_l\psi_{mijk}) \\
&\quad + (\nabla_m h_{il})\varphi_{mjk} + (\nabla_m h_{jl})\varphi_{imk} + (\nabla_m h_{kl})\varphi_{ijm} \\
&\quad - (\nabla_i h_{lm})\varphi_{mjk} - (\nabla_j h_{lm})\varphi_{imk} - (\nabla_k h_{lm})\varphi_{ijm} \\
&\quad + (\nabla_l X_m)\psi_{mijk}.
\end{aligned} \tag{3.1.15}$$

Proof. This is proved in [Kar09, Lemma 3.7]. □

We derive the evolution of the torsion tensor.

Theorem 3.1.13. *Along* (3.1.1)*, the torsion* T_{pq} *evolves by*

$$\frac{\partial}{\partial t}T_{pq} = T_{pl}h_{lq} + T_{pl}X_m\varphi_{mlq} + (\nabla_k h_{ip})\varphi_{kiq} + \nabla_p X_q. \tag{3.1.16}$$

Proof. We know that

$$T_{pq} = \frac{1}{24}(\nabla_p \varphi_{abc})\psi_{qabc}$$

and we know the evolution of all the quantities on the right hand side. Substituting the expressions from (3.1.3), (3.1.14) and (3.1.15) and simplifying gives (3.1.16). □

3.2 Parabolic equations and short-time existence

In this section we first see the notion of *principal symbol* of a linear partial differential operator and then use it to define parabolic PDEs. We then see DeTurck's trick for a general flow of G_2 structures. We start by describing a parabolic PDE of second order and the existence and uniqueness of solutions of such PDEs.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected subset and consider a PDE for a function $u : \Omega \to \mathbb{R}$

$$\frac{\partial u}{\partial t} = a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu, \tag{3.2.1}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $a_{ij},\ b_i,\ c : \Omega \to \mathbb{R}$ are smooth coefficients. We say that (3.2.1) is *parabolic* if a_{ij} is uniformly positive definite, i.e., if there exists $\lambda > 0$ such that

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2,$$

for all $\xi \in \mathbb{R}^n$.

Suppose M is a closed manifold and consider the PDE for $u : M \to \mathbb{R}$ given by

$$\frac{\partial u}{\partial t} = L(u) \tag{3.2.2}$$

where $L : C^\infty(M) \to C^\infty(M)$ is a linear differential operator of second order and can be written as

$$L(u) = a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu$$

in local coordinates $\{x^i\}$. Here $a_{ij},\ b_i,\ c$ are locally defined smooth real coefficients. We say that (3.2.2) is parabolic if a_{ij} is positive definite for all $x \in M$. The importance of parabolic PDEs is that they have a good theory of existence and uniqueness of solutions. More precisely, given a smooth function $u_0 : M \to \mathbb{R}$ there exists a smooth solution $u : M \times [0, \infty) \to \mathbb{R}$ to

$$\begin{cases} \dfrac{\partial u}{\partial t} = L(u) & \text{on } M \times [0, \infty) \\ u(0) = u_0 & \text{on } M. \end{cases}$$

Moreover, suppose that $\frac{\partial u}{\partial t} = L(u)$ and $\frac{\partial v}{\partial t} = L(v)$ on $M \times [0, \infty)$. If either $u(0) = v(0)$ or $u(T) = v(T)$ then $u(t) = v(t)$ for all $t \in [0, T]$, i.e., given an initial or final data, the solution to (3.2.2) is unique (see [Eva98, Chapter 7]).

To check whether a differential operator L is parabolic or not, we compute its **principal symbol** which captures algebraically those analytic properties of L which only depend on its highest derivative. For L as in (3.2.2), define its principal symbol $\sigma(L) : T^*M \to \mathbb{R}$ by

$$\sigma(L)(x, \xi) = a_{ij}(x)\xi_i\xi_j. \tag{3.2.3}$$

Thus the PDE (3.2.2) is parabolic if $\sigma(L)(x, \xi) > 0$ for all $(x, \xi) \in T^*M$ with $\xi \neq 0$.

Remark 3.2.1. Some authors use $i\xi$ in place of ξ in computing the principal symbol where $i = \sqrt{-1}$.

We now generalize the above to partial differential equations of sections of vector bundles and nonlinear PDEs of any order. Let E and F be vector bundles over M and $\Gamma(E)$ and $\Gamma(F)$ be the spaces of smooth sections of E and F respectively. For $u \in \Gamma(E)$, consider the equation

$$\frac{\partial u}{\partial t} = L(u) \tag{3.2.4}$$

where L is a linear differential operator of order k (and not necessarily 2), i.e., it is a map

$$L : \Gamma(E) \to \Gamma(F)$$

given by

$$L(S) = \sum_{|\alpha| \le k} L_\alpha \partial^\alpha S$$

where $S \in \Gamma(E)$ and $L_\alpha \in \mathrm{Hom}(E, F)$ for every multi-index α. Let $\xi \in \Gamma(T^*M)$. The **principal symbol** of L in the direction of ξ is the bundle homomorphism

$$\sigma[L] : E \to F$$

given by

$$\sigma[L](\xi)S = \sum_{|\alpha|=k} \xi^\alpha L_\alpha S. \tag{3.2.5}$$

We say that (3.2.4) is parabolic if $\sigma[L]$ is an isomorphism for all $\xi \neq 0$. The basic property of the principal symbol which is used frequently is the following. Let G be another vector bundle over M and let

$$K : \Gamma(F) \to \Gamma(G)$$

be another differential operator of order l. Then the symbol of $K \circ L$ in the direction ξ is the bundle homomorphism

$$\sigma[K \circ L](\xi) = \sigma[K](\xi) \circ \sigma[L](\xi) : E \to G$$

of degree at most $k + l$ in ξ.

A nonlinear PDE is parabolic at a section $w \in \Gamma(E)$ if its linearization is parabolic. If

$$\frac{\partial u}{\partial t} = P(u)$$

is parabolic at w then there exist $\epsilon > 0$ and a smooth family $u(t) \in \Gamma(E)$ for $t \in [0, \epsilon]$, such that

$$\begin{cases} \dfrac{\partial u}{\partial t} = P(u) & t \in [0, \epsilon] \\ u(0) = w. \end{cases}$$

Moreover, we have uniqueness as well, as described before.

We now discuss DeTurck's trick for a general flow of G_2 structures. Consider the flow of G_2 structures

$$\frac{\partial \varphi}{\partial t} = h \diamond \varphi + X \lrcorner \psi \tag{3.2.6}$$

where h is a family of time-dependent symmetric 2-tensors and X is a time-dependent vector field on M. For an explicit flow of G_2 structures, h and X will be associated to φ. For example, in [DGK] we show that h could be the Ricci tensor of the metric induced by the G_2 structure and X could be the divergence of the torsion tensor $\operatorname{div} T$. In such a case, due to the diffeomorphism invariance of the quantities involved, i.e., h and X, the principal symbol of (3.2.6) will have a non-trivial kernel and hence will not be parabolic. Thus the standard theory of short-time existence and uniqueness of solutions cannot be applied to (3.2.6). We can still prove the short-time existence and uniqueness of the solution by using the DeTurck's trick which was given by DeTurck [DeT83] for the Ricci flow of metrics but we will adapt the same idea in the context of flow of G_2 structures. We consider the modified flow

$$\frac{\partial \varphi}{\partial t} = h \diamond \varphi + X \lrcorner \psi + \mathcal{L}_W \varphi \tag{3.2.7}$$

where W is a time-dependent vector field.

Consider the family of diffeomorphism $\Phi_t : M \to M$ by

$$\begin{cases} \dfrac{\partial \Phi_t(p)}{\partial t} = -W(\Phi_t(p), t) \\ \Phi_0 = \mathrm{id}_M. \end{cases}$$

If (3.2.7) is parabolic then it has a unique solution $\bar{\varphi}(t)$ for short time. Consider

$$\varphi(t) = \Phi_t^*(\bar{\varphi}(t)).$$

Note from the definition of Φ_t that

$$\frac{\partial}{\partial s}\Big|_{s=0} (\Phi_t^{-1} \circ \Phi_{t+s}) = (\Phi_t^{-1})_* \Big(\frac{\partial}{\partial s}\Big|_{s=0} \Phi_{t+s} \Big) = -(\Phi_t^{-1})_* W(t)$$

Then

$$\begin{aligned} \frac{\partial}{\partial t}(\Phi_t^*(\bar{\varphi}(t))) &= \frac{\partial}{\partial s}\Big|_{s=0} (\Phi_{t+s}^* \bar{\varphi}(t+s)) \\ &= \Phi_t^* \Big(\frac{\partial}{\partial t} \bar{\varphi}(t) \Big) + \frac{\partial}{\partial s}\Big|_{s=0} (\Phi_{t+s}^* \bar{\varphi}(t)) \\ &= \Phi_t^*(h \diamond \varphi(t) + X \lrcorner \psi(t) + \mathcal{L}_{W(t)} \varphi(t)) + \frac{\partial}{\partial s}\Big|_{s=0} \Big[(\Phi_t^{-1} \circ \Phi_{t+s})^* \Phi_t^* \bar{\varphi}(t) \Big] \\ &= h \diamond \Phi_t^*(\varphi(t)) + X \lrcorner \Phi_t^*(\psi(t)) + \Phi_t^*(\mathcal{L}_{W(t)} \varphi(t)) - \mathcal{L}_{[(\Phi_t^{-1})_* W(t)]} (\Phi_t^* \bar{\varphi}(t)) \\ &= h \diamond \Phi_t^*(\varphi(t)) + X \lrcorner \Phi_t^*(\psi(t)). \end{aligned}$$

where we have used the definition of Φ_t to cancel the last two terms in the penultimate line. Thus $\varphi(t) = \Phi_t^*(\bar{\varphi}(t))$ is a solution of (3.2.6). The precise expression of W will depend on the geometric flow under consideration.

In the above computation, we have also used the fact that in (3.1.1), $h(t)$ and $X(t)$ are diffeomorphism invariant quantities related to a G_2 structure in the sense that for any diffeomorphism F of M we have

$$F^*(h(\varphi)) = h(F^*(\varphi)) \qquad \text{and} \qquad F^*(X(\varphi)) = X(F^*(\varphi)) \tag{3.2.8}$$

For instance, in [DGK] we examine different possibilities for h and X and they all satisfy (3.2.8).

3.3 Maximum principles

One of the most powerful tools in the study of any geometric flow are the maximum principles. In this section, we review some of the maximum principles which are used in later chapters. We start with the maximum principles for scalar equations both with a linear and a non-linear reaction term. We will then discuss the maximum principles for tensor equations.

3.3.1 Weak maximum principles for scalar equations

The prototype for parabolic equations is the heat equation and it satisfies the maximum principle. Let M be a manifold and $g(t)$ be a 1-parameter family of metrics, $X(t)$ be a 1-parameter family of vector fields and $\beta : M^n \times [0, T) \to \mathbb{R}$ is a given function. We first give the following

Definition 3.3.1. We say $u : M \times [0, T) \to \mathbb{R}$ is a **supersolution** to the linear heat equation

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle + \beta v \tag{3.3.1}$$

at $(x, t) \in M \times [0, T)$ if

$$\frac{\partial u}{\partial t}(x, t) \geq (\Delta_{g(t)} u)(x, y) + \langle X, \nabla u \rangle (x, t) + \beta(x, t) u(x, t).$$

We have the following proposition.

Proposition 3.3.2 (Scalar maximum principle with linear reaction term). *Let $u : M \times [0,T) \to \mathbb{R}$ be a C^2 supersolution to (3.3.1) on a closed manifold. Suppose that for each $\tau \in [0,T)$, there exists a constant $C_\tau < \infty$ such that $\beta(x,t) \leq C_\tau$ for all $x \in M$ and $t \in [0,\tau]$. If $u(x,0) \geq 0$ for all $x \in M$ then $u(x,t) \geq 0$ for all $x \in M$ and $t \in [0,T)$.*

We do not prove Proposition 3.3.2, whose proof can be found in [CK04, Chapter 4, Theorem 4.2]. Instead, we use it to give a proof of the maximum principle for scalar equations with a nonlinear reaction term. Consider the semilinear heat equation

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle + F(v) \tag{3.3.2}$$

where $g(t)$ is a family of metrics on a closed manifold M, $X(t)$ is a family of vector fields and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. We say that u is a supersolution of (3.3.2) if

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u)$$

and a subsolution if

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u).$$

Theorem 3.3.3 (Scalar maximum principle: ODE gives pointwise bounds for PDE). *Let $u : M \times [0,T) \to \mathbb{R}$ be a C^2 supersolution to (3.3.2) on a closed manifold. Suppose that there exists $C \in \mathbb{R}$ such that $u(x,0) \geq C$ for all $x \in M$ and let α be the solution to the associated ordinary differential equation*

$$\frac{d\alpha}{dt} = F(\alpha)$$

satisfying

$$\alpha(0) = C.$$

Then

$$u(x,t) \geq \alpha(t),$$

for all $x \in M$ and $t \in [0,T)$ such that $\alpha(t)$ exists.

Remark 3.3.4. We can state a similar theorem for subsolutions u of (3.3.2). If $u(x,0) \leq C$ for all $x \in M$ and $\alpha_1(t)$ is a solution of the ODE

$$\frac{d\alpha_1}{dt} = F(\alpha_1),$$

$$\alpha_1(0) = C.$$

Then $u(x,t) \leq \alpha_1(t)$ for all $x \in M$ and $t \in [0,T)$ such that $\alpha_1(t)$ exists.

Proof. We follow the proof in [CK04, Theorem 4.4]. We observe that

$$\frac{\partial}{\partial t}(u-\alpha) \geq \Delta(u-\alpha) + \langle X, \nabla(u-\alpha)\rangle + F(u) - F(\alpha).$$

Since $\alpha(0) = C$ hence at $t = 0$, we have $u - \alpha \geq 0$ We want to prove that $u - \alpha \geq 0$ for all $t \in [0,T)$. Let $\tau \in (0,T)$ be a given time. Since M is compact, there exists a constant $C_\tau < \infty$ such that $|u(x,t)| \leq C_\tau$ and $|\alpha(t)| \leq C_\tau$ for all $(x,t) \in M \times [0,\tau]$. Since F is locally Lipschitz, there exists a constant $L_\tau < \infty$ such that

$$|F(a) - F(b)| \leq L_\tau |a-b|$$

for all $a, b \in [-C_\tau, C_\tau]$. Hence we have

$$\frac{\partial}{\partial t}(u-\alpha) \geq \Delta(u-\alpha) + \langle X, \nabla(u-\alpha)\rangle - L_\tau \mathrm{sgn}(u-\alpha)\cdot(u-\alpha)$$

on $M \times [0,\tau]$, where $\mathrm{sgn}(\cdot) \in \{-1,0,1\}$ denotes the signum function. We apply Proposition 3.3.2 with $\beta = -L_\tau \mathrm{sgn}(u-\alpha)$ which is a linear function, to get that

$$u - \alpha \geq 0$$

on $M \times [0,\tau]$. Since $\tau \in (0,T)$ was arbitrary, the theorem follows. □

3.3.2 Weak maximum principles for systems

One of the main advantages of the maximum principle is that it is extremely robust: it applies to general classes of second-order parabolic equations. We apply it to the systems of geometric flows of G_2 structures in Chapter 6. We first state and prove the maximum principle for symmetric 2-tensors which was originally proved in [Ham82]. Recall that a symmetric 2-tensor A is said to be *non-negative* and write $A \geq 0$ if the quadratic form induced by A is positive semidefinite.

Theorem 3.3.5 (Tensor maximum principle: non-negativity is preserved). *Suppose $g(t)$ is a family of Riemannian metrics on a closed Riemannian manifold M and let $A(t) \in \Gamma(T^*M \otimes_S T^*M)$ be a 1-parameter family of symmetric 2-tensors satisfying the semilinear heat equation*

$$\frac{\partial}{\partial t} A \geq \Delta_{g(t)} A + \beta \tag{3.3.3}$$

where $\beta = \beta(A, g, t)$ is a symmetric 2-tensor which is locally Lipschitz in all its arguments and satisfies the **null eigenvector assumption** *that is*

$$\beta(V, V)(x, t) = (\beta_{ij} V^i V^j)(x, t) \geq 0,$$

whenever $V(x, t)$ is a null eigenvector of $A(t)$, that is

$$(A_{ij} V^j)(x, t) = 0.$$

If $A(0) \geq 0$ then $A(t) \geq 0$ for all t such that the solution exists.

Proof. Given any $\tau \in (0, T)$, we show that there exists $\delta \in (0, \tau]$ such that for all $t_0 \in [0, \tau - \delta]$, if $A \geq 0$ at t_0 then $A \geq 0$ on $M \times [t_0, t_0 + \delta]$. Since $[0, \tau - \delta]$ is compact and τ is arbitrary, the theorem would follow.

Let $t_0 \in [0, \tau - \delta]$ be fixed. For $0 < \epsilon \leq 1$ consider the modified 2-tensor A_ϵ given by

$$A_\epsilon(x, t) = A(x, t) + \epsilon[\delta + (t - t_0)] \cdot g(x, t)$$

where $\delta > 0$ is to be chosen later. Note that the term $\epsilon\delta g$ makes A_ϵ strictly positive definite at $t = t_0$ as g is positive definite. Also

$$\frac{\partial}{\partial t} A_\epsilon = \frac{\partial}{\partial t} A + \epsilon\delta\partial_t g + \epsilon g + \epsilon(t - t_0)\partial_t g.$$

So for $t \in [t_0, t_0 + \delta]$ choose $\delta > 0$ sufficiently small, depending on

$$\max_{M \times [0,\tau]} \left| \frac{\partial}{\partial t} g \right|$$

so that

$$\frac{\partial}{\partial t} A_\epsilon > \frac{\partial}{\partial t} A.$$

Since $\Delta A_\epsilon = \Delta A$ and A satisfies (3.3.3) we get

$$\frac{\partial}{\partial t} A_\epsilon \geq \Delta A_\epsilon + \beta + \epsilon g + \epsilon[\delta + (t - t_0)]\partial_t g$$

which can be re-written as

$$\frac{\partial}{\partial t} A_\epsilon \geq \Delta A_\epsilon + \beta(A_\epsilon, g, t) + [\beta(A, g, t) - \beta(A_\epsilon, g, t)] + \epsilon g + \epsilon[\delta + (t - t_0)]\partial_t g. \tag{3.3.4}$$

We first choose $\delta_0 > 0$ depending on $g(t)$, $t \in [0, \tau]$, to be small enough such that

$$\partial_t g \geq -\frac{1}{4\delta_0} g$$

which implies that on $M \times [t_0, t_0 + \delta_0]$ we have

$$\epsilon g + \epsilon[\delta_0 + (t - t_0)]\partial_t g \geq \frac{1}{2}\epsilon g \tag{3.3.5}$$

as $\epsilon[\delta_0 + (t - t_0)]\partial_t g \geq \epsilon\delta_0\partial_t g \geq -\frac{1}{4}\epsilon g$. Since β is locally Lipschitz, there exists a constant K depending on the bounds for A and g on $M \times [0, \tau]$ (and not on ϵ) which is large enough so that on $M \times [t_0, t_0 + \delta_0]$

$$\beta(A, g, t) - \beta(A_\epsilon, g, t) \geq -K\epsilon[\delta_0 + (t - t_0)]g \geq -2K\epsilon\delta_0 g.$$

Choose $\delta \in (0, \delta_0)$ small enough so that

$$\delta < \frac{1}{4K}$$

and so that on $M \times [t_0, t_0 + \delta]$ we have

$$\beta(A, g, t) - \beta(A_\epsilon, g, t) > -\frac{1}{2}\epsilon g. \tag{3.3.6}$$

Thus combining (3.3.4), (3.3.5) and (3.3.6) we get

$$\frac{\partial}{\partial t} A_\epsilon > \Delta A_\epsilon + \beta(A_\epsilon, g, t). \tag{3.3.7}$$

We claim that $A_\epsilon > 0$ on $M \times [t_0, t_0 + \delta]$. Suppose the claim is false. Then there exists a point $(x_1, t_1) \in M \times (t_0, t_0 + \delta]$ and a non-zero vector $v \in T_{x_1}M$ such that $A_\epsilon > 0$ for all times $t_0 \leq t < t_1$ but

$$(A_\epsilon)_{ij} v^j (x_1, t_1) = 0.$$

We extend v to a vector field V defined in a space-time neighborhood of the (x_1, t_1) such that $V(x_1, t_1) = v$ and

$$\frac{\partial V}{\partial t}(x_1, t_1) = 0, \tag{3.3.8}$$

$$\nabla V(x_1, t_1) = 0, \tag{3.3.9}$$

$$\Delta V(x_1, t_1) = 0. \tag{3.3.10}$$

This can be done by parallel translating v with respect to $g(t_1)$ along geodesic rays emanating from x_1 and then taking V to be independent of time. This construction gives (3.3.8) and (3.3.9). To see (3.3.10), choose any frame $\{e_i \in T_{x_1}M\}_{i=1}^n$ which is orthonormal with respect to $g(t_1)$ and parallel translate it in a space-time neighborhood along geodesic rays emanating from x_1. In this frame, we have

$$\begin{aligned}\Delta V(x_1, t_1) &= \sum_{i=1}^{n} [\nabla_{e_i}(\nabla_{e_1} V) - \nabla_{\nabla_{e_i} e_i} V](x_1, t_1) \\ &= \sum_{i=1}^{n} [\nabla_{e_i} \vec{0} - \nabla_{\vec{0}} V](x_1, t_1) = 0\end{aligned}$$

thus giving (3.3.10). So we have

$$(A_\epsilon)_{ij} V^j (x_1, t_1) = 0.$$

Then (3.3.6) and the null eigenvector assumption imply that at (x_1, t_1), we have

$$\begin{aligned}0 \geq \frac{\partial}{\partial t}\Big((A_\epsilon)_{ij} V^i V^j\Big) &= \Big(\frac{\partial}{\partial t} A_\epsilon\Big)_{ij} V^i V^j \\ &> \Big[(\Delta A_\epsilon)_{ij} + \beta_{ij}(A_\epsilon, g, t)\Big] V^i V^j \\ &= \Delta\Big((A_\epsilon)_{ij} V^i V^j\Big) + \beta_{ij}(A_\epsilon, g, t) V^i V^j \geq 0\end{aligned}$$

which is a contradiction. Here the first inequality is due to the fact that for a fixed point in M, A_ϵ is decreasing in time, the equality in the last line is due to (3.3.10) and the final inequality is due to the fact that since $A_\epsilon = 0$ for a vector in $T_{x_1}M$, hence $\Delta((A_\epsilon)_{ij} V^i V^j) = 0$ and β satisfies the null eigenvector condition.

This contradiction proves the claim. Since $\delta > 0$ depends only on

$$\max_{M \times [0,\tau]} \Big|\frac{\partial}{\partial t} g\Big|$$

and K and is independent of ϵ, we let $\epsilon \searrow 0$, thus proving the theorem. $\square$

Chapter 4

Special hypersurfaces in nearly G_2 manifolds

4.1 Introduction

Recall from Table 2.1 that a G_2 structure on a manifold M is a *nearly* G_2 structure if

$$d\varphi = \lambda\psi$$

for some non-zero constant λ. In this case, (M, φ) is a nearly G_2 manifold. In fact, $\lambda = \frac{4}{7}\operatorname{tr} T$. To see that λ is indeed a constant when M is connected (if M is not connected then we look at any of its connected component), differentiate the above equation and note that $d\psi = 0$ to get

$$0 = d\lambda \wedge \psi.$$

Since wedge product with ψ is an isomorphism between $\Omega^1(M)$ and $\Omega^5_1(M)$, we get $d\lambda = 0$ and hence λ is a constant. Nearly G_2 manifolds were studied in detail in [FKMS97]. The authors in [FKMS97] call such structures nearly parallel G_2 structures but we simply call them nearly G_2 structures. Since for nearly G_2 manifolds, T_1 is the only non-zero component of the torsion tensor, from (2.4.2) and the fact that $\lambda = \frac{4}{7}\operatorname{tr} T$ is a constant we see that for a nearly G_2 manifold

$$R_{ij} = \frac{6}{49}(\operatorname{tr} T)^2 g_{jk} = \frac{6}{7}|T_1|^2 g_{jk} \tag{4.1.1}$$

and hence nearly G_2 manifolds are always *positive Einstein*. The scalar curvature is given by

$$R = 6|T_1|^2. \tag{4.1.2}$$

We remark that S^7 with the round metric and also the squashed S^7 are examples of manifolds with nearly G_2 structure (see [FKMS97] for more on nearly G_2 structures.) In particular, S^7 with radius 1 has scalar curvature 42, so comparing with (4.1.2) we get that $|T_1|^2 = 7$. In this chapter we will study hypersurfaces of a manifold with a nearly G_2 structure. This chapter is based on [Dwi19].

Let $(\overline{M}^7, \overline{g})$ be a Riemannian manifold with a vector cross product B. Then as we saw in §2.1, they induce a G_2 structure φ on $\overline{M}$. Let M^6 be a hypersurface of $\overline{M}$ with the induced metric g from $\overline{g}$ and denote by N the unit normal vector field of M in $\overline{M}$. If we define $\xi : TM \to TM$ by $\xi(X) = B(N, X)$, where $X \in \Gamma(TM)$ and B is the vector cross product, then ξ is a metric compatible almost complex structure on M (cf. Proposition 4.3.1). More generally, if (L, g, J) is an almost Hermitian manifold with an almost complex structure J, then we have the following

Definition 4.1.1. Let (L, g, J) be an almost Hermitian manifold with an almost complex structure J. Then L is called a *nearly Kähler* manifold if ∇J is a skew-symmetric tensor, i.e.,

$$(\nabla_X J)X = 0, \quad \text{for all} \quad X \in \Gamma(TL). \tag{4.1.3}$$

So a natural question is to find conditions on the oriented hypersurface M such that with respect to the almost complex structure ξ, (M, g, ξ) is a nearly Kähler manifold. Our first result is a characterization of nearly Kähler hypersurfaces of manifolds with a nearly G_2 structure. In §4.3, we prove the following (cf. Theorem 4.3.8)

Theorem 4.1.2. *Let M be an oriented hypersurface of a nearly G_2 manifold $(\overline{M}, \varphi)$. Then (M, g, ξ) is a nearly Kähler structure if and only if M is totally umbilic, i.e., for all $X \in \Gamma(TM)$*

$$AX = \alpha X, \tag{4.1.4}$$

where A is the shape operator of M in $\overline{M}$ and $\alpha \in C^\infty(M)$.

We note that Theorem 4.1.2 was already proved in [Gra69, Theorem 4.8]. However, for our proof of Theorem 4.1.2, we define new quantities related to a manifold with a nearly G_2 structure which have analogs in the study of manifolds with a nearly Kähler structure and which we hope will be of further use in the study of submanifolds of manifolds with a nearly G_2 structure.

In a different but related direction, suppose M^n is a closed *minimal* hypersurface of *constant scalar curvature* in the unit sphere S^{n+1} and let A be its shape operator. A famous rigidity theorem due to the combined works of Simons [Sim68], Lawson[Law69] and Chern–doCarmo–Kobayashi [CdCK70] states that if $|A|^2 \leq n$ then $|A|^2 = 0$ or $|A|^2 = n$, where $|A|^2$ is the squared length of the shape operator. If $|A|^2 = 0$, then M is isometric to the totally geodesic equatorial sphere S^n in S^{n+1} and if $|A|^2 = n$, then M is isometric to the Clifford torus $S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$. Following on his study of subsequent gaps for the scalar curvature of such hypersurfaces M, Chern asked the following question (cf. [Yau82, pg.693])

Question 4.1.3. [Chern] *Consider the set of all compact minimal hypersurfaces in S^{n+1} with constant scalar curvature. Think of the scalar curvature as a function on this set. Is the image of the scalar curvature function a discrete set of positive numbers ?*

Since for any minimal hypersurface M^n with scalar curvature R in S^{n+1}, $R = n(n-1) - |A|^2$ (cf. (4.2.7) in §4.2), the above question asks whether the set of $|A|^2$ for such hypersurfaces M is a discrete set.

The first two values of $|A|^2$ are known to be 0 and n. For the third value of $|A|^2$, Peng and Terng [PT83] proved that if $|A|^2 > n$, then there exists a positive constant $\delta(n)$ such that $|A|^2 > n + \delta(n)$. Also, for $n = 3$ they proved that $|A|^2 \geq 6$ and they conjectured that the third value of $|A|^2$ should be equal to $2n$. Yang and Cheng in [YC94] improved the constant $\delta(n)$ by proving that $\delta(n) > \frac{2}{7}n - \frac{9}{14}$ and in [YC98] they further improved this result by proving that if $|A|^2 > n$ then $|A|^2 > \frac{1}{3}(4n+1)$. In [Des10], Deshmukh used the nearly Kähler structure on S^6 to prove the following theorem

Theorem 4.1.4. [Deshmukh, [Des10]] *Let M be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^6. If the shape operator A of M satisfies $|A|^2 > 5$, then there exists an eigenvalue $\lambda > 10$ of the Laplace operator on M satisfying $|A|^2 = \lambda - 5$.*

The round unit sphere S^7 has a nearly G_2 structure, so a natural question is whether we can say anything about the third value of $|A|^2$ for compact minimal hypersurfaces with constant scalar curvature in S^7 by using the nearly G_2 structure on it. Our next result is an analog of Theorem 4.1.4 for minimal hypersurfaces with constant scalar curvature in S^7. More precisely we prove the following

Theorem 4.1.5. *Let M^6 be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^7. If the shape operator A of M satisfies $|A|^2 > 6$, then there exists an eigenvalue $\lambda > 12$ of the Laplace operator on M such that $|A|^2 = \lambda - 6$.*

This puts a restriction on possible examples of compact minimal hypersurfaces of constant scalar curvature in S^7 which have $|A|^2 > 6$ as they must have an eigenvalue $\lambda > 12$ of the Laplacian operator such that $|A|^2 = \lambda - 6$.

In §4.2 we discuss some notions from the geometry of submanifolds. In §4.3 we start by defining several quantities associated to a hypersurface of a nearly G_2 manifold and then prove various relations among them. Using that we prove Theorem 4.1.2. We note that several of the results in §4.3 are already known. However, we prove them using our notations to make the chapter self contained and give references to the original results accordingly. Finally in §4.4, we prove Theorem 4.1.5.

4.2 Geometry of submanifolds

In this section, we briefly recall the geometry of submanifolds. More details can be found, for example in [Lee97]. Let $(\overline{M}, \overline{g})$ be Riemannian manifold and (M, g) be an immersed orientable submanifold of $\overline{M}$ with induced metric. Then for $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + II(X, Y), \tag{4.2.1}$$

where $\overline{\nabla}$ is the covariant derivative on $\overline{M}$, ∇ is the covariant derivative on M and $II : TM \times TM \to NM$ is the second fundamental form of M. Here NM is the normal bundle of M in $\overline{M}$.

If M is an oriented hypersurface of $\overline{M}$ and we denote by N the unit normal vector field of M in $\overline{M}$ corresponding to this orientation, then the second fundamental form is a multiple of N and is given by the *shape operator*, which we denote by A. Here $A : TM \to TM$ is a self-adjoint linear map and (4.2.1) becomes

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N. \tag{4.2.2}$$

We also have the Weingarten equation

$$\overline{\nabla}_X N = -AX. \tag{4.2.3}$$

If $\overline{\mathrm{Rm}}$ denotes the Riemann curvature tensor on $(\overline{M}, \overline{g})$ and Rm denotes the Riemann curvature tensor on (M, g), then the Gauss equation for M is

$$\overline{\mathrm{Rm}}(X,Y,Z,W) = \mathrm{Rm}(X,Y,Z,W) - g(AX,W)g(AY,Z) + g(AX,Z)g(AY,W). \quad (4.2.4)$$

Now suppose $\overline{M}$ is the unit sphere S^7 with the round metric. Then $\overline{\mathrm{Rm}}$ as a $(3,1)-$tensor is given by $\overline{\mathrm{Rm}}(X,Y)Z = \overline{g}(Y,Z)X - \overline{g}(X,Z)Y$. In this case (4.2.4) becomes

$$\mathrm{Rm}(X,Y)Z = \overline{g}(Y,Z)X - \overline{g}(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY. \quad (4.2.5)$$

If M is also a *minimal hypersurface* of S^7 (i.e., the mean curvature vector $H = 0$) then by taking the trace of (4.2.5), the Ricci and the scalar curvature of M are

$$\mathrm{Ric}(X,Y) = 5g(X,Y) - g(AX,AY), \quad (4.2.6)$$
$$R = 30 - |A|^2 \quad (4.2.7)$$

where $|A|^2$ is the square of the length of the shape operator of M. We also have the Codazzi equation, which in this case is

$$\nabla_X(AY) - \nabla_Y(AX) = A([X,Y]). \quad (4.2.8)$$

Finally, we define totally umbilic hypersurface.

Definition 4.2.1. A hypersurface M of a Riemannian manifold $\overline{M}$ is called totally umbilic at $x \in M$ if the shape operator A of M is a multiple of the identity map of T_xM. Moreover M is called totally umbilic if it is totally umbilic at each of its points.

Remark 4.2.2. Throughout the chapter, all quantities associated to the ambient manifold $\overline{M}$ will have a bar on them, for example the metric on $\overline{M}$ is $\overline{g}$ whereas those of the hypersurface are written without any bar.

4.3 Proof of Theorem 4.1.2

We start this section by defining various quantities for hypersurfaces (not necessarily minimal) of a manifold with a nearly G_2 structure which have analogs for hypersurfaces of a manifold with a nearly Kähler structure. Being motivated from the notion of a *characteristic vector field* on a manifold with an almost complex structure, we define a $(1,1)$ tensor

ξ on M^6, induced from the octonionic multiplication on a manifold with a G_2 structure $(\overline{M}^7, \varphi)$, as follows

$$\xi(X) = B(N, X), \tag{4.3.1}$$

where $X \in \Gamma(TM)$ and $B(.,.)$ is the cross product and N is the unit normal to M^6 in $\overline{M}$. We have the following

Proposition 4.3.1. *The tensor ξ is a metric compatible almost complex structure on (M^6, g).*

Proof. For $X \in \Gamma(TM)$, we have

$$\begin{aligned}\xi^2(X) &= \xi(B(N, X)) = B(N, B(N, X)) \\ &= -|N|^2 X + \overline{g}(N, X)N = -X\end{aligned}$$

where the equality in the second line is from the identity (2.1.4) for the cross product. Hence $\xi^2(X) = -X$. Also,

$$\begin{aligned}g(\xi(X), \xi(Y)) &= g(B(N, X), B(N, Y)) \\ &= g(B(B(N, X), N), Y) \\ &= -g(B(N, B(N, X)), Y) = g(X, Y)\end{aligned}$$

where we have used (2.1.3) in going from the first to the second line, the anti-commutativity of B in the first equality and (2.1.4) and the fact that N is a unit vector in the second equality of the third line. $\square$

Remark 4.3.2. The previous proposition is a special case of Theorem 2.6 in [Gra69].

Again, from the motivation from nearly Kähler geometry, we define a $(3, 1)$ tensor field G as follows

$$G(X, Y, Z) = (\overline{\nabla}_X B)(Y, Z) \tag{4.3.2}$$

for $X, Y, Z \in \Gamma(T\overline{M})$.

Now we prove some results about G and relationships between G and B for manifolds with a nearly G_2 structure. The next proposition is a special case of Lemma 3.7 in [Sem03].

Proposition 4.3.3. *Let $\psi = \star\varphi$ denotes the 4-form on $(\overline{M}, \varphi)$ with a nearly G_2 structure. Then for any vector fields X, Y, Z, W*

$$\overline{g}(G(X, Y, Z), W) = \frac{\operatorname{tr} T}{7} \psi(X, Y, Z, W) \tag{4.3.3}$$

where T is the torsion tensor.

Proof. If φ is a G_2 structure then

$$\varphi(X, Y, Z) = \overline{g}(B(X, Y), Z) \tag{4.3.4}$$

Then from (4.3.4) we have

$$\begin{aligned}
\overline{g}(G(X, Y, Z), W) &= \overline{g}((\overline{\nabla}_X B)(Y, Z), W) \\
&= \overline{g}(\overline{\nabla}_X(B(Y, Z)) - B(\overline{\nabla}_X Y, Z) - B(Y, \overline{\nabla}_X Z), W) \\
&= \overline{\nabla}_X(\varphi(Y, Z, W)) - \varphi(\overline{\nabla}_X Y, Z, W) - \varphi(Y, \overline{\nabla}_X Z, W) \\
&\quad - \varphi(Y, Z, \overline{\nabla}_X W) \\
&= (\overline{\nabla}_X \varphi)(Y, Z, W) \\
&= \frac{\operatorname{tr} T}{7} \psi(X, Y, Z, W)
\end{aligned}$$

where we have used $\overline{g}(\overline{\nabla}_X(B(Y, Z)), W) = \overline{\nabla}_X(\overline{g}(B(Y, Z), W)) - \overline{g}(B(Y, Z), \overline{\nabla}_X(W))$ in going from the second to the third equality, $\nabla_i \varphi_{jkl} = T_{im} \psi_{mjkl}$ and the fact that for a nearly G_2 structure, $T_{ij} = \frac{\operatorname{tr} T}{7} g_{ij}$ in the last equality. □

Remark 4.3.4. From (4.3.3), we see that G is skew-symmetric in all of its entries.

Proposition 4.3.5. *For any vector fields X, Y, Z, W, we have*

$$\begin{aligned}
G(B(W, Z), X, Y) = \frac{\operatorname{tr} T}{7} [&\overline{g}(X, Z) B(W, Y) + \overline{g}(Y, Z) B(X, W) - \overline{g}(W, X) B(Z, Y) \\
&- \overline{g}(W, Y) B(X, Z) + \varphi(X, Y, W) Z - \varphi(X, Y, Z) W].
\end{aligned} \tag{4.3.5}$$

Proof. We know from Proposition 4.3.3 that

$$G(X, Y, Z) = \frac{\operatorname{tr} T}{7} \psi(X, Y, Z, \cdot)$$

so

$$G(B(X, Y), Z, W) = \frac{\operatorname{tr} T}{7} \psi(B(X, Y), Z, W, \cdot)^{\#}.$$

In local coordinates $\{x_1, x_2, \dots, x_7\}$, we have $\overline{g}(B(\partial_k, \partial_l), \partial_n) = \varphi_{kln}$. So

$$\begin{aligned}
G(B(\partial_k, \partial_l), \partial_i, \partial_j) &= \frac{\operatorname{tr} T}{7} \psi(B(\partial_k, \partial_l), \partial_i, \partial_j, \cdot)^{\#} \\
&= \frac{\operatorname{tr} T}{7} \psi(\varphi_{kl \cdot}{}^{\#}, \partial_i, \partial_j, .)^{\#} \\
&= -\frac{\operatorname{tr} T}{7} \psi_{ij \cdot n} \varphi_{kln \cdot}
\end{aligned} \tag{4.3.6}$$

Using the identity in (2.1.12)

$$\psi_{ijmn}\varphi_{kln} = \overline{g}_{ki}\varphi_{ljm} + \overline{g}_{kj}\varphi_{ilm} + \overline{g}_{km}\varphi_{ijl} - \overline{g}_{li}\varphi_{kjm} - \overline{g}_{lj}\varphi_{ikm} - \overline{g}_{lm}\varphi_{ijk}$$

we get the proposition. □

Proposition 4.3.6. *For any vector fields X, Y, Z, W, we have*

$$B(G(X,Y,Z),W) = -G(B(X,Y),Z,W). \tag{4.3.7}$$

Proof. In local coordinates $\{x_1,\dots,x_7\}$, we have $G(\partial_k,\partial_l,\partial_m) = \frac{\operatorname{tr} T}{7}\psi^{\#}_{klm\cdot}$ and $B(\partial_k,\partial_l) = \varphi^{\#}_{kl\cdot}$, so

$$B(G(\partial_k,\partial_l,\partial_m),\partial_n) = \frac{\operatorname{tr} T}{7}\varphi(\psi^{\#}_{klm\cdot},\partial_n,\cdot)^{\#} = \frac{\operatorname{tr} T}{7}\varphi^{\#}_{n\cdot p}\overline{g}^{ps}\psi_{klms}.$$

The proposition now follows from the last line of (4.3.6). □

We require the expression for $\nabla_X\xi$ later, so we have the following proposition which is a special case of Proposition 4.7 of [Gra69] for the 2-fold VCP.

Proposition 4.3.7. *Let M be an oriented hypersurface of $(\overline{M},\varphi)$ and ξ be as defined in (4.3.1). Then for any vector field $X \in \Gamma(TM)$, we have*

$$(\nabla_X\xi)(Y) = G(X,N,Y) - \varphi(N,Y,AX)N - B(AX,Y). \tag{4.3.8}$$

Proof. We calculate

$$\begin{aligned}
(\nabla_X\xi)(Y) &= \nabla_X(\xi(Y)) - \xi(\nabla_X Y)\\
&= \overline{\nabla}_X(B(N,Y)) - g(AX,B(N,Y))N - \xi(\nabla_X Y)\\
&= (\overline{\nabla}_X B)(N,Y) + B(\overline{\nabla}_X N,Y) + B(N,\overline{\nabla}_X Y) - g(AX,B(N,Y))N\\
&\quad - \xi(\nabla_X Y)\\
&= G(X,N,Y) - B(AX,Y) + B(N,\nabla_X Y) + g(AX,Y)B(N,N)\\
&\quad - g(AX,B(N,Y))N - \xi(\nabla_X Y)\\
&= G(X,N,Y) - \varphi(N,Y,AX)N - B(AX,Y)
\end{aligned} \tag{4.3.9}$$

where we have used (4.2.2) in the second equality, (4.2.3) and (4.3.2) in the fourth equality and the fact that $B(N,N) = 0$ in the last equality. □

Now we prove Theorem 4.1.2 mentioned in §4.1, namely, we give a necessary and sufficient condition for an oriented hypersurface of a nearly G_2 manifold to be nearly Kähler. We restate the theorem.

Theorem 4.3.8. *Let M be an oriented hypersurface of a nearly G_2 manifold $(\overline{M}, \varphi)$. Then (M, g, ξ) is a nearly Kähler structure if and only if M is totally umbilic, i.e., for all $X \in \Gamma(TM)$*

$$AX = \alpha X, \tag{4.3.10}$$

where A is the shape operator of M in $\overline{M}$ and $\alpha \in C^\infty(M)$.

Proof. We know from (4.1.3) that if J is a metric compatible almost complex structure on M then (M, J, g) is nearly Kähler if and only if for all $X \in \Gamma(TM)$, we have $(\nabla_X J)X = 0$. From Proposition 4.3.1, we know that ξ is a metric compatible almost complex structure on M. Denote by $B(X, Y)^T$, the tangential component of $B(X, Y)$. Using (4.3.8) from Proposition 4.3.7, for $X \in \Gamma(TM)$

$$\begin{aligned} (\nabla_X \xi)(X) = 0 &\iff \\ G(X, N, X) - \varphi(N, X, AX)N - B(AX, X) = 0 &\iff \\ \varphi(N, X, AX)N + B(AX, X)^T + g(B(AX, X), N)N = 0 &\iff \\ B(AX, X)^T + \varphi(AX, X, N)N + \varphi(N, X, AX)N = 0 &\iff \\ B(AX, X)^T = 0 & \end{aligned} \tag{4.3.11}$$

where we used the fact that G is skew-symmetric in all of its entries in going from the second line to the third.

If $X = 0$ then from (4.3.11), the theorem is true. So we assume that $X \neq 0$. Now if $AX = \alpha X$ then

$$\begin{aligned} B(AX, X)^T &= B(\alpha X, X)^T \\ &= 0. \end{aligned} \tag{4.3.12}$$

Thus (4.3.11) and (4.3.12) proves one direction of the theorem.

Now suppose $B(AX, X)^T = 0$. Since AX is tangent to M so we write $AX = \alpha X + Y$ where α is a function which might depend on X and $g(X, Y) = 0$. So $B(Y, X)^T = 0$. Suppose

$$B(Y, X) = fN$$

for some function f.

Then from (2.1.4) we have

$$B(B(Y,X),X) = -|X|^2 Y,$$

Also, $B(B(Y,X),X) = fB(N,X) = f\xi(X)$, so we get

$$Y = -\frac{f}{|X|^2}\xi(X)$$

and hence

$$AX = \alpha X + \beta \xi(X), \tag{4.3.13}$$

where $\beta = -\dfrac{f}{|X|^2}$. Now we prove that $\beta = 0$. Indeed, for any $Z \in \Gamma(TM)$,

$$\begin{aligned} g(AX, Z) &= g(\alpha X + \beta\xi(X), Z) \\ &= \alpha g(X,Z) - \beta g(X, \xi(Z)) \end{aligned}$$

and similarly $g(X, AZ) = \alpha g(X,Z) + \beta g(X, \xi(Z))$. But since A is self-adjoint we get that $2\beta g(X, \xi(Z)) = 0$. So choosing Z such that $B(X,Z) = N$, we get that $\beta = 0$. This proves the other direction. □

Remark 4.3.9. Note that the proof of Theorem 4.3.8 remains unchanged if $G = 0$. So the above theorem also holds for hypersurfaces of G_2 manifolds, i.e., manifolds with torsion free G_2 structures.

Remark 4.3.10. Theorem 4.3.8 was proved in [Gra69, Theorem 4.8] where Gray proved that $\beta = 0$ by using the fact that any nearly Kähler structure J is *quasi-Kähler*, i.e., for all $X, Y \in \Gamma(TM)$, $(\nabla_X J)(Y) + (\nabla_{JX} J)(JY) = 0$. We gave a direct proof that $\beta = 0$.

Remark 4.3.11. Koiso proved in [Koi81, Theorem B] that if (M, g) is a totally umbilic Einstein hypersurface in a complete Einstein manifold $(\overline{M}, \overline{g})$ and g has positive Ricci curvature then both g and $\overline{g}$ have constant sectional curvature. This restricts the possibility for new examples of hypersurfaces which are totally umbilic. It would be interesting to find examples of hypersurfaces in a manifold with a nearly G_2 structure which are nearly Kähler with respect to ξ but are not totally umbilic.

We need the following lemma in §4.4 which is a special case of Theorem 4.10 in [Gra69].

Lemma 4.3.12. *Let M be an oriented hypersurface of a nearly G_2 manifold $(\overline{M}, \varphi)$ and let ξ be as in (4.3.1). Then* $\operatorname{div} \xi = 0$.

Proof. Since A is a self-adjoint operator, we choose an orthonormal frame $\{e_1, ..., e_6\}$ at a point $p \in M$ which diagonalizes A, i.e., $Ae_i = a_i e_i$, $\forall\ i$. Then for $v \in T_pM$ we compute using Proposition 4.3.7

$$
\begin{aligned}
(\operatorname{div}\xi)_p(v) &= \sum_{i=1}^{6} g((\nabla_{e_i}\xi)(v), e_i) \\
&= \sum_{i=1}^{6} g(G(e_i, N, v) - \varphi(N, v, Ae_i)N - B(Ae_i, v), e_i) \\
&= -\sum_{i=1}^{6} g(B(Ae_i, v), e_i) = -\sum_{i=1}^{6} \varphi(Ae_i, v, e_i) = -\sum_{i=1}^{6} \varphi(a_i e_i, v, e_i) \\
&= 0
\end{aligned} \tag{4.3.14}
$$

where we used (4.3.8) in the second equality, Remark 4.3.4 in the third equality and the fact that φ is a 3-from in the last equality. □

4.4 Proof of Theorem 4.1.5

In this section we prove Theorem 4.1.5, stated in §4.1. Let (L, g) be a Riemannian manifold. A vector field X on L is said to be a *conformal vector field* if

$$
\mathcal{L}_X g = 2fg \tag{4.4.1}
$$

for some $f \in C^\infty(L)$, which is called the *potential* of X. Here $\mathcal{L}_X g$ denotes the Lie derivative of g with respect to X. If $f \equiv 0$, then X is a Killing vector field. There are many non-Killing conformal vector fields on the unit sphere S^n with the round metric $\overline{g}$. In particular, if Y is a non-zero constant vector field on $\mathbb{R}^{n+1}$, $\overline{N}$ is the unit normal of S^n in $\mathbb{R}^{n+1}$ and $Y = X + f\overline{N}$, where X is the tangential component of Y, then using (4.2.2) and (4.2.3) and the fact that for S^n as a hypersurface in $\mathbb{R}^{n+1}$, $A = -I$, we see that $\nabla f = X$ and $\nabla_W X = -fW$, and hence $\mathcal{L}_X \overline{g} = -2f\overline{g}$, so X is a conformal vector field with potential $-f$. In fact, all non-Killing conformal vector fields on the unit S^n arise in this manner. (see [MO])

Let M be an oriented compact minimal hypersurface of S^7 satisfying the hypotheses of Theorem 4.1.5, i.e., M is of constant scalar curvature and the shape operator A of M satisfies $|A|^2 > 6$. Let $V, \tilde{V}$ be two non-Killing conformal vector fields on S^7 with potential functions $f, \tilde{f}$ respectively, arising from two linearly independent constant vector fields on $\mathbb{R}^8$. Let $W, \tilde{W}$ be the tangential components on M of V and $\tilde{V}$ respectively. Then we have $V = W + sN$ and $\tilde{V} = \tilde{W} + \tilde{s}N$, where $s, \tilde{s} : M \to \mathbb{R}$.

Using (4.2.2) and (4.2.3), for $X \in \Gamma(TM)$ we get

$$\begin{aligned} \nabla_X W &= \overline{\nabla}_X V - \overline{\nabla}_X(sN) \\ &= -fX + sAX, \end{aligned} \tag{4.4.2}$$

$$\nabla f = W, \tag{4.4.3}$$

$$\nabla s = -AW. \tag{4.4.4}$$

Similarly, we get

$$\nabla_X \tilde{W} = -\tilde{f}X + \tilde{s}AX, \qquad \nabla \tilde{f} = \tilde{W} \quad \text{and} \quad \nabla \tilde{s} = -A\tilde{W}. \tag{4.4.5}$$

Now we define the function $h : M \to \mathbb{R}$ as

$$h = g(\xi W, \tilde{W}). \tag{4.4.6}$$

We are interested in finding $\Delta_M h$. Note that *only in this section* Δ_M will denote the rough Laplacian on M which is a negative operator and hence the spectrum of Δ_M is positive. We compute

$$\begin{aligned} \nabla_X h &= \nabla_X g(\xi W, \tilde{W}) \\ &= g((\nabla_X \xi)W, \tilde{W}) + g(\xi(\nabla_X W), \tilde{W}) + g(\xi W, \nabla_X \tilde{W}) \\ &= g\big(G(X, N, W) - \varphi(N, W, AX)N - B(AX, W)^T, \tilde{W}\big) \\ &\quad + g(\xi(-fX + sAX), \tilde{W}) + g(\xi W, -\tilde{f}X + \tilde{s}AX) \\ &= -g(G(N, W, \tilde{W}), X) - g(B(W, \tilde{W})^T, AX) + g(f\xi\tilde{W}, X) \\ &\quad - g(s\xi\tilde{W}, AX) - g(\tilde{f}\xi W, X) + g(\tilde{s}\xi W, AX), \end{aligned} \tag{4.4.7}$$

so we get

$$\nabla h = -G(N, W, \tilde{W}) - AB(W, \tilde{W})^T + f\xi\tilde{W} - sA\xi\tilde{W} - \tilde{f}\xi W + \tilde{s}A\xi W. \tag{4.4.8}$$

We use (4.4.2), (4.4.3), (4.4.4) and (4.4.5) to calculate the divergence of each term in (4.4.8). For that, we choose a local orthonormal frame $\{e_1, ..., e_6\}$ at $p \in M$ such that $Ae_i = a_i e_i$, for each i.

$$
\begin{aligned}
\operatorname{div}(f\xi\tilde{W}) &= g(\nabla f, \xi\tilde{W}) + f\sum_{i=1}^{6}[g((\nabla_i\xi)\tilde{W}, e_i) + g(\xi(\nabla_i\tilde{W}), e_i)] \\
&= g(W, \xi\tilde{W}) + \sum_{i=1}^{6} g(\xi(-\tilde{f}e_i + \tilde{s}Ae_i), e_i) \\
&= g(W, \xi\tilde{W}) + \sum_{i=1}^{6}[-fg(\xi e_i, e_i) + \tilde{s}a_i g(\xi e_i, e_i)] \\
&= -h \qquad (4.4.9)
\end{aligned}
$$

where we have used Lemma 4.3.12 in the second equality and the definition of ξ to eliminate the terms inside the summation in the third equality.

Similarly

$$
\operatorname{div}(\tilde{f}\xi W) = h. \qquad (4.4.10)
$$

We continue to calculate

$$
\begin{aligned}
\operatorname{div}(sA\xi\tilde{W}) &= g(\nabla s, A\xi\tilde{W}) + s\sum_{i=1}^{6} g(\nabla_i(A\xi\tilde{W}), e_i) \\
&= -g(AW, A\xi\tilde{W}) + s\sum_{i=1}^{6}[\nabla_i g(A\xi\tilde{W}, e_i) - g(A\xi\tilde{W}, \nabla_{e_i}e_i)] \\
&= -g(AW, A\xi\tilde{W}) + s\sum_{i=1}^{6}[g(\nabla_i\xi\tilde{W}, Ae_i) + g(\xi\tilde{W}, \nabla_{e_i}Ae_i) \\
&\qquad - g(\xi\tilde{W}, A\nabla_{e_i}e_i)] \\
&= -g(AW, A\xi\tilde{W}) + s\sum_{i=1}^{6}[g((\nabla_{e_i}\xi)\tilde{W}, Ae_i) + g(\xi(\nabla_{e_i}\tilde{W}), Ae_i)] \\
&= -g(AW, A\xi\tilde{W}) \qquad (4.4.11)
\end{aligned}
$$

where in the third equality we have used that $\sum_i(\nabla A)(e_i, e_i) = \sum_i(\nabla_{e_i}Ae_i - A\nabla_{e_i}e_i) = 0$ which follows from the Codazzi identity (4.2.8) and the fact that M is minimal, (4.3.8), (4.4.5) and $Ae_i = a_i e_i$ to eliminate the terms inside the summation in the second last equality.

Similarly

$$\operatorname{div}(\tilde{s}A\xi W) = -g(A\tilde{W}, A\xi W). \tag{4.4.12}$$

For calculating $\operatorname{div}(AB(W,\tilde{W})^T)$, we repeatedly use (4.2.2) and (4.2.3) to first compute

$$\begin{aligned}
(\overline{\nabla}_Z B)(X,Y) &= \overline{\nabla}_Z(B(X,Y)) - B(\overline{\nabla}_Z X, Y) - B(X, \overline{\nabla}_Z Y) \\
&= \overline{\nabla}_Z\big(B(X,Y)^T + \overline{g}(B(X,Y),N)N\big) - B(\nabla_Z X, Y) \\
&\quad - g(AZ,X)B(N,Y) - B(X,\nabla_Z Y) - g(AZ,Y)B(X,N) \\
&= \nabla_Z(B(X,Y)^T) + g(AZ, B(X,Y)^T)N + \overline{g}(\overline{\nabla}_Z(B(X,Y)),N)N \\
&\quad - \overline{g}(B(X,Y),AZ)N - \overline{g}(B(X,Y),N)AZ - B(\nabla_Z X, Y) \\
&\quad - g(AZ,X)B(N,Y) - B(X,\nabla_Z Y) - g(AZ,Y)B(X,N),
\end{aligned} \tag{4.4.13}$$

where we have written $B(X,Y)$ as a sum of its tangential and normal components in the first term in second equality and then used (4.2.2) in the third equality. So we get

$$\begin{aligned}
(\overline{\nabla}_Z B)(X,Y) &= \nabla_Z(B(X,Y)^T) + \overline{g}((\overline{\nabla}_Z B)(X,Y),N)N + \overline{g}(B(\nabla_Z X,Y),N)N \\
&\quad + g(AZ,X)\overline{g}(B(N,Y),N)N + \overline{g}(B(X,\nabla_Z Y),N)N \\
&\quad + g(AZ,Y)\overline{g}(B(X,N),N)N - \overline{g}(B(X,Y),N)AZ - B(\nabla_Z X,Y) \\
&\quad - g(AZ,X)B(N,Y) - B(X,\nabla_Z Y) - g(AZ,Y)B(X,N) \\
&= \nabla_Z(B(X,Y)^T) + \overline{g}((\overline{\nabla}_Z B)(X,Y),N)N + \overline{g}(B(\nabla_Z X,Y),N)N \\
&\quad + \overline{g}(B(X,\nabla_Z Y),N)N - \overline{g}(B(X,Y),N)AZ - B(\nabla_Z X,Y) \\
&\quad - B(X,\nabla_Z Y) - g(AZ,X)B(N,Y) - g(AZ,Y)B(X,N),
\end{aligned} \tag{4.4.14}$$

where we have used $\overline{g}(B(N,V),N) = \varphi(N,V,N) = 0$, for all V in going from fourth to fifth equality. Now using (4.3.2), we see that (4.4.14) is

$$\begin{aligned}
\nabla_Z(B(X,Y)^T) &= G(Z,X,Y)^T - \overline{g}(B(\nabla_Z X,Y),N)N - \overline{g}(B(X,\nabla_Z Y),N)N \\
&\quad + \overline{g}(B(X,Y),N)AZ + B(\nabla_Z X,Y) + B(X,\nabla_Z Y) \\
&\quad + g(AZ,X)B(N,Y) + g(AZ,Y)B(X,N).
\end{aligned} \tag{4.4.15}$$

Using (4.4.15) we calculate

$$
\begin{aligned}
\operatorname{div}(AB(W,\tilde{W})^T) &= \sum_{i=1}^{6}[g((\nabla_i A)(B(W,\tilde{W})^T),e_i)+g(\nabla_{e_i}(B(W,\tilde{W})^T),Ae_i)] \\
&= \sum_{i=1}^{6} g\big((G(e_i,W,\tilde{W})^T-\overline{g}(B(\nabla_{e_i}W,\tilde{W}),N)N \\
&\quad -\overline{g}(B(W,\nabla_{e_i}\tilde{W}),N)N+\overline{g}(B(W,\tilde{W}),N)Ae_i+B(\nabla_{e_i}W,\tilde{W}) \\
&\quad +B(W,\nabla_{e_i}\tilde{W})+g(Ae_i,W)B(N,\tilde{W})+g(Ae_i,\tilde{W})B(W,N)),Ae_i\big) \\
&= \sum_{i=1}^{6}[\overline{g}(B(N,W),\tilde{W})g(Ae_i,Ae_i)+g(B(-fe_i+sAe_i,\tilde{W}),Ae_i) \\
&\quad +g(B(W,-\tilde{f}e_i+\tilde{s}Ae_i),Ae_i)+g(e_i,AW)g(B(N,\tilde{W}),Ae_i) \\
&\quad +g(e_i,A\tilde{W})g(B(W,N),Ae_i)] \\
&= |A|^2h+g(AW,A\xi\tilde{W})-g(A\tilde{W},A\xi W) \qquad (4.4.16)
\end{aligned}
$$

where we have used Remark 4.3.4 to eliminate the first term inside the summation in the second equality, Proposition 4.3.3 in the third equality and the facts that $\overline{g}(B(a,b),c)=\varphi(a,b,c)$ and $Ae_i=a_ie_i$ in going from the third to last equality.

For calculating $\operatorname{div}(G(N,W,\tilde{W}))$, we first of all note that due to Proposition 4.3.3, $G(N,X,Y)$ is tangent to M for any $X,Y\in\Gamma(TM)$. We calculate

$$
\begin{aligned}
\operatorname{div}(G(N,W,\tilde{W})) &= \sum_{i=1}^{6} g(\nabla_i(G(N,W,\tilde{W})),e_i) \\
&= \sum_{i=1}^{6}[\nabla_i(g(G(N,W,\tilde{W}),e_i))-g(G(N,W,\tilde{W}),\nabla_ie_i)] \\
&= \frac{\operatorname{tr}T}{7}\sum_{i=1}^{6}[(\nabla_i\psi)(N,W,\tilde{W},e_i)+\psi(\nabla_iN,W,\tilde{W},e_i)+\psi(N,\nabla_iW,\tilde{W},e_i) \\
&\quad +\psi(N,W,\nabla_i\tilde{W},e_i)+\psi(N,W,\tilde{W},\nabla_ie_i)-\psi(N,W,\tilde{W},\nabla_ie_i)],
\end{aligned}
$$

where we have used Proposition 4.3.3 in the third equality, (4.4.2), (4.4.5), $Ae_i=a_ie_i$ and the fact that ψ is a 4-form to eliminate the $\psi(N,\nabla_iW,\tilde{W},e_i)$ and $\psi(N,W,\nabla_i\tilde{W},e_i)$ in the last equality.

Thus we get

$$
\begin{aligned}
\operatorname{div}(G(N,W,\tilde{W})) &= \frac{\operatorname{tr} T}{7}\sum_{i=1}^{6}[(\nabla_i\psi)(N,W,\tilde{W},e_i) - \psi(Ae_i,W,\tilde{W},e_i)] \\
&= \frac{\operatorname{tr} T}{7}\sum_{i=1}^{6}[\frac{\operatorname{tr} T}{7}(-g(e_i,N)\varphi(W,\tilde{W},e_i) + g(e_i,W)\varphi(N,\tilde{W},e_i) \\
&\quad - g(e_i,\tilde{W})\varphi(N,W,e_i) + g(e_i,e_i)\varphi(N,W,\tilde{W}))] \\
&= \frac{(\operatorname{tr} T)^2}{49}[g(\xi\tilde{W},W) - g(\xi W,\tilde{W}) + \sum_{i=1}^{6} g(e_i,e_i)g(\xi W,\tilde{W})] \\
&= \frac{4(\operatorname{tr} T)^2}{49}h, \qquad (4.4.17)
\end{aligned}
$$

where we used (2.3.3) (expression for $\nabla_i\psi_{jklm}$) and the fact that for nearly G_2 structures $T_{ij} = \frac{\operatorname{tr} T}{7}g_{ij}$ in the fifth equality.

Using the fact that for the unit S^7, $|T_1|^2 = \dfrac{(\operatorname{tr} T)^2}{7} = 7$, (4.4.8), (4.4.9), (4.4.10), (4.4.11), (4.4.12), (4.4.16) and (4.4.17) we see that

$$
\begin{aligned}
\Delta_M h = -4h - |A|^2 h - g(AW, A\xi\tilde{W}) + g(A\tilde{W}, A\xi W) - h + g(AW, A\xi\tilde{W}) \\
- h - g(A\tilde{W}, A\xi W), \qquad (4.4.18)
\end{aligned}
$$

so

$$
\Delta_M h = -(|A|^2 + 6)h. \qquad (4.4.19)
$$

We note that since M is a compact minimal hypersurface of constant scalar curvature in S^7, $|A|^2$ is constant by (4.2.7) and hence $|A|^2 + h$ is an eigenvalue of Δ_M. Now if h is a constant function then (4.4.19) implies that $h = 0$, i.e., $g(\xi(W),\tilde{W}) = 0$. Recall that $\tilde{W}$ is the tangential component of a non-Killing conformal vector field $\tilde{V}$ on S^7 where $\tilde{V}$ is the tangential component of any constant vector field on $\mathbb{R}^8$. The vector field W was obtained in a similar manner by taking the tangential component of a non-Killing conformal vector field V on S^7 which was obtained as the tangential component of a constant vector field on $\mathbb{R}^8$ which was linearly independent from the constant vector field which gives $\tilde{V}$.

So if $g(\xi(W),\tilde{W}) = 0$ for all $\tilde{W}$, we get $\xi(W) = 0$, i.e., $B(N,W) = 0$. This is a contradiction because ξ is invertible and N is a unit vector. Hence there exists W, $\tilde{W}$ such that h is not constant and (4.4.19) implies that h is an eigenfunction of Δ_M corresponding to the eigenvalue $\lambda = |A|^2 + 6$. So if $|A|^2 > 6$ then $\lambda > 12$ with $|A|^2 = \lambda - 6$. The proof of Theorem 4.1.5 is now complete.

Chapter 5

Some results on Ricci-Bourguignon solitons and almost solitons

5.1 Introduction

In this chapter we study some rigidity properties of the solitons and almost solitons of the Ricci-Bourguignon flow. This chapter is based on [Dwi18]. The author was led to the study of the Ricci-Bourguignon flow as a result of his study of the Laplacian flow for closed G_2 structures.

Ricci solitons play a major role in Ricci flow where they correspond to self-similar solutions of the flow. Thus, given a geometric flow it is natural to study the solitons associated to that flow.

A family of metrics $g(t)$ on an n-dimensional Riemannian manifold (M^n, g) is said to evolve by the *Ricci-Bourguignon* flow (RB flow for short) if $g(t)$ satisfies the following evolution equation

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} - \rho R g), \tag{5.1.1}$$

where Ric is the Ricci tensor of the metric, R is the scalar curvature and $\rho \in \mathbb{R}$ is a constant. The flow in equation (5.1.1) was first introduced by Bourguignon [Bou81], building on some unpublished work of Lichnerowicz and a paper of Aubin [Aub70]. We note that (5.1.1) is precisely the Ricci flow if $\rho = 0$. In particular, the right hand side of the evolution equation (5.1.1) is of special interest for different values of ρ, for example

- $\rho = \frac{1}{2}$, the Einstein tensor $\mathrm{Ric} - \frac{R}{2}g$,
- $\rho = \frac{1}{n}$, the traceless Ricci tensor $\mathrm{Ric} - \frac{R}{n}g$,
- $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $\mathrm{Ric} - \frac{R}{2(n-1)}g$,
- $\rho = 0$, the Ricci tensor Ric.

A systematic study of the parabolic theory of the RB flow was initiated in [CCDMM17]. In that paper, the authors proved, along with many other results, the short time existence of the flow (5.1.1) on any closed n-dimensional manifold starting with an arbitrary initial metric g_0 for $\rho < \frac{1}{2(n-1)}$. As in the Ricci flow case, we make the following

Definition 5.1.1. A Ricci-Bourguignon soliton (RB soliton for short) is a Riemannian manifold (M^n, g) endowed with a vector field X on M that satisfies

$$R_{ij} + \frac{1}{2}(\mathcal{L}_X g)_{ij} = \lambda g_{ij} + \rho R g_{ij} \tag{5.1.2}$$

where $\mathcal{L}_X g$ denotes the Lie derivative of the metric g with respect to the vector field X and $\lambda \in \mathbb{R}$ is a constant.

When $X = \nabla f$ for some smooth $f : M \to \mathbb{R}$ then (M, g) is called a *gradient* RB soliton. The soliton is called

1. expanding when $\lambda < 0$,
2. steady when $\lambda = 0$,
3. shrinking when $\lambda > 0$.

RB solitons correspond to self-similar solutions of the RB flow. An RB soliton is called *trivial* if X is a Killing vector field, i.e., $\mathcal{L}_X g = 0$. We remark that even though the short time existence result for the flow (5.1.1) is for $\rho < \frac{1}{2(n-1)}$, any value of ρ is possible for the considerations of self-similar solutions of the flow.

Gradient RB solitons were studied in detail, for example in [CM16] and [CMM15] where the authors called them *gradient ρ-Einstein solitons*. Various classification and rigidity results about gradient RB solitons were proved in those papers and we refer the reader to those papers for precise statements and proofs of the results.

The notion of Ricci *almost* solitons was introduced in [PRRS11] where the authors modified the definition of a Ricci soliton by considering the parameter λ in the definition of a Ricci soliton to be a *function* rather than a constant. Motivated by the Ricci flow case we make the following

Definition 5.1.2. A Riemannian manifold (M^n, g) is a Ricci-Bourguignon *almost* soliton (RB almost soliton for short) if there is a vector field X and a soliton function $\lambda : M \to \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g. \tag{5.1.3}$$

An RB almost soliton is called a *gradient* RB almost soliton if $X = \nabla f$ for some smooth function f on M and is expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. We note that if X is a Killing vector field then a RB almost soliton is just a RB soliton as it forces λ to be a constant.

Recall that a vector field Y on a Riemannian manifold (M, g) is called a conformal vector field if there exists a function $\psi : M \to \mathbb{R}$ such that

$$\mathcal{L}_Y g = 2\psi g.$$

The conformal vector field is non-trivial if $\psi \neq 0$.

Some characterization results for compact Ricci and Ricci almost solitons were obtained in [ABR11] and [BR12] respectively. In this chapter we generalize the results obtained in those papers to RB and RB almost solitons. More precisely, in §5.3 we prove the following theorems.

Theorem 5.1.3. *Let $(M^n, g, X, \lambda, \rho)$ be a RB soliton with $n \geq 3$ and suppose that the vector field X is a conformal vector field.*

1. *If M is compact then X is a Killing vector field and hence $(M^n, g, X, \lambda, \rho)$ is a trivial RB soliton.*
2. *If M is non-compact, complete and gradient RB soliton then either X is a Killing vector field or $(M^n, g, X, \lambda, \rho)$ is isometric to the Euclidean space.*

This generalizes Theorem 3 in [ABR11] and characterizes compact RB solitons when X is a conformal vector field. The following corollary gives a lower bound for the first eigenvalue of the Laplacian on a compact RB soliton when X is a conformal vector field and generalizes Theorem 4 in [ABR11].

Corollary 5.1.4. *Let $(M^n, g, X, \lambda, \rho)$ be a compact RB soliton with X a conformal vector field. If $n \geq 3$ and $\lambda + \rho R > 0$ then the first eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 \geq (\lambda + \rho R)\frac{n}{n-1}$. Moreover, equality occurs if and only if M^n is isometric to a standard sphere.*

The next theorem characterizes compact RB almost solitons with X a conformal vector field and generalizes Theorem 2 in [BR12].

Theorem 5.1.5. *Let $(M^n, g, X, \lambda, \rho)$ be a compact RB almost soliton with $n \geq 3$. If X is a nontrivial conformal vector field then M^n is isometric to an Euclidean sphere.*

The next theorem generalizes Theorem 3 in [BR12] obtained for compact Ricci almost solitons, which is the case when $\rho = 0$.

Theorem 5.1.6. *Let $(M^n, g, X, \lambda, \rho)$ be a compact RB almost soliton with $n \geq 3$. If $\rho \neq \frac{1}{n}$ and*

$$\int_M [\mathrm{Ric}(X,X) + \frac{n\rho}{n\rho - 1}\nabla_X \operatorname{div} X - 2\rho g(\nabla R, X) - \frac{(n(2\rho+1)-2)}{n\rho - 1} g(\nabla\lambda, X)] \operatorname{vol} \leq 0, \tag{5.1.4}$$

then X is a Killing vector field and M^n is a trivial RB soliton.

Since every RB almost soliton is also a RB soliton for constant λ, using $\nabla\lambda = 0$ we get the following corollary for compact RB soliton

Corollary 5.1.7. *Let $(M^n, g, X, \lambda, \rho)$ be a compact RB soliton with $n \geq 3$. If $\rho \neq \frac{1}{n}$ and*

$$\int_M [\mathrm{Ric}(X,X) + \frac{n\rho}{(n\rho - 1)}\nabla_X \operatorname{div} X - 2\rho g(\nabla R, X)] \operatorname{vol} \leq 0, \tag{5.1.5}$$

then X is a Killing vector field and M^n is a trivial RB soliton.

Remark 5.1.8. Corollary 5.1.7 is an analog of Theorem 1.1 in [PW09] which was for the case of compact Ricci solitons. We obtain Petersen–Wylie's result from ours by setting $\rho = 0$. In fact, the condition in (5.1.5) is analogous to the condition in [PW09, Theorem 1.1], which is obtained when $\rho = 0$ in (5.1.5).

Finally, we obtain an integral formula for compact gradient RB almost solitons generalizing corresponding result for compact gradient Ricci almost solitons from [BR12]

Theorem 5.1.9. *Let $(M^n, g, \nabla f, \lambda, \rho)$ be a compact gradient RB almost soliton. Then*

$$\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 \mathrm{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f)\, \mathrm{vol}, \tag{5.1.6}$$

$$\int_M \left|\mathrm{Ric} - \frac{R}{n} g\right|^2 \mathrm{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f)\, \mathrm{vol}\,. \tag{5.1.7}$$

As an application of the previous theorem we state some conditions for a compact gradient RB almost soliton to be isometric to an Euclidean sphere.

Corollary 5.1.10. *A nontrivial compact gradient RB almost soliton $(M^n, g, \nabla f, \lambda, \rho)$, $n \geq 3$ is isometric to an Euclidean sphere if any one of the following holds*

1. *M^n has constant scalar curvature.*
2. *$\int_M g(\nabla R, \nabla f)\, \mathrm{vol} \leq 0$.*
3. *M^n is a homogenous manifold.*

This chapter is organized as follows. In §5.2 we state and prove some identities for RB solitons and RB almost solitons which are used to prove the main results. The corresponding analogs for Ricci and Ricci almost solitons can be found, for example, in [ABR11] and [BR12] respectively. In §5.3 we prove the main theorems and their corollaries.

5.2 Preliminaries

In this section we prove some general results about RB and RB almost solitons. The proofs of some of these results in the compact gradient case can also be found in [CM16] or [CMM15]. Let us first recall the Ricci identities (1.1.1) for a $(0,2)$-tensor α:

$$\nabla_i \nabla_j \alpha_{kl} - \nabla_j \nabla_i \alpha_{kl} = -R_{ijkm} \alpha_{ml} - R_{ijlm} \alpha_{km}$$

where R_{ijkl} is the Riemann curvature tensor.

We start with the following

Proposition 5.2.1. *Let $(M^n, g, \nabla f, \lambda, \rho)$ be a gradient RB almost soliton. Then the following identities hold*

$$
\begin{aligned}
(1 - n\rho)R + \Delta f &= n\lambda, && (5.2.1)\\
(1 - 2\rho(n-1))\nabla_i R &= 2R_{il}\nabla_l f + 2(n-1)\nabla_i \lambda, && (5.2.2)\\
\nabla_j R_{ik} - \nabla_k R_{ij} &= R_{jkil}\nabla_l f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij})\\
&\quad + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}), && (5.2.3)\\
\nabla_i\big[(1 - 2\rho(n-1))R + |\nabla f|^2 - 2(n-1)\lambda\big] &= (2\rho R + 2\lambda)\nabla_i f. && (5.2.4)
\end{aligned}
$$

Proof. For a gradient RB almost soliton we have

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij} + \rho R g_{ij}. \tag{5.2.5}$$

Taking trace of the above equation gives (5.2.1).

Taking the covariant derivative of (5.2.1) in an orthonormal frame gives

$$(1 - n\rho)\nabla_i R + \nabla_i \nabla_j \nabla_j f = n\nabla_i \lambda.$$

Commuting covariant derivatives and using the contracted second Bianchi identity give

$$
\begin{aligned}
(1 - n\rho)\nabla_i R &= -\nabla_j \nabla_i \nabla_j f + R_{il}\nabla_l f + n\nabla_i \lambda\\
&= -\nabla_j(-R_{ij} + \lambda g_{ij} + \rho R g_{ij}) + R_{il}\nabla_l f + n\nabla_i \lambda\\
&= \frac{1}{2}\nabla_i R - \rho\nabla_i R - \nabla_i \lambda + R_{il}\nabla_l f + n\nabla_i \lambda
\end{aligned}
$$

and hence

$$\Big(\frac{1}{2} - \rho(n-1)\Big)\nabla_i R = R_{il}\nabla_l f + (n-1)\nabla_i \lambda \tag{5.2.6}$$

which proves (5.2.2).

For proving (5.2.3), we use (5.2.5) and commute covariant derivatives to get

$$\begin{aligned}
\nabla_j R_{ik} - \nabla_k R_{ij} &= (\nabla_k \nabla_i \nabla_j f - \nabla_j \nabla_i \nabla_k f) + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) \\
&\quad + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \\
&= (\nabla_k \nabla_j \nabla_i f - \nabla_j \nabla_k \nabla_i f) + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) \\
&\quad + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \\
&= R_{jkil} \nabla_l f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}). \qquad (5.2.7)
\end{aligned}$$

Finally from (5.2.2) we get

$$\begin{aligned}
(1 - 2\rho(n-1))\nabla_i R &= 2\nabla_l f(-\nabla_i \nabla_l f + \lambda g_{il} + \rho R g_{il}) + 2(n-1)\nabla_i \lambda \\
&= -2\nabla_l f \nabla_i \nabla_l f + 2\lambda \nabla_i f + 2\rho R \nabla_i f + 2(n-1)\nabla_i \lambda \\
&= -\nabla_i |\nabla_l f|^2 + 2\lambda \nabla_i f + 2\rho R \nabla_i f + 2(n-1)\nabla_i \lambda,
\end{aligned}$$

so we get

$$\nabla_i \big[(1 - 2\rho(n-1))R + |\nabla f|^2 - 2(n-1)\lambda \big] = (2\rho R + 2\lambda)\nabla_i f \qquad (5.2.8)$$

which proves (5.2.4).

□

Remark 5.2.2. The analogous identities for gradient RB solitons $(M^n, g, \nabla f, \lambda, \rho)$ are

$$\begin{aligned}
(1 - n\rho)R + \Delta f &= n\lambda, \qquad (5.2.9) \\
(1 - 2\rho(n-1))\nabla_i R &= 2R_{il} \nabla_l f, \qquad (5.2.10) \\
\nabla_j R_{ik} - \nabla_k R_{ij} &= R_{jkil} \nabla_l f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}), \qquad (5.2.11) \\
\nabla_i \big[(1 - 2\rho(n-1))R + |\nabla f|^2 - 2\lambda f \big] &= 2\rho R \nabla_i f. \qquad (5.2.12)
\end{aligned}$$

The proofs of these identities are special cases of the previous result as $\nabla \lambda = 0$.

We recall the following lemma from [PW09, Lemma 2.1]

Lemma 5.2.3. *Let X be a vector field on a Riemannian manifold (M^n, g). Then*

$$\operatorname{div}(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta |X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + \nabla_X \operatorname{div} X. \qquad (5.2.13)$$

When $X = \nabla f$ and Z is any vector field then

$$\operatorname{div}(\mathcal{L}_{\nabla f} g)(Z) = 2Ric(Z, \nabla f) + 2\nabla_Z \operatorname{div} \nabla f. \qquad (5.2.14)$$

We use the preceding lemma to prove the following

Lemma 5.2.4. *Let $(M^n, g, X, \lambda, \rho)$ be a RB almost soliton. Then*

$$\frac{(1-n\rho)}{2}\Delta|X|^2 = (1-n\rho)|\nabla X|^2 + (n\rho-1)\mathrm{Ric}(X,X) + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X) \tag{5.2.15}$$

and

$$\frac{(1-n\rho)}{2}(\Delta - \nabla_X)|X|^2 = (1-n\rho)|\nabla X|^2 + \lambda(n\rho-1)|X|^2 + \rho(n\rho-1)R|X|^2 + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X). \tag{5.2.16}$$

Proof. We first notice that (5.1.3) gives

$$2\operatorname{div}\mathrm{Ric} + \operatorname{div}(\mathcal{L}_X g) = 2\nabla\lambda + 2\rho\nabla R. \tag{5.2.17}$$

Taking the trace of (5.1.3) gives $(1-n\rho)R + \operatorname{div} X = n\lambda$ and thus

$$(1-n\rho)\nabla_X R + \nabla_X(\operatorname{div} X) = n\nabla_X\lambda. \tag{5.2.18}$$

So using (5.2.13), (5.2.17), (5.2.18) and the contracted second Bianchi identity, we get

$$\begin{aligned}
\nabla_X(\operatorname{div} X) &= (n\rho-1)\nabla_X R + ng(\nabla\lambda, X)\\
&= 2(n\rho-1)\operatorname{div}\mathrm{Ric}(X) + ng(\nabla\lambda, X)\\
&= -(n\rho-1)\operatorname{div}(\mathcal{L}_X g)(X) + 2\rho(n\rho-1)g(\nabla R, X) + 2(n\rho-1)g(\nabla\lambda, X)\\
&\quad + ng(\nabla\lambda, X)\\
&= (1-n\rho)\Big(\frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \mathrm{Ric}(X,X) + \nabla_X \operatorname{div} X\Big)\\
&\quad + 2\rho(n\rho-1)g(\nabla R, X) + (n(2\rho+1)-2)g(\nabla\lambda, X)\\
&= \frac{(1-n\rho)}{2}\Delta|X|^2 - (1-n\rho)|\nabla X|^2 + (1-n\rho)\mathrm{Ric}(X,X)\\
&\quad + (1-n\rho)\nabla_X \operatorname{div} X + 2\rho(n\rho-1)g(\nabla R, X) + (n(2\rho+1)-2)g(\nabla\lambda, X)
\end{aligned}$$

which gives

$$\frac{(1-n\rho)}{2}\Delta|X|^2 = (1-n\rho)|\nabla X|^2 + (n\rho-1)\mathrm{Ric}(X,X) + n\rho\nabla_X \operatorname{div} X \\ + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X), \tag{5.2.19}$$

thus proving (5.2.15).

Using (5.1.3) to write $\mathrm{Ric}(X,X) = -\frac{1}{2}(\mathcal{L}_X g)(X,X) + \lambda|X|^2 + \rho R|X|^2$ in (5.2.15) we get

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta|X|^2 &= (1-n\rho)|\nabla X|^2 + (n\rho-1)\Big(-\frac{1}{2}(\mathcal{L}_X g)(X,X) + \lambda|X|^2 + \rho R|X|^2\Big) \\ &\quad + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X) \\ &= (1-n\rho)|\nabla X|^2 + \frac{(1-n\rho)}{2}\nabla_X|X|^2 + \lambda(n\rho-1)|X|^2 + \rho(n\rho-1)R|X|^2 \\ &\quad + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X)\end{aligned}$$

which gives

$$\begin{aligned}\frac{(1-n\rho)}{2}(\Delta-\nabla_X)|X|^2 &= (1-n\rho)|\nabla X|^2 + \lambda(n\rho-1)|X|^2 + \rho(n\rho-1)R|X|^2 \\ &\quad + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X) \\ &\quad - (n(2\rho+1)-2)g(\nabla\lambda, X),\end{aligned} \tag{5.2.20}$$

proving (5.2.16). □

If we consider the diffusion operator $\Delta_X = \Delta - \nabla_X$ then the previous lemma with $X = \nabla f$ and $\Delta_f = \Delta - \nabla_{\nabla f}$ gives the following corollary

Corollary 5.2.5. *For a gradient RB almost soliton* $(M, g, \nabla f, \lambda, \rho)$ *we have*

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta_f|\nabla f|^2 &= (1-n\rho)|\nabla^2 f|^2 + \lambda(n\rho-1)|\nabla f|^2 + \rho(n\rho-1)R|\nabla f|^2 \\ &\quad + n\rho\nabla_{\nabla f}(\Delta f) + 2\rho(1-n\rho)g(\nabla R, \nabla f) \\ &\quad - (n(2\rho+1)-2)g(\nabla\lambda, \nabla f).\end{aligned} \tag{5.2.21}$$

Remark 5.2.6. The analogs of (5.2.15) and (5.2.16) for a RB soliton $(M^n, g, X, \lambda, \rho)$ are

$$\frac{(1-n\rho)}{2}\Delta|X|^2 = (1-n\rho)|\nabla X|^2 + (n\rho-1)\mathrm{Ric}(X,X) + n\rho\nabla_X \operatorname{div} X \\ + 2\rho(1-n\rho)g(\nabla R, X) \tag{5.2.22}$$

and

$$\frac{(1-n\rho)}{2}(\Delta - \nabla_X)|X|^2 = (1-n\rho)|\nabla X|^2 + \lambda(n\rho-1)|X|^2 + \rho(n\rho-1)R|X|^2 \\ + n\rho\nabla_X \operatorname{div} X + 2\rho(1-n\rho)g(\nabla R, X). \tag{5.2.23}$$

The proofs are special cases of the proof of Lemma 5.2.4 with $\nabla\lambda = 0$.

5.3 Proofs of the results

We start this section by proving the following lemma which is used in the proof of Theorem 5.1.3 and Theorem 5.1.5.

Lemma 5.3.1. *Let $(M^n, g, X, \lambda, \rho)$ be a RB almost soliton with $n \geq 3$. If X is a nontrivial conformal vector field with $\mathcal{L}_X g = 2\psi g$ then R and $\lambda - \psi$ are constant.*

Proof. The soliton equation is

$$R_{ij} + \frac{1}{2}(\mathcal{L}_X g)_{ij} = \lambda g_{ij} + \rho R g_{ij} \tag{5.3.1}$$

where $\lambda : M \to \mathbb{R}$ is a function. If X is a nontrivial conformal vector field then we have

$$\mathcal{L}_X g = 2\psi g \tag{5.3.2}$$

for some function $\psi : M \to \mathbb{R}$, $\psi \neq 0$. So (5.3.1) becomes

$$R_{ij} = (\lambda - \psi + \rho R)g_{ij}. \tag{5.3.3}$$

Taking the divergence of (5.3.3) we get

$$\begin{aligned} \nabla_i R_{ij} &= \nabla_i(\lambda - \psi + \rho R) g_{ij}, \\ \implies (\frac{1}{2} - \rho)\nabla_j R &= \nabla_j(\lambda - \psi). \end{aligned} \tag{5.3.4}$$

On the other hand, tracing (5.3.3) and taking the covariant derivative we get

$$(1 - n\rho)\nabla_j R = n\nabla_j(\lambda - \rho). \tag{5.3.5}$$

So from (5.3.4) and (5.3.5) we get

$$(1 - n\rho)\nabla_j R = n(\frac{1}{2} - \rho)\nabla_j R. \tag{5.3.6}$$

So if M is connected then R is a constant and hence $\lambda - \psi$ is a constant. $\square$

Remark 5.3.2. If $(M^n, g, X, \lambda, \rho)$ is a RB soliton with $n \geq 3$ and X is a conformal vector field with $\mathcal{L}_X g = 2\psi g$ for some function $\psi : M \to \mathbb{R}$ then the proof of Lemma 5.3.1 shows that R and ψ are constant as in this case $\nabla\lambda = 0$.

We now prove Theorem 5.1.3 which we restate here

Theorem 5.3.3. *Let $(M^n, g, X, \lambda, \rho)$ be a RB soliton with $n \geq 3$ and suppose that the vector field X is a conformal vector field.*

1. *If M is compact then X is a Killing vector field and hence $(M^n, g, X, \lambda, \rho)$ is a trivial RB soliton.*
2. *If M is non-compact, complete and a gradient RB soliton then either X is a Killing vector field or $(M^n, g, X, \lambda, \rho)$ is isometric to the Euclidean space.*

Proof. Suppose X is a conformal vector field with potential $\psi : M \to \mathbb{R}$, i.e.,

$$\mathcal{L}_X g = 2\psi g \tag{5.3.7}$$

then from Remark 5.3.2 we know that R and ψ are constant.

Taking trace of (5.3.7) we get

$$2 \operatorname{div} X = 2n\psi$$

which upon integration over compact M gives

$$0 = \int_M 2\,\mathrm{div}\, X \,\mathrm{vol} = 2n\,\mathrm{Vol}(M)\psi, \tag{5.3.8}$$

i.e., $\psi = 0$. So X is a Killing vector field and hence $(M^n, X, g, \lambda, \rho)$ is a trivial RB soliton.

If M is noncompact and a gradient RB soliton with $X = \nabla f$, then X being conformal implies

$$\nabla_i \nabla_j f = \psi g_{ij}$$

and by Remark 5.3.2, ψ is constant. If $\psi = 0$ then X is a Killing vector field and M is a trivial RB soliton. If $\psi \neq 0$, then from [Tas65, Theorem 2], we conclude that M^n is isometric to the Euclidean space.

□

Next we prove Corollary 5.1.4.

Proof. Since M^n is compact, from Theorem 5.1.3 we know that X is a Killing vector field and hence $\mathrm{Ric} = (\lambda + \rho R)g$. So we can apply a classical theorem due to Lichnerowicz [Lic58] which states that if $\mathrm{Ric} \geq k$ where $k > 0$ is a constant then the first eigenvalue of the Laplacian λ_1 satisfies $\lambda_1 \geq \frac{n}{n-1}k$. So we get

$$\lambda_1 \geq (\lambda + \rho R)\frac{n}{n-1}.$$

Moreover, for the equality case we can apply Obata's theorem [Oba62] to conclude that equality occurs in the above inequality if and only if M^n is isometric to a sphere of constant curvature $\dfrac{\lambda + \rho R}{n-1}$. □

We now prove Theorem 5.1.5 which we restate here

Theorem 5.3.4. *Let $(M^n, g, X, \lambda, \rho)$ be a compact RB almost soliton with $n \geq 3$. If X is a nontrivial conformal vector field then M^n is isometric to an Euclidean sphere.*

Proof. Suppose X is a nontrivial conformal vector field with potential function $\psi : M \to \mathbb{R}$, i.e.,

$$\mathcal{L}_X g = 2\psi g$$

with $\psi \neq 0$. Since $(M^n, g, X, \lambda, \rho)$ is a compact RB almost soliton with $n \geq 3$, Lemma 5.3.1 tells us that R and $\lambda - \psi$ are constant. So from Lemma 2.3 in [Yan70, pg.52] we conclude that $R \neq 0$ or else ψ would be 0. Taking the Lie derivative of (5.3.3) we get

$$\mathcal{L}_X \mathrm{Ric} = \mathcal{L}_X(\lambda - \psi + \rho R) g$$

and since $(\lambda - \psi)$, ρ and R are all constant so we get

$$\mathcal{L}_X \mathrm{Ric} = 2(\lambda - \psi + \rho R)\psi g. \tag{5.3.9}$$

Now we can apply Theorem 4.2 of [Yan70, pg. 54] to conclude that M is isometric to an Euclidean sphere. □

We proceed to the proof of Theorem 5.1.6.

Proof. We see from (5.2.15) of Lemma 5.2.4 that

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta|X|^2 = (1-n\rho)|\nabla X|^2 + (n\rho - 1)\mathrm{Ric}(X,X) + n\rho\nabla_X \operatorname{div} X \\ + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X).\end{aligned}$$

Integrating above over compact M we get

$$\begin{aligned}0 = \int_M [(1-n\rho)|\nabla X|^2 + (n\rho - 1)\mathrm{Ric}(X,X) + n\rho\nabla_X \operatorname{div} X \\ + 2\rho(1-n\rho)g(\nabla R, X) - (n(2\rho+1)-2)g(\nabla\lambda, X)] \operatorname{vol}.\end{aligned} \tag{5.3.10}$$

Since $\rho \neq \frac{1}{n}$, we get

$$\begin{aligned}\int_M |\nabla X|^2 \operatorname{vol} = \int_M [\mathrm{Ric}(X,X) + \frac{n\rho}{n\rho - 1}\nabla_X \operatorname{div} X - 2\rho g(\nabla R, X) \\ - \frac{(n(2\rho+1)-2)}{n\rho - 1} g(\nabla\lambda, X)] \operatorname{vol},\end{aligned} \tag{5.3.11}$$

so if (5.1.4) holds then $|\nabla X|^2 = 0$ and hence X is a Killing vector field. Thus $(M^n, g, X, \lambda, \rho)$ is trivial.

□

The proof of Corollary 5.1.7 is a special case of the proof of Theorem 5.1.6 where we use (5.2.22) of Remark 5.2.6.

Next, we prove Theorem 5.1.9 which we restate here

Theorem 5.3.5. *Let $(M^n, g, \nabla f, \lambda, \rho)$ be a compact gradient RB almost soliton. Then*

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 \,\mathrm{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) \,\mathrm{vol}, \tag{5.3.12}$$

$$\int_M |\mathrm{Ric} - \frac{R}{n} g|^2 \,\mathrm{vol} = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) \,\mathrm{vol}\,. \tag{5.3.13}$$

Proof. For proving (5.3.12) we take the divergence of (5.2.4) of Proposition 5.2.1 to get

$$\begin{aligned}(1 - 2\rho(n-1))\Delta R + \Delta|\nabla f|^2 - 2(n-1)\Delta\lambda = 2\rho g(\nabla R, \nabla f) + 2g(\nabla\lambda, \nabla f)\\ + (2\rho R + 2\lambda)\Delta f.\end{aligned} \tag{5.3.14}$$

By commuting covariant derivatives we have

$$\begin{aligned}\nabla_i\nabla_i(g(\nabla_j f, \nabla_j f)) &= 2\nabla_i(g(\nabla_i\nabla_j f, \nabla_j f))\\ &= 2g(\nabla_i\nabla_i\nabla_j f, \nabla_j f) + 2|\nabla^2 f|^2\\ &= 2g(\nabla_j\nabla_i\nabla_i f - R_{ijil}\nabla_l f, \nabla_j f) + 2|\nabla^2 f|^2\\ &= 2g(\nabla(\Delta f), \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\nabla^2 f|^2\end{aligned}$$

so (5.3.14) becomes

$$\begin{aligned}(1 - 2\rho(n-1))\Delta R + 2g(\nabla(\Delta f), \nabla f) + 2\mathrm{Ric}(\nabla f, \nabla f) + 2|\nabla^2 f|^2 - 2(n-1)\Delta\lambda =\\ 2\rho g(\nabla R, \nabla f) + 2g(\nabla\lambda, \nabla f) + (2\rho R + 2\lambda)\Delta f.\end{aligned} \tag{5.3.15}$$

From (5.2.1) of Proposition 5.2.1 we know that $\Delta f = n\lambda + (n\rho - 1)R$ which on differentiation and using (5.2.5) becomes

$$
\begin{aligned}
0 &= \nabla_i \Delta f + (1-n\rho)\nabla_i R - n\nabla_i \lambda \\
&= (1-n\rho)\nabla_i R + \nabla_j \nabla_i \nabla_j f - R_{il}\nabla_l f - n\nabla_i \lambda \\
&= (1-n\rho)\nabla_i R + \nabla_j(-R_{ij} + \lambda g_{ij} + \rho R g_{ij}) - R_{il}\nabla_l f - n\nabla_i \lambda \\
&= (\frac{1}{2} - \rho(n-1))\nabla_i R - R_{il}\nabla_l f + (1-n)\nabla_i \lambda
\end{aligned}
$$

and hence

$$
2\mathrm{Ric}(\nabla f, \nabla f) = (1-2\rho(n-1))g(\nabla R, \nabla f) + 2(1-n)g(\nabla\lambda, \nabla f). \tag{5.3.16}
$$

So using (5.3.16) and $\Delta f = n\lambda + (n\rho - 1)R$, the left hand side of (5.3.15) becomes

$$
(1-2\rho(n-1))\Delta R + 2|\nabla^2 f|^2 - 2(n-1)\Delta\lambda + 2g(\nabla\lambda, \nabla f) + (2\rho - 1)g(\nabla R, \nabla f)
$$

and hence (5.3.15) becomes

$$
(1-2\rho(n-1))\Delta R + 2|\nabla^2 f|^2 - 2(n-1)\Delta\lambda = g(\nabla R, \nabla f) + (2\rho R + 2\lambda)\Delta f. \tag{5.3.17}
$$

Since $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}$, equation (5.3.17) becomes

$$
\begin{aligned}
(1-2\rho(n-1))\Delta R + 2|\nabla^2 f - \frac{\Delta f}{n} g|^2 &= g(\nabla R, \nabla f) + (2\rho R + 2\lambda)\Delta f - 2\frac{(\Delta f)^2}{n} \\
&\quad + 2(n-1)\Delta\lambda \\
&= g(\nabla R, \nabla f) + (2\rho R + 2\lambda)\Delta f \\
&\quad - 2\frac{(\Delta f)}{n}(n\lambda + (n\rho - 1)R) + 2(n-1)\Delta\lambda \\
&= g(\nabla R, \nabla f) + \frac{2}{n} R\Delta f + 2(n-1)\Delta\lambda. \qquad (5.3.18)
\end{aligned}
$$

Integrating (5.3.18) over compact M gives

$$
\begin{aligned}
\int_M 2|\nabla^2 f - \frac{\Delta f}{n} g|^2 \,\mathrm{vol} &= \int_M \left[g(\nabla R, \nabla f) + \frac{2}{n} R\Delta f \right] \mathrm{vol} \\
&= \frac{(n-2)}{n} \int_M g(\nabla R, \nabla f) \,\mathrm{vol} \qquad (5.3.19)
\end{aligned}
$$

where we have used integration by parts in the second equality. This proves (5.3.12).

For proving (5.3.13) note that

$$\begin{aligned}\mathrm{Ric} - \frac{R}{n}g &= -\nabla^2 f + \lambda g + \rho R g - \frac{R}{n}g \\ &= -\nabla^2 f + (\lambda + \rho R - \frac{R}{n})g \\ &= -\nabla^2 f + \frac{\Delta f}{n}g \end{aligned} \tag{5.3.20}$$

and then (5.3.13) follows from (5.3.12). □

Remark 5.3.6. Since a gradient RB soliton is a special case of a gradient RB almost soliton, the proof of Theorem 5.1.9, with $\nabla\lambda = 0$, shows that the same integral formulas (5.3.12) and (5.3.13) hold for a compact gradient RB soliton too.

Finally, using Theorem 5.1.9 we prove Corollary 5.1.10.

Proof. Observe that any of the assumptions of Corollary 5.1.10 enable us to conclude that the right hand side of (5.3.13) is less than or equal to zero and hence $\mathrm{Ric} = \frac{R}{n}g$. So from (5.2.5) we see that

$$\nabla_i \nabla_j f = (\lambda + R(\rho - \frac{1}{n}))g$$

and hence ∇f is a nontrivial conformal vector field so from Theorem 5.1.5 we get that M^n is isometric to an Euclidean sphere. □

Chapter 6

A gradient flow of isometric G_2 structures

6.1 Introduction

The existence of torsion-free G_2-structures on a manifold is a challenging problem. Geometric flows are a powerful tool to tackle such questions and one hopes that a suitable flow of G_2-structures might help in establishing the existence of torsion-free G_2-structures. There has been a lot of work in this direction. General flows of G_2-structures were considered by Karigiannis in [Kar09]. Earlier in [Bry06], Bryant introduced the Laplacian flow of closed G_2-structures. Several foundational results for the Laplacian flow for closed G_2-structures were established in a series of papers [LW17; LW19b; LW19a] by Lotay–Wei. The Laplacian flow for co-closed G_2-structures was introduced by Karigiannis–McKay–Tsui in [KMT12] and a modified co-flow was studied by Grigorian [Gri13]. An approach via gradient flow of energy-type functionals was introduced by Weiss–Witt [WW12] and Ammann–Weiss–Witt in [AWW16].

In the present chapter, we study a different but related problem, in that we use a particular geometric flow to look for a G_2-structure which is in some sense optimal. Specifically, we consider a flow $\varphi(t)$ of G_2-structures on a manifold M that *preserves the Riemannian metric*, which we call the *isometric flow* of G_2-structures. This flow is the negative gradient flow of a natural energy functional restricted to the set of G_2-structures inducing a fixed metric. The flow seeks a G_2-structure amongst those G_2-structures inducing the same fixed metric which has minimal L^2 norm of torsion.

One possible motivation for studying this isometric flow of G_2-structures is that it can be coupled with "Ricci flow" of G_2-structures, which is a flow of G_2-structures that induces *precisely* the Ricci flow on metrics, in contrast to the Laplacian flow which induces Ricci flow plus lower order terms involving the torsion. In effect, one may hope to first flow the 3-form in a way that improves the metric, and then flow the 3-form in a way that preserves the metric but still decreases the torsion. More generally, the isometric flow is a particular geometric flow of G_2-structures distinct from the Laplacian flow, and both fit into a broader class of geometric flows of G_2-structures with good analytic properties. A detailed study of a general class of flows that includes both the Laplacian flow and the isometric flow is undertaken in [DGK].

We develop a comprehensive foundational theory for the isometric flow. A summary of the main results of the chapter is as follows.

In §6.2 we discuss preliminary results on the isometric flow, including the gradient of the energy functional, short-time existence, parabolic rescaling, and solitons.

In §6.3 we prove Shi-type estimates for the flow (Theorem 6.3.3). We also prove local derivative estimates in Theorem 6.3.7. Using these we show that the flow (6.2.8) has a solution as long as the torsion tensor T remains bounded along the flow (Theorem 6.3.8). We also derive a compactness theorem for solutions along the flow (Theorem 6.3.13).

In §6.4, we briefly summarize the rest of the results from [DGK19].

We note that the paper by Grigorian [Gri19] studies the same flow and he obtained similar results by using the theory of octonion bundles whereas in the present chapter we use a more traditional geometric flows approach. Another closely related preprint is by Loubeau–Sà Earp [LE19], in which they consider the more general context of harmonic G-structures for a fixed Riemannian metric.

6.2 Preliminary results on the Isometric Flow

In this section we discuss several preliminary properties of the isometric flow. This includes a derivation of the fact that it is the negative gradient flow of the energy functional, short-time existence, and parabolic rescaling which we use frequently as a crucial tool. We also discuss solitons for the isometric flow.

6.2.1 The isometric flow of $\mathbf{G_2}$-structures

In this section we define the isometric flow, and establish that it is a negative gradient flow.

Definition 6.2.1 (Isometric $\mathbf{G_2}$-structures)**.** Two G_2-structures φ_1 and φ_2 on M are called *isometric* if they induce the same Riemannian metric, that is if $g_{\varphi_1} = g_{\varphi_2}$. We will denote the space of G_2-structures that are isometric to a given G_2-structure φ by $[\![\varphi]\!]$.

Remark 6.2.2. The space of *torsion-free* G_2-structures that induce the same Riemannian metric was studied by Lin [Lin18]. We do not restrict to torsion-free G_2-structures in the present paper.

Fix an initial G_2-structure φ_0 on M.

Definition 6.2.3. Define the *energy functional* E on the set $[\![\varphi_0]\!]$ by

$$E(\varphi) = \frac{1}{2}\int_M |T_\varphi|^2 \operatorname{vol}_\varphi \tag{6.2.1}$$

where T_φ is the torsion of φ.

Note that E is the same functional considered in [WW12], but here we only allow φ to vary in the class $[\![\varphi_0]\!]$ of isometric G_2-structures, whereas in [WW12] the functional was considered on the space of *all* G_2-structures.

The functional E in (6.2.1) was considered by Grigorian in [Gri17] in the context of "octonionic bundles" over M where he showed that the critical points of the functional are precisely the G_2-structures with divergence-free torsion, that is, $\operatorname{div} T = 0$. Note that the underlying metric here is the same for all G_2-structures in $[\![\varphi_0]\!]$, so the divergence is unambiguously defined. A very natural question arises: given any initial G_2-structure φ_0 on M what is the 'best' G_2-structure in the class $[\![\varphi_0]\!]$. An obvious way to study this question is to consider the negative gradient flow of the functional (6.2.1). (In fact it is more convenient to take the negative gradient flow of $4E$. See Proposition 6.2.5.)

Before we can describe this flow, recall from (2.2.11) that if h be a symmetric 2-tensor on M then we can define a 3-form $h \diamond \varphi$ on M by the formula

$$(h \diamond \varphi)_{ijk} = h_{ip}\varphi_{pjk} + h_{jp}\varphi_{ipk} + h_{kp}\varphi_{ijp}. \tag{6.2.2}$$

Note from (6.2.2) that if $h = g$ is the metric, we get

$$g \diamond \varphi = 3\varphi. \tag{6.2.3}$$

Then from (3.1.1) the most general flow of G_2-structures is given by

$$\frac{\partial \varphi}{\partial t} = h \diamond \varphi + X \lrcorner \psi \tag{6.2.4}$$

where h is a time-dependent symmetric 2-tensor and X is a time-dependent vector field. In this case the flow of the metric g is given by

$$\frac{\partial g}{\partial t} = 2h. \tag{6.2.5}$$

To begin we consider the first variation of the torsion T with respect to variations of the G_2-structure that preserve the metric.

Lemma 6.2.4. *Let $(\varphi_t)_{t\in(-\delta,\delta)}$ be a smooth family of G_2-structures in the class $[\![\varphi]\!]$ with $\varphi_0 = \varphi$. By equations (6.2.4) and (6.2.5), we can write $\frac{\partial}{\partial t}\big|_{t=0} \varphi_t = X \lrcorner \psi$ for some vector field X. Let T_t be the torsion of φ_t. Then we have*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (T_t)_{ij} = \nabla_i X_j + X_l T_{im} \varphi_{lmj}. \tag{6.2.6}$$

Proof. Since $g_t = g$ for all $t \in (-\delta, \delta)$, the covariant derivative ∇ is independent of T. Since $\frac{\partial}{\partial t}\big|_{t=0} \varphi_t = X \lrcorner \psi$, by [Kar09, Theorem 3.5] we have $\frac{\partial}{\partial t}\big|_{t=0} \psi_t = -X \wedge \varphi$. That is, we have

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t)_{ijk} &= X_p \psi_{pijk}, \\
\left.\frac{\partial}{\partial t}\right|_{t=0} (\psi_t)_{ijkl} &= -X_i \varphi_{jkl} + X_j \varphi_{ikl} - X_k \varphi_{ijl} + X_l \varphi_{ijk}.
\end{aligned}$$

From these observations and equation (2.3.2), we compute

$$\begin{aligned}
24 \left.\frac{\partial}{\partial t}\right|_{t=0} (T_t)_{ij} &= \left.\frac{\partial}{\partial t}\right|_{t=0} (\nabla_i (\varphi_t)_{abc} (\psi_t)_{jabc}) \\
&= \nabla_i \Big(\left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t)_{abc} \Big) \psi_{jabc} + \nabla_i \varphi_{abc} \Big(\left.\frac{\partial}{\partial t}\right|_{t=0} (\psi_t)_{jabc} \Big) \\
&= \nabla_i (X_p \psi_{pabc}) \psi_{jabc} + \nabla_i \varphi_{abc} (-X_j \varphi_{abc} + X_a \varphi_{jbc} - X_b \varphi_{jac} + X_c \varphi_{jab}).
\end{aligned}$$

Using (2.3.1), (2.3.3) and the contraction identities (2.1.12) and (2.1.15), the above becomes

$$\begin{aligned}
24 \left.\frac{\partial}{\partial t}\right|_{t=0} (T_t)_{ij} &= \nabla_i X_p \psi_{pabc}\psi_{jabc} + X_p \nabla_i \psi_{pabc}\psi_{jabc} \\
&\quad + T_{ip}\psi_{pabc}(-X_j\varphi_{abc} + X_a\varphi_{jbc} - X_b\varphi_{jac} + X_c\varphi_{jab}) \\
&= 24\nabla_i X_p g_{pj} + X_p(-T_{ip}\varphi_{abc} + T_{ia}\varphi_{pbc} - T_{ib}\varphi_{pac} + T_{ic}\varphi_{pab})\psi_{jabc} \\
&\quad - 0 + 3T_{ip}X_a\varphi_{jbc}\psi_{pabc} \\
&= 24\nabla_i X_j - 0 + 3T_{ia}X_p\varphi_{pbc}\psi_{jabc} + 3T_{ip}X_a(-4\varphi_{jpa}) \\
&= 24\nabla_i X_j + 3T_{ia}X_p(-4\varphi_{pja}) - 12X_aT_{ip}\varphi_{jpa} \\
&= 24\nabla_i X_j + 24X_aT_{ip}\varphi_{apj},
\end{aligned}$$

which is precisely (6.2.6). □

Now let E be the energy functional from Definition 6.2.3, restricted to the set $[\![\varphi]\!]$ of G_2-structures inducing the same metric as φ.

Proposition 6.2.5. *The* gradient *of* $4E : [\![\varphi]\!] \to \mathbb{R}$ *at the point* φ *is* $-\operatorname{div} T \lrcorner \psi$*, where* T *is the torsion of* φ *and* $\psi = \star\varphi$*. That is, if* $(\varphi_t)_{t\in(-\delta,\delta)}$ *is a smooth family in the class* $[\![\varphi]\!]$ *with* $\varphi_0 = \varphi$ *and* $\left.\frac{d}{dt}\right|_{t=0} \varphi_t = \eta$*, then*

$$\left.\frac{d}{dt}\right|_{t=0} 4E(\varphi_t) = -\int_M \langle \operatorname{div} T \lrcorner \psi, \eta\rangle \operatorname{vol}_g .$$

Proof. Using Lemma 6.2.4 compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) &= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}\int_M (T_t)_{ij}(T_t)_{ij} \operatorname{vol}_g \\
&= \int_M T_{ij}(\nabla_i X_j + X_l T_{im}\varphi_{lmj}) \operatorname{vol}_g .
\end{aligned}$$

The second term vanishes because $T_{ij}T_{im}$ is symmetric in j, m and φ_{lmj} is skew in j, m. We integrate by parts on the first term to obtain

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M X_j \nabla_i T_{ij} \operatorname{vol}_g = -\int_M \langle X, \operatorname{div} T\rangle \operatorname{vol}_g .$$

Equation (2.1.16) implies that $\langle X \lrcorner \psi, Y \lrcorner \psi\rangle = \frac{1}{6} X_p \psi_{pabc} Y_q \psi_{qabc} = 4X_pY_p = 4\langle X, Y\rangle$, so the above equation becomes

$$\left.\frac{d}{dt}\right|_{t=0} 4E(\varphi_t) = -4\int_M X_j \nabla_i T_{ij} \operatorname{vol}_g = -\int_M \langle X \lrcorner \psi, \operatorname{div} T \lrcorner \psi\rangle \operatorname{vol}_g . \tag{6.2.7}$$

The space of 3-forms decomposes into the pointwise orthogonal splitting

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where $\Omega^3_7 = \{Y \lrcorner \psi : Y \in \Gamma(TM)\}$. Using this observation, the result follows immediately from (6.2.7). $\square$

We can now define the isometric flow.

Definition 6.2.6 (The isometric flow)**.** Let (M^7, φ_0) be a compact manifold with a G_2-structure. Consider the negative gradient flow of the functional $4E$ restricted to the class $[\![\varphi]\!]$. By Proposition 6.2.5, this evolution of φ is given by

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \operatorname{div} T \lrcorner \psi, \\ \varphi(0) = \varphi_0. \end{cases} \tag{6.2.8}$$

We call (6.2.8) the *isometric flow* of G_2-structures. Note from (6.2.4) that $h \equiv 0$ for the isometric flow and hence (6.2.8) is indeed a flow of isometric G_2-structures.

6.2.2 Short time existence

The isometric flow (6.2.8) has short time existence and uniqueness, because it is equivalent to a strictly parabolic flow. This was first proved by Bagaglini in [Bag19] using spinorial methods. A proof is also given in Grigorian [Gri19, Section 5] using octonion algebra. In this section we explain how to derive the equivalent strictly parabolic flow, avoiding the use of spinors or octonions. The full details are quite laborious and unenlightening. We need to make extensive use of the various contraction identities of φ with φ and φ with ψ. We present just enough details so that the interested reader can fill in the gaps on their own.

Note: In this section only, for brevity, we use $\dot{A}$ to denote the time derivative of A.

The starting point is the following result of Bryant.

Proposition 6.2.7 ([Bry06, Equation (3.6)])**.** *Let (M, φ) be a manifold with G_2-structure such that φ induces the Riemannian metric g. Then all the other G_2-structures on M inducing the same metric g can be parametrized by a pair (f, X) where f is a function and X is a vector field satisfying $f^2 + |X|^2 = 1$. The explicit formula for the G_2-structure $\varphi_{(f,X)}$ corresponding to the pair (f, X) is*

$$\varphi_{(f,X)} = (f^2 - |X|^2)\varphi - 2fX \lrcorner \psi + 2X \wedge (X \lrcorner \varphi), \tag{6.2.9}$$

where $\psi = \star_g \varphi$ and the norm of X is taken with respect to g. Note that the pair $(-f, -X)$ induces the same G_2-structure as (f, X) so in fact the G_2-structures on M inducing the metric g correspond to sections of an $\mathbb{RP}^7$-bundle over M.

Fix a pair (f, X) with $f^2 + |X|^2 = 1$ and write $\widetilde{\varphi}$ for $\varphi_{f,X}$. In terms of a local orthonormal frame, equation (6.2.9) is

$$\begin{aligned} \widetilde{\varphi}_{ijk} = (1 - 2|X|^2)\varphi_{ijk} - 2f X_m \psi_{mijk} \\ + 2X_i X_m \varphi_{mjk} + 2X_j X_m \varphi_{imk} + 2X_k X_m \varphi_{ijm}. \end{aligned} \tag{6.2.10}$$

Since $\widetilde{\varphi}$ induces the same metric g as φ, they have the same Hodge star operator $\star$, so we have $\psi_{(f,X)} = \star\varphi_{(f,X)}$. Using equation (6.2.9) and the identity $\star(X \wedge \alpha) = (-1)^k X \lrcorner \star\alpha$ for α a k-form, we obtain

$$\begin{aligned} \psi_{(f,X)} &= (1 - 2|X|^2) \star \varphi - 2f \star (X \lrcorner \psi) + 2 \star (X \wedge (X \lrcorner \varphi)) \\ &= (1 - 2|X|^2)\psi + 2f X \wedge \varphi + 2X \lrcorner \star (X \lrcorner \varphi) \\ &= (1 - 2|X|^2)\psi + 2f X \wedge \varphi + 2X \lrcorner (X \wedge \psi). \end{aligned}$$

Using the fact that $\lrcorner$ is a derivation, this becomes

$$\begin{aligned} \psi_{(f,X)} &= (1 - 2|X|^2)\psi + 2f X \wedge \varphi + 2|X|^2\psi - 2X \wedge (X \lrcorner \psi) \\ &= \psi + 2f X \wedge \varphi - 2X \wedge (X \lrcorner \psi). \end{aligned}$$

In a local frame this is

$$\begin{aligned} \widetilde{\psi}_{qjkl} = \psi_{qjkl} + 2f(X_q \varphi_{jkl} - X_j \varphi_{qkl} + X_k \varphi_{qjl} - X_l \varphi_{qjk}) \\ - 2(X_q X_m \psi_{mjkl} + X_j X_m \psi_{qmkl} + X_k X_m \psi_{qjml} + X_l X_m \psi_{qjkm}). \end{aligned} \tag{6.2.11}$$

Note that all the contractions above are taken with respect to the fixed metric g that is induced by both φ and $\widetilde{\varphi}$.

Now suppose that φ_t is evolving by the isometric flow (6.2.8). Since the metric is constant, this time-dependent G_2-structure will correspond by (6.2.9) to a time-dependent pair (f, X). We write $\widetilde{\varphi}$ for φ_t, with torsion $\widetilde{T} = T_t$. The initial condition $\varphi_0 = \varphi$ corresponds to initial conditions $f_0 = 1$ and $X_0 = 0$.

Proposition 6.2.8. *Under the isometric flow, the pair (f, X) evolves by*

$$\begin{aligned} \dot{f} &= \frac{1}{2}\langle X, \operatorname{div} \widetilde{T} \rangle, \\ \dot{X} &= -\frac{1}{2} f \operatorname{div} \widetilde{T} + \frac{1}{2}(\operatorname{div} \widetilde{T}) \times X, \end{aligned} \tag{6.2.12}$$

where $\times$ is the cross product with respect to the initial G_2-structure φ, given by $(Y \times X)_k = Y_a X_b \varphi_{abk}$, and $\langle \cdot, \cdot \rangle$ is the inner product given by the metric g.

Proof. Let $\gamma = \dot{\varphi}_t$. Since φ and ψ in equation (6.2.10) are constant in time, differentiating with respect to t we get

$$\begin{aligned}\gamma_{ajk} = &-4\langle X, \dot{X}\rangle \varphi_{ajk} - 2\dot{f} X_m \psi_{majk} - 2f\dot{X}_m \psi_{majk} \\ &+ 2\dot{X}_a X_m \varphi_{mjk} + 2\dot{X}_j X_m \varphi_{amk} + 2\dot{X}_k X_m \varphi_{ajm} \\ &+ 2X_a \dot{X}_m \varphi_{mjk} + 2X_j \dot{X}_m \varphi_{amk} + 2X_k \dot{X}_m \varphi_{ajm}.\end{aligned}$$

Let $\sigma = \operatorname{div} \widetilde{T} \lrcorner \widetilde{\psi}$. Using (6.2.11) we have

$$\begin{aligned}\sigma_{ajk} &= (\operatorname{div} \widetilde{T})_m \widetilde{\psi}_{majk} \\ &= (\operatorname{div} \widetilde{T})_m \psi_{majk} + 2f(\operatorname{div} \widetilde{T})_m (X_m \varphi_{ajk} - X_a \varphi_{mjk} + X_j \varphi_{mak} - X_k \varphi_{maj}) \\ &\quad - 2(\operatorname{div} \widetilde{T})_m (X_m X_p \psi_{pajk} + X_a X_p \psi_{mpjk} + X_j X_p \psi_{mapk} + X_k X_p \psi_{majp}).\end{aligned}$$

Under the flow we have $\gamma = \dot{\varphi}_t = \operatorname{div} \widetilde{T} \lrcorner \widetilde{\psi} = \sigma$, so we must have $\gamma_{ajk} = \sigma_{ajk}$. Contracting both sides of this equation with φ_{ijk} gives an equivalent equation, as the map $\alpha_{ajk} \mapsto \varphi_{ijk}\alpha_{ajk}$ is a linear isomorphism from $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ onto $\mathrm{Sym}^2 \oplus \Lambda^2_7$, the space of 2-tensors with no Λ^2_{14} component. (See [Kar09] for details.) Now using the contraction identities (2.1.9) and (2.1.12), one can compute that

$$\varphi_{ijk}\gamma_{ajk} = -16\langle X, \dot{X}\rangle g_{ia} + 8(X_i \dot{X}_a + X_a \dot{X}_i) - 8(\dot{f} X_p + f\dot{X}_p)\varphi_{pia}, \tag{6.2.13}$$

and similarly that

$$\begin{aligned}\varphi_{ijk}\sigma_{ajk} = &\,4(\operatorname{div} \widetilde{T})_p \varphi_{pia} + 8f\langle X, \operatorname{div} \widetilde{T}\rangle g_{ia} - 8f(\operatorname{div} \widetilde{T})_i X_a + 4f(\operatorname{div} \widetilde{T})_p X_q \psi_{pqia} \\ &4(\operatorname{div} \widetilde{T} \times X)_i X_a + 4(\operatorname{div} \widetilde{T} \times X)_a X_i - 4\langle X, \operatorname{div} \widetilde{T}\rangle X_p \varphi_{pia} - 4|X|^2 (\operatorname{div} \widetilde{T})_p \varphi_{pia}.\end{aligned} \tag{6.2.14}$$

Thus from $\gamma = \sigma$, the right hand sides of equations (6.2.13) and (6.2.14) must be equal. If we take the trace of both sides, we find that

$$\langle X, \dot{X}\rangle = -\frac{f}{2}\langle X, \operatorname{div} \widetilde{T}\rangle. \tag{6.2.15}$$

On the other hand, if we contract both sides with φ_{iak}, we find that

$$\dot{f} X_k + f\dot{X}_k = -\frac{f^2}{2}(\operatorname{div} \widetilde{T})_k + \frac{f}{2}((\operatorname{div} \widetilde{T}) \times X)_k + \frac{1}{2}\langle X, \operatorname{div} \widetilde{T}\rangle X_k. \tag{6.2.16}$$

Multiplying (6.2.16) with X_k and summing over k, we get

$$\dot{f}|X|^2 + f\langle X, \dot{X}\rangle = -\frac{f^2}{2}\langle \operatorname{div}\widetilde{T}, X\rangle + 0 + \frac{1}{2}\langle X, \operatorname{div}\widetilde{T}\rangle |X|^2.$$

Substituting (6.2.15) into the above, we obtain the first equation in (6.2.12). Then substituting that back into (6.2.16) gives the second equation in (6.2.12). Thus the two equations in (6.2.12) are necessary consequences of $\gamma = \sigma$. However, substituting both equations in (6.2.12) back into (6.2.13) and (6.2.14) shows that these are in fact sufficient to ensure $\gamma = \sigma$. Thus the proof is complete. $\square$

In fact, from $f^2 = 1 - |X|^2$, it is easy to check that the first equation in (6.2.12) is a consequence of the second equation in (6.2.12). Thus the isometric flow (6.2.8) is completely determined by the single equation $\dot{X} = -\frac{1}{2} f \operatorname{div}\widetilde{T} + \frac{1}{2}(\operatorname{div}\widetilde{T}) \times X$. In order to establish that this equation is strictly parabolic, we need to express the torsion $T_t = \widetilde{T}$ and its divergence in terms of (f, X).

Lemma 6.2.9. *The torsion $\widetilde{T}$ of $\widetilde{\varphi} = \varphi_{(f,X)}$ is*

$$\begin{aligned} \widetilde{T}_{pq} = (1 - 2|X|^2)T_{pq} + 2T_{pm}X_mX_q + 2fT_{pm}X_l\varphi_{mlq} \\ - 2\nabla_pX_mX_l\varphi_{mlq} + 2\nabla_pfX_q - 2f\nabla_pX_q. \end{aligned} \tag{6.2.17}$$

Proof. Taking ∇_p of (6.2.10) gives

$$\begin{aligned} \nabla_p\widetilde{\varphi}_{ijk} = {} & -4\nabla_pX_mX_m\varphi_{ijk} + (1 - 2|X|^2)\nabla_p\varphi_{ijk} \\ & - 2\nabla_pfX_m\psi_{mijk} - 2f\nabla_pX_m\psi_{mijk} - 2fX_m\nabla_p\psi_{mijk} \\ & + 2\nabla_pX_iX_m\varphi_{mjk} + 2\nabla_pX_jX_m\varphi_{imk} + 2\nabla_pX_kX_m\varphi_{ijm} \\ & + 2X_i\nabla_pX_m\varphi_{mjk} + 2X_j\nabla_pX_m\varphi_{imk} + 2X_k\nabla_pX_m\varphi_{ijm} \\ & + 2X_iX_m\nabla_p\varphi_{mjk} + 2X_jX_m\nabla_p\varphi_{imk} + 2X_kX_m\nabla_p\varphi_{ijm}. \end{aligned}$$

We now substitute the expressions for $\nabla\varphi$ and $\nabla\psi$ from (2.3.1) and (2.3.3) into the above expression, and use (2.3.2) to write

$$24\widetilde{T}_{pq} = \nabla_p\widetilde{\varphi}_{ijk}\widetilde{\psi}_{qijk}.$$

After an extremely lengthy computation using the various identities in (2.1.9)-(2.1.12), one indeed obtains the result (6.2.17). We omit the details. $\square$

Corollary 6.2.10. *The divergence* $\operatorname{div} \widetilde{T}_q = \nabla_p \widetilde{T}_{pq}$ *of the torsion* $\widetilde{T}$ *of* $\widetilde{\varphi} = \varphi_{(f,X)}$ *is*

$$\begin{aligned}\operatorname{div} \widetilde{T}_q = (1 - 2|X|^2)(\operatorname{div} T)_q &- 4X_m \nabla_p X_m T_{pq} + 2(\operatorname{div} T)_m X_m X_q + 2T_{pm} \nabla_p X_m X_q \\ &+ 2T_{pm} X_m \nabla_p X_q + 2\nabla_p f T_{pl} X_m \varphi_{lmq} + 2f(\operatorname{div} T)_l X_m \varphi_{lmq} + 2f T_{pl} \nabla_p X_m \varphi_{lmq} \\ &- 2\nabla_p \nabla_p X_l X_m \varphi_{lmq} - 2\nabla_p X_l X_m T_{ps} \psi_{slmq} + 2\nabla_p \nabla_p f X_q - 2f \nabla_p \nabla_p X_q.\end{aligned} \tag{6.2.18}$$

Proof. This again follows by applying ∇_p to equation (6.2.17) and using the various identities in (2.1.9)-(2.1.12). We omit the details. $\square$

We can now apply the above result as follows.

Proposition 6.2.11. *Under the isometric flow, the vector field* X *evolves by*

$$\begin{aligned}\dot{X}_q = \Delta X_k + f X_m \nabla_p X_m T_{pq} &- f T_{pm} \nabla_p X_m X_q - T_{pl} \nabla_p X_m \varphi_{lmq} \\ &+ |\nabla f|^2 X_q + |\nabla X|^2 X_q - |X|^2 \nabla_p f T_{pq} + T_{pl} \nabla_p f X_l X_q \\ &+ T_{ps} \nabla_p X_l X_a X_q \varphi_{sla} - \frac{f}{2}(\operatorname{div} T)_q + \frac{1}{2}(X \times (\operatorname{div} T))_q.\end{aligned} \tag{6.2.19}$$

Proof. Once again this follows from equations (6.2.12) and (6.2.18) after a lengthy calculation, using also the relation $f^2 + |X|^2 = 1$. $\square$

Equation (6.2.19) is just a heat equation for the vector field X with lower order terms, and is thus strictly parabolic. Using classical parabolic theory, we have therefore established the following result.

Theorem 6.2.12. *Let* (M^7, φ_0) *be a compact manifold with* G_2*-structure. The flow* (6.2.8) *has a unique solution for a short time* $t \in [0, \varepsilon)$*.*

6.2.3 Parabolic rescaling

As is usual for geometric evolution equations, the natural 'parabolic rescaling' of the problem involves scaling the t by $c^2 t$ when we scale the space variables by c. In this section we make this precise, as we will crucially use this property frequently later in the chapter.

Lemma 6.2.13. *Let* $c > 0$ *be a constant. If* $\varphi(t)$ *is a solution of the isometric flow* (6.2.8) *with* $\varphi(0) = \varphi$*, then* $\widetilde{\varphi}(\widetilde{t}) = c^3 \varphi(c^2 t)$ *is a solution of* (6.2.8) *with* $\widetilde{\varphi}(0) = c^3 \varphi$*.*

Proof. Define a new G_2-structure $\widetilde{\varphi} = c^3\varphi$. Then it follows [Kar09, Theorem 2.23] that $\widetilde{g} = c^2 g$ and $\widetilde{\psi} = c^4\psi$. Hence from (2.3.2) we have $\widetilde{T} = cT$. (Recall that we are suppressing the writing of the g^{-1} terms because we are using an orthonormal frame.) Therefore as a 1-form, $\operatorname{div}_{\widetilde{g}} \widetilde{T} = c^{-1}\operatorname{div}_g T$, and so converting to vector fields using the metric, we have $(\operatorname{div}_{\widetilde{g}} \widetilde{T}) \lrcorner \widetilde{\psi} = c^{-1}c^{-2}c^4(\operatorname{div}_g T) \lrcorner \psi = c(\operatorname{div}_g T) \lrcorner \psi$. But then it is clear from (6.2.8) that with $\widetilde{t} = c^2 t$, we obtain the desired conclusion. $\square$

We note here for later use that if $\widetilde{\varphi} = c^3\varphi$, then we also have

$$|\widetilde{\nabla}^j \widetilde{\mathrm{Rm}}|_{\widetilde{g}} = c^{-(2+j)}|\nabla^j \mathrm{Rm}|_g, \qquad |\widetilde{\nabla}^j \widetilde{T}|_{\widetilde{g}} = c^{-(1+j)}|\nabla^j T|_g. \tag{6.2.20}$$

6.2.4 Solitons for the isometric flow

In this section we study the relation between self-similar solutions and solitons for the isometric flow.

Let $\mathcal{L}_Y$ denote the Lie derivative with respect to Y. Consider the identity

$$(\mathcal{L}_Y\varphi)_{ijk} = (\nabla_Y\varphi)_{ijk} + \nabla_i Y_p \varphi_{pjk} + \nabla_j Y_p \varphi_{ipk} + \nabla_k Y_p \varphi_{ijp}.$$

Using equations (2.3.1) and (6.2.2) we can rewrite the above as

$$(\mathcal{L}_Y\varphi)_{ijk} = Y_l T_{lp}\psi_{pijk} + \big((\nabla Y)\diamond\varphi\big)_{ijk}.$$

The second term above can be written as $h\diamond\varphi + Z\lrcorner\psi$ where $h_{ij} = \frac{1}{2}(\nabla_i Y_j + \nabla_j Y_i) = \frac{1}{2}(\mathcal{L}_Y g)_{ij}$ and Z is a vector field on M such that $Z_p\psi_{pijk}$ is the Ω^3_7 component of $(\nabla Y)\diamond\varphi$. Because $\Omega^3_1 \oplus \Omega^3_{27}$ is the kernel of $\gamma \mapsto \gamma_{ijk}\psi_{mijk}$, from the contraction identities (2.1.15) and (2.1.12) we deduce that

$$\begin{aligned} 24 Z_m = Z_l \psi_{lijk}\psi_{mijk} &= (\nabla_i Y_p \varphi_{pjk} + \nabla_j Y_p \varphi_{ipk} + \nabla_k Y_p \varphi_{ijp})\psi_{mijk} \\ &= 3\nabla_i Y_p \varphi_{pjk}\psi_{mijk} = -12\nabla_i Y_p \varphi_{pmi}. \end{aligned}$$

Thus we have $Z_m = -\frac{1}{2}\nabla_i Y_j \varphi_{ijm} = -\frac{1}{2}(\operatorname{curl} Y)_m$. (See [Kar10] for more about the curl operator.)

Combining these observations we can write

$$(\mathcal{L}_Y\varphi)_{ijk} = (Y\lrcorner T)_p\psi_{pijk} - \tfrac{1}{2}(\operatorname{curl} Y)_p\psi_{pijk} + \tfrac{1}{2}(\mathcal{L}_Y g)\diamond\varphi. \tag{6.2.21}$$

Definition 6.2.14. Let $(\varphi(t))_{t\in(\alpha,\beta)}$ be a solution of the isometric flow (6.2.8) where $0 \in (\alpha, \beta)$. We say that it is a *self-similar solution* if there exist a function $a(t)$ with $a(0) = 1$, a G_2-structure φ_0, and a family of diffeomorphisms $f_t : M \to M$ with $f_0 = \mathrm{id}_M$ such that

$$\varphi(t) = (a(t))^3 f_t^* \varphi_0$$

for all $t \in (\alpha, \beta)$. Since $\varphi(t)$ is a solution to the isometric flow, we have

$$g(t) := g_{\varphi(t)} = g_{\varphi(0)} = f_0^* g_{\varphi_0} = g(0).$$

Lemma 6.2.15. *Given a self-similar solution $(\varphi(t))_{t\in(\alpha,\beta)}$ of the isometric flow, there is a family $X(t)$ of vector fields such that*

$$\operatorname{div} T_{\varphi(t)} = -\frac{1}{2}\operatorname{curl}_{\varphi(t)}(X(t)) + X(t) \lrcorner T_{\varphi(t)}.$$

In particular, there is a vector field X_0 such that φ_0 satisfies

$$\operatorname{div} T_{\varphi_0} = -\frac{1}{2}\operatorname{curl}_{\varphi_0}(X_0) + X_0 \lrcorner T_{\varphi_0}.$$

Proof. Set $\varphi_0 = \varphi(0)$ and $g_0 = g_{\varphi_0}$, and let $W(t)$ be the infinitesimal generator of f_t. That is,

$$\frac{\partial}{\partial t} f_t = W(t) \circ f_t.$$

With $X(t) = (f_t^{-1})_* W(t)$ we compute

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t) &= 3a'(t)(a(t))^2 f_t^*\varphi_0 + (a(t))^3 f_t^*(\mathcal{L}_{W(t)}\varphi_0)\\ &= 3a'(t)(a(t))^2 f_t^*\varphi_0 + (a(t))^3 \mathcal{L}_{(f_t^{-1})_*W(t)} f_t^*\varphi_0\\ &= 3a'(t)(a(t))^{-1}\varphi(t) + \mathcal{L}_{X(t)}\varphi(t).\end{aligned} \tag{6.2.22}$$

From (6.2.21) we also have

$$\mathcal{L}_{X(t)}\varphi(t) = \frac{1}{2}\mathcal{L}_{X(t)} g(t) \diamond \varphi(t) + \left(-\frac{1}{2}\operatorname{curl}_{\varphi(t)} X(t) + X(t) \lrcorner T\right) \lrcorner \psi(t). \tag{6.2.23}$$

On the other hand, since $g(t) = g_{\varphi(t)} = (a(t))^2 f_t^* g_0$ we find that

$$\begin{aligned}0 = \frac{\partial}{\partial t} g(t) &= 2a'(t)a(t) f_t^* g_0 + (a(t))^2 f_t^*(\mathcal{L}_{W(t)} g_0)\\ &= 2a'(t)a(t) f_t^* g_0 + (a(t))^2 \mathcal{L}_{(f_t^{-1})^*W(t)} g_0\\ &= 2a'(t)(a(t))^{-1} g(t) + \mathcal{L}_{X(t)} g(t).\end{aligned} \tag{6.2.24}$$

Hence, combining (6.2.23) and (6.2.24), and using also (6.2.3), the expression (6.2.22) becomes

$$
\begin{aligned}
\frac{\partial}{\partial t}\varphi(t) &= \operatorname{div} T_{\varphi(t)} \lrcorner \psi(t) \\
&= 3a'(t)(a(t))^{-1}\varphi(t) + \frac{1}{2}\mathcal{L}_{X(t)}g(t) \diamond \varphi(t) + \left(-\frac{1}{2}\operatorname{curl}_{\varphi(t)} X(t) + X(t) \lrcorner T\right) \lrcorner \psi(t) \\
&= 3a'(t)(a(t))^{-1}\varphi(t) - a'(t)(a(t))^{-1}g(t) \diamond \varphi(t) + \Big(-\frac{1}{2}\operatorname{curl}_{\varphi(t)} X(t) + X(t) \lrcorner T\Big) \lrcorner \psi(t) \\
&= \Big(-\frac{1}{2}\operatorname{curl}_{\varphi(t)} X(t) + X(t) \lrcorner T\Big) \lrcorner \psi(t)
\end{aligned}
$$

as claimed. □

Definition 6.2.16. An *isometric soliton* on (M, g_0) is defined to be a triple (φ_0, X_0, c) where φ_0 is a G_2-structure on M inducing the Riemannian metric g_0, and X_0 is a vector field satisfying

$$\mathcal{L}_{X_0} g_0 = c g_0$$

for some constant $c \in \mathbb{R}$ and

$$\operatorname{div} T_{\varphi_0} = -\frac{1}{2}\operatorname{curl}_{\varphi_0} X_0 + X_0 \lrcorner T_{\varphi_0}.$$

Moreover, it is called shrinking, steady, or expanding, depending on whether c is positive, zero, or negative, respectively.

We now relate isometric solitons to self-similar solutions of the isometric flow.

Lemma 6.2.17. *Let φ_0 be a G_2 structure on M with $g_{\varphi_0} = g_0$, let $c \in \{-1, 0, 1\}$, and let X be a vector field such that*

$$
\begin{aligned}
\mathcal{L}_X g_0 &= c g_0, \\
\operatorname{div}_{g_0} T_{\varphi_0} &= -\frac{1}{2}\operatorname{curl}_{\varphi_0} X + X \lrcorner T_{\varphi_0},
\end{aligned}
\tag{6.2.25}
$$

That is, (φ_0, X_0, c) is an isometric soliton.

- *If $c = 1$, let $t < 0$ and let $f_t : M \to M$ be a 1-parameter family of diffeomorphisms such that*

$$
\begin{aligned}
\frac{\partial}{\partial t} f_t &= -\frac{1}{t} X \circ f_t, \\
f_{-1} &= \operatorname{id}_M .
\end{aligned}
$$

Then

$$\varphi(t) = |t|^{\frac{3}{2}} f_t^* \varphi_0$$

is a self-similar solution of the isometric flow, with $\varphi(-1) = \varphi_0$*. Moreover,* $(\varphi(t), |t|^{-1}X)$ *satisfies*

$$\begin{aligned} \mathcal{L}_{|t|^{-1}X} g_0 &= |t|^{-1} g_0, \\ \operatorname{div}_{g_0} T_{\varphi(t)} &= -\frac{1}{2} \operatorname{curl}_{\varphi(t)} \left(|t|^{-1}X\right) + \left(|t|^{-1}X\right) \lrcorner\, T_{\varphi(t)}. \end{aligned}$$

- *If* $c = 0$*, let* $t \in \mathbb{R}$ *and let* $f_t : M \to M$ *be a 1-parameter family of diffeomorphisms such that*

$$\begin{aligned} \frac{d}{dt} f_t &= X \circ f_t, \\ f_0 &= \mathrm{id}_M\,. \end{aligned}$$

Then

$$\varphi(t) = f_t^* \varphi_0$$

is a self-similar solution of the isometric flow, with $\varphi(0) = \varphi_0$*. Moreover,* $(\varphi(t), |t|^{-1}X)$ *satisfies*

$$\begin{aligned} \mathcal{L}_{|t|^{-1}X} g_0 &= 0, \\ \operatorname{div}_{g_0} T_{\varphi(t)} &= -\frac{1}{2} \operatorname{curl}_{\varphi(t)} X + X \lrcorner\, T_{\varphi(t)}. \end{aligned}$$

- *If* $c = -1$*, let* $t > 0$ *and let* $f_t : M \to M$ *be a 1-parameter family of diffeomorphisms such that*

$$\begin{aligned} \frac{d}{dt} f_t &= \frac{1}{t} X \circ f_t, \\ f_1 &= \mathrm{id}_M\,. \end{aligned}$$

Then

$$\varphi(t) = |t|^{\frac{3}{2}} f_t^* \varphi_0$$

is a self-similar solution of the isometric flow, with $\varphi(1) = \varphi_0$*. Moreover,* $(\varphi(t), |t|^{-1}X)$ *satisfies*

$$\begin{aligned} \mathcal{L}_{|t|^{-1}X} g_0 &= -|t|^{-1} g_0, \\ \operatorname{div}_{g_0} T_{\varphi(t)} &= -\frac{1}{2} \operatorname{curl}_{\varphi(t)} \left(|t|^{-1}X\right) + \left(|t|^{-1}X\right) \lrcorner\, T_{\varphi(t)}. \end{aligned}$$

In particular, the vector fields $X(t)$ in Lemma 6.2.15 are $|t|^{-1}X$ or X, in the shrinking/expanding or steady case respectively.

Proof. We only prove the case $c = 1$, $t < 0$, since the other cases are similar. In this case we have

$$\begin{aligned}\frac{\partial}{\partial t} f_t^* \varphi_0 &= -f_t^*(\mathcal{L}_{t^{-1}X}\varphi_0),\\ \frac{\partial}{\partial t} f_t^* g_0 &= -f_t^*(\mathcal{L}_{t^{-1}X} g_0).\end{aligned}$$

Now $g(t) = |t| f_t^* g_0$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t} g(t) &= -f_t^* g_0 - |t| f_t^*(\mathcal{L}_{t^{-1}X} g_0)\\ &= -f_t^* g_0 + f_t^* g_0 = 0.\end{aligned}$$

Moreover, if $\varphi(t) = |t|^{\frac{3}{2}} f_t^* \varphi_0$ then

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t) &= -\frac{3}{2}|t|^{\frac{1}{2}} f_t^*\varphi_0 + |t|^{\frac{3}{2}}\frac{\partial}{\partial t} f_t^*\varphi_0\\ &= -\frac{3}{2|t|}\varphi(t) + |t|^{\frac{1}{2}} f_t^*(\mathcal{L}_X\varphi_0).\end{aligned}$$

Using (6.2.21), (6.2.3) and $\mathcal{L}_X g = g$ from (6.2.25), we get

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t) &= -\frac{3}{2|t|}\varphi(t) + |t|^{\frac{1}{2}} f_t^*\Big(\frac{1}{2}\mathcal{L}_X g \diamond \varphi_0 + \big(-\frac{1}{2}\operatorname{curl}_{\varphi_0} X + X \lrcorner T_{\varphi_0}\big)\lrcorner \psi_0\Big)\\ &= -\frac{3}{2|t|}\varphi(t) + |t|^{\frac{1}{2}} f_t^*\Big(\frac{3}{2}\varphi_0 + \big(-\frac{1}{2}\operatorname{curl}_{\varphi_0} X + X \lrcorner T_{\varphi_0}\big)\lrcorner \psi_0\Big)\\ &= -\frac{3}{2|t|}\varphi(t) + \frac{3}{2|t|}\varphi(t) + |t|^{\frac{1}{2}} f_t^*\Big(\big(-\frac{1}{2}\operatorname{curl}_{\varphi_0} X + X \lrcorner T_{\varphi_0}\big)\lrcorner \psi_0\Big).\end{aligned}$$

From the hypothesis (6.2.25) and the rescaling Lemma 6.2.13 we thus obtain

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t) &= |t|^{\frac{1}{2}} f_t^*\left(\operatorname{div} T_{\varphi_0}\lrcorner \psi_0\right)\\ &= \operatorname{div} T_{\varphi(t)}\lrcorner \psi(t).\end{aligned}$$

We conclude that $\varphi(t)$ is a self-similar isometric flow, with $\varphi(-1) = \varphi_0$.

Finally, again by Lemma 6.2.13 and the hypothesis (6.2.25) we have

$$\begin{aligned}
\operatorname{div}_{g_0} T_{\varphi(t)} &= \operatorname{div}_{|t| f_t^* g_0} T_{|t|^{\frac{3}{2}} f_t^* \varphi_0} \\
&= |t|^{-\frac{1}{2}} f_t^* (\operatorname{div}_{g_0} T_{\varphi_0}) \\
&= |t|^{-\frac{1}{2}} f_t^* \Big(-\frac{1}{2} \operatorname{curl}_{\varphi_0} X + X \lrcorner T_{\varphi_0} \Big) \\
&= |t|^{1/2} f_t^* \left(-\frac{1}{2} \operatorname{curl}_{\varphi_0} |t|^{-1} X + |t|^{-1} X \lrcorner T_{\varphi_0} \right) \\
&= -\frac{1}{2} \operatorname{curl}_{\varphi(t)} ((f_t^{-1})_* |t|^{-1} X) + ((f_t^{-1})_* |t|^{-1} X) \lrcorner T_{\varphi(t)}.
\end{aligned} \tag{6.2.26}$$

We observe that

$$\frac{\partial}{\partial t} (f_t^{-1})_* X = (f_t^{-1})_* \left(\mathcal{L}_{|t|^{-1} X} X \right) = 0,$$

hence $(f_t^{-1})_* X = X$ for all $t < 0$. This, together with (6.2.26), gives that

$$\operatorname{div}_{g_0} T_{\varphi(t)} = -\frac{1}{2} \operatorname{curl}_{\varphi(t)} (|t|^{-1} X) + (|t|^{-1} X) \lrcorner T_{\varphi(t)},$$

completing the proof. □

Remark 6.2.18. If M is compact then every steady soliton in fact satisfies

$$\operatorname{div} T = 0.$$

This is because $\varphi(t) = f_t^* \varphi_0$ satisfies $E(\varphi(t)) = E(\varphi_0)$ for all t, and therefore by Proposition 6.2.5 we have

$$\frac{d}{dt} 4E(\varphi(t)) = -\int_M |\operatorname{div} T|^2 d\mu_g = 0.$$

It is unclear if there exist any nontrivial expanding or shrinking solitons in the compact case. This is an important question for future study.

We now restrict to the special case when $M = \mathbb{R}^7$ and $g = g_{\mathrm{Eucl}}$.

Proposition 6.2.19. *Let (φ, Y, c) be a soliton for the isometric flow on $\mathbb{R}^7$ with the Euclidean metric g_{Eucl}. Then $Y = \frac{c}{2} x + Y_0$, where $x = x^i \frac{\partial}{\partial x^i}$ is the position (radial) vector field on $\mathbb{R}^7$ and Y_0 is a Killing vector field on $(\mathbb{R}^7, g_{\mathrm{Eucl}})$. That is, Y_0 induces an isometry of Euclidean space.*

Proof. In terms of the global coordinates $x^1, \dots, x^7$ on $\mathbb{R}^7$, the equation $\mathcal{L}_Y g_{\mathrm{Eucl}} = c g_{\mathrm{Eucl}}$ becomes $\partial_i Y_j + \partial_j Y_i = c\delta_{ij}$. It is straightforward to verify that the only solutions are $Y_i = \frac{c}{2}x^i + a_{ij}x^j + b_i$ where a_{ij} is skew-symmetric. Thus $Y_0 = a_{ij}x^j \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial x^i}$ generates a rigid motion of $(\mathbb{R}^7, g_{\mathrm{Eucl}})$. □

A special class of solitons on $(\mathbb{R}^7, g_{\mathrm{Eucl}})$ are those for which $Y_0 = 0$. In this case, we have $Y = \frac{x}{2} = \frac{x^i}{2}\frac{\partial}{\partial x^i}$, so $(\operatorname{curl} Y)_m = \frac{1}{2}\nabla_i x^j \varphi_{ijm} = \frac{1}{2}\delta_{ij}\varphi_{ijm} = 0$. Hence, by Lemma 6.2.17 the special class of isometric *shrinking solitons* (φ, Y) on $(\mathbb{R}^7, g_{\mathrm{Eucl}})$ for which $Y_0 = 0$ are precisely those φ which satisfy the equation

$$\operatorname{div} T = \frac{x}{2} \lrcorner T. \tag{6.2.27}$$

The particular special case of shrinking isometric solitons of the form (6.2.27) arises in Theorem 6.4.2 when we prove the almost monotonicity formula for the quantity Θ.

It would be interesting to investigate whether any nontrivial examples of this special type of isometric soliton on $\mathbb{R}^7$ actually exist. One would need to solve the underdetermined equations (6.2.27) on $\mathbb{R}^7$ under the additional constraint that $g_\varphi = g_{\mathrm{Eucl}}$. Such solitons are important in the study of Type I singularities for the isometric flow. See Theorem 6.4.9 for more details.

6.3 Derivative estimates, blow-up time, and compactness

In this section we first derive the global and local derivative estimates for the torsion T (also known as Bando–Bernstein–Shi estimates) for the flow. We prove a doubling time estimate for the torsion (Proposition 6.3.2), under the isometric flow which demonstrates that the assumption of a torsion bound is reasonable. Using the derivative estimates, in §6.3.3, we prove that any solution of the isometric flow exists as long as the torsion remains bounded, and we obtain a lower bound for the blow-up rate of the torsion. Finally, in §6.3.4 we prove a Cheeger–Gromov type compactness theorem for the solutions of the isometric flow.

6.3.1 Global derivative estimates of torsion

Let (M^7, φ) be a compact manifold with G_2-structure and consider the evolution of φ by the isometric flow (6.2.8)

$$\frac{\partial \varphi}{\partial t} = \operatorname{div} T \lrcorner \psi.$$

We first determine the evolution of the torsion under the flow (6.2.8).

Lemma 6.3.1. *Let $\varphi(t)$ be an isometric flow on M. Then the torsion evolves by*

$$\frac{\partial T_{pq}}{\partial t} = \Delta T_{pq} - \nabla_i T_{pb} T_{ia} \varphi_{abq} + F(\varphi, T, \mathrm{Rm}, \nabla \mathrm{Rm}) \tag{6.3.1}$$

where

$$F(\varphi, T, \mathrm{Rm}, \nabla \mathrm{Rm})_{pq} = \nabla_a R_{bp} \varphi_{abq} + R_{ipqm} T_{im} - \frac{1}{2} R_{ipab} T_{im} \psi_{mabq} - R_{pm} T_{mq}. \tag{6.3.2}$$

Proof. Recall from (3.1.16) that for a general flow of G_2-structures

$$\frac{\partial \varphi}{\partial t} = h \diamond \varphi + X \lrcorner \psi$$

we have

$$\frac{\partial T_{pq}}{\partial t} = T_{pl} h_{lq} + T_{pl} X_k \varphi_{klq} + \nabla_k h_{ip} \varphi_{kiq} + \nabla_p X_q.$$

Hence for (6.2.8), where $h = 0$ and $X = \operatorname{div} T$, we get

$$\begin{aligned} \frac{\partial T_{pq}}{\partial t} &= T_{pl} (\operatorname{div} T)_k \varphi_{klq} + \nabla_p (\operatorname{div} T)_q \\ &= T_{pl} \nabla_i T_{ik} \varphi_{klq} + \nabla_p \nabla_i T_{iq}. \end{aligned} \tag{6.3.3}$$

We first compute ΔT_{pq}. Using the G_2-Bianchi identity (2.4.1) and the fact that $T_{ia} T_{im}$ is symmetric in a, m, we get

$$\begin{aligned} \nabla_i \nabla_i T_{pq} &= \nabla_i (\nabla_p T_{iq} + T_{ia} T_{pb} \varphi_{abq} + \frac{1}{2} R_{ipab} \varphi_{abq}) \\ &= \nabla_i \nabla_p T_{iq} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} + T_{ia} T_{pb} T_{im} \psi_{mabq} \\ &\quad + \frac{1}{2} \nabla_i R_{ipab} \varphi_{abq} + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq} \\ &= \nabla_i \nabla_p T_{iq} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} + \frac{1}{2} \nabla_i R_{abip} \varphi_{abq} + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq}. \end{aligned}$$

Applying the Riemannian second Bianchi identity to the fourth term above, we get

$$
\begin{aligned}
\nabla_i \nabla_i T_{pq} &= \nabla_i \nabla_p T_{iq} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} + \frac{1}{2}(-\nabla_a R_{biip} - \nabla_b R_{iaip}) \varphi_{abq} \\
&\quad + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq} \\
&= \nabla_i \nabla_p T_{iq} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} + \frac{1}{2}(\nabla_b R_{ap} - \nabla_a R_{bp}) \varphi_{abq} \\
&\quad + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq} \\
&= \nabla_i \nabla_p T_{iq} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} - \nabla_a R_{bp} \varphi_{abq} + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq}.
\end{aligned}
$$

Commuting covariant derivatives for the first term above with the Ricci identity (1.1.1), we get

$$
\begin{aligned}
\Delta T_{pq} &= \nabla_p \nabla_i T_{iq} + R_{pm} T_{mq} - R_{ipqm} T_{im} + \nabla_i T_{ia} T_{pb} \varphi_{abq} + T_{ia} \nabla_i T_{pb} \varphi_{abq} \\
&\quad - \nabla_a R_{bp} \varphi_{abq} + \frac{1}{2} R_{ipab} T_{im} \psi_{mabq}.
\end{aligned} \tag{6.3.4}
$$

Combining equations (6.3.4) and (6.3.3), we deduce that

$$
\frac{\partial T_{pq}}{\partial t} = \Delta T_{pq} - \nabla_i T_{pb} T_{ia} \varphi_{abq} + \nabla_a R_{bp} \varphi_{abq} + R_{ipqm} T_{im} - \frac{1}{2} R_{ipab} T_{im} \psi_{mabq} - R_{pm} T_{mq}, \tag{6.3.5}
$$

as claimed. □

We write equation (6.3.5) schematically as

$$
\frac{\partial}{\partial t} T = \Delta T + \nabla T * T * \varphi + \nabla \mathrm{Rm} * \varphi + \mathrm{Rm} * T + \mathrm{Rm} * T * \psi. \tag{6.3.6}
$$

For a solution $\varphi(t)$ of the isometric flow (6.2.8), define

$$
\mathcal{T}(t) = \sup_M |T(x,t)| \tag{6.3.7}
$$

where $T(t)$ is the torsion of $\varphi(t)$. We next prove a doubling time estimate for the quantity $\mathcal{T}(t)$, which roughly says that $\mathcal{T}(t)$ cannot blow up too quickly and therefore the assumption that $|T|$ is bounded for a short time is a reasonable one. Note that if $\mathcal{T}(0) = 0$, then $\varphi(0)$ is torsion-free, and does not flow under (6.2.8). Thus in the following proposition we can assume that $\mathcal{T}(0) > 0$.

Proposition 6.3.2 (Doubling-time estimate)**.** *Let $\varphi(t)$ be a solution to (6.2.8) on a compact 7-manifold M for $t \in [0, \tau]$. Then there exists $\delta > 0$ such that*

$$\mathcal{T}(t) \leq 2\mathcal{T}(0) \quad \textit{for all } 0 \leq t \leq \delta.$$

Moreover, δ satisfies $\delta \leq \min\{\tau, \frac{1}{C\mathcal{T}(0)^2}\}$ for some $C > 0$.

Proof. If $|T| \leq 1$ at time 0, then by continuity we have $|T| \leq 1 + \varepsilon$ for some small ε for $0 \leq t \leq \delta < \tau$, and since $1 + \varepsilon \leq 2$, the assertion holds. Thus we can assume that $|T| > 1$ at time 0, and thus by continuity we can assume that $|T| > 1$ for all $0 \leq t \leq \delta'$ for some $0 < \delta' < \tau$.

We first compute a differential inequality for $\mathcal{T}(t)$ and then use the maximum principle. Since the metric is not evolving under (6.2.8), we have

$$\frac{\partial}{\partial t}|T|^2 = \frac{\partial}{\partial t}(T_{ij}T_{pq}g^{ip}g^{jq}) = 2T_{pq}\frac{\partial T_{pq}}{\partial t},$$

so using (6.3.6), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}|T|^2 &= 2\langle T, \frac{\partial}{\partial t}T\rangle \\ &\leq \Delta|T|^2 - 2|\nabla T|^2 + C|\nabla T||T|^2 + C|\nabla \mathrm{Rm}||T| + C|\mathrm{Rm}||T|^2 \end{aligned} \tag{6.3.8}$$

where C is a constant. Now since the metric is not evolving and M is compact, both $|\mathrm{Rm}|$ and $|\nabla\mathrm{Rm}|$ are bounded by some constant which we still call C. Thus we have

$$\frac{\partial}{\partial t}|T|^2 \leq \Delta|T|^2 - 2|\nabla T|^2 + C|\nabla T||T|^2 + C|T| + C|T|^2. \tag{6.3.9}$$

Notice from (6.3.5) that the third term in (6.3.9) is due to the $T * (\nabla T * T * \varphi)$ term. We need to estimate this term by using the *explicit* expression for $\nabla T * T * \varphi$ rather than the schematic expression. Using the skew-symmetry of φ_{abq} in a, q and the G_2-Bianchi identity (2.4.1), we have

$$\begin{aligned}T_{pq}\nabla_i T_{pb}T_{ia}\varphi_{abq} &= \frac{1}{2}T_{pq}(\nabla_i T_{pb} - \nabla_p T_{ib})T_{ia}\varphi_{abq} \\ &= \frac{1}{2}T_{pq}(T_{im}T_{pn}\varphi_{mnb} + \frac{1}{2}R_{ipmn}\varphi_{mnb})T_{ia}\varphi_{abq},\end{aligned}$$

and hence (6.3.9) becomes

$$\frac{\partial}{\partial t}|T|^2 \leq \Delta|T|^2 - 2|\nabla T|^2 + C|T|^4 + C|T| + C|T|^2. \tag{6.3.10}$$

Since we have $|T| > 1$ for all $0 \leq t \leq \delta'$, we have $|T| < |T|^4$ and $|T|^2 < |T|^4$ and hence (6.3.10) becomes

$$\frac{\partial}{\partial t}|T|^2 \leq \Delta|T|^2 - 2|\nabla T|^2 + C|T|^4. \tag{6.3.11}$$

Recall that $\mathcal{T}(t) = \sup_M |T(x,t)|$ is a Lipschitz function, so applying the maximum principle to (6.3.11), we get

$$\frac{d}{dt}\mathcal{T} \leq \frac{C}{2}\mathcal{T}^3$$

in the sense of the lim sup of forward difference quotients. Thus we have $\mathcal{T}^{-3}\frac{\partial}{\partial t}\mathcal{T} \leq \frac{C}{2}$. Integrating the inequality above from 0 to t we deduce that

$$\mathcal{T}(t) \leq \frac{\mathcal{T}(0)}{\sqrt{1 - C\mathcal{T}(0)^2 t}} \tag{6.3.12}$$

and hence $\mathcal{T}(t) \leq 2\mathcal{T}(0)$ for all $0 \leq t \leq \delta$ if we take $\delta = \min\left\{\delta', \dfrac{3}{4C\mathcal{T}(0)^2}\right\}$. □

Next we derive the Shi-type estimates for the flow in (6.2.8).

Theorem 6.3.3. *Suppose that $K > 0$ is a constant and $\varphi(t)$ is a solution to the isometric flow on a closed manifold M^7 with $t \in [0, \frac{1}{K^2}]$. For all $m \in \mathbb{N}$, there exists a constant C_m* depending only on (M, g) *such that if*

$$\mathcal{T} \leq K \text{ and } |\nabla^j \mathrm{Rm}| \leq B_j K^{2+j} \quad \text{for all } j \geq 0 \text{ on } M^7 \times [0, \tfrac{1}{K^2}], \tag{6.3.13}$$

then for all $t \in [0, \frac{1}{K^2}]$ we have

$$|\nabla^m T| \leq C_m t^{-\frac{m}{2}} K. \tag{6.3.14}$$

Before we give the proof of Theorem 6.3.3, we remark that the form of the assumed bounds on $\nabla^j\mathrm{Rm}$ in (6.3.13) is precisely as required by the rescaling properties of the curvature in equation (6.2.20).

Proof of Theorem 6.3.3. Since the proof is quite long, we first summarize the strategy of the proof. The proof is by induction on m. We first define a function $f_m(x,t)$ (see (6.3.36) for the precise expression) for each m, just as in the case of Ricci flow, which satisfies a parabolic differential inequality, and then we use the maximum principle.

For $m = 1$ case, we define

$$f = t|\nabla T|^2 + \beta|T|^2 \tag{6.3.15}$$

where β is a constant to be determined later. Note that $f(x,0) \leq \beta K^2$. To calculate the evolution of f, we first need to calculate the evolution of $|\nabla T|^2$.

Because the metric is not evolving, by differentiating (6.3.6) we have that

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla T &= \nabla(\Delta T + \nabla T * T * \varphi + \nabla \mathrm{Rm} * \varphi + \mathrm{Rm} * T + \mathrm{Rm} * T * \psi) \\
&= \nabla\Delta T + \nabla(\nabla T * T * \varphi) + \nabla^2 \mathrm{Rm} * \varphi + \nabla \mathrm{Rm} * \nabla\varphi + \nabla \mathrm{Rm} * T + \mathrm{Rm} * \nabla T \\
&\quad + \nabla \mathrm{Rm} * T * \psi + \mathrm{Rm} * \nabla T * \psi + \mathrm{Rm} * T * \nabla\psi \\
&= \Delta\nabla T + \nabla(\nabla T * T * \varphi) + \nabla^2 \mathrm{Rm} * \varphi + \nabla \mathrm{Rm} * \nabla\varphi + \nabla \mathrm{Rm} * T + \mathrm{Rm} * \nabla T \\
&\quad + \nabla \mathrm{Rm} * T * \psi + \mathrm{Rm} * \nabla T * \psi + \mathrm{Rm} * T * \nabla\psi
\end{aligned}$$

where we have used the Ricci identity in the last equality. Thus we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla T|^2 = 2\langle \nabla T, \frac{\partial}{\partial t}\nabla T\rangle &= \nabla T * \Big(\Delta\nabla T + \nabla(\nabla T * T * \varphi) + \nabla^2 \mathrm{Rm} * \varphi + \nabla \mathrm{Rm} * \nabla\varphi \\
&\quad + \nabla \mathrm{Rm} * T + \mathrm{Rm} * \nabla T + \nabla \mathrm{Rm} * T * \psi + \mathrm{Rm} * \nabla T * \psi \\
&\quad + \mathrm{Rm} * T * \nabla\psi\Big) \\
&= \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 + \nabla(\nabla T * T * \varphi) * \nabla T + \nabla T * \nabla^2 \mathrm{Rm} * \varphi \\
&\quad + \nabla \mathrm{Rm} * \nabla\varphi * \nabla T + \nabla \mathrm{Rm} * T * \nabla T + \mathrm{Rm} * \nabla T * \nabla T \\
&\quad + \nabla \mathrm{Rm} * T * \psi * \nabla T + \mathrm{Rm} * \nabla T * \nabla T * \psi \\
&\quad + \mathrm{Rm} * T * \nabla\psi * \nabla T.
\end{aligned} \tag{6.3.16}$$

From (2.3.1) and (2.3.3) we have

$$\nabla\varphi = T * \psi, \qquad \nabla\psi = T * \varphi$$

and hence

$$|\nabla\varphi| \leq CK, \qquad |\nabla\psi| \leq CK. \tag{6.3.17}$$

Using (6.3.17) and the hypotheses (6.3.13) of the theorem, the estimate (6.3.16) becomes

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla T|^2 &\leq \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 + C|\nabla^2 T||T||\nabla T| + C|\nabla T|^3 + C|\nabla T|^2|T|^2 \\
&\quad + C|\nabla T||\nabla^2 \mathrm{Rm}| + C|\nabla \mathrm{Rm}||\nabla\varphi||\nabla T| + C|\nabla \mathrm{Rm}||T||\nabla T| + C|\mathrm{Rm}||\nabla T|^2 \\
&\quad + C|\mathrm{Rm}||T||\nabla\psi||\nabla T| \\
&\leq \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 + CK|\nabla^2 T||\nabla T| + C|\nabla T|^3 + CK^2|\nabla T|^2 + C|\nabla T|K^4
\end{aligned} \tag{6.3.18}$$

for some constant C depending only on the dimension and the order of the derivative. Consider the third term in the right hand side of the inequality (6.3.18). By Young's inequality, for all $\varepsilon > 0$, we have

$$2K|\nabla^2 T||\nabla T| \leq \frac{1}{\varepsilon}K^2|\nabla T|^2 + \varepsilon|\nabla^2 T|^2.$$

Substituting this into (6.3.18) gives

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 - (2 - C\varepsilon)|\nabla^2 T|^2 + CK^4|\nabla T| + CK^2|\nabla T|^2 + C|\nabla T|^3. \tag{6.3.19}$$

We pause here for an important remark. In the Shi-type estimates for the Laplacian flow of Lotay–Wei [LW17], they assume a bound on $|\nabla T|$. In contrast, we only assume a bound on $|T|$, not $|\nabla T|$. This remark has the following consequence. It turns out that the third and fourth terms in (6.3.19) can be dealt with easily, which we do below. However, the presence of the $|\nabla T|^3$ term on the right hand side of (6.3.19) would cause problems in trying to apply the maximum principle to the function f and cannot be dealt with easily, so we have to work harder. Notice from (6.3.16) that the $|\nabla T|^3$ term comes from the $\nabla T * \nabla(\nabla T * T * \varphi)$ term. We get rid of the problematic term by considering the *explicit* expression for $\nabla T * \nabla(\nabla T * T * \varphi)$ rather than the schematic one, and using the G_2-Bianchi identity (2.4.1) to get a lower order term. Specifically, the expression for $\nabla T * T * \varphi$ is $\nabla_i T_{pb} T_{ia} \varphi_{abq}$. So we have

$$\begin{aligned}\nabla T * \nabla(\nabla T * T * \varphi) &= \nabla_j T_{pq} \nabla_j(\nabla_i T_{pb} T_{ia} \varphi_{abq}) \\ &= \nabla_j T_{pq} \nabla_j \nabla_i T_{pb} T_{ia} \varphi_{abq} + \nabla_j T_{pq} \nabla_i T_{pb} \nabla_j T_{ia} \varphi_{abq} \\ &\quad + \nabla_j T_{pq} \nabla_i T_{pb} T_{ia} \nabla_j \varphi_{abq}.\end{aligned}$$

Since the first and the last term in the above equation do not cause any problems in (6.3.19), we focus on the second term. Using the fact that φ_{abq} is skew-symmetric in a, q, and the G_2-Bianchi identity (2.4.1), we have

$$\begin{aligned}\nabla_j T_{pq} \nabla_i T_{pb} \nabla_j T_{ia} \varphi_{abq} &= \frac{1}{2}\nabla_j T_{pq} \nabla_j T_{ia}(\nabla_i T_{pb} - \nabla_p T_{ib})\varphi_{abq} \\ &= \frac{1}{2}\nabla_j T_{pq} \nabla_j T_{ia}(T_{im} T_{pn} \varphi_{mnb} + \frac{1}{2}R_{ipmn}\varphi_{mnb})\varphi_{abq}.\end{aligned} \tag{6.3.20}$$

Thus from (6.3.19) and (6.3.20) and using Young's inequality as before we get

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 - (2 - C\varepsilon)|\nabla^2 T|^2 + CK^4|\nabla T| + CK^2|\nabla T|^2. \tag{6.3.21}$$

Hence, with a suitably chosen ε we have

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 + CK^2|\nabla T|^2 + CK^4|\nabla T|. \tag{6.3.22}$$

From (6.3.8) and (6.3.22), we get

$$\begin{aligned}\frac{\partial f}{\partial t} \leq \Delta f &+ t(CK^4|\nabla T| + CK^2|\nabla T|^2)\\ &+ \beta(-2|\nabla T|^2 + C|\nabla T||T|^2 + C|\nabla \mathrm{Rm}||T| + C|\mathrm{Rm}||T|^2).\end{aligned}$$

Using the hypotheses that $\mathcal{T} = \sup_M T(x,t) \leq K$, $|\nabla^j \mathrm{Rm}| \leq K^{2+j}$, and $tK^2 \leq 1$, and using Young's inequality on the $|\nabla T||T|^2$ term, the above inequality becomes

$$\frac{\partial f}{\partial t} \leq \Delta f + CK^2|\nabla T| + C|\nabla T|^2 - (2-\varepsilon)\beta|\nabla T|^2 + C\beta K^4.$$

Using Young's inequality again on the second term above we get

$$\frac{\partial f}{\partial t} \leq \Delta f + (C - (2-\varepsilon)\beta)|\nabla T|^2 + C\beta K^4.$$

Now choose β large enough so that $C - (2-\varepsilon)\beta \leq 0$, so we have

$$\frac{\partial f}{\partial t} \leq \Delta f + C\beta K^4.$$

From (6.3.15) we have $f(x,0) \leq \beta K^2$. Thus, applying the maximum principle to the above inequality and using $tK^2 \leq 1$, we get

$$\sup_{x \in M} f(x,t) \leq \beta K^2 + C\beta t K^4 \leq CK^2. \tag{6.3.23}$$

From the definition (6.3.15) of f, we conclude that

$$|\nabla T| \leq CKt^{-\frac{1}{2}}$$

and thus the base case of the induction is complete.

Next we prove the estimate for $m \geq 2$ by induction. Suppose $|\nabla^j T| \leq C_j K t^{-\frac{j}{2}}$ holds for all $1 \leq j < m$. Looking at the definition of f_m in (6.3.36) below, it is clear that we need to first determine the evolution equation for $|\nabla^m T|^2$. Since the metric is not evolving, by differentiating (6.3.6) we have that

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m T &= \nabla^m(\Delta T + \nabla T * T * \varphi + \nabla \mathrm{Rm} * \varphi + \mathrm{Rm} * T + \mathrm{Rm} * T * \psi) \\ &= \nabla^m \Delta T + \nabla^m(\nabla T * T * \varphi) + \sum_{i=0}^{m} \nabla^{m+1-i}\mathrm{Rm} * \nabla^i \varphi \\ &\quad + \sum_{i=0}^{m} \nabla^{m-i} T * \nabla^i \mathrm{Rm} + \sum_{i=0}^{m} \nabla^{m-i}(\mathrm{Rm} * T) * \nabla^i \psi.\end{aligned}$$

Using the identity (1.1.2) with $S = T$, we can write the above equation as

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m T &= \Delta \nabla^m T + \sum_{i=0}^{m} \nabla^{m-i} T * \nabla^i \mathrm{Rm} + \nabla^m(\nabla T * T * \varphi) \\ &\quad + \sum_{i=0}^{m} \nabla^{m+1-i}\mathrm{Rm} * \nabla^i \varphi + \sum_{i=0}^{m} \nabla^{m-i}(\mathrm{Rm} * T) * \nabla^i \psi. \end{aligned} \tag{6.3.24}$$

Thus we find that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m T|^2 = 2\langle \nabla^m T, \frac{\partial}{\partial t}\nabla^m T\rangle &= \Delta |\nabla^m T|^2 - 2|\nabla^{m+1} T|^2 + \sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm} \\ &\quad + \nabla^m T * \nabla^m(\nabla T * T * \varphi) \\ &\quad + \sum_{i=0}^{m} \nabla^m T * \nabla^{m+1-i}\mathrm{Rm} * \nabla^i \varphi \\ &\quad + \sum_{i=0}^{m} \nabla^m T * \nabla^{m-i}(\mathrm{Rm} * T) * \nabla^i \psi. \end{aligned} \tag{6.3.25}$$

Using the induction hypothesis, we estimate each term in (6.3.25) as follows.

Consider the third term $\sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm}$. When $i = 0$ we get

$$|\nabla^m T * \nabla^m T * \mathrm{Rm}| \leq CK^2 |\nabla^m T|^2.$$

When $1 \leq i \leq m$, using $K^2 t \leq 1$ and the induction hypothesis, we get

$$\begin{aligned}\Big|\sum_{i=1}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm}\Big| &\leq C|\nabla^m T| \sum_{i=1}^{m} |\nabla^{m-i} T||\nabla^i \mathrm{Rm}| \\ &\leq C|\nabla^m T| \sum_{i=1}^{m} K t^{-\frac{(m-i)}{2}} K^{i+2} \\ &\leq C K^3 t^{-\frac{m}{2}} |\nabla^m T|.\end{aligned}$$

Thus the third term in (6.3.25) can be estimated as

$$\Big|\sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm}\Big| \leq C K^2 |\nabla^m T|^2 + C K^3 |\nabla^m T| t^{-\frac{m}{2}}. \tag{6.3.26}$$

For the moment we skip the fourth term in (6.3.25) and consider the fifth and sixth terms. We need to first estimate the quantities $\nabla^i \psi$ and $\nabla^i \varphi$. From (2.3.3) we have $\nabla \psi = T * \varphi$, and thus

$$|\nabla \psi| \leq CK.$$

Schematically have

$$\nabla^2 \psi = \nabla T * \varphi + T * T * \psi$$

and hence

$$|\nabla^2 \psi| \leq C(|\nabla T| + |T|^2) \leq C(K t^{-\frac{1}{2}} + K^2) = CK(t^{-\frac{1}{2}} + K).$$

Using the same equations again, we have

$$\nabla^3 \psi = \nabla^2 T * \varphi + \nabla T * T * \psi + T * T * T * \varphi$$

and therefore

$$|\nabla^3 \psi| \leq C(|\nabla^2 T| + |\nabla T||T| + |T|^3) \leq CK(t^{-1} + K t^{-\frac{1}{2}} + K^2).$$

Similarly, we have

$$\nabla^4 \psi = \nabla^3 T * \varphi + \nabla^2 T * T * \psi + \nabla T * \nabla T * \psi + \nabla T * T * T * \varphi + T * T * T * T * \psi$$

thus yielding, using the induction hypothesis, that

$$\begin{aligned}|\nabla^4 \psi| &\leq C(|\nabla^3 T| + |\nabla^2 T||T| + |\nabla T|^2 + |\nabla T||T|^2 + |T|^4) \\ &\leq CK(t^{-\frac{3}{2}} + K t^{-1} + K^2 t^{-\frac{1}{2}} + K^3).\end{aligned}$$

A straightforward induction argument which we omit then shows that for $i \geq 1$ we have

$$|\nabla^i \psi| \leq C \sum_{j=1}^{i} K^j t^{\frac{j-i}{2}}. \tag{6.3.27}$$

Because φ is the Hodge star of ψ, and the Hodge star is both parallel and an isometry, we deduce the same estimates for $|\nabla^i \varphi|$ for $i \geq 1$. That is, we have

$$|\nabla^i \varphi| \leq C \sum_{j=1}^{i} K^j t^{\frac{j-i}{2}}. \tag{6.3.28}$$

Using the hypotheses (6.3.13) on $|\nabla^j \mathrm{Rm}|$, equation (6.3.28), and $K^2 t \leq 1$, the fifth term in (6.3.25) can thus be estimated as

$$\Big| \sum_{i=0}^{m} \nabla^m T * \nabla^{m+1-i} \mathrm{Rm} * \nabla^i \varphi \Big| \leq C K^3 |\nabla^m T| t^{-\frac{m}{2}}. \tag{6.3.29}$$

Next consider the expression $\nabla^{m-i}(\mathrm{Rm} * T)$, which is part of the sixth term of (6.3.25). Using the induction hypothesis and $|\mathrm{Rm}| \leq K^2$, for $i = 0$ we get

$$|\nabla^m (\mathrm{Rm} * T)| = \Big| \sum_{j=0}^{m} \nabla^{m-j} \mathrm{Rm} * \nabla^j T \Big| \leq C K^2 |\nabla^m T| + C K^3 t^{-\frac{m}{2}}$$

and for $1 \leq i \leq m$ we get

$$|\nabla^{m-i} (\mathrm{Rm} * T)| \leq \Big| \sum_{j=0}^{m-i} \nabla^{m-i-j} \mathrm{Rm} * \nabla^j T \Big| \leq C K^3 t^{\frac{(i-m)}{2}}.$$

Hence, using (6.3.27) and the above two estimates, we get

$$\begin{aligned} \Big| \sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} (\mathrm{Rm} * T) * \nabla^i \psi \Big| &\leq C K^2 |\nabla^m T|^2 + C K^3 |\nabla^m T| t^{-\frac{m}{2}} \\ &\quad + C |\nabla^m T| \sum_{i=1}^{m} \Big(K^3 t^{\frac{(i-m)}{2}} \sum_{j=1}^{i} K^j t^{\frac{(j-i)}{2}} \Big). \end{aligned}$$

Using $K^2 t \leq 1$ on the above, the sixth term in (6.3.25) can be estimated as

$$\Big| \sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} (\mathrm{Rm} * T) * \nabla^i \psi \Big| \leq C K^2 |\nabla^m T|^2 + C K^3 |\nabla^m T| t^{-\frac{m}{2}}. \tag{6.3.30}$$

Finally we return to the fourth term in (6.3.25). We have

$$|\nabla^m T * \nabla^m(\nabla T * T * \varphi)| = \Big|\nabla^m T * \sum_{i=0}^{m} \nabla^{m+1-i} T * \nabla^i (T * \varphi)\Big|.$$

We break up the sum over i into four terms: $i = 0$, $i = 1$, $1 < i < m$, and $i = m$. Thus we have

$$\begin{aligned}|\nabla^m T * \nabla^m(\nabla T * T * \varphi)| \le{}& |\nabla^m T * \nabla^{m+1} T * T * \varphi| + |\nabla^m T * \nabla^m T * (\nabla T * \varphi + T * T * \psi)| \\ &+ \Big|\nabla^m T * \sum_{i=2}^{m-1} \nabla^{m+1-i} T * (\sum_{j=0}^{i} \nabla^{i-j} T * \nabla^j \varphi)\Big| \\ &+ \Big|\nabla^m T * \nabla T * \sum_{i=0}^{m} \nabla^{m-i} T * \nabla^i \varphi\Big|.\end{aligned}$$

Using the induction hypothesis and equation (6.3.28) on the above, the fourth term in (6.3.25) can be estimated as

$$\begin{aligned}|\nabla^m T * \nabla^m(\nabla T * T * \varphi)| \le{}& CK|\nabla^m T||\nabla^{m+1} T| + CKt^{-\frac{1}{2}}|\nabla^m T|^2 + CK^2|\nabla^m T|^2 \\ &+ CK^2 t^{-\frac{(m+1)}{2}}|\nabla^m T|. \end{aligned} \tag{6.3.31}$$

Combining the estimates (6.3.26), (6.3.29), (6.3.30), and (6.3.31), equation (6.3.25) thus becomes

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m T|^2 \le{}& \Delta|\nabla^m T|^2 - 2|\nabla^{m+1} T|^2 + CK^2|\nabla^m T|^2 + CK|\nabla^{m+1} T||\nabla^m T| \\ &+ CK^3|\nabla^m T|t^{-\frac{m}{2}} + CKt^{-\frac{1}{2}}|\nabla^m T|^2 + CK^2 t^{-\frac{(m+1)}{2}}|\nabla^m T|. \end{aligned} \tag{6.3.32}$$

Using Young's inequality for the fourth term in (6.3.32), we know that for any $\varepsilon > 0$ we have

$$K|\nabla^{m+1} T||\nabla^m T| \le \frac{K^2}{2\varepsilon}|\nabla^m T|^2 + \frac{\varepsilon}{2}|\nabla^{m+1} T|^2$$

and hence

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m T|^2 \le{}& \Delta|\nabla^m T|^2 - (2 - \frac{C\varepsilon}{2})|\nabla^{m+1} T|^2 + CK^2|\nabla^m T|^2 \\ &+ CK^3|\nabla^m T|t^{-\frac{m}{2}} + CKt^{-\frac{1}{2}}|\nabla^m T|^2 + CK^2 t^{-\frac{(m+1)}{2}}|\nabla^m T|. \end{aligned} \tag{6.3.33}$$

Hence for suitably chosen ε, we deduce that

$$\frac{\partial}{\partial t}|\nabla^m T|^2 \leq \Delta|\nabla^m T|^2 - |\nabla^{m+1}T|^2 + CK^2|\nabla^m T|^2 + CK^3|\nabla^m T|t^{-\frac{m}{2}} + CKt^{-\frac{1}{2}}|\nabla^m T|^2 + CK^2 t^{-\frac{(m+1)}{2}}|\nabla^m T|. \tag{6.3.34}$$

The derivation of (6.3.34) in fact holds for m replaced by $m-k$ for any $1 \leq k \leq m-1$. That is, we also have

$$\frac{\partial}{\partial t}|\nabla^{m-k} T|^2 \leq \Delta|\nabla^{m-k} T|^2 - |\nabla^{m+1-k}T|^2 + CK^2|\nabla^{m-k} T|^2 + CK^3|\nabla^{m-k} T|t^{-\frac{m-k}{2}} + CKt^{-\frac{1}{2}}|\nabla^{m-k} T|^2 + CK^2 t^{-\frac{(m+1-k)}{2}}|\nabla^{m-k} T|$$

for $1 \leq k \leq m-1$. Using the induction hypothesis that (6.3.14) holds for all $1 \leq k \leq m-1$, the above inequality becomes

$$\frac{\partial}{\partial t}|\nabla^{m-k} T|^2 \leq \Delta|\nabla^{m-k} T|^2 - |\nabla^{m+1-k}T|^2 + CK^4 t^{-(m-k)} + CK^3 t^{-\frac{1}{2}} t^{-(m-k)} \tag{6.3.35}$$

for all $k < m$. We emphasize here that we needed to use the induction hypothesis to get our simplified evolution inequality (6.3.35) when $1 \leq k \leq m-1$.

With these computations in hand, we define

$$f_m = t^m|\nabla^m T|^2 + \beta_m \sum_{k=1}^{m} \alpha_k^m t^{m-k}|\nabla^{m-k}T|^2 \tag{6.3.36}$$

for some positive constants β_m to be chosen later, where $\alpha_k^m = \frac{(m-1)!}{(m-k)!}$.

Using (6.3.34) and (6.3.35) we compute that

$$\begin{aligned}
\frac{\partial}{\partial t} f_m &= t^m \frac{\partial}{\partial t}|\nabla^m T|^2 + mt^{m-1}|\nabla^m T|^2 + \beta_m \sum_{k=1}^{m} \alpha_k^m t^{m-k} \frac{\partial}{\partial t}|\nabla^{m-k} T|^2 \\
&\quad + \beta_m \sum_{k=1}^{m} (m-k)\alpha_k^m t^{m-k-1}|\nabla^{m-k} T|^2 \\
&\le t^m \Big(\Delta|\nabla^m T|^2 - |\nabla^{m+1} T|^2 + CK^2|\nabla^m T|^2 + CK^3|\nabla^m T| t^{-\frac{m}{2}} \\
&\qquad + CKt^{-\frac{1}{2}}|\nabla^m T|^2 + CK^2 t^{-\frac{(m+1)}{2}}|\nabla^m T| \Big) \\
&\quad + mt^{m-1}|\nabla^m T|^2 + \beta_m \sum_{k=1}^{m} (m-k)\alpha_k^m t^{m-k-1}|\nabla^{m-k} T|^2 \\
&\quad + \beta_m \sum_{k=1}^{m} \alpha_k^m t^{m-k} (\Delta|\nabla^{m-k} T|^2 - |\nabla^{m+1-k} T|^2 + CK^4 t^{-(m-k)} + CK^3 t^{-\frac{1}{2}} t^{-(m-k)}).
\end{aligned}$$

Observe that in the first summation above, the term for $k = m$ vanishes. We reindex the second term in the last line above to sum from $k = 0$ to $k = m - 1$, and throw away the negative term corresponding to $k = 0$. Collecting terms, the above then becomes

$$\begin{aligned}
\frac{\partial}{\partial t} f_m &\le \Delta f_m + (CK^2 t^m + mt^{m-1} + CKt^{\frac{(2m-1)}{2}} - \beta_m \alpha_1^m t^{m-1})|\nabla^m T|^2 + CK^3 |\nabla^m T| t^{\frac{m}{2}} \\
&\quad + CK^2 t^{\frac{(m-1)}{2}} |\nabla^m T| + \beta_m \sum_{k=1}^{m-1} \big((m-k)\alpha_k^m - \alpha_{k+1}^m \big) t^{m-k-1} |\nabla^{m-k} T|^2 \\
&\quad + C\beta_m \sum_{k=1}^{m} \alpha_k^m (K^4 + K^3 t^{-\frac{1}{2}}).
\end{aligned}$$

Using Young's inequality on the third and the fourth terms above, we have

$$CK^3 |\nabla^m T| t^{\frac{m}{2}} \le CK^4 + CK^2 |\nabla^m T|^2 t^m$$

and

$$CK^2 t^{\frac{(m-1)}{2}} |\nabla^m T| \le Ct^{m-1} |\nabla^m T|^2 + CK^4,$$

and hence we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} f_m &\le \Delta f_m + (CK^2 t^m + mt^{m-1} + CKt^{\frac{(2m-1)}{2}} + Ct^{m-1} - \beta_m \alpha_1^m t^{m-1})|\nabla^m T|^2 \\
&\quad + \beta_m \sum_{k=1}^{m} \big((m-k)\alpha_k^m - \alpha_{k+1}^m \big) t^{m-k-1} |\nabla^{m-k} T|^2 + C\beta_m \sum_{k=1}^{m} \alpha_k^m (K^4 + K^3 t^{-\frac{1}{2}}).
\end{aligned}$$

Now we choose β_m sufficiently large and use the fact that $(m-k)\alpha_k^m - \alpha_{k+1}^m = 0$ for $1 \le k \le m-1$ to deduce that

$$\frac{\partial}{\partial t} f_m \le \Delta f_m + CK^4 + CK^3 t^{-\frac{1}{2}}. \tag{6.3.37}$$

Since $m \ge 1$, from the definition (6.3.36) of f_m we have that $f_m(0) = \beta_m \alpha_m^m |T|^2 \le \beta_m \alpha_m^m K^2$, so applying the maximum principle to (6.3.37) and using $K^2 t \le 1$ gives

$$\sup_{x \in M} f_m(x,t) \le \beta_m \alpha_m^m K^2 + CK^4 t + CK^3 t^{\frac{1}{2}} \le CK^2.$$

From the definition (6.3.36) of f_m, we finally conclude that

$$|\nabla^m T| \le CKt^{-\frac{m}{2}},$$

and the inductive step is complete. $\square$

One of our goals is to study the long-time existence of the flow. We seek a criterion that characterizes the blow-up time for the flow. This will be established in Theorem 6.3.8 later. In order to prove Theorem 6.3.8 later, we require the following corollary to Theorem 6.3.3, whose proof is an adaptation of the argument in the case of Ricci flow, and can be found in [CK04, §6.7].

Corollary 6.3.4. *Let $(M^7, \varphi(t))$ be a solution to the isometric flow. Suppose there exists $K > 0$ such that*

$$|T(x,t)|_g \le K \qquad \text{for all } x \in M \text{ and } t \in [0,\tau],$$

where $\tau > \frac{1}{K^2}$ and $|\nabla^j \mathrm{Rm}| \le C_j K^{2+j}$ for all $j \ge 0$. Then for all $m \in \mathbb{N}$ there exists a constant C_m depending only on (M,g) *such that*

$$|\nabla^m T(x,t)|_g \le C_m K^{1+m} \qquad \text{for all } x \in M \text{ and } t \in [\tfrac{1}{K^2}, \tau]. \tag{6.3.38}$$

Proof. Fix $t_0 \in \left[\frac{1}{K^2}, \tau\right]$ and let $\tau_0 = t_0 - \frac{1}{K^2}$. Let $\bar{t} = t - \tau_0$ and let $\bar{\varphi}(\bar{t})$ solve the Cauchy problem

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\varphi}(\bar{t}) &= \operatorname{div} \bar{T} \lrcorner \bar{\psi}, \\ \bar{\varphi}(0) &= \varphi(\tau_0).\end{aligned}$$

Then by the uniqueness of solutions to the isometric flow given in Theorem 6.2.12, we deduce that $\bar{\varphi}(\bar{t}) = \varphi(\bar{t} + \tau_0) = \varphi(t)$ for $\bar{t} \in [0, \frac{1}{K^2}]$. So by the hypothesis on the solution $\varphi(t)$, we have

$$|\bar{T}(x, \bar{t})| \leq K \qquad \text{for all } x \in M \text{ and } \bar{t} \in [0, \tfrac{1}{K^2}].$$

Applying Theorem 6.3.3 we have constants $\bar{C}_m$ depending only on m such that

$$|\bar{\nabla}^m \bar{T}(x, \bar{t})| \leq \bar{C}_m K \bar{t}^{-\frac{m}{2}}$$

for all $x \in M$ and $\bar{t} \in (0, \frac{1}{K^2}]$.

Now when $\bar{t} \in [\frac{1}{2K^2}, \frac{1}{K^2}]$ then

$$\bar{t}^{\frac{m}{2}} \geq 2^{-\frac{m}{2}} K^{-m},$$

so taking $\bar{t} = \frac{1}{K^2}$, we find that

$$|\nabla^m T(x, t_0)| \leq 2^{\frac{m}{2}} \bar{C}_m K^{1+m} \qquad \text{for all } x \in M.$$

Since $t_0 \in [\frac{1}{K^2}, \tau]$ was arbitrary, we obtain (6.3.38). □

6.3.2 Local estimates of the torsion

In this section we prove the local estimates on the derivatives of the torsion. The proof is similar to the local bounds on the higher derivatives of a solution of the harmonic map heat flow by Grayson–Hamilton [GH96] and to the local derivative estimates of the curvature for the Yang-Mills flow which was proved by Weinkove [Wei04]. We first define the parabolic cylinder

$$P_r(x_0, t_0) = \{(x, t) \in M \times \mathbb{R} \mid d(x, x_0) \leq r, t_0 - r^2 \leq t \leq t_0\}.$$

We need the following lemma, which is proved in [GH96, Lemma 2.1]. We state the particular version that is given in [Wei04, Lemma 2.1].

Lemma 6.3.5. *Let M be a compact manifold and F be a smooth function on $M \times [0, \infty)$. Let $x_0 \in M$ and $t_0 \geq 0$. There exists a constant $s > 0$ and for every $\gamma < 1$ a constant C_γ, such that the following holds. Let $r \leq s$. If at any point in the parabolic cylinder $P_r(x_0, t_0)$ for which $F \geq 0$, we have*

$$\frac{\partial F}{\partial t} \leq \Delta F - F^2,$$

then

$$F \leq \frac{C_\gamma}{r^2}$$

on the smaller parabolic cylinder $P_{\gamma r}(x_0, t_0)$.

Remark 6.3.6. From the proof of [GH96, Lemma 2.1] we deduce that Lemma 6.3.5 in fact also holds when M is complete, noncompact, with *bounded geometry*. That is, we require that there are $D_m < +\infty$ for $m \geq 0$, and $i_0 > 0$ such that

$$\begin{aligned} |\nabla^m \mathrm{Rm}| &\leq D_m \qquad \text{in } M, \\ \mathrm{inj}(M, g) &\geq i_0. \end{aligned}$$

This observation is used for the noncompact case of the almost monotonicity of Θ. See [DGK19, §5] for more details.

We now state and prove the local estimates for the derivatives of the torsion.

Theorem 6.3.7. *Let $\varphi(t)$ be a solution to the isometric flow on M^7. Let $x_0 \in M$ and $t_0 \geq 0$ such that $\varphi(t)$ is defined at least up to time t_0. There exists a constant $s > 0$ and constants C_m for $m \geq 1$ such that the following holds. Whenever $|T| \leq K$ and $|\nabla^j \mathrm{Rm}| \leq B_j K^{2+j}$ for all $j \geq 0$ in some parabolic cylinder $P_r(x_0, t_0)$ with $r \leq s$ and $K \geq \frac{1}{r^2}$, then we have*

$$|\nabla^m T| \leq C_m K^{m+1} \tag{6.3.39}$$

on the much smaller parabolic cylinder $P_{\frac{r}{2^m}}(x_0, t_0)$.

Proof. The proof is similar to the proof of Theorem 6.3.3 and is by induction on m. We have already derived all the evolution equations required for the proof in §6.3.1. By the discussion between the statement and the proof of Theorem 6.3.3, we can assume that $K \geq 1$.

We first prove the $m = 1$ case. Define the function

$$h = (8K^2 + |T|^2)|\nabla T|^2. \tag{6.3.40}$$

Applying Young's inequality to the third term of (6.3.21), we get

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 - (2 - C\varepsilon)|\nabla^2 T|^2 + CK^2|\nabla T|^2 + CK^6. \tag{6.3.41}$$

Now using (6.3.41) and (6.3.11), and the fact that $|T| \leq K$, we find from (6.3.40) that

$$\begin{aligned} \frac{\partial h}{\partial t} \leq{}& (\Delta|T|^2 - 2|\nabla T|^2 + CK^4)|\nabla T|^2 \\ &+ (8K^2 + |T|^2)(\Delta|\nabla T|^2 - (2 - C\varepsilon)|\nabla^2 T|^2 + CK^2|\nabla T|^2 + CK^6). \end{aligned} \tag{6.3.42}$$

Observe that

$$\nabla|T|^2 = 2\langle T, \nabla T\rangle \le 2|T||\nabla T| \le 2K|\nabla T|$$

and similarly

$$\nabla|\nabla T|^2 = 2\langle \nabla T, \nabla^2 T\rangle \le 2|\nabla T||\nabla^2 T|.$$

Combining the above two estimates and Cauchy-Schwarz gives

$$\langle \nabla|T|^2, \nabla|\nabla T|^2\rangle \ge -|\nabla|T|^2|\,|\nabla|\nabla T|^2| \ge -4K|\nabla T|^2|\nabla^2 T|.$$

Using the above we compute directly from (6.3.40) that

$$\begin{aligned}\Delta h &= (\Delta|T|^2)|\nabla T|^2 + (8K^2 + |T|^2)\Delta|\nabla T|^2 + 2\langle \nabla|T|^2, \nabla|\nabla T|^2\rangle \\ &\ge (\Delta|T|^2)|\nabla T|^2 + (8K^2 + |T|^2)\Delta|\nabla T|^2 - 8K|\nabla T|^2|\nabla^2 T|.\end{aligned} \tag{6.3.43}$$

From (6.3.42) and (6.3.43) and $|T| \le K$, we get

$$\begin{aligned}\frac{\partial h}{\partial t} &\le \Delta h - 2|\nabla T|^4 + CK^4|\nabla T|^2 + (8K^2 + |T|^2)(-(2 - C\varepsilon)|\nabla^2 T|^2 + CK^2|\nabla T|^2 + CK^6) \\ &\qquad + 8K|\nabla T|^2|\nabla^2 T| \\ &\le \Delta h - 2|\nabla T|^4 + CK^4|\nabla T|^2 - (2 - C\varepsilon)(8K^2 + |T|^2)|\nabla^2 T|^2 + CK^8 + 8K|\nabla T|^2|\nabla^2 T|.\end{aligned} \tag{6.3.44}$$

We want to use Young's inequality on both the $CK^4|\nabla T|^2$ and the $8K|\nabla T|^2|\nabla^2 T|$ terms above, so that the net amount of $|\nabla T|^4$ terms that remain are still strictly negative and the net amount of $|\nabla^2 T|^2$ terms that remain are also negative and can be discarded. This is a delicate balancing act. Explicitly, let $\delta, \gamma > 0$ and write

$$\begin{aligned}CK^4|\nabla T|^2 &\le \frac{C\delta}{2}|\nabla T|^4 + \frac{C}{2\delta}K^8, \\ 8K|\nabla T|^2|\nabla^2 T| &\le \frac{4}{\gamma}K^2|\nabla T|^4 + 4\gamma|\nabla^2 T|^2.\end{aligned}$$

Then (6.3.44) becomes

$$\begin{aligned}\frac{\partial h}{\partial t} &\le \Delta h - 2|\nabla T|^4 + \frac{C\delta}{2}|\nabla T|^4 + \frac{C}{2\delta}K^8 - (2 - C\varepsilon)(8K^2 + |T|^2)|\nabla^2 T|^2 + CK^8 \\ &\qquad + \frac{4}{\gamma}K^2|\nabla T|^4 + 4\gamma|\nabla^2 T|^2 \\ &\le \Delta h + \Big(-2 + \frac{C\delta}{2} + \frac{4K^2}{\gamma}\Big)|\nabla T|^4 + \Big(4\gamma - (2 - C\varepsilon)(8K^2 + |T|^2)\Big)|\nabla^2 T|^2 + \tilde{C}K^8.\end{aligned}$$

We want to ensure that

$$-2+\frac{C\delta}{2}+\frac{4K^2}{\gamma}<-\frac{1}{2}, \qquad \text{and} \qquad 4\gamma-(2-C\varepsilon)(8K^2+|T|^2)<0. \tag{6.3.45}$$

The second inequality in (6.3.45) is satisfied if we choose

$$\gamma<(2-C\varepsilon)2K^2. \tag{6.3.46}$$

Then, assuming $C\delta<3$, the first inequality in (6.3.45) and (6.3.46) can be combined to yield

$$\frac{8K^2}{3-C\delta}<\gamma<(2-C\varepsilon)2K^2.$$

It is clear that if δ and ε are chosen sufficiently small then γ will exit satisfying the above condition.

With these choices of ε, γ, and δ, we can discard the $|\nabla^2 T|^2$ term (which now has a negative coefficient), and we are left with

$$\frac{\partial h}{\partial t}\leq \Delta h-\frac{1}{2}|\nabla T|^4+CK^8. \tag{6.3.47}$$

From (6.3.40) and $|T|\leq K$, we have $h\leq 9K^2|\nabla T|^2$, so (6.3.47) finally becomes

$$\frac{\partial h}{\partial t}\leq \Delta h-\frac{h^2}{CK^4}+CK^8. \tag{6.3.48}$$

Now define, for *the same constant C as above*, the function

$$F=\frac{h}{CK^4}-K^2. \tag{6.3.49}$$

We compute using (6.3.49) and (6.3.48) that

$$\begin{aligned}\frac{\partial F}{\partial t}&\leq \frac{1}{CK^4}(\Delta h-\frac{h^2}{CK^4}+CK^8)\\&=\Delta F-(F+K^2)^2+K^4\\&\leq \Delta F-F^2.\end{aligned} \tag{6.3.50}$$

Let $(x,t)\in P_r(x_0,t_0)$. If $F(x,t)\leq 0$, then by the definition of F in (6.3.49) we have $|\nabla T|^2\leq \frac{C}{8}K^4\leq CK^4$ at such a point. If $F(x,t)\geq 0$, then since (6.3.50) holds, by Lemma 6.3.5 with $\gamma=\frac{1}{2}$ we have

$$F\leq \frac{C_\gamma}{r^2}\leq C_\gamma K \qquad \text{on } P_{\frac{r}{2}}(x_0,t_0).$$

Using the above, along with equation (6.3.49) and our assumption that $K \geq 1$, we deduce that

$$h \leq CK^4(C_\gamma K + K^2) \leq \tilde{C} K^6$$

and thus from (6.3.40) that

$$|\nabla T| \leq CK^2 \qquad \text{on } P_{\frac{r}{2}}(x_0, t_0),$$

which establishes the base case of the induction.

Now assume inductively that (6.3.39) holds for all $k < m$. We prove the theorem for m. Choose B to be a constant such that

$$K^m \leq B \leq CK^m \qquad \text{and} \qquad |\nabla^{m-1} T| \leq B \tag{6.3.51}$$

for some $C > 1$. (We can take $B = C_{m-1}K^m$ if we take $C_{m-1} > 1$.) Using this B, define a function h_m by

$$h_m = (8B^2 + |\nabla^{m-1}T|^2)|\nabla^m T|^2. \tag{6.3.52}$$

We estimate each term in the evolution (6.3.25) of $|\nabla^m T|^2$ using the induction hypothesis (6.3.39) for $k < m$. For the third term on the right hand side of (6.3.25), we get

$$\begin{aligned} \Big|\sum_{i=0}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm}\Big| &\leq |\nabla^m T * \nabla^m T * \mathrm{Rm}| + \Big|\sum_{i=1}^{m} \nabla^m T * \nabla^{m-i} T * \nabla^i \mathrm{Rm}\Big| \\ &\leq CK^2|\nabla^m T|^2 + CK^{m+3}|\nabla^m T| \end{aligned} \tag{6.3.53}$$

where we have used the hypothesis on $|\nabla^j \mathrm{Rm}|$ and the induction hypothesis in the last inequality. Note that following the same procedure that lead to (6.3.27) with assumption (6.3.39) instead we get

$$|\nabla^i \varphi| \leq CK^i \qquad \text{and} \qquad |\nabla^i \psi| \leq CK^i. \tag{6.3.54}$$

Thus for the fourth term on the right hand side of (6.3.25) is

$$\nabla^m T * \nabla^m(\nabla T * T * \varphi) = \sum_{i=0}^{m} \nabla^{m+1-i} T * \nabla^i (T * \varphi).$$

We decompose the sum above into four parts, corresponding to $i = 0$, $i = 1$, $2 \leq i \leq m-1$, and $i = m$. Then using (6.3.54) we compute

$$\begin{aligned}
|\nabla^m T * \nabla^m(\nabla T * T * \varphi)| &= |\nabla^m T * \nabla^{m+1} T * T * \varphi| + |\nabla^m T * \nabla^m T * (\nabla T * \varphi + T * T * \psi)| \\
&\quad + \Big|\nabla^m T * \sum_{i=2}^{m-1} \nabla^{m+1-i} T * (\sum_{j=0}^{i} \nabla^{i-j} T * \nabla^j \varphi)\Big| \\
&\quad + \Big|\nabla^m T * \nabla T * \sum_{i=0}^{m} \nabla^{m-i} T * \nabla^i \varphi\Big| \\
&\leq CK|\nabla^m T||\nabla^{m+1} T| + CK^2|\nabla^m T|^2 + CK^{m+3}|\nabla^m T|.
\end{aligned} \tag{6.3.55}$$

For the fifth term on the right hand side of (6.3.25), using (6.3.54) we have

$$\Big|\nabla^m T * \nabla^{m+1-i} \mathrm{Rm} * \nabla^i \varphi\Big| \leq CK^{m+3}|\nabla^m T|. \tag{6.3.56}$$

Similarly, for the last term on the right hand side of (6.3.25) we have

$$\sum_{i=0}^{m} \nabla^m T * \nabla^{m-i}(\mathrm{Rm} * T) * \nabla^i \psi = \sum_{i=0}^{m} \nabla^m T * \Big(\sum_{j=0}^{m-i} \nabla^{m-i-j} \mathrm{Rm} * \nabla^j T\Big) * \nabla^i \psi.$$

We split the double sum above into two parts, the first part corresponding to $i = 0, j = m$ and the second part corresponding to the rest. Then using the hypothesis on $|\nabla^j \mathrm{Rm}|$, the induction hypothesis, and (6.3.54) we have

$$\Big|\sum_{i=0}^{m} \nabla^m T * \nabla^{m-i}(\mathrm{Rm} * T) * \nabla^i \psi\Big| \leq CK^2|\nabla^m T|^2 + CK^{m+3}|\nabla^m T|. \tag{6.3.57}$$

Substituting the estimates (6.3.53), (6.3.55), (6.3.56) and (6.3.57) into (6.3.25) we get

$$\frac{\partial}{\partial t}|\nabla^m T|^2 \leq \Delta|\nabla^m T|^2 - 2|\nabla^{m+1} T|^2 + CK|\nabla^m T||\nabla^{m+1} T| + CK^2|\nabla^m T|^2 + CK^{m+3}|\nabla^m T|.$$

Now we use Young's inequality on the third term and the last term above to write

$$\begin{aligned}
K|\nabla^m T||\nabla^{m+1} T| &\leq \frac{K^2|\nabla^m T|^2}{2\varepsilon} + \frac{\varepsilon|\nabla^{m+1} T|^2}{2}, \\
K^{m+3}|\nabla^m T| &\leq \frac{K^{2m+4}}{2} + \frac{K^2|\nabla^m T|^2}{2}.
\end{aligned}$$

Substituting these into the expression for $\frac{\partial}{\partial t}|\nabla^m T|^2$ above gives

$$\frac{\partial}{\partial t}|\nabla^m T|^2 \leq \Delta|\nabla^m T|^2 - (2 - C\varepsilon)|\nabla^{m+1}T|^2 + CK^2|\nabla^m T|^2 + CK^{2m+4}. \tag{6.3.58}$$

The derivation of (6.3.58) in fact holds for m replaced by $m-1$. That is, we also have

$$\frac{\partial}{\partial t}|\nabla^{m-1}T|^2 \leq \Delta|\nabla^{m-1}T|^2 - (2 - C\varepsilon)|\nabla^m T|^2 + CK^2|\nabla^{m-1}T|^2 + CK^{2m+2}.$$

Using the induction hypothesis, the above inequality becomes

$$\frac{\partial}{\partial t}|\nabla^{m-1}T|^2 \leq \Delta|\nabla^{m-1}T|^2 - (2 - C\varepsilon)|\nabla^m T|^2 + CK^{2m+2}. \tag{6.3.59}$$

From (6.3.58) and (6.3.59) and the definition (6.3.52) of h_m, we have

$$\begin{aligned}\frac{\partial}{\partial t}h_m \leq{}& (8B^2 + |\nabla^{m-1}T|^2)(\Delta|\nabla^m T|^2 - (2 - C\varepsilon)|\nabla^{m+1}T|^2 + CK^2|\nabla^m T|^2 + CK^{2m+4}) \\ &+ |\nabla^m T|^2(\Delta|\nabla^{m-1}T|^2 - (2 - C\varepsilon)|\nabla^m T|^2 + CK^{2m+2}).\end{aligned}$$

Using (6.3.51) and throwing away some but not all of the negative terms, this inequality becomes

$$\begin{aligned}\frac{\partial}{\partial t}h_m \leq{}& (8B^2 + |\nabla^{m-1}T|^2)\Delta|\nabla^m T|^2 + |\nabla^m T|^2\Delta|\nabla^{m-1}T|^2 - (2 - C\varepsilon)8B^2|\nabla^{m+1}T|^2 \\ &- (2 - C\varepsilon)|\nabla^m T|^4 + CK^{2m+2}|\nabla^m T|^2 + CK^{4m+4}.\end{aligned} \tag{6.3.60}$$

Observe that from the inductive hypothesis (6.3.39) for $k < m$ and (6.3.51) we have

$$\nabla|\nabla^{m-1}T|^2 = 2\langle\nabla^{m-1}T, \nabla^m T\rangle \leq 2|\nabla^{m-1}T||\nabla^m T| \leq 2B|\nabla^m T|$$

and also that

$$\nabla|\nabla^m T|^2 = 2\langle\nabla^m T, \nabla^{m+1}T\rangle \leq 2|\nabla^m T||\nabla^{m+1}T|.$$

Combining the above two estimates and Cauchy-Schwarz gives

$$\langle\nabla^{m-1}|T|^2, \nabla^m|\nabla T|^2\rangle \geq -|\nabla^{m-1}|T|^2|\,|\nabla|\nabla^m T|^2| \geq -4B|\nabla^m T|^2|\nabla^{m+1}T|.$$

Using the above we compute directly from (6.3.52) that

$$\begin{aligned}\Delta h_m &= (\Delta|\nabla^{m-1}T|^2)|\nabla^m T|^2 + (8B^2 + |\nabla^{m-1}T|^2)\Delta|\nabla^m T|^2 + 2\langle\nabla|\nabla^{m-1}T|^2, \nabla|\nabla^m T|^2\rangle \\ &\geq |\nabla^m T|^2\Delta|\nabla^{m-1}T|^2 + (8B^2 + |\nabla^{m-1}T|^2)\Delta|\nabla^m T|^2 - 8B|\nabla^m T|^2|\nabla^{m+1}T|.\end{aligned} \tag{6.3.61}$$

From (6.3.60) and (6.3.61) we get

$$\begin{aligned}\frac{\partial}{\partial t}h_m \leq \Delta h_m &- (2-C\varepsilon)8B^2|\nabla^{m+1}T|^2 - (2-C\varepsilon)|\nabla^m T|^4 + CK^{2m+2}|\nabla^m T|^2 + CK^{4m+4}\\ &+ 8B|\nabla^m T|^2|\nabla^{m+1}T|.\end{aligned}$$

Applying Young's inequality on the final term we have

$$\begin{aligned}\frac{\partial}{\partial t}h_m \leq \Delta h_m &- (2-C\varepsilon)8B^2|\nabla^{m+1}T|^2 - (2-C\varepsilon)|\nabla^m T|^4 + CK^{2m+2}|\nabla^m T|^2 + CK^{4m+4}\\ &+ 4B^2\delta|\nabla^{m+1}T|^2 + \frac{4}{\delta}|\nabla^m T|^4.\end{aligned}$$

Just as in the base case, we now have a delicate balancing act. We want to choose δ and ε above that the net amount of $|\nabla^m T|^4$ terms that remain are still strictly negative and the net amount of $|\nabla^{m+1}T|^2$ terms that remain are also negative and can be discarded. Explicitly, we demand that

$$-(2-C\varepsilon)8B^2 + 4B^2\delta < 0, \qquad \text{and} \qquad -(2-C\varepsilon) + \frac{4}{\delta} < -\frac{3}{4}.$$

These can be rearranged to yield

$$\frac{16}{5-4C\varepsilon} < \delta < 4 - 2C\varepsilon.$$

It is clear that if ε is chosen sufficiently small then δ will exit satisfying the above condition.

With these choices of ε and δ, we are left with

$$\frac{\partial h}{\partial t} \leq \Delta h_m - \frac{3}{4}|\nabla^m T|^4 + CK^{2m+2}|\nabla^m T|^2 + CK^{4m+4}.$$

Using Young's inequality on the third term, the above becomes

$$\frac{\partial h}{\partial t} \leq \Delta h_m - \frac{1}{2}|\nabla^m T|^4 + CK^{4m+4}. \tag{6.3.62}$$

From (6.3.52) and $|\nabla^{m-1}T| \leq B \leq CK^m$ in (6.3.51), we have $h_m \leq CK^{2m}|\nabla^m T|^2$, so (6.3.62) finally becomes

$$\frac{\partial}{\partial t}h_m \leq \Delta h_m - \frac{h_m^2}{CK^{4m}} + CK^{4m+4}. \tag{6.3.63}$$

As in the $m = 1$ case, for *the same constant C as above*, define the function

$$F = \frac{h_m}{CK^{4m}} - K^2. \tag{6.3.64}$$

We compute using (6.3.64) and (6.3.63) that

$$\begin{aligned}\frac{\partial F}{\partial t} &\leq \frac{1}{CK^{4m}}(\Delta h_m - \frac{h_m^2}{CK^{4m}} + CK^{4m+4}) \\ &= \Delta F - (F + K^2)^2 + K^4 \\ &\leq \Delta F - F^2. \end{aligned} \tag{6.3.65}$$

Let $(x,t) \in P_r(x_0, t_0)$. If $F(x,t) \leq 0$, then by the definition of F in (6.3.64) and $K^m \leq B$ in (6.3.51) we have $|\nabla^m T|^2 \leq \frac{B^{-2}}{8} h_m \leq CK^{-2m}K^{4m+2} = CK^{2m+2}$ at such a point. If $F(x,t) \geq 0$, then since (6.3.65) holds, by Lemma 6.3.5 with $\gamma = \frac{1}{2}$ we have

$$F \leq \frac{C_\gamma}{r^2} \leq C_\gamma K \qquad \text{on } P_{\frac{r}{2}}(x_0, t_0).$$

Using the above, along with equation (6.3.64) and our assumption that $K \geq 1$, we deduce that

$$h_m \leq CK^{4m}(C_\gamma K + K^2) \leq \tilde{C}K^{4m+2}$$

and thus from (6.3.52) and $K^m \leq B$ in (6.3.51) that

$$|\nabla^m T| \leq CK^{m+1} \qquad \text{on } P_{\frac{r}{2^m}}(x_0, t_0),$$

which establishes the inductive step. □

6.3.3 Characterization of the blow-up time

Let M be a compact 7-manifold and let φ_0 be a G_2-structure on M. Then starting with φ_0, there exists a unique solution $\varphi(t)$ of the isometric flow on a maximal time interval $[0, \tau)$ where *maximal* means that either $\tau = \infty$ or $\tau < \infty$. The case $\tau < \infty$ means that there does not exist any $\varepsilon > 0$ such that $\bar{\varphi}(t)$ is a solution of the isometric flow for $t \in [0, \tau + \varepsilon)$ with $\bar{\varphi}(t) = \varphi(t)$ for $t \in [0, \tau)$. We call τ the *singular time* for the flow.

In this section, we use the global derivative estimates (6.3.14) to prove that the quantity $\mathcal{T}(t)$ defined in (6.3.7) must blow up at a finite time singularity along the flow. Explicitly, we prove the following result.

Theorem 6.3.8. *Let M^7 be compact and let $\varphi(t)$ be a solution to the isometric flow (6.2.8) in a maximal time interval $[0, \tau)$. If $\tau < \infty$, then $\mathcal{T}$ satisfies*

$$\lim_{t \nearrow \tau} \mathcal{T}(t) = \infty \tag{6.3.66}$$

and there is a lower bound on the blow-up rate of $\mathcal{T}(t)$ given by

$$\mathcal{T}(t) \geq \frac{1}{\sqrt{C(\tau - t)}} \tag{6.3.67}$$

for some constant $C > 0$.

Proof. We prove the contrapositive of the theorem. That is, we show that if $\mathcal{T}$ remains bounded along a sequence of times approaching τ, then the solution can be extended past τ. Let $\varphi(t)$ be a solution to the isometric flow which exists on a maximal time interval $[0, \tau]$. We first prove by contradiction that

$$\limsup_{t \nearrow \tau} \mathcal{T}(t) = \infty. \tag{6.3.68}$$

Suppose that (6.3.68) does not hold, so there exists a constant $K > 0$ such that

$$\sup_{M \times [0,\tau]} \mathcal{T}(t) = \sup_{M \times [0,\tau]} |T(x,t)|_g \leq K. \tag{6.3.69}$$

Note that since the metric does not evolve along the flow, we use the metric g induced by the initial G_2-structure. We have from (6.3.14) and (6.3.69) that

$$\Big|\frac{\partial}{\partial t}\varphi\Big|_g = |\operatorname{div} T \lrcorner \psi|_g \leq CKt^{-\frac{1}{2}} \tag{6.3.70}$$

for some uniform positive constant C. For any $0 < t_1 < t_2 < \tau$, we have

$$|\varphi(t_2) - \varphi(t_1)|_g \leq \int_{t_1}^{t_2} \Big|\frac{\partial}{\partial t}\varphi\Big| dt \leq CK(\sqrt{t_2} - \sqrt{t_1}) \tag{6.3.71}$$

which implies that $\varphi(t)$ converges to a 3-form $\varphi(\tau)$ continuously as $t \to \tau$. Since $\varphi(t)$ is a G_2-structure, we know that for all $t \in [0, \tau)$ we have

$$g(u,v)\operatorname{vol}_g = -\frac{1}{6}(u \lrcorner \varphi(t)) \wedge (v \lrcorner \varphi(t)) \wedge \varphi(t) \tag{6.3.72}$$

where vol_g is the volume form of g. Since g and vol_g do not change along the flow, as $t \to \tau$ the left hand side of (6.3.72) tends to a positive definite 7-form valued bilinear form and thus the limit 3-form is a *positive* 3-form and so is a G_2-structure. Moreover from the right hand side of (6.3.72) we see that the limit $\varphi(\tau)$ induces the same metric g. Thus, the solution $\varphi(t)$ of the isometric flow can be extended continuously to the time interval $[0, \tau]$. We now show that the extension is actually smooth, which gives our required contradiction.

We pause to prove the following.

Claim 6.3.9. *For all $m \in \mathbb{N}$, there exist constants C_m such that*

$$\sup_{M \times [0,\tau)} \Big|\nabla^m \varphi(t)\Big|_g \leq C_m.$$

Proof of Claim 6.3.9. The proof is by induction on m. For $m = 1$, at any $(x, t) \in M \times [0, \tau)$, we have

$$\frac{\partial}{\partial t}\nabla\varphi = \nabla\frac{\partial}{\partial t}\varphi = \nabla(\operatorname{div} T \lrcorner\, \psi) = \nabla(\operatorname{div} T) * \psi + \operatorname{div} T * T * \varphi.$$

Here we are again using the fact that the metric does not evolve along the flow. We know from (6.3.69) and Corollary 6.3.4 that both $|\nabla(\operatorname{div} T)| \leq A$ and $|\operatorname{div} T| \leq A$ on the time interval $(\frac{1}{K^2}, \tau)$ for some $A = A(m, K)$. Since $|\nabla(\operatorname{div} T)|$ and $|\operatorname{div} T|$ are bounded on $[0, \frac{1}{K^2}]$ by some constant $B = B(K)$ we get that

$$\Big|\frac{\partial}{\partial t}\nabla\varphi\Big| \leq \max\,(CA, CB) = \tilde{C},$$

and thus by integration we have

$$|\nabla\varphi(t)|_g \leq |\nabla\varphi(0)|_g + \int_0^\tau \Big|\frac{\partial}{\partial t}\nabla\varphi(t)\Big| dt \leq |\nabla\varphi(0)|_g + \tilde{C}\tau \leq C_1$$

because $\tau < \infty$. (This is where we crucially use that the maximal existence time is finite.) We have thus established the $m = 1$ case of the claim.

For the general case of the claim, we have

$$\Big|\frac{\partial}{\partial t}\nabla^m\varphi\Big|_g = |\nabla^m(\operatorname{div} T \lrcorner\, \psi)|_g \leq C\sum_{i=0}^{m} |\nabla^{m-i}(\operatorname{div} T)||\nabla^i\psi|. \tag{6.3.73}$$

By the induction hypothesis, we may assume that $\Big|\frac{\partial}{\partial t}\nabla^p\varphi\Big|$ and hence $|\nabla^p(\operatorname{div} T \lrcorner\, \psi)|$ has been estimated for all $0 \leq p < m$. Since $\nabla^i\psi$ contains $\nabla^{i-1}T$ as the highest order term, we just need to estimate the $|\nabla^m(\operatorname{div} T)|$ term. But again it follows from (6.3.69) and Corollary 6.3.4 that $|\nabla^m(\operatorname{div} T)| \leq A$ for some $A = A(m, K)$ on $(\frac{1}{K^2}, \tau)$ and $|\nabla^m(\operatorname{div} T)| \leq B$ for some $B(m, K)$ on $[0, \frac{1}{K^2}]$. Thus from (6.3.73) we get that

$$\Big|\frac{\partial}{\partial t}\nabla^m\varphi\Big|_g \leq C'_m, \tag{6.3.74}$$

and the inductive step now follows from (6.3.74) by integration. This completes the proof of Claim 6.3.9. □

We now return to the proof of Theorem 6.3.8. Let U be the domain of a fixed local coordinate chart. We know that $\varphi(\tau)$ is a continuous limit of G_2-structures and in U it satisfies

$$\varphi_{ijk}(\tau) = \varphi_{ijk}(t) + \int_t^\tau (\operatorname{div} T(s) \lrcorner\, \psi(s))_{ijk} ds. \tag{6.3.75}$$

Let $\alpha = (a_1, ..., a_r)$ be any multi-index with $|\alpha| = a_1 + \cdots + a_r = m \in \mathbb{N}$. We know from Claim 6.3.9 and (6.3.74) that

$$\frac{\partial^m}{\partial x^\alpha}\varphi_{ijk} \qquad \text{and} \qquad \frac{\partial^m}{\partial x^\alpha}(\operatorname{div} T \lrcorner\, \psi)_{ijk}$$

are uniformly bounded on $U \times [0, \tau)$. So from (6.3.75) we have that $\frac{\partial^m}{\partial x^\alpha}\varphi_{ijk}(\tau)$ is bounded on U and hence $\varphi(\tau)$ is a smooth G_2-structure. Moreover, from (6.3.75) we have

$$\Big|\frac{\partial^m}{\partial x^\alpha}\varphi_{ijk}(\tau) - \frac{\partial^m}{\partial x^\alpha}\varphi_{ijk}(t)\Big| \leq C(\tau - t)$$

and thus $\varphi(t) \to \varphi(\tau)$ uniformly in any C^m norm as $t \to \tau$, for $m \geq 2$.

Now, since $\varphi(\tau)$ is smooth, Theorem 6.2.12 gives a solution $\bar{\varphi}(t)$ of the isometric flow with $\bar{\varphi}(0) = \varphi(\tau)$ for a short time $0 \leq t < \varepsilon$. Since $\varphi(t) \to \varphi(\tau)$ smoothly as $t \to \tau$, it follows that

$$\bar{\varphi}(t) = \begin{cases} \varphi(t) & 0 \leq t < \tau \\ \bar{\varphi}(t - \tau) & \tau \leq t < \tau + \varepsilon \end{cases}$$

is a solution of the isometric flow which is smooth and satisfies $\bar{\varphi}(0) = \varphi(0)$. This contradicts the maximality of τ. Thus we indeed have

$$\limsup_{t \nearrow \tau} \mathcal{T}(t) = \infty, \tag{6.3.76}$$

which is equation (6.3.68). Thus, if $\lim_{t \nearrow \tau} \mathcal{T}(t)$ exists, it must be ∞.

Next we show that in fact (6.3.66) is true. Suppose not. Then there exists $K_0 < \infty$ and a sequence of times $t_i \nearrow \tau$ such that $\mathcal{T}(t_i) \leq K_0$. By the doubling time estimate in Proposition 6.3.2, we get that

$$\mathcal{T}(t) \leq 2\mathcal{T}(t_i) \leq 2K_0$$

for all times $t \in [t_i, \min\{\tau, t_i + \frac{1}{CK_0^2}\}]$. Since $t_i \nearrow \tau$ as $i \to \infty$, there exists i_0 large enough such that $t_{i_0} + \frac{C}{K_0^2} \geq \tau$. (Here again we crucially use the fact that τ is assumed to be finite.) But this implies that

$$\sup_{M \times [t_{i_0}, \tau]} \mathcal{T}(x,t) \leq 2K_0$$

which cannot happen as we have already shown above that this leads to a contradiction to the maximality of τ. This completes the proof of (6.3.66).

To obtain the lower bound of the blow-up rate (6.3.67), we apply the maximum principle to (6.3.11). We get

$$\frac{d}{dt}\mathcal{T}(t)^2 \leq C\mathcal{T}(t)^4$$

which implies that

$$\frac{d}{dt}\mathcal{T}(t)^{-2} \geq -C. \tag{6.3.77}$$

Since we proved above that $\lim_{t\to\tau} \mathcal{T}(t) = \infty$, we have

$$\lim_{t\to\tau} \mathcal{T}(t)^{-2} = 0.$$

Integrating (6.3.77) from t to $t_0 \in (t, \tau)$ and taking the limit as $t_0 \to \tau$, we get

$$\mathcal{T}(t) \geq \frac{1}{\sqrt{C(\tau - t)}}.$$

This completes the proof of Theorem 6.3.8. □

Combining Proposition 6.3.2 and Theorem 6.3.8, we deduce the following result about the minimal existence time.

Corollary 6.3.10. *Let φ_0 be a G_2-structure on a compact 7-manifold M with*

$$\mathcal{T} \leq K$$

for some constant K. Then the unique solution of the isometric flow with initial G_2-structure φ_0 exists at least for time $t \in [0, \frac{1}{CK^2}]$ where C is the uniform constant from Proposition 6.3.2.

6.3.4 Compactness

In this section, we prove a Cheeger–Gromov type compactness theorem for solutions to the isometric flow for G_2-structures. We also give a local version of the compactness theorem. Recall the following definition from [LW17].

Definition 6.3.11. Let (M_i, φ_i, p_i) be a sequence of 7-manifolds with G_2-structures φ_i and $p_i \in M_i$ for each i. Suppose the metric g_i on M_i associated to the G_2-structure φ_i is complete for each i. Let M be a 7-manifold with $p \in M$ and φ be a G_2-structure on M. We say that the sequence (M_i, φ_i, p_i) converges to (M, φ, p) in the Cheeger–Gromov sense and write

$$(M_i, \varphi_i, p_i) \to (M, \varphi, p) \qquad \text{as } i \to \infty$$

if there exists a sequence of compact subsets $\Omega_i \subset M$ exhausting M with $p \in \operatorname{int}(\Omega_i)$ for each i, a sequence of diffeomorphisms $F_i : \Omega_i \to F_i(\Omega_i) \subset M_i$ with $F_i(p) = p_i$ such that

$$F_i^* \varphi_i \to \varphi \qquad \text{as } i \to \infty$$

in the sense that $F_i^* \varphi_i - \varphi$ and its covariant derivatives of all orders (with respect to any fixed metric) converge uniformly to zero on every compact subset of M.

Lotay–Wei proved the following very general compactness theorem for G_2-structures in [LW17, Theorem 7.1].

Theorem 6.3.12. *Let M_i be a sequence of smooth 7-manifolds and for each i we let $p_i \in M_i$ and φ_i be a G_2-structure on M_i such that the metric g_i on M_i induced by φ_i is complete on M_i. Suppose that*

$$\sup_i \sup_{x \in M_i} \left(|\nabla_{g_i}^{k+1} T_i(x)|_{g_i}^2 + |\nabla_{g_i}^k \mathrm{Rm}_{g_i}(x)|_{g_i}^2 \right)^{\frac{1}{2}} < \infty \tag{6.3.78}$$

for all $k \geq 0$ and

$$\inf_i \operatorname{inj}(M_i, g_i, p_i) > 0,$$

where T_i, Rm_{g_i} are the torsion and the Riemann curvature tensor of φ_i and g_i respectively and $\operatorname{inj}(M_i, g_i, p_i)$ denotes the injectivity radius of (M_i, g_i) at p_i.

Then there exists a 7-manifold M, a G_2-structure φ on M and a point $p \in M$ such that, after passing to a subsequence, we have

$$(M_i, \varphi_i, p_i) \to (M, \varphi, p) \qquad \textit{as } i \to \infty.$$

The idea of the proof is to use Cheeger–Gromov compactness theorem [Ham95, Theorem 2.3] for complete pointed Riemannian manifolds to get a complete Riemannian 7-manifold (M, g) and $p \in M$ such that, after passing to a subsequence

$$(M_i, g_i, p_i) \to (M, g, p) \qquad \text{as } i \to \infty.$$

That is, there exist nested compact sets $\Omega_i \subset M$ exhausting M with $p \in \operatorname{int}(\Omega_i)$ for all i and diffeomorphisms $F_i : \Omega_i \to F_i(\Omega_i) \subset M_i$ with $F_i(p) = p_i$ such that $F_i^* g \to g$ smoothly as $i \to \infty$ on any compact subset of M. We then use the diffeomorphisms from the above convergence to pull-back the G_2-structure to get G_2-structures φ_i on Ω_i and using (6.3.78) we show that covariant derivatives of all orders of φ_i are uniformly bounded. The Arzelá–Ascoli theorem [AH11, Corollary 9.14] then implies that there is a 3-form φ such that after passing to a subsequence, $\varphi_i \to \varphi$ as $i \to \infty$. We then show that φ is a G_2-structure and it induces the metric g and hence we get that $(M_i, \varphi_i, p_i) \to (M, \varphi, p)$ as $i \to \infty$.

We note that if all the metrics in the sequence (M_i, φ_i, g_i) are the same then the limiting G_2-structure φ induces the same metric.

We now state and prove the compactness theorem for the isometric flow of G_2-structures.

Theorem 6.3.13. *Let M_i be a sequence of compact 7-manifolds and let $p_i \in M_i$ for each i. Let $\varphi_i(t)$ be a sequence of solutions to the isometric flow (6.2.8) for G_2-structures on M_i for $t \in (a, b)$, where $-\infty \leq a < 0 < b \leq \infty$. Suppose that*

$$\sup_i \sup_{x \in M_i, t \in (a,b)} |T_i(x, t)|_{g_i} < \infty \tag{6.3.79}$$

where T_i denotes the torsion of $\varphi_i(t)$, and the injectivity radius satisfies

$$\inf_i \operatorname{inj}(M_i, g_i(0), p_i) > 0. \tag{6.3.80}$$

Suppose further that there are uniform constants C_k, for all $k \geq 0$, such that

$$\sup_i |\nabla^k \mathrm{Rm}_i|_{g_i} \leq C_k. \tag{6.3.81}$$

Then there exists a 7-manifold M, a point $p \in M$ and a solution $\varphi(t)$ of the flow (6.2.8) on M for $t \in (a, b)$ such that, after passing to a subsequence,

$$(M_i, \varphi_i(t), p_i) \to (M, \varphi(t), p) \qquad \text{as } i \to \infty.$$

The proof is similar in spirit to the compactness theorem for the Ricci flow by Hamilton [Ham95]. See also the compactness theorem for the Laplacian flow for closed G_2-structures by Lotay–Wei [LW17]. The idea is to show that the bounds on the G_2-structure and on covariant derivatives and time derivatives of the G_2-structure at time $t = 0$ extend to bounds on the G_2-structures and covariant derivatives of the G_2-structures at subsequent times in the presence of bounds on the torsion and covariant derivatives of the torsion for all time.

Proof of Theorem 6.3.13. From the derivative estimates (6.3.14), Corollary 6.3.4 and (6.3.79), we have

$$|\nabla^m_{g_i(t)} T_i(x,t)| \leq C_m. \tag{6.3.82}$$

Since M_i is compact for each i, we know $|\mathrm{Rm}_i|_{g_i}$ is bounded. Assumption (6.3.80) allows us to use Theorem 6.3.12 for $t = 0$ to extract a subsequence of $(M_i, \varphi_i(0), p_i)$ which converges to a complete limit $(M, \varphi_\infty(0), p)$. So there exist compact subsets $\Omega_i \subset M$ exhausting M with $p \in \mathrm{int}(\Omega_i)$ for each i and diffeomorphisms $F_i : \Omega_i \to F_i(\Omega_i) \subset M_i$ with $F_i(p) = p_i$ such that $F_i^* g_i(0) \to g_\infty(0)$ and $F_i^* \varphi_i(0) \to \varphi_\infty(0)$ smoothly on any compact subset $\Omega \subset M$ as $i \to \infty$. Fix a compact subset $\Omega \times [c, d] \subset M \times (a, b)$ and let i be sufficiently large so that $\Omega \subset \Omega_i$. Let $\bar{g}_i(t) = F_i^* g_i(t)$. Now since $\varphi_i(t)$ are all solutions to the isometric flow, we have $g_i(t) = g_i(0)$ for each i. Thus we trivially have

$$\sup_{\Omega\times[c,d]} |\nabla^m_{\bar{g}_i(0)} \bar{g}_i(t)|_{\bar{g}_i(0)} = 0.$$

Since the limit metric $g_\infty(0)$ is uniformly equivalent to $g_i(0)$, we get

$$\sup_{\Omega\times[c,d]} |\nabla^m_{\bar{g}_i(\infty)} \bar{g}_i(t)|_{\bar{g}_i(\infty)} \leq C_m$$

for some positive constants C_m and similarly

$$\sup_{\Omega\times[c,d]} \left| \frac{\partial^l}{\partial t^l} \nabla^m_{\bar{g}_\infty(0)} \bar{g}_i(t) \right|_{\bar{g}_\infty(0)} \leq C_{m,l}$$

for some positive constants $C_{m,l}$.

Now let $\bar{\varphi}_i(t) = F_i^* \varphi_i(t)$. Then $\bar{\varphi}_i(t)$ is a sequence of solutions of the isometric flow on $\Omega \subset M$ for $t \in [c, d]$. Using (6.3.82) and proceeding in a similar way as in the proof of Claim 6.3.9, we deduce that there exist constants C_m, independent of i, such that

$$\sup_{\Omega\times[c,d]} |\nabla^m_{\bar{g}_i(0)} \bar{\varphi}_i(t)|_{\bar{g}_i(0)} \leq C_m \tag{6.3.83}$$

and since $\bar{g}_i(0)$ and $\bar{\varphi}(0)$ converge uniformly to $g_\infty(0)$ and $\bar{\varphi}_\infty(0)$ with all their covariant derivatives on Ω, we have

$$\sup_{\Omega\times[c,d]} |\nabla^m_{\bar{g}_\infty(0)}\bar{\varphi}_i(t)|_{\bar{g}_\infty(0)} \leq C_m. \tag{6.3.84}$$

Moreover, because the time derivatives can be written in terms of the spatial derivatives using the evolution equations of the isometric flow, we get for some uniform constants $C_{m,l}$ that

$$\sup_{\Omega\times[c,d]} \left|\frac{\partial^l}{\partial t^l}\nabla^m_{\bar{g}_\infty(0)}\bar{\varphi}_i(t)\right|_{\bar{g}_\infty(0)} \leq C_{m,l}. \tag{6.3.85}$$

It now follows from the Arzelá–Ascoli theorem that there exists a subsequence of $\bar{\varphi}_i(t)$ that converges smoothly on $\Omega \times [c,d]$. A diagonal subsequence argument then produces a subsequence that converges smoothly on any compact subset of $M \times (a,b)$ to a solution $\bar{\varphi}_\infty(t)$ of the isometric flow. $\square$

The compactness theorem for the Ricci flow has natural applications in the analysis of singularities of the Ricci flow. We would also like to have a similar application for the isometric flow. The idea is to consider shorter and shorter time intervals leading up to a singularity of the isometric flow and to rescale the solutions on each of these time intervals to obtain solutions with uniformly bounded torsion. By doing this we hope that the limiting manifold will tell us something about the nature of the singularity and more information, such as whether the singularity is modelled on a soliton.

More precisely, suppose M^7 is a compact manifold and let $\varphi(t)$ be a solution to the isometric flow on a maximal time interval $[0,\tau)$ with $\tau < \infty$. Theorem 6.3.8 then implies that $\mathcal{T}(t)$ defined in (6.3.7) satisfies $\lim_{t\nearrow\tau}\mathcal{T}(t) = \infty$. Consider a sequence of points (x_i, t_i) with $t_i \nearrow \tau$ and

$$\mathcal{T}(x_i,t_i) = \sup_{x\in M,\ t\in[0,t_i]} |T(x,t)|_g.$$

Define a sequence of parabolic dilations of the isometric flow

$$\varphi_i(t) = \mathcal{T}(x_i,t_i)^3\varphi(t_i + \mathcal{T}(x_i,t_i)^{-2}t) \tag{6.3.86}$$

and define

$$\mathcal{T}_{\varphi_i}(x,t) = |T_i(x,t)|_{g_i}. \tag{6.3.87}$$

If $\widetilde{\varphi} = c^3\varphi$ then, as explained in the proof of Lemma 6.2.13, we have

$$\widetilde{\operatorname{div}}\widetilde{T}\lrcorner\widetilde{\psi} = c^3 \operatorname{div} T\lrcorner\psi.$$

Hence, for each i, we have that $(M, \varphi_i(t))$ is a solution of the isometric flow (6.2.8) on the time interval $[-t_i \mathcal{T}(x_i, t_i)^2, (\tau - t_i)\mathcal{T}(x_i, t_i)^2]$. Note that for each i and for all $t \leq 0$ we have

$$\sup_M |\mathcal{T}_{\varphi_i}(x,t)| = \frac{|T_i(x,t)|_{g_i}}{\mathcal{T}(x_i, t_i)} \leq 1$$

by the definition of $\mathcal{T}(x_i, t_i)$. Thus by the doubling time estimate Proposition 6.3.2 and Corollary 6.3.10, there exists a uniform $b > 0$ such that

$$\sup_i \sup_{M \times (a,b)} |\mathcal{T}_{\varphi_i}(x,t)| \leq 2$$

for any $a < 0$. Thus, if we have $\inf_i \operatorname{inj}(M, g_i(0), x_i) > 0$, then using the compactness Theorem 6.3.13, we can extract a subsequence of $(M, \varphi_i(t), x_i)$ that converges to a solution $(M_\infty, \varphi_\infty(t), x_\infty)$ of the isometric flow.

Just as in the Ricci flow (see [CCGGIIKLLN07, §3.1]), from the proof of the compactness theorem for the isometric flow, we can prove a local version of Theorem 6.3.13 without much difficulty.

Theorem 6.3.14 (Local compactness). *Let $\{(M_i, \varphi_i(t), x_i)\}_{i \in \mathbb{N}}$, $x_i \in M_i$ and $t \in (a,b)$ be a sequence of compact pointed solutions of the isometric flow. If there exist $\rho > 0$, $C_0 < \infty$ independent of i such that*

$$|T_i|_{g_i} \leq C_0 \qquad \text{in } B_{g_i}(x_i, \rho) \times (a,b)$$

and

$$\operatorname{inj}_{g_i}(x_i) > 0,$$

and if there exist uniform constants C_k, for all $k \geq 0$, such that

$$|\nabla^k \mathrm{Rm}_i|_{g_i} \leq C_k \qquad \text{in } B_{g_i}(x_i, \rho) \times (a,b),$$

then there exists a subsequence such that $\{(B_{g_i}(x_i, \rho), \varphi_i(t), x_i)\}_{i \in \mathbb{N}}$ converges as $i \to \infty$ to a pointed solution $(B_\infty, \varphi_\infty(t), x_\infty)$, $t \in (a,b)$ of the isometric flow, smoothly on any compact subset of $B_\infty \times (a,b)$. Furthermore, B_∞ is an open manifold and the metric $g_\infty(t)$ of $\varphi_\infty(t)$ is complete on the closed ball $\overline{B_{g_\infty}(x_\infty, r)}$ for all $r < \rho$.

6.4 Summary of remaining results from [DGK19]

In this section we briefly summarize the remaining results from [DGK19]. These include the following.

(1) An Uhlenbeck-type trick which together with a modification of the underlying connection yields a nice reaction-diffusion equation for the torsion along the flow.

(2) Defining a quantity Θ for any solution of the flow and proving that it is *almost monotonic* along the flow. We also prove an ε-regularity result associated to Θ.

(3) Inspired by work of Colding–Minicozzi in [CM12] and Boling–Kelleher–Streets on the harmonic map heat flow [BKS17] and work of Kelleher–Streets on the Yang–Mills flow [KS16] we define an *entropy functional* and use it to establish that, if we have sufficiently small entropy, then we have long time existence and convergence of the flow to a G_2-structure φ_∞ with small divergence-free torsion.

(4) When the entropy is not small the flow may develop singularities in finite time. However, we prove that we can only have singularities of co-dimension at least 2. Finally, we prove that if the singularity is of Type-I then a sequence of blow-ups of the flow has a subsequence that converges to a shrinking soliton of the flow.

We elaborate on (2), (3) and (4) below.

Given a complete Riemannian manifold (M, g) with *bounded curvature* and $(x_0, t_0) \in M \times \mathbb{R}$, we denote by $u_{(x_0,t_0)}$ the *kernel of the backwards heat equation* on M starting with δ_{x_0} at time t_0. Explicitly, for $t \in (-\infty, t_0)$ we have

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + \Delta_g\Big) u_{(x_0,t_0)} &= 0, \\ \lim_{t \to t_0-} u_{(x_0,t_0)} &= \delta_{x_0}. \end{aligned} \tag{6.4.1}$$

We also define the smooth function $f_{(x_0,t_0)}$ by the relation

$$u_{(x_0,t_0)} = \frac{e^{-f_{(x_0,t_0)}}}{(4\pi(t_0 - t))^{\frac{7}{2}}}. \tag{6.4.2}$$

Definition 6.4.1. Let $(\varphi(t))_{t \in [0,t_0)}$ be an isometric flow on M inducing the metric g and define

$$\Theta_{(x_0,t_0)}(\varphi(t)) = (t_0 - t) \int_M |T_{\varphi(t)}|^2 u_{(x_0,t_0)} \operatorname{vol}_g. \tag{6.4.3}$$

From the discussion in Section 6.2.3, it follows that the quantity $\Theta_{(x_0,t_0)}$ is invariant under parabolic rescaling. In what follows we will simply write u for $u_{(x_0,t_0)}$.

One can think of Θ as a kind of "localized energy", but we will not use this terminology.

An *almost monotonicity formula* for $\Theta_{(x_0,t_0)}(\varphi(t))$ was proved in [DGK19, Theorem 5.3] which we state below.

Theorem 6.4.2 (Almost monotonicity formula)**.** *This theorem has two versions, as follows.*

(1) Let (M^7, g) be compact and let $(\varphi(t))_{t\in[0,t_0)}$ be an isometric flow inducing the metric g. Then for any $x_0 \in M$ and $\max\{0, t_0 - 1\} < \tau_1 < \tau_2 < t_0$, there exist $K_1, K_2 > 0$ depending only on the geometry of (M, g) such that the following monotonicity formula holds:

$$\Theta_{(x_0,t_0)}(\varphi(\tau_2)) \leq K_1\Theta_{(x_0,t_0)}(\varphi(\tau_1)) + K_2(\tau_2 - \tau_1)(E(\varphi(0)) + 1). \tag{6.4.4}$$

(2) Let $(M, g) = (\mathbb{R}^7, g_{\mathrm{Eucl}})$ and let $(\varphi(t))_{t\in[0,t_0)}$ be an isometric flow inducing g_{Eucl}. Then for any $x_0 \in \mathbb{R}^7$ and $0 \leq \tau_1 < \tau_2 < t_0$ we have strict monotonicity

$$\Theta_{(x_0,t_0)}(\varphi(\tau_2)) \leq \Theta_{(x_0,t_0)}(\varphi(\tau_1))$$

with equality if and only if for all $t \in [\tau_1, \tau_2]$

$$\operatorname{div} T_{\varphi(t)} = \frac{x - x_0}{2(t_0 - t)} \lrcorner\, T_{\varphi(t)}.$$

Remark 6.4.3. We note that in Theorem 6.4.2 (2), the case of equality corresponds to a particular special type of *shrinking isometric soliton* on $(\mathbb{R}^7, g_{\mathrm{Eucl}})$, as described in (6.2.27).

The energy functional, although quite natural, has the disadvantage that it is not scale invariant. As a result, it is not strong enough to control the small scale behaviour of a G_2-structure φ. In this section, motivated by analogous functionals for the mean curvature flow [CM12], the high dimensional Yang-Mills flow [KS16] and the harmonic map heat flow [BKS17], we introduce an *entropy* functional, and use it and the almost monotonicity formula above to establish an ε-regularity result, as well as to prove long time existence and convergence given small entropy.

Definition 6.4.4. Let (M, φ) be a compact manifold with G_2-structure inducing the Riemannian metric g. Let $u_{(x,t)}(y, s) = u^g_{(x,t)}(y, s)$ denote the backwards heat kernel, with respect to g, that becomes $\delta_{(x,t)}$ as $s \to t$. For $\sigma > 0$ we define

$$\lambda(\varphi, \sigma) = \max_{(x,t)\in M\times(0,\sigma]} \left\{ t \int_M |T_\varphi|^2(y) u_{(x,t)}(y, 0) \operatorname{vol}_g(y) \right\}. \tag{6.4.5}$$

We call $\lambda(\varphi, \sigma)$ the *entropy* of (M, φ). The precise value of σ is not important, only that $\sigma > 0$. One should think of σ as the "scale" at which we are analyzing the flow.

We prove the following ε-regularity theorem in [DGK19, Theorem 5.7].

Theorem 6.4.5 ($\boldsymbol{\varepsilon}$-regularity)**.** *Given (M, g) compact and $E_0 < +\infty$ there exist $\varepsilon, \bar{\rho} > 0$ such that for every $\rho \in (0, \bar{\rho}]$ there exist $r \in (0, \rho)$ and $C < +\infty$ such that the following holds:*

If $(M, \varphi(t))_{t \in [0, t_0)}$ is an isometric flow with $g_{\varphi(t)} = g$ and $E(\varphi(0)) \leq E_0$, and if $x_0 \in M$ is such that

$$\Theta_{(x_0, t_0)}(\varphi(t_0 - \rho^2)) < \varepsilon$$

then

$$\Lambda_r(x, t) |T_\varphi(x, t)| \leq C r^{-1}$$

in $B(x_0, r) \times [t_0 - r^2, t_0]$, where

$$\Lambda_r(x, t) = \min \left(1 - r^{-1} d_g(x_0, x), \sqrt{1 - r^{-2}(t_0 - t)}\right).$$

The following long time existence theorem for the isometric flow is proved in [DGK19, Theorem 5.15].

Theorem 6.4.6 (Low entropy convergence)**.** *Let (M^7, φ_0) be a compact manifold with G_2-structure inducing the metric g. Then, there exist constants $C_k < +\infty$* depending only on *(M, g) such that for every small $\delta > 0$ and $\sigma > 0$, there exists $\varepsilon(g, \delta, \sigma) > 0$ such that if*

$$\lambda(\varphi_0, \sigma) < \varepsilon, \tag{6.4.6}$$

then the isometric flow starting at φ_0 exists for all time and converges smoothly to a G_2-structure φ_∞ satisfying

$$\begin{aligned} \operatorname{div} T_{\varphi_\infty} &= 0, \\ |T_{\varphi_\infty}| &< \delta, \end{aligned}$$

and

$$|\nabla^k T_{\varphi_\infty}| \leq C_k,$$

for all $k \geq 1$.

If the entropy λ is not small then the flow may develop finite time singularities. Fixing the constants $\varepsilon, \bar{\rho} > 0$ of the ε-regularity Theorem 6.4.5 we define the *singular set*

$$S = \{x \in M : \Theta_{(x,\tau)}(\varphi(\tau - \rho^2)) \geq \varepsilon, \text{ for all } \rho \in (0, \bar{\rho}]\}. \tag{6.4.7}$$

The following theorem (cf. [DGK19, Theorem 5.18]) establishes an upper bound on the "size" of the singular set S.

Theorem 6.4.7 (Singularity structure)**.** *Let φ_0 be a* G_2*-structure inducing the metric g with*

$$E(\varphi_0) = \frac{1}{2} \int_M |T_{\varphi_0}|^2 \operatorname{vol}_g \leq E_0 \tag{6.4.8}$$

and consider the maximal smooth isometric flow $(\varphi(t))_{t \in [0,\tau)}$ with $\varphi(0) = \varphi_0$.

Suppose that $\tau < +\infty$. Then as $t \to \tau$ the flow converges smoothly to a G_2*-structure φ_τ outside a closed set S with finite 5-dimensional Hausdorff measure satisfying*

$$\mathcal{H}^5(S) \leq C E_0,$$

for some constant $C < \infty$ depending on g. In particular the Hausdorff dimension of S is at most 5.

Remark 6.4.8. Theorem 6.4.7 says that the singular set S is *at most* 5-dimensional. It would be interesting to find a geometric interpretation of the singular set S in terms of G_2 geometry. If such a description exists, then it is likely that S would be at most 4-dimensional, as there are no distinguished 5-dimensional subspaces in G_2 geometry.

Finally, we proved in [DGK19, Theorem 5.20] that if a singularity is of Type-I then a sequence of blow-ups of the flow admits a subsequence that converges to a shrinking soliton of the flow.

Theorem 6.4.9 (Type-I singularities)**.** *Let φ_0 be a* G_2*-structure inducing the metric g on a compact 7-manifold M, and consider the maximal smooth isometric flow $(\varphi(t))_{t \in [0,\tau)}$, with $\varphi(0) = \varphi_0$. Suppose that $\tau < +\infty$ and the flow encounters a Type I singularity. That is,*

$$\max_M |T_{\varphi(t)}| \leq \frac{C}{\sqrt{\tau - t}}.$$

Let $x \in M$ and $\mu_i \searrow 0$ and consider the rescaled sequence $\varphi_i(t) = \mu_i^{-3} \varphi(\tau - \mu_i^2 t)$. Then, after possibly passing to a subsequence, $(M, \varphi_i(t), x)$ converges smoothly to an ancient isometric flow $(\varphi_\infty(t))_{t<0}$ on $(\mathbb{R}^7, g_{\mathrm{Eucl}})$ induced by a shrinking soliton. That is,

$$\operatorname{div} T_{\varphi_\infty}(x,t) = -\frac{x}{2t} \lrcorner T_{\varphi_\infty}.$$

Moreover $x \in M \setminus S$ *if and only if* $\varphi_\infty(t)$ *is the stationary flow induced by a torsion-free* G_2*-structure* φ_∞ *on* $(\mathbb{R}^7, g_{\mathrm{Eucl}})$.

Remark 6.4.10. It is an interesting open problem whether there exist any nontrivial shrinking solitons on the Euclidean $\mathbb{R}^7$. If there do not exist any such solitons, then Theorem 6.4.9 would imply that no Type-I singularities can occur along the isometric flow.

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