Weak Moment Maps in Multisymplectic Geometry

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We introduce the notion of a weak (homotopy) moment map associated to a Lie group action on a multisymplectic manifold. We show that the existence/uniqueness theory governing these maps is a direct generalization from symplectic geometry.

We use weak moment maps to extend Noether's theorem from Hamiltonian mechanics by exhibiting a correspondence between multisymplectic conserved quantities and continuous symmetries on a multi-Hamiltonian system. We find that a weak moment map interacts with this correspondence in a way analogous to the moment map in symplectic geometry.

We define a multisymplectic analog of the classical momentum and position functions on the phase space of a physical system by introducing momentum and position forms. We show that these differential forms satisfy generalized Poisson bracket relations extending the classical bracket relations from Hamiltonian mechanics. We also apply our theory to derive some identities on manifolds with a torsion-free G_2 structure.

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1 Introduction

Multisymplectic geometry is the natural generalization of symplectic geometry in which the symplectic 2-form is replaced by an 'n-plectic' form, where n is any positive integer. Manifolds that are Kähler, hyper-Kähler, G_2, and Spin(7) are all naturally multisymplectic. In physics, n-plectic manifolds are used to describe n-dimensional covariant field theories (see e.g. [8]) and the case $n = 2$ is relevant to string theory (see e.g. [2]).

The main idea this thesis is concerned with is generalizing moment maps from symplectic geometry to multisymplectic geometry, and the corresponding applications. In particular, we introduce a weak (homotopy) moment map, defined in equation (1.2) below.

Recall that for a symplectic manifold (M, ω), a Lie algebra $\mathfrak{g}$ is said to act symplectically if $\mathcal{L}_{V_\xi}\omega = 0$, for all $\xi \in \mathfrak{g}$, where V_ξ is its infinitesimal generator. A symplectic group action is called Hamiltonian if one can find a moment map, that is, a map $f : \mathfrak{g} \to C^\infty(M)$ satisfying

$$df(\xi) = V_\xi \lrcorner\, \omega,$$

for all $\xi \in \mathfrak{g}$.

In multisymplectic geometry, ω is replaced by a closed, non-degenerate $(n+1)$-form, where $n \geq 1$. The pair (M, ω) is called an n-plectic manifold. A Lie algebra action is called multisymplectic if $\mathcal{L}_{V_\xi}\omega = 0$ for each $\xi \in \mathfrak{g}$. A generalization of moment maps from symplectic to multisymplectic geometry is given by a (homotopy) moment map. These maps are discussed in detail in [3]. A homotopy moment map is a collection of maps, $f_k : \Lambda^k\mathfrak{g} \to \Omega^{n-k}(M)$, with $1 \leq k \leq n+1$, satisfying

$$df_k(p) = -f_{k-1}(\partial_k(p)) + (-1)^{\frac{k(k+1)}{2}} V_p \lrcorner\, \omega, \tag{1.1}$$

for all $p \in \Lambda^k\mathfrak{g}$, where V_p is an infinitesimal generator (see Definition 3.10) and ∂_k is the k-th Lie algebra homology differential $\partial_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}$, defined by

$$\partial_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g} \qquad \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j}[\xi_i, \xi_k] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k,$$

for $k \geq 1$ and $\xi_1, \cdots, \xi_k \in \mathfrak{g}$. A weak (homotopy) moment map is a collection of maps $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}(M)$ satisfying

$$df_k(p) = (-1)^{\frac{k(k+1)}{2}} V_p \lrcorner\, \omega, \tag{1.2}$$

for $p \in \mathcal{P}_{\mathfrak{g},k}$. Here $\mathcal{P}_{\mathfrak{g},k}$ is the Lie kernel, which is defined to be the kernel of ∂_k. We call the k-th component f_k of a weak moment map a weak k-moment map.

We see that any collection of functions satisfying equation (1.1) must also satisfy (1.2). That is, any homotopy moment map induces a weak homotopy moment map.

Also note that the n-th component of a homotopy moment map coincides with the multi-moment maps of Madsen and Swann introduced in [15] and [16].

Of central importance in this thesis is the applications of these weak moment maps to multi-Hamiltonian systems. We define a multi-Hamiltonian system to be a triple (M, ω, H) where (M, ω) is n-plectic and H is a 'Hamiltonian' $(n-1)$-form. This means that there exists a vector field $X_H \in \Gamma(TM)$ such that $X_H \lrcorner\, \omega = -dH$. In analogy to Hamiltonian mechanics, the integral curves of the vector field are to be interpreted as the motions in the relevant physical system.

One of our main results is a generalization of Noether's theorem to multisymplectic geometry. In order to state our version of Noether's theorem, we first develop a multisymplectic 'Poisson' bracket $\{\cdot, \cdot\}$, which was introduced in [6], and a notion of multisymplectic conserved quantity and symmetry.

Multisymplectic conserved quantities were introduced in [20]. They defined three types: A differential form α is called a local, global, or strict conserved quantity if it is Hamiltonian and $\mathcal{L}_{X_H}\alpha$ is closed, exact, or zero respectively. Here Hamiltonian means that there exists a multivector field X_α such that $-d\alpha = X_\alpha \lrcorner\, \omega$. By adding in this requirement, we are then able to study how the extended 'Poisson' bracket interacts with the conserved quantities. We find that, analogous to the case of Hamiltonian mechanics, the Poisson bracket of two conserved quantities is always strictly conserved. That is,

Proposition 1.1. *Let α and β be two (local, global or strict) conserved quantities on a multi-Hamiltonian system (M, ω, H). Then $\{\alpha, \beta\}$ is strictly conserved, meaning $\mathcal{L}_{X_H}\{\alpha, \beta\} = 0$.*

From this proposition we show that the conserved quantities, modulo closed forms, constitute a graded Lie algebra. We also show that when restricted to a certain subspace, namely the Lie n-algebra of observables (see Definition 3.6), the conserved quantities form an L_∞-algebra.

Similarly, we find that our continuous symmetries also generate a graded Lie algebra. As an extension from Hamiltonian mechanics, we define a symmetry to be a Hamiltonian multivector field with respect to which the Lie derivative of the Hamiltonian has a specific form. Just as for the conserved quantities, we have three types of continuous symmetry.

Namely, a multivector field X is a local, global, or strict symmetry on (M, ω, H) if $\mathcal{L}_X\omega = 0$ and $\mathcal{L}_X H$ is closed, exact, or zero respectively. A generalization from Hamiltonian mechanics is

Proposition 1.2. *Given any two (local, global, strict) continuous symmetries X and Y, their Schouten bracket $[X, Y]$ is a continuous symmetry of the same type.*

From this proposition we show that the continuous symmetries, modulo elements in the kernel of ω, form a graded Lie algebra.

Our first generalization of Noether's theorem says that there is a correspondence between these notions of symmetry and conserved quantity on a multisymplectic manifold.

Theorem 1.3. *If α is a (local or global) conserved quantity, then every corresponding Hamiltonian multivector field X_α is a (local or global) continuous symmetry. Conversely, if X is a (local or global) continuous symmetry, then every corresponding Hamiltonian form is a (local or global) conserved quantity.*

As in symplectic geometry, this correspondence is not one-to-one. Indeed, for a Hamiltonian form, any two of its corresponding Hamiltonian multivector fields differ by an element in the kernel of ω. Conversely, any two Hamiltonian forms corresponding to a Hamiltonian multivector field differ by a closed form:

Let

$$\Omega_{\mathrm{Ham}}(M) = \{\alpha \in \Omega^\bullet(M); d\alpha = X \lrcorner\, \omega \text{ for some } X \in \Gamma(\Lambda^\bullet(TM))\}$$

denote the graded vector space of Hamiltonian forms, and let $\widetilde{\Omega}_{\mathrm{Ham}}(M)$ denote the quotient of $\Omega_{\mathrm{Ham}}(M)$ by closed forms. Similarily, we let

$$\mathfrak{X}_{\mathrm{Ham}}(M) = \{X \in \Gamma(\Lambda^\bullet(TM)); X \lrcorner\, \omega \text{ is exact}\}$$

denote the graded vector space of Hamiltonian multivector fields and $\widetilde{\mathfrak{X}}_{\mathrm{Ham}}(M)$ denote the quotient of $\mathfrak{X}_{\mathrm{Ham}}(M)$ by elements in the kernel of ω. We slightly improve on the results of [5] and show that $\{\cdot, \cdot\}$ descends to a well defined graded Poisson bracket on $\widetilde{\Omega}_{\mathrm{Ham}}(M)$. Then we show that

Theorem 1.4. *There is a natural isomorphism of graded Lie algebras between $(\widetilde{\mathfrak{X}}_{Ham}(M), [\cdot, \cdot])$ and $(\widetilde{\Omega}_{Ham}(M), \{\cdot, \cdot\})$.*

As a consequence of this theorem, we then show that our symmetries and conserved quantities, after appropriate quotients, are in one-to-one correspondence. In particular, we let

$\mathcal{C}_{\mathrm{loc}}(X_H)$, $\mathcal{C}(X_H)$, $\mathcal{C}_{\mathrm{str}}(X_H)$ denote the spaces of local, global, and strict conserved quantities respectively, and $\widetilde{\mathcal{C}}_{\mathrm{loc}}(X_H), \widetilde{\mathcal{C}}(X_H)$, and $\widetilde{\mathcal{C}}_{\mathrm{str}}(X_H)$ their quotients by closed forms. Similarily, we let $\mathcal{S}_{\mathrm{loc}}(H)$, $\mathcal{S}(H)$, and $\mathcal{S}_{\mathrm{str}}(H)$ denote the space of local, global, and strict continuous symmetries respectively, and $\widetilde{\mathcal{S}}_{\mathrm{loc}},(H), \widetilde{\mathcal{S}}(H)$, and $\widetilde{\mathcal{S}}_{\mathrm{str}}(H)$ their quotient by elements in the kernel of ω. We obtain:

Theorem 1.5. *There exists an isomorphism of graded Lie algebras from* $(\widetilde{\mathcal{S}}(H), [\cdot,\cdot])$ *and* $(\widetilde{\mathcal{C}}(X_H), \{\cdot,\cdot\})$ *and from* $(\widetilde{\mathcal{S}}_{loc}(H), [\cdot,\cdot])$ *to* $(\widetilde{\mathcal{C}}_{loc}(X_H), \{\cdot,\cdot\})$. *Moreover, there exists an injective graded Lie algebra homomorphism from* $(\widetilde{\mathcal{S}}_{str}(H), [\cdot,\cdot])$ *to* $(\widetilde{\mathcal{C}}(X_H), \{\cdot,\cdot\})$ *and from* $(\widetilde{\mathcal{C}}_{str}(X_H), \{\cdot,\cdot\})$ *to* $(\widetilde{\mathcal{S}}(H), [\cdot,\cdot])$.

Furthermore, we show that under certain assumptions for a group action on M, a weak moment map (f) gives rise to a whole family of conserved quantities and continuous symmetries. Specifically, a group action on a multi-Hamiltonian system (M, ω, H) is called locally, globally, or strictly H preserving if the Lie derivative of H under each infinitesimal generator from $\mathfrak{g}$ is closed, exact, or zero respectively. Under a locally or globally H preserving action, it was shown in [20] that for any $p \in \mathcal{P}_{\mathfrak{g},k}$, the k-th Lie kernel (see Definition 2.12), $f_k(p)$ is locally conserved, and if the group strictly preserves H then $f_k(p)$ is globally conserved.

We add to this result by showing that under the above assumptions V_p is a local or global continuous symmetry. In particular, let $S_k = \{V_p; p \in \mathcal{P}_{\mathfrak{g},k}\}$ denote the infinitesimal generators coming from the Lie kernel. Then $S = \oplus S_k$ is a differential graded Lie algebra. Let $C_k = \{f_k(p); p \in \mathcal{P}_{\mathfrak{g},k}\}$ denote the image of the moment map. We set $C = \oplus C_k$ and show that $C \cap L_\infty(M,\omega)$ is an L_∞-subalgebra of $L_\infty(M,\omega)$, the Lie n-algebra of observables. We then obtain

Theorem 1.6. *For any H preserving action, a homotopy moment map induces an L_∞-morphism from S to $C \cap L_\infty(M,\omega)$.*

We present two applications of our results:

First, we briefly recall the classical momentum and position functions discussed in Chapter 5.4 of [1]. Let N be a manifold and $M = T^*N$. Then M has a canonical symplectic form $\omega = -d\theta$, and the pair (M,ω) is called the phase space (see Example 3.13). Given a function $f \in C^\infty(N)$, by pulling it back to M we obtain a function $\widetilde{f} \in C^\infty(M)$, called the classical position function corresponding to f. Given a vector field $X \in \Gamma(TN)$, we define $P(X) = X^\sharp \lrcorner\, \theta$, where $X^\sharp$ is the complete lift of X (see Definition 7.3). We call the function $P(X) \in C^\infty(M)$ the classical momentum function corresponding to X. These position and momentum functions satisfy the following bracket relations: For $X, Y \in \Gamma(TN)$ and $f, g \in C^\infty(N)$,

$$\{P(X), P(Y)\} = P([X,Y]), \tag{1.3}$$

$$\{\widetilde{f}, \widetilde{g}\} = 0, \tag{1.4}$$

and

$$\{\widetilde{f}, P(X)\} = \widetilde{Xf}. \tag{1.5}$$

These bracket relations form the bridging gap from classical to quantum mechanics.

In Section 7 we generalize these bracket relations in the following way: Given a manifold N, we let $M = \Lambda^k(T^*N)$, for $k \geq 1$. Then M has a canonical k-plectic structure $\omega = -d\theta$ and the pair (M, ω) is called the multisymplectic phase space (see Example 3.13). For $\alpha \in \Omega^{k-s}(N)$, we denote its pullback to M by $\widetilde{\alpha}$, and call it the classical position form. Note that $\widetilde{\alpha}$ is in $\Omega^{k-s}(M)$. Moreover, given $X \in \Gamma(\Lambda^s(TN))$ we define its classical momentum form to be

$$P(X) = -(-1)^{\frac{(s+1)(s+2)}{2}} X^\sharp \lrcorner\, \theta,$$

a $(k-s)$-form on M, where again $X^\sharp$ is the complete lift of X (see Definition 7.3 for details). Letting $\mathfrak{g} = \Gamma(TN)$, we find that for $X \in \mathcal{P}_{\mathfrak{g},s}$, $Y \in \mathcal{P}_{\mathfrak{g},t}$ and $\alpha \in \Omega^{k-s}(N)$, $\beta \in \Omega^{k-t}(N)$,

$$\{P(X), P(Y)\} = -(-1)^{ts+s+t} P([X,Y]) - (-1)^{\frac{(s+1)(s+2)+(t+1)(t+2)}{2}} d(X^\sharp \lrcorner\, Y^\sharp \lrcorner\, \theta), \tag{1.6}$$

$$\{\widetilde{\alpha}, \widetilde{\beta}\} = 0, \tag{1.7}$$

and

$$\{\widetilde{\alpha}, P(Y)\} = -(-1)^{\frac{t(t+1)}{2}} (\widetilde{Y \lrcorner\, d\alpha}). \tag{1.8}$$

Notice that these equations are generalization of equations (1.3), (1.4), and (1.5) respectively.

In Section 7, we also apply our work to manifolds with closed G_2-structure. In particular, we derive some identities and extend Example 6.7 of [15] by obtaining a homotopy moment map for a T^2 action on a closed G_2-manifold.

Lastly, in Section 4, we study the existence and uniqueness of weak homotopy moment maps and show that the theory can be generalized directly from symplectic geometry. We also show that the equivariance of a weak moment map can be characterized in terms of $\mathfrak{g}$-module morphisms, analogous to the case of symplectic geometry. To state our results, recall that in symplectic geometry we have the following well-known results on the existence and uniqueness of moment maps, whose proofs can be found in [4].

Proposition 1.7. *Consider the symplectic action of a connected Lie group G acting on a* *symplectic manifold* (M, ω).

- *If the first Lie algebra cohomology vanishes, i.e.* $H^1(\mathfrak{g}) = 0$, *then a not necessarily equivariant moment map exists.*
- *If the second Lie algebra cohomology vanishes, i.e.* $H^2(\mathfrak{g}) = 0$, *then any non-equivariant moment map can be made equivariant.*
- *If the first Lie algebra cohomology vanishes, i.e.* $H^1(\mathfrak{g}) = 0$, *then equivariant moment maps are unique,*

and combining these results,

- *If both the first and second Lie algebra cohomology vanish, i.e.* $H^1(\mathfrak{g}) = 0$ *and* $H^2(\mathfrak{g}) = 0$, *then there exists a unique equivariant moment map.*

We generalize these results with the following theorems. Letting $\Omega^{n-k}_{\mathrm{cl}}$ denote the set of closed $(n-k)$-forms on M, we get the above propositions, in their respective order, by taking $n = k = 1$.

Theorem 1.8. *If* $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$ *then there exists a not necessarily equivariant weak homotopy k-moment map.*

Theorem 1.9. *If* $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{cl}) = 0$, *then any non-equivariant weak homotopy k-moment map can be made equivariant.*

Theorem 1.10. *If* $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{cl}) = 0$ *then an equivariant weak homotopy k-moment map is unique.*

Combining these results, we obtain from the Kunneth formula (see e.g. [21]):

Theorem 1.11. *If* $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$ *and* $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$, *then there exists a unique equivariant weak k-moment map* $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}$. *Moreover, if* $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$ *and* $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$ *for all* $1 \leq k \leq n$, *then a full equivariant weak moment map exists and is unique.*

We also show that the morphism properties of moment maps and their relationship to equivariance in multisymplectic geometry is analogous to the case of symplectic geometry. More specifically, recall that in symplectic geometry the equivariance of a moment map

$f : \mathfrak{g} \to C^\infty(M)$ is characterized by whether or not f is a Lie algebra morphism. That is, f is equivariant if and only if

$$f([\xi,\eta]) = \{f(\xi), f(\eta)\},$$

for all $\xi, \eta \in \mathfrak{g}$. However, as shown in Theorem 4.2.8 of [1] it is always true that f induces a Lie algebra morphism between $\mathfrak{g}$ and $C^\infty(M)$/constant, because $df([\xi,\eta]) = d\{f(\xi), f(\eta)\}$.

We generalize these results to multisymplectic geometry by showing that:

Theorem 1.12. *For any $1 \leq k \leq n$, a weak k-moment map is always a $\mathfrak{g}$-module morphism from $\mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\mathrm{Ham}}(M)/closed$. A weak k-moment map is equivariant if and only if it is a $\mathfrak{g}$-module morphism from $\mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\mathrm{Ham}}(M)$.*

This thesis is a synthesis of the results obtained in [10] and [11]. If not stated otherwise, we will always assume our manifold to be connected.

2 Background

We start by recalling some basic concepts from symplectic geometry.

2.1 Symplectic Geometry

Let M be a manifold and let $\omega \in \Omega^2(M)$ be a 2-form. By definition, for each $p \in M$ we have that $\omega(p) := \omega_p$ is a skew-symmetric bilinear map $\omega_p : T_pM \times T_pM \to \mathbb{R}$.

Definition 2.1. A 2-form $\omega \in \Omega^2(M)$ is said to be symplectic if it is closed and if ω_p is non-degenerate for each $p \in M$. Non-degeneracy of ω means that the map

$$T_pM \to T_p^*M \qquad V_p \mapsto V_p \lrcorner\, \omega_p \tag{2.1}$$

is an isomorphism for each $p \in M$. In such a case, the pair (M, ω) is called a symplectic manifold.

Definition 2.2. A triple (M, ω, H) where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$ is called a Hamiltonian system. The function H is called the associated Hamiltonian function.

Definition 2.3. Given $f \in C^\infty(M)$, we let X_f represent the unique vector field in $\Gamma(TM)$ satisfying

$$X_f \lrcorner\, \omega = df. \tag{2.2}$$

The vector field X_f is called the Hamiltonian vector field associated to f.

Hamiltonian vector fields allow us to define the Poisson bracket on $C^\infty(M)$.

Definition 2.4. Given $f, g \in C^\infty(M)$ their Poisson bracket is defined to be

$$\{f, g\} := X_g \lrcorner\, X_f \lrcorner\, \omega = \omega(X_f, X_g). \tag{2.3}$$

The Poisson bracket turns $C^\infty(M)$ into a Lie algebra. To show this $\{\cdot,\cdot\}$, we use the fact from (2.2) and (2.3) that $\{f, g\}$ is also equal to $X_f g$. Moreover, one can verify the Leibniz rule:

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

A straightforward computation shows that the map $C^\infty(M) \to \Gamma(TM)$ given by $f \mapsto X_f \in \Gamma(TM)$ is a Lie algebra anti-homomorphism. That is

$$X_{\{f,g\}} = -[X_f, X_g].$$

2.2 Differential Graded Lie Algebras

Recall the definition of a differential graded Lie algebra:

Definition 2.5. A differential graded Lie algebra is a $\mathbb{Z}$-graded vector space $L = \oplus_{i\in\mathbb{Z}} L_i$ together with a bracket $[\cdot,\cdot] : L_i \otimes L_j \to L_{i+j}$ and a differential $d : L_i \to L_{i-1}$. The bilinear map $[\cdot,\cdot]$ is graded skew symmetric:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

and satisfies the graded Jacobi identity:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

Lastly, the differential and bilinear map satisfiy the graded Leibniz rule:

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy].$$

Here we have let x, y, and z be arbitrary homogeneous elements in L of degrees $|x|, |y|$ and $|z|$ respectively.

Definition 2.6. A Gerstenhaber algebra is a $\mathbb{Z}$-graded algebra $A = \oplus_{i\in\mathbb{Z}} A_i$ that is graded commutative and has a bilinear map $[\cdot,\cdot] : A \otimes A \to A$ along with the following properties:

- $|ab| = |a| + |b|$,
- $|[a, b]| = |a| + |b| - 1$ (the bilinear map has degree -1),
- $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c]$ (the bilinear map satisfies the Poisson identity),
- $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$,

and lastly, the bilinear map satisfies the Jacobi identity:

- $(-1)^{(|a|-1)(|c|-1)}[a, [b, c]] + (-1)^{(|b|-1)(|a|-1)}[b, [c, a]] + (-1)^{(|c|-1)(|b|-1)}[c, [a, b]] = 0.$

Here we have let $|a|$ denote the degree of $a \in A$, and ab the product of a and b in A.

Let $(V, [\cdot,\cdot])$ be a Lie algebra. The Schouten bracket turns $\Lambda^\bullet V$, the exterior algebra of V, into a Gerstenhaber algebra. We quickly recall some properties of the Schouten bracket. A more detailed discussion can be found in [17].

On decomposable multivectors $X = X_1 \wedge \cdots \wedge X_k \in \Lambda^k V$ and $Y = Y_1 \wedge \cdots \wedge Y_l \in \Lambda^l V$ the Schouten bracket is given by

$$[X,Y] := \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge Y_l. \tag{2.4}$$

Proposition 2.7. *The Schouten bracket is the unique bilinear map* $[\cdot,\cdot] : \Lambda^\bullet V \times \Lambda^\bullet V \to \Lambda^\bullet V$ *satisfying the following properties:*

- *If* $\deg X = k$ *and* $\deg Y = l$ *then* $\deg([X,Y]) = k + l - 1$.
- $[X,Y] = -(-1)^{(k+1)(l+1)}[Y,X]$.
- *It coincides with the Lie bracket on* V.
- *It satisfies the graded Leibniz rule: For* X, Y, *and* Z *of degree* k, l, *and* m *respectively,*

$$[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(k-1)l} Y \wedge [X,Z].$$

- *It satisfies the graded Jacobi identity: For* X, Y, *and* Z *of degree* k, l *,and* m *respectively,*

$$\sum_{\text{cyclic}} (-1)^{(k-1)(m-1)} [X,[Y,Z]] = 0.$$

Proof. We leave the details of the proof to the reader, but note that existence can be proved by showing the expression given in (2.4) satisfies the desired properties. In particular, that the Schouten bracket satisfies the graded Jacobi identity follows from the Jacobi identity for the Lie bracket and the graded Leibniz rule. The uniqueness of the bracket follows from the requirement that the bracket is $\mathbb{R}$-bilinear. More details can be found in [17], for example. □

We now let M be a manifold and consider the Gerstenhaber algebra $(\Gamma(\Lambda^\bullet(TM)), \wedge, [\cdot,\cdot])$.

Definition 2.8. For a decomposable multivector field $X = X_1 \wedge \cdots \wedge X_k$ in $\Gamma(\Lambda^k(TM))$ and a differential form τ, we define the contraction of τ by X to be

$$X \lrcorner \tau := X_k \lrcorner \cdots \lrcorner X_1 \lrcorner \tau,$$

and extend by linearity to all multivector fields. We define the Lie derivative of τ in the direction of X to be

$$\mathcal{L}_X \tau := d(X \lrcorner \tau) - (-1)^k X \lrcorner d\tau. \tag{2.5}$$

Note that this is the usual Lie derivative when $k = 1$.

Throughout the thesis we will make extensive use of the following propositions.

Proposition 2.9. *Let $X \in \Gamma(\Lambda^k(TM))$ and $Y \in \Gamma(\Lambda^l(TM))$ be arbitrary. For a differential form τ, the following identities hold:*

$$d\mathcal{L}_X\tau = (-1)^{k+1}\mathcal{L}_X d\tau \tag{2.6}$$

$$[X,Y] \lrcorner \tau = (-1)^{(k+1)l}\mathcal{L}_X(Y \lrcorner \tau) - Y \lrcorner (\mathcal{L}_X\tau) \tag{2.7}$$

$$\mathcal{L}_{[X,Y]}\tau = (-1)^{(k+1)(l+1)}\mathcal{L}_X\mathcal{L}_Y\tau - \mathcal{L}_Y\mathcal{L}_X\tau \tag{2.8}$$

$$\mathcal{L}_{X\wedge Y}\tau = (-1)^l Y \lrcorner (\mathcal{L}_X\tau) + \mathcal{L}_Y(X \lrcorner \tau) \tag{2.9}$$

Proof. A full proof is given in Proposition A.3 of [6]. Note that equation (2.6) follows by taking the differential of both sides of (2.5) and using that $d^2 = 0$. Equation (2.7) is proved by induction on the tensor degrees of the multivector fields (see [6]). Using equation (2.7) we provide a proof of equation (2.8). We have that

$$\begin{aligned}\mathcal{L}_{[X,Y]}\tau &= d([X,Y] \lrcorner \tau) + (-1)^{k+l}[X,Y] \lrcorner d\tau \\ &= (-1)^{(k+1)l}d\mathcal{L}_X(Y \lrcorner \tau) - d(Y \lrcorner \mathcal{L}_X\tau) + (-1)^{k(l+1)}\mathcal{L}_X(Y \lrcorner d\tau) - (-1)^{k+l}Y \lrcorner \mathcal{L}_X d\tau.\end{aligned}$$

Now using equation (2.6), this is equal to

$$(-1)^{(k+1)(l+1)}\mathcal{L}_X(d(Y \lrcorner \tau)) - d(Y \lrcorner \mathcal{L}_X\tau) + (-1)^{k(l+1)}\mathcal{L}_X(Y \lrcorner d\tau) + (-1)^l Y \lrcorner d\mathcal{L}_X\tau$$

which by (2.5) is equal to

$$(-1)^{(k+1)(l+1)}\mathcal{L}_X\mathcal{L}_Y\tau - \mathcal{L}_Y\mathcal{L}_X\tau$$

as desired. Lastly, equation (2.9) can be proved directly. We have

$$\begin{aligned}\mathcal{L}_{X\wedge Y}\tau &= d(Y \lrcorner X \lrcorner \tau) - (-1)^{k+l}Y \lrcorner X \lrcorner d\tau \\ &= d(Y \lrcorner X \lrcorner \alpha) - (-1)^l Y \lrcorner d(X \lrcorner \tau) + (-1)^l Y \lrcorner d(X \lrcorner \tau) - (-1)^{k+l}Y \lrcorner X \lrcorner d\tau \\ &= \mathcal{L}_Y(X \lrcorner \tau) + (-1)^l Y \lrcorner \mathcal{L}_X\tau.\end{aligned}$$

$\square$

Another formula for the interior product by the Schouten bracket is given by the next proposition.

Proposition 2.10. *For $X \in \Gamma(\Lambda^k(TM))$ and $Y \in \Gamma(\Lambda^l(TM))$ we have that interior product with their Schouten bracket satisfies*

$$i[X,Y] = [-[i(Y),d],i(X)],$$

where the bracket on the right hand side is the graded commutator. Written out fully, this says that for an arbitrary form τ,

$$[X,Y]\lrcorner\tau = -Y\lrcorner d(X\lrcorner\tau) + (-1)^l d(Y\lrcorner X\lrcorner\tau) + (-1)^{kl+k} X\lrcorner Y\lrcorner d\tau - (-1)^{kl+k+l} X\lrcorner d(Y\lrcorner\tau) \quad (2.10)$$

Proof. This is Proposition 4.1 of [17]. It can also be derived directly from equations (2.5) and (2.7). We state it as a separate proposition because equation (2.10) will be used frequently in the rest of the thesis. $\square$

Next we recall the Chevalley-Eilenberg complex. We start with a Lie algebra $\mathfrak{g}$ and its exterior algebra $\Lambda^\bullet\mathfrak{g}$. The Gerstenhaber algebra $(\Lambda^\bullet\mathfrak{g}, \wedge, [\cdot,\cdot])$ is turned into a differential algebra by the following differential.

Definition 2.11. For a Lie algebra $\mathfrak{g}$, consider the differential

$$\partial_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}, \qquad \xi_1\wedge\cdots\wedge\xi_k \mapsto \sum_{1\le i<j\le k} (-1)^{i+j}[\xi_i,\xi_j]\wedge\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\widehat{\xi_j}\wedge\cdots\wedge\xi_k$$

for $k \ge 1$, and extend by linearity to non-decomposables. Define $\Lambda^{-1}\mathfrak{g} = \{0\}$ and ∂_0 to be the zero map. It follows from the graded Jacobi identity that $\partial^2 = 0$. The differential Gerstenhaber algebra $(\Lambda^k\mathfrak{g}, \wedge, \partial, [\cdot,\cdot])$ is called the Chevalley-Eilenberg complex.

Definition 2.12. We follow the terminology and notation of [15] and call $\mathcal{P}_{\mathfrak{g},k} = \ker\partial_k$ the k-th Lie kernel, which is a vector subspace of $\Lambda^k\mathfrak{g}$. Let $\mathcal{P}_\mathfrak{g}$ denote the direct sum of all the Lie kernels:

$$\mathcal{P}_\mathfrak{g} = \oplus_{k=0}^{\dim(\mathfrak{g})}\mathcal{P}_{\mathfrak{g},k}.$$

Note that if the Lie algebra is abelian then $\mathcal{P}_{\mathfrak{g},k} = \Lambda^k\mathfrak{g}$.

A straightforward computation gives the following lemma.

Lemma 2.13. For arbitrary $p \in \Lambda^k\mathfrak{g}$ and $q \in \Lambda^l\mathfrak{g}$ we have that

$$\partial(p \wedge q) = \partial(p) \wedge q + (-1)^k p \wedge \partial(q) + (-1)^k [p, q].$$

From this lemma we get the following.

Proposition 2.14. *We have $(\mathcal{P}_{\mathfrak{g}}, \partial, [\cdot, \cdot])$ is a differential graded subalgebra of the Chevalley-Eilenberg complex, with $\partial = 0$.*

Proof. The only nontrivial thing we need to show is that the Schouten bracket preserves the Lie kernel. While this follows immediately from the fact that ∂ is a graded derivation of the Schouten bracket, we can also show it using Lemma 2.13. Indeed, for $p \in \mathcal{P}_{\mathfrak{g},l}$ and $q \in \mathcal{P}_{\mathfrak{g},l}$ we have that

$$\begin{aligned}\partial(p \wedge q) &= \partial(p) \wedge q + (-1)^k p \wedge \partial(q) + [p, q] \\ &= [p, q].\end{aligned}$$

Hence, $[p, q]$ is exact and therefore closed. □

We now define a differential graded Lie algebra consisting of multivector fields.

Let G be a connected Lie group acting on a manifold M. For $\xi \in \mathfrak{g}$ let $V_\xi \in \Gamma(TM)$ denote the infinitesimal generator of the induced action on M by the one-parameter subgroup of G generated by ξ. For decomposable $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k\mathfrak{g}$ we introduce the notation $V_p := V_{\xi_1} \wedge \cdots \wedge V_{\xi_k}$ for the associated multivector field, and extend by linearity. Let

$$S_k = \{V_p \ : \ p \in \mathcal{P}_{\mathfrak{g},k}\} \tag{2.11}$$

and set

$$S = \oplus_{k=0}^{\dim(\mathfrak{g})} S_k. \tag{2.12}$$

Proposition 2.15. *We have that $(S, [\cdot, \cdot])$ is a graded Lie algebra. Moreover, for $p \in \Lambda^k\mathfrak{g}$, we have that $\partial V_p = -V_{\partial p}$. Note that we have abused notation and let ∂ denote the Chevalley-Eilenberg differentials for both the Lie algebras $(\Gamma(TM), [\cdot, \cdot])$ and $(\mathfrak{g}, [\cdot, \cdot])$.*

Proof. We first show that $V_{[p,q]} = -[V_p, V_q]$. Let $p = \xi_1 \wedge \cdots \wedge \xi_k$ and $q = \eta_1 \wedge \cdots \wedge \eta_l$. Then we have that

$$
\begin{aligned}
V_{[p,q]} &= \sum_{i,j} (-1)^{i+j} V_{[\xi_i,\eta_j]} \wedge V_{\xi_1} \wedge \cdots \widehat{V}_{\xi_i} \cdots \widehat{V}_{\eta_j} \wedge \cdots \wedge V_{\eta_l} \\
&= \sum_{i,j} -(-1)^{i+j} [V_{\xi_i}, V_{\eta_j}] \wedge V_{\xi_1} \wedge \cdots \widehat{V}_{\xi_i} \cdots \widehat{V}_{\eta_j} \wedge \cdots \wedge V_{\eta_l} \\
&= -[V_p, V_q].
\end{aligned}
$$

Now extend by linearity to all $p, q \in \Lambda^k \mathfrak{g}$. Here we used the standard result of group actions that $[V_\xi, V_\eta] = -V_{[\xi,\eta]}$. The first claim now follows since $(\mathcal{P}_{\mathfrak{g}}, [\cdot,\cdot])$ is a graded Lie algebra by Proposition 2.14. Moreover, we have that

$$
\begin{aligned}
\partial V_p &= \partial(V_{\xi_1} \wedge \cdots \wedge V_{\xi_k}) \\
&= \sum_{1 \le i < j \le k} (-1)^{i+j} [V_{\xi_i}, V_{\xi_j}] \wedge V_{\xi_1} \wedge \cdots \wedge \widehat{V}_{\xi_i} \wedge \cdots \wedge \widehat{V}_{\xi_j} \wedge \cdots \wedge V_{\xi_k} \\
&= -\sum_{1 \le i < j \le k} (-1)^{i+j} V_{[\xi_i,\xi_j]} \wedge V_{\xi_1} \wedge \cdots \wedge \widehat{V}_{\xi_i} \wedge \cdots \wedge \widehat{V}_{\xi_j} \wedge \cdots \wedge V_{\xi_k} \\
&= -V_{\partial p}
\end{aligned}
$$

Now extend by linearity to all $p \in \Lambda^k \mathfrak{g}$. In particular then, if p is in the Lie kernel, we have that $\partial V_p = -V_{\partial p} = 0$.

□

The following lemma will be used repeatedly in the rest of the thesis. We remark that it holds for arbitrary multivector fields; however, for our purposes it will suffice to consider the restriction to elements of S.

Lemma 2.16. **(Extended Cartan Lemma)** For decomposable $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k \mathfrak{g}$ and differential form τ we have that

$$
(-1)^k d(V_p \lrcorner\, \tau) = V_{\partial p} \lrcorner\, \tau + \sum_{i=1}^{k} (-1)^i (V_{\xi_1} \wedge \cdots \wedge \widehat{V}_{\xi_i} \wedge \cdots \wedge V_{\xi_k}) \lrcorner\, \mathcal{L}_{V_{\xi_i}} \tau + V_p \lrcorner\, d\tau.
$$

Proof. This is Lemma 3.4 of [15] or Lemma 2.18 of [20]. □

Let $\Phi : G \times M \to M$ be a Lie group action on M.

Definition 2.17. For $A \in \Gamma(TM)$ we let $\Phi_g^* A$ denote the vector field given by the push-forward of A by Φ_g^{-1}. That is,

$$(\Phi_g^*(A))_x := (\Phi_{g^{-1}})_{*,\Phi_g(x)}(A_{\Phi_{g(x)}}),$$

where $x \in M$. For a decomposable multivector field $Y = Y_1 \wedge \cdots \wedge Y_k$ in $\Gamma(\Lambda^k(TM))$ we will let $\mathrm{Ad}_g Y$ denote the extended adjoint action

$$\mathrm{Ad}_g Y = \mathrm{Ad}_g Y_1 \wedge \cdots \wedge \mathrm{Ad}_g Y_k$$

and we will let $\Phi_g^* Y$ denote the multivector field

$$\Phi_g^* Y = \Phi_g^* Y_1 \wedge \cdots \wedge \Phi_g^* Y_k.$$

We also extend ad to a map $\mathrm{ad} : \mathfrak{g} \times \Lambda^k \mathfrak{g} \to \Lambda^k \mathfrak{g}$ by

$$\mathrm{ad}_\xi(Y_1 \wedge \cdots \wedge Y_k) = \sum_{i=1}^k Y_1 \wedge \cdots \wedge \mathrm{ad}_\xi(Y_i) \wedge \cdots \wedge Y_k. \tag{2.13}$$

For an arbitrary multivector field $Y \in \Gamma(\Lambda^k(TM))$, the above definitions are all extended by linearity. Notice that for arbitrary $p \in \Lambda^k \mathfrak{g}$, we have $\mathrm{ad}_\xi(p) = [\xi, p]$.

The next proposition shows that the infinitesimal generator of the extended adjoint action agrees with the pull back action.

Proposition 2.18. *Let $\Phi : G \times M \to M$ be a group action. For every $g \in G$ and $p \in \Lambda^k \mathfrak{g}$ we have that*

$$V_{\mathrm{Ad}_g p} = \Phi_{g^{-1}}^* V_p.$$

Equivalently, the map $\Lambda^k \mathfrak{g} \to \Gamma(\Lambda^k(TM))$ given by $\xi_1 \wedge \cdots \wedge \xi_k \mapsto V_{\xi_1} \wedge \cdots \wedge V_{\xi_k}$ is equivariant with respect to the extended adjoint and pull back action.

Proof. Fix $q \in M$, $g \in G$. First suppose that $\xi \in \mathfrak{g}$. Then by Proposition 4.1.26 of [1] we have that

$$V_{\mathrm{Ad}_g \xi} = \Phi_{g^{-1}}^* V_\xi.$$

The claim now follows since for $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k \mathfrak{g}$,

$$\begin{aligned} V_{\mathrm{Ad}_g p} &:= V_{\mathrm{Ad}_g \xi_1} \wedge \cdots \wedge V_{\mathrm{Ad}_g \xi_k} \\ &= \Phi_{g^{-1}}^* V_{\xi_1} \wedge \cdots \wedge \Phi_{g^{-1}}^* V_{\xi_k} \end{aligned}$$

$$= \Phi^*_{g^{-1}} V_p \qquad \text{by definition.}$$

□

While in this thesis we will mostly be concerned with differential graded Lie algebras, we will also have the need to consider the more general structure of an L_∞-algebra.

2.3 L_∞-Algebras

We only state the definition of an L_∞-algebra and do not go into detail. More detail can be found in [18], for example.

Definition 2.19. An L_∞-algebra is a graded vector space $L = \oplus_{i=-\infty}^{\infty} L_i$ together with a collection of graded skew-symmetric linear maps $\{l_k : L^{\otimes k} \to L\ ;\ k \geq 1\}$, with $\deg(l_k) = k-2$, satisfying the following identity for all $m \geq 1$:

$$\sum_{\substack{i+j=m+1 \\ \sigma \in \mathrm{Sh}(i,m-i)}} (-1)^{\sigma} \epsilon(\sigma) (-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}) = 0.$$

Here σ is a permutation of m letters, $(-1)^\sigma$ is the sign of σ, and $\epsilon(\sigma)$ is the Koszul sign. We will not have need for the Koszul sign in this thesis and direct the reader to [18] for its definition. The subset $\mathrm{Sh}(p,q)$ of permutations on $p+q$ letters is the set of (p,q)-unshuffles. A permutation σ of $p+q$ letters is called a (p,q)-unshuffle if $\sigma(i) < \sigma(i+1)$ for $i \neq p$.

An L_∞-algebra $(L, \{l_k\})$ is called a Lie n-algebra if $L_i = 0$ for $i \geq n$.

L_∞ algebras are not central to this thesis; however, they are needed to define the L_∞ algebra of observables from [18] and to state Theorem 1.6.

Since any differential graded Lie algebra is a Lie n-algebra (indeed, just take $l_1 = \partial$, $l_2 = [\cdot,\cdot]$ and $l_k = 0$ for $k \geq 3$), Propositions 2.14 and 2.15 show that the spaces S and $\mathcal{P}_{\mathfrak{g}}$ have L_∞-algebra structures.

Next we recall the basic notions from group and Lie algebra cohomology that will be needed in this thesis.

2.4 Group Cohomology

Let G be a group and S a G-module. For $g \in G$ and $s \in S$, let $g \cdot s$ denote the action of G on S. Let $C^k(G,S)$ denote the space of smooth alternating functions from G^k to S and consider the differential $\delta_k : C^k(G,S) \to C^{k+1}(G,S)$ defined as follows. For $\sigma \in C^k(G,S)$ and $g_1, \cdots, g_{k+1} \in G$ define

$$\delta_k\sigma(g_1,\cdots,g_{k+1}) := \\ g_1 \cdot \sigma(g_2,\cdots,g_{k+1}) + \sum_{i=1}^{k}(-1)^i\sigma(g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+2},\cdots,g_{k+1}) - (-1)^k\sigma(g_1,\cdots,g_k). \tag{2.14}$$

A computation shows that $\delta^2 = 0$ so that $C^0(G,S) \to C^1(G,S) \to \cdots$ is a cochain complex. This cohomology is known as the differentiable cohomology of G with coefficients in S. We let $H^k(G,S)$ denote the k-th cohomology group and will call an equivalence class representative a k-cocycle.

2.5 Lie Algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra and R a $\mathfrak{g}$-module. Given $\xi \in \mathfrak{g}$ and $r \in R$, let $\xi \cdot r$ denote the action of $\mathfrak{g}$ on R. We let $C^k(\mathfrak{g},R)$ denote the space of multilinear alternating functions from $\mathfrak{g}^k$ to R and consider the differential $\delta_k : C^k(\mathfrak{g},R) \to C^{k+1}(\mathfrak{g},R)$ defined as follows. For $f \in C^k(\mathfrak{g},R)$ and $\xi_1, \cdots \xi_{k+1} \in \mathfrak{g}$ define

$$\delta_k f(\xi_1,\cdots,\xi_{k+1}) := \\ \sum_i(-1)^{i+1}\xi_i \cdot f(\xi_1,\cdots,\widehat{\xi_i},\cdots,\xi_{k+1}) + \sum_{i<j}(-1)^{i+j}f([\xi_i,\xi_j],\xi_1,\cdots,\widehat{\xi_i},\cdots,\widehat{\xi_j},\cdots,\xi_{k+1}). \tag{2.15}$$

A computation shows that $\delta^2 = 0$. We let $H^k(\mathfrak{g},R)$ denote the k-th cohomology group and call an equivalence class representative a (Lie algebra) k-cocycle. Note that for $k = 0$ the map $\delta_0 : R \to C^1(\mathfrak{g},R)$ is given by $(\delta_0 r)(\xi) = \xi \cdot r$, where $r \in R$ and $\xi \in \mathfrak{g}$. Thus, by definition,

$$H^0(\mathfrak{g},R) = \{r \in R;\ \xi \cdot r = 0 \text{ for all } \xi \in \mathfrak{g}\}.$$

For $k = 1$ the map $\delta_1 : C^1(\mathfrak{g},R) \to C^2(\mathfrak{g},R)$ is given by

$$\delta_1(f)(\xi_1,\xi_2) = \xi_1 \cdot f(\xi_2) - \xi_2 \cdot f(\xi_1) - f([\xi_1,\xi_2]),$$

where $f \in C^1(\mathfrak{g},R)$ and ξ_1 and ξ_2 are in $\mathfrak{g}$.

The standard example of Lie algebra cohomology is given when $R = \mathbb{R}$:

Example 2.20. (Exterior algebra of $\mathfrak{g}^*$) Consider the trivial $\mathfrak{g}$-action on $\mathbb{R}$. Then

$C^k(\mathfrak{g},\mathbb{R}) = \Lambda^k\mathfrak{g}^*$, and the Lie algebra cohomology differential $\delta_k : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ is given by

$$\delta_k\alpha(\xi_1 \wedge \cdots \wedge \xi_k) := \alpha\left(\sum_{1\le i<j\le k}(-1)^{i+j}[\xi_i,\xi_k] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k\right) \tag{2.16}$$

where $\alpha \in \Lambda^k\mathfrak{g}^*$, and $\xi_1\wedge\cdots\wedge\xi_k$ is a decomposable element of $\Lambda^k\mathfrak{g}$, and extended by linearity to non-decomposables. It is easy to check that $\delta^2 = 0$. We will also make frequent reference to the corresponding Lie algebra homology differential which is given by

$$\partial_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g} \qquad \xi_1\wedge\cdots\wedge\xi_k \mapsto \sum_{1\le i<j\le k}(-1)^{i+j}[\xi_i,\xi_k]\wedge\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\widehat{\xi_j}\wedge\cdots\wedge\xi_k, \tag{2.17}$$

for $k \ge 1$. We define $\Lambda^{-1}\mathfrak{g} = \{0\}$ and ∂_0 to be the zero map.

3 Multisymplectic Geometry

In this section we develop the necessary background in multisymplectic geometry.

3.1 Multi-Hamiltonian Systems

Multisymplectic manifolds are the natural generalization of symplectic manifolds.

Definition 3.1. A manifold M equipped with a closed $(n+1)$-form ω is called a pre-multisymplectic (or pre-n-plectic) manifold. If in addition the map $T_pM \to \Lambda^n T_p^*M$, $V \mapsto V \lrcorner\, \omega$ is injective, then (M, ω) is called a multisymplectic or n-plectic manifold.

The next example is a generalization of the phase space in Hamiltonian mechanics. It comes up frequently in this thesis.

Example 3.2. (Multisymplectic Phase Space)

Let N be a manifold and let $M = \Lambda^k(T^*N)$. Then $\pi : M \to N$ is a vector bundle over N with canonical k-form $\theta \in \Omega^k(M)$ defined by

$$\theta_{\mu_x}(Z_1, \cdots, Z_k) := \mu_x(\pi_*(Z_1), \cdots, \pi_*(Z_k)),$$

for $x \in N$, $\mu_x \in \Lambda^k(T_x^*N)$, and $Z_1, \cdots, Z_k \in T_{\mu_x}M$. The $(k+1)$-form $\omega \in \Omega^{k+1}(M)$ defined by $\omega = -d\theta$ is the canonical $(k+1)$-form. The pair (M, ω) is a k-plectic manifold. Notice that for $k = 1$ we recover the usual symplectic structure on the cotangent bundle.

Definition 3.3. If for $\alpha \in \Omega^{n-1}(M)$ there exists $X_\alpha \in \Gamma(TM)$ such that $d\alpha = -X_\alpha \lrcorner\, \omega$ then we call α a Hamiltonian $(n-1)$-form and X_α a corresponding Hamiltonian vector field to α. We let $\Omega^{n-1}_{\mathrm{Ham}}(M)$ denote the space of Hamiltonian $(n-1)$-forms.

Remark 3.4. If ω is n-plectic then the Hamiltonian vector field X_α is unique. If ω is pre-n-plectic then Hamiltonian vector fields are unique up to an element in the kernel of ω. Also, notice that in the 1-plectic (i.e. symplectic) case, every function is Hamiltonian.

Definition 3.5. In analogy to Hamiltonian mechanics, for a fixed n-plectic form ω and Hamiltonian $(n-1)$-form H, we call (M, ω, H) a multi-Hamiltonian system. We denote the Hamiltonian vector field of H by X_H.

There are many examples of multi-Hamiltonian systems and we refer the reader to Section 3.1 of [20] for some results on their existence.

In [18] it was shown that to any multisymplectic manifold one can associate the following L_∞-algebra.

Definition 3.6. The Lie n-algebra of observables, $L_\infty(M,\omega)$ is the following L_∞-algebra. Let $L = \oplus_{i=0}^{n} L_i$ where $L_0 = \Omega^{n-1}_{\mathrm{Ham}}(M)$ and $L_i = \Omega^{n-1-i}(M)$ for $1 \leq i \leq n-1$. The maps $l_k : L^{\otimes k} \to L$ of degree $k-2$ are defined as follows: For $k=1$,

$$l_1(\alpha) = \begin{cases} d\alpha & \text{if deg } \alpha > 0, \\ 0 & \text{if deg } \alpha = 0. \end{cases}$$

For $k > 1$,

$$l_k(\alpha_1, \cdots, \alpha_k) = \begin{cases} \zeta(k) X_{\alpha_k} \lrcorner \cdots X_{\alpha_1} \lrcorner\, \omega & \text{if deg } \alpha_1 \otimes \cdots \otimes \alpha_k = 0, \\ 0 & \text{if deg } \alpha_1 \otimes \cdots \otimes \alpha_k > 0. \end{cases}$$

Here $\zeta(k)$ is defined to equal $-(-1)^{\frac{k(k+1)}{2}}$. We introduce this notation as this sign comes up frequently.

Remark 3.7. It is easily verified that $\zeta(k)\zeta(k+1) = (-1)^{k+1}$. For future reference we also note that $\zeta(k)\zeta(l)\zeta(k+l-1) = -(-1)^{k+l+kl}$ and $\zeta(k)\zeta(l) = -(-1)^{lk}\zeta(k+l)$.

The following lemma from [18] will be useful later on in the thesis.

Lemma 3.8. Let $\alpha_1, \ldots, \alpha_m \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ be arbitrary Hamiltonian $(n-1)$-forms on a multisymplectic manifold (M,ω). Let $X_1, \ldots, X_m$ denote the associated Hamiltonian vector fields. Then

$$d(X_m \lrcorner \cdots \lrcorner\, X_1 \lrcorner\, \omega) = (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} X_m \lrcorner \cdots \lrcorner\, \widehat{X}_j \lrcorner \cdots \lrcorner\, \widehat{X}_i \lrcorner \cdots \lrcorner\, X_1 \lrcorner\, [X_i, X_j] \lrcorner\, \omega.$$

Proof. This is Lemma 3.7 of [18]. □

Lastly we recall the terminology for group actions on a multisymplectic manifold.

Definition 3.9. A Lie group action $\Phi : G \times M \to M$ is called multisymplectic if $\Phi_g^*\omega = \omega$. A Lie algebra action $\mathfrak{g} \times \Gamma(TM) \to \Gamma(TM)$ is called multisymplectic if $\mathcal{L}_{V_\xi}\omega = 0$ for all $\xi \in \mathfrak{g}$. We remark that a multisymplectic Lie group action induces a multisymplectic Lie algebra action. Conversely, a multisymplectic Lie algebra action induces a multisymplectic group action if the Lie group is simply-connected. Moreover, as in [20], we will call a Lie group action on a multi-Hamiltonian system (M,ω,H) locally, globally, or strictly H-preserving if it is multisymplectic and if $\mathcal{L}_{V_\xi}H$ is closed, exact, or zero respectively, for all $\xi \in \mathfrak{g}$.

3.2 Weak Homotopy Moment Maps

For a group acting on a symplectic manifold M, a moment map is a Lie algebra morphism between $(\mathfrak{g}, [\cdot,\cdot])$ and $(C^\infty(M), \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ is the Poisson bracket. In multisymplectic geometry, the n-plectic form no longer provides a Lie algebra structure on the space of smooth functions. However, as we saw in the previous section, the n-plectic structure does define an L_∞-algebra, namely the Lie n-algebra of observables. A homotopy moment map is an L_∞-morphism from $\mathfrak{g}$ to the Lie n-algebra of observables. We explain what this means in the following definition and refer the reader to [3] for further information on L_∞-morphisms.

For the rest of this section, we assume a multisymplectic action of a Lie algebra $\mathfrak{g}$ on (M, ω).

Definition 3.10. A (homotopy) moment map is an L_∞-morphism (f) between $\mathfrak{g}$ and the Lie n-algebra of observables. This means that (f) is a collection of maps $f_1 : \Lambda^1\mathfrak{g} \to \Omega^{n-1}_{\mathrm{Ham}}(M)$ and $f_k : \Lambda^k\mathfrak{g} \to \Omega^{n-k}(M)$ for $k \geq 2$ satisfying,

$$df_1(\xi) = -V_\xi \lrcorner\, \omega$$

for $\xi \in \mathfrak{g}$ and

$$-f_{k-1}(\partial p) = df_k(p) + \zeta(k) V_p \lrcorner\, \omega, \tag{3.1}$$

for $p \in \Lambda^k\mathfrak{g}$ and $k \geq 1$. A moment map is called equivariant if each component $f_i : \Lambda^i\mathfrak{g} \to \Omega^{n-i}(M)$ is equivariant with respect to the adjoint and pullback actions respectively.

Definition 3.11. A weak (homotopy) moment map, is a collection of maps (f) with $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\mathrm{Ham}}(M)$ satisying

$$df_k(p) = -\zeta(k) V_p \lrcorner\, \omega. \tag{3.2}$$

Remark 3.12. Notice that any collection of functions satisfying (3.1) also satisfies (3.2). That is, any homotopy moment map induces a weak homotopy moment map. Moreover, the multi-moment maps of Madsen and Swann (in [15] and [16]) are given precisely by the n-th component of our weak homotopy moment maps. Lastly, notice that when $n = 1$, both full and weak homotopy moment maps reduce to the standard definition in symplectic geometry.

We conclude this section with some examples of weak moment maps. In Hamiltonian mechanics, the phase space of a manifold M is the symplectic manifold $(T^*M, \omega = -d\theta)$. The next example generalizes this to the setting of multisymplectic geometry.

Example 3.13. (Multisymplectic Phase Space) As in Example 3.2, let N be a manifold and let $M = \Lambda^k(T^*N)$, with $\pi : M \to N$ the projection map. Let θ and $\omega = -d\theta$ denote the canonical k and $(k+1)$-forms respectively. Let G be a group acting on N and lift this action to M in the standard way. Such an action on M necessarily preserves θ. We define a weak homotopy moment map by

$$f_l(p) := -\zeta(l+1)V_p \lrcorner \theta,$$

for $p \in \mathcal{P}_{\mathfrak{g},l}$.

We now show that (f) is a weak homotopy moment map. For $l \geq 1$, first consider a decomposable element $p = A_1 \wedge \cdots \wedge A_l$ in $\Lambda^l \mathfrak{g}$. Then, using Lemma 2.16 and the fact that G preserves ω, we find

$$\begin{aligned} df_l(p) &= -\zeta(l+1)d(V_p \lrcorner \theta) \\ &= -\zeta(l+1)(-1)^l \left(\partial V_p \lrcorner \theta + \sum_i^l (-1)^i A_1 \wedge \cdots \wedge \widehat{A}_i \wedge \cdots \wedge A_l \lrcorner \mathcal{L}_{A_i}\theta - V_p \lrcorner \omega \right) \\ &= \zeta(l) \left(\partial V_p \lrcorner \theta - V_p \lrcorner \omega \right). \end{aligned}$$

By linearity, we thus see that this equation holds for an arbitrary element in $\Lambda^l \mathfrak{g}$. That is, for all $p \in \Lambda^l \mathfrak{g}$,

$$df_l(p) = \zeta(l) \left(\partial V_p \lrcorner \theta - V_p \lrcorner \omega \right).$$

If we assume now that $p \in \mathcal{P}_{\mathfrak{g},l}$, it then follows from Proposition 2.15 that

$$df_l(p) = -\zeta(l)V_p \lrcorner \omega.$$

Thus by equation (3.2) we see (f) is a weak homotopy moment map.

Remark 3.14. In symplectic geometry, symmetries on the phase space T^*M have an important relationship with the classical momentum and position functions (see Chapter 4.3 of [1]). These momentum and position functions satisfy specific commutation relations which play an important role in connecting classical and quantum mechanics. Once we extend the Poisson bracket to multisymplectic manifolds and discuss a generalized notion of symmetry, we will come back to this multisymplectic phase space and give a generalization of these classical momentum and position functions (see Section 6.1).

The next two examples will be used when we look at manifolds with a torsion-free G_2 structure.

Example 3.15. **($\mathbb{C}^3$ with the standard holomorphic volume form)** Consider $\mathbb{C}^3$ with standard coordinates z_1, z_2, z_3. Let $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ denote the standard holomorphic volume form. Let

$$\alpha = \mathrm{Re}(\Omega) = \frac{1}{2}(dz_1 \wedge dz_2 \wedge dz_3 + d\overline{z}_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3).$$

It follows that α is a 2-plectic form on $\mathbb{C}^3$. We consider the diagonal action by the maximal torus $T^2 \subset SU(3)$ given by $(e^{i\theta}, e^{i\eta}) \cdot (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{i\eta} z_2, e^{-i(\theta+\eta)} z_3)$. We have $\mathfrak{t}^2 = \mathbb{R}^2$ and that the infinitesimal generators of $(1,0)$ and $(0,1)$ are

$$A = \frac{i}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3} - \overline{z}_1 \frac{\partial}{\partial \overline{z}_1} + \overline{z}_3 \frac{\partial}{\partial \overline{z}_3}\right)$$

and

$$B = \frac{i}{2}\left(z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - \overline{z}_2 \frac{\partial}{\partial \overline{z}_2} + \overline{z}_3 \frac{\partial}{\partial \overline{z}_3}\right)$$

respectively.

A computation then shows that

$$A \lrcorner\, \alpha = \frac{1}{2} d(\mathrm{Im}(z_1 z_3 dz_2))$$

and

$$B \lrcorner\, \alpha = \frac{1}{2} d(\mathrm{Im}(z_1 z_3 dz_1)).$$

Moreover,

$$B \lrcorner\, A \lrcorner\, \alpha = -\frac{1}{4} d(\mathrm{Re}(z_1 z_2 z_3))$$

Since $G = T^2$ is abelian we have that $\mathcal{P}_{\mathfrak{g},2} = \Lambda^2 \mathfrak{g}$. Thus, by equation (3.2) we see a weak homotopy moment map is given by

$$f_1(A) = \frac{1}{2}(\mathrm{Im}(z_1 z_3 dz_2)) \qquad\qquad f_1(B) = \frac{1}{2}(\mathrm{Im}(z_1 z_3 dz_1))$$

and

$$f_2(A \wedge B) = \frac{1}{4}(\mathrm{Re}(z_1 z_2 z_3)).$$

If instead of $\mathrm{Re}(\Omega)$ we were to consider $\mathrm{Im}(\Omega)$ then in the above expressions for f_1 and f_2 we would just swap the roles of Re and Im.

Example 3.16. **($\mathbb{C}^3$ with the standard Kahler form)** Working with the same set up as Example 3.15, now consider the standard Kahler form $\omega = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3)$.

This is a 1-plectic (i.e. symplectic) form on $\mathbb{C}^3$. A computation shows that

$$A\lrcorner\,\omega = -\frac{1}{4}d(|z_1|^2 - |z_3|^2)$$

and

$$B\lrcorner\,\omega = -\frac{1}{4}d(|z_2|^2 - |z_3|^2).$$

Thus, by equation (3.2) a weak homotopy moment map is given by

$$f_1(A) = -\frac{1}{4}(|z_1|^2 - |z_3|^2) \qquad\qquad f_1(B) = -\frac{1}{4}(|z_2|^2 - |z_3|^2).$$

For our last example of this section, we consider a multi-Hamiltonian system which models the motion of a particle, with unit mass, under no external net force.

Example 3.17. (Motion in a conservative system under translation)

Consider $\mathbb{R}^3$ with the standard metric g and standard coordinates q^1, q^2, q^3. Let q^1, q^2, q^3, p_1, p_2, p_3 denote the induced coordinates on $T^*\mathbb{R}^3 = \mathbb{R}^6$. The motion of a particle in $\mathbb{R}^3$, subject to no external force, is given by a geodesic. That is, the path γ of the particle is an integral curve for the geodesic spray S, a vector field on $T\mathbb{R}^3 = \mathbb{R}^6$. Using the metric to identify $T\mathbb{R}^3$ and $T^*\mathbb{R}^3$, the geodesic spray is given by

$$S = g^{kj}p_j\frac{\partial}{\partial q^k} - \frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}p_ip_j\frac{\partial}{\partial p_k},$$

as shown in Example 5.21 of [9]. This vector field S is the standard Hamiltonian vector field on the phase space. Since we our working with the standard metric, the geodesic spray is just

$$S = \sum_{i=1}^{3} p_i\frac{\partial}{\partial q^i}.$$

Let $M = T^*\mathbb{R}^3 = \mathbb{R}^6$ and consider the multi-Hamiltonian system (M, ω, H) where

$$\omega = \mathrm{vol} = dq^1dq^2dq^3dp_1dp_2dp_3$$

is the canonical volume form, and

$$H = \frac{1}{2}\left((p_1q^2dq^3 - p_1q^3dq^2) + (p_2q^1dq^3 - p_2q^3dq^1) + (p_3q^1dq^2 - p_3q^2dq^1)\right)dp_1dp_2dp_3.$$

Then

$$S\lrcorner\,\omega = dH$$

so that the $X_H = S$. That is, the Hamiltonian vector field in this multi-Hamiltonian system is the geodesic spray. Consider the translation action of $G = \mathbb{R}^3$ on $\mathbb{R}^3$ and pull this back to an action on M. The infinitesimal generators of e_1, e_2, e_3 on M are $\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}$ and $\frac{\partial}{\partial q^3}$ respectively. We compute the moment map for this action:

Since $\frac{\partial}{\partial q^1} \lrcorner \omega = dq^2 dq^3 dp_1 dp_2 dp_3$ it follows that $f_1(e_1) = \frac{1}{2}(q^2 dq^3 - q^3 dq^2) dp_1 dp_2 dp_3$ satisfies $df(e_1) = V_{e_1} \lrcorner \omega$. Similar computations show that the following is a homotopy moment map for the translation action on (M, ω, H):

$$f_1(e_1) = \frac{1}{2}(q^2 dq^3 - q^3 dq^2) dp_1 dp_2 dp_3,$$

$$f_1(e_2) = \frac{1}{2}(q^1 dq^3 - q^3 dq^1) dp_1 dp_2 dp_3,$$

$$f_1(e_3) = \frac{1}{2}(q^1 dq^2 - q^2 dq^1) dp_1 dp_2 dp_3,$$

$$f_2(e_1 \wedge e_2) = q^3 dp_1 dp_2 dp_3, \qquad f_2(e_1 \wedge e_3) = q^2 dp_1 dp_2 dp_3, \qquad f_2(e_2 \wedge e_3) = q^1 dp_1 dp_2 dp_3,$$

and

$$f_3(e_1 \wedge e_2 \wedge e_3) = \frac{1}{3}\left(p_1 dp_2 dp_3 + p_2 dp_3 dp_1 + p_3 dp_1 dp_2\right).$$

Remark 3.18. In Section 5.3 we will come back to Example 3.17 and consider the multi-symplectic symmetries and conserved quantities coming from this homotopy moment map.

4 Existence and Uniqueness of Weak Moment Maps

In this section we show that the classical results on the existence and uniqueness of moment maps in symplectic geometry generalize directly to weak homotopy moment maps in multisymplectic geometry. In particular, we show that their existence and uniqueness is governed by a Lie algebra cohomology complex which reduces to the Chevalley-Eilenberg complex in the symplectic setup.

4.1 Equivariance in Multisymplectic Geometry

Definition 4.1. A homotopy moment map is called equivariant if each component $f_k : \Lambda^k\mathfrak{g} \to \Omega^{n-k}(M)$ is equivariant with respect to the adjoint and pullback actions respectively. That is, (f) is equivariant if for all $g \in G$, $p \in \Lambda^k\mathfrak{g}$, and $1 \leq k \leq n$

$$f_k(\mathrm{Ad}^*_{g^{-1}}p) = \Phi^*_g f_k(p). \tag{4.1}$$

Similarly, a weak homotopy moment map is equivariant if equation (4.1) holds for all $p \in \mathcal{P}_{\mathfrak{g},k}$.

Next we recall the cohomology theory governing equivariance from symplectic geometry, without proof, and then generalize to the multisymplectic setting. The results from symplectic geometry can all be found in Chapter 4.2 of [1] for example. We will provide more general proofs later on in this section. Let (M,ω) be a symplectic manifold, and $\Phi : G \times M \to M$ a symplectic Lie group action by a connected Lie group G . We consider the induced symplectic Lie algebra action $\mathfrak{g} \times \Gamma(TM) \to \Gamma(TM)$. Suppose that a moment map $f : \mathfrak{g} \to C^\infty(M)$ exists. That is, $df(\xi) = V_\xi \lrcorner \omega$ for all $\xi \in \mathfrak{g}$. By definition, f is equivariant if

$$f(\mathrm{Ad}_{g^{-1}}\xi) = \Phi^*_g f(\xi).$$

Following Chapter 4.2 of [1], for $g \in G$ and $\xi \in \mathfrak{g}$ define $\psi_{g,\xi} \in C^\infty(M)$ by

$$\psi_{g,\xi}(x) := f(\xi)(\Phi_g(x)) - f(\mathrm{Ad}_{g^{-1}}\xi)(x). \tag{4.2}$$

Proposition 4.2. *For each $g \in G$ and $\xi \in \mathfrak{g}$, the function $\psi_{g,\xi} \in C^\infty(M)$ is constant.*

Since $\psi_{g,\xi}$ is constant, we may define the map $\sigma : G \to \mathfrak{g}^*$ by

$$\sigma(g)(\xi) := \psi_{g,\xi},$$

where the right hand side is the constant value of $\psi_{g,\xi}$.

Proposition 4.3. *The map $\sigma : G \to \mathfrak{g}^*$ is a cocycle in the chain complex*

$$\mathfrak{g}^* \to C^1(G, \mathfrak{g}^*) \to C^2(G, \mathfrak{g}^*) \to \cdots .$$

That is, $\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h)$ for all $g, h \in G$.

The map σ is called the cocycle corresponding to f. The following proposition shows that for any symplectic group action, the cocycle gives a well defined cohomology class.

Proposition 4.4. *For any symplectic action of G on M admitting a moment map, there is a well defined cohomology class. More specifically, if f_1 and f_2 are two moment maps, then their corresponding cocycles σ_1 and σ_2 are in the same cohomology class, i.e. $[\sigma_1] = [\sigma_2]$.*

By definition, we see that σ is measuring the equivariance of f. That is, $\sigma = 0$ if and only if f is equivariant. Moreover, if the cocycle corresponding to a moment map vanishes in cohomology, the next proposition shows that we can modify the original moment map to make it equivariant.

Proposition 4.5. *Suppose that f is a moment map with corresponding cocycle σ. If $[\sigma] = 0$ then $\sigma = \partial\theta$ for some $\theta \in \mathfrak{g}^*$ and $f + \theta$ is an equivariant moment map.*

We now show how this theory generalizes to multisymplectic geometry. For the rest of this section we let (M, ω) denote an n-plectic manifold and $\Phi : G \times M \to M$ a multisymplectic connected group action. We consider the induced multisymplectic Lie algebra action $\mathfrak{g} \times \Gamma(TM) \to \Gamma(TM)$. Assume that we have a weak homotopy moment map (f), i.e. a collection of maps $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\mathrm{Ham}}(M)$ satisfying equation (3.2).

To extend equation (4.2) to multisymplectic geometry, for $g \in G$ and $p \in \mathcal{P}_{\mathfrak{g},k}$, we define the following $(n-k)$-form:

$$\psi^k_{g,p} := f_k(p) - \Phi^*_{g^{-1}} f_k(\mathrm{Ad}_{g^{-1}}(p)). \tag{4.3}$$

The following proposition generalizes Proposition 4.2.

Proposition 4.6. *The $(n-k)$-form $\psi^k_{g,p}$ is closed.*

Proof. Since Φ^*_g is injective and commutes with the differential, our claim is equivalent to showing that $\Phi^*_g(\psi^k_{g,p})$ is closed. Indeed we have that

$$d(\Phi^*_g(\psi^k_{g,p})) = d(\Phi^*_g f_k(p) - f_k(\mathrm{Ad}_{g^{-1}} p))$$

$$
\begin{aligned}
&= \Phi_g^*(df_k(p)) - d(f_k(\mathrm{Ad}_{g^{-1}}p)) \\
&= -\zeta(k)\Phi_g^*(V_p \lrcorner\, \omega) + \zeta(k)V_{\mathrm{Ad}_{g^{-1}}p} \lrcorner\, \omega && \text{since } (f) \text{ is moment map} \\
&= -\zeta(k)\Phi_g^*(V_p \lrcorner\, \omega) + \zeta(k)(\Phi_g^* V_p) \lrcorner\, \omega && \text{by Proposition 2.18} \\
&= -\zeta(k)\Phi_g^*(V_p \lrcorner\, \omega) + \zeta(k)\Phi_g^*(V_p \lrcorner\, \omega) && \text{since } G \text{ preserves } \omega \\
&= 0.
\end{aligned}
$$

□

In analogy to symplectic geometry, we now see each component of a weak moment map gives a cocycle.

Definition 4.7. We call the map $\sigma_k : G \to \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$ defined by

$$\sigma_k(g)(p) := \psi^k_{g,p}$$

the cocycle corresponding to f_k.

As a generalization of Proposition 4.3 we obtain:

Proposition 4.8. *The map σ_k is a 1-cocycle in the chain complex*

$$\mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}} \to C^1(G, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) \to C^2(G, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) \to \cdots,$$

where the action of G on $\mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$ is given by the tensor product of the co-adjoint and pullback actions. The induced infinitesimal action of $\mathfrak{g}$ on $\mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$ is defined as follows: for $f \in \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{Ham}}$, $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$,

$$(\xi \cdot f)(p) := f(\mathrm{ad}_\xi(p)) + \mathcal{L}_{V_\xi} f(p). \tag{4.4}$$

Proof. By equation (2.14) we know that $(\partial(\sigma)(g,h))(p) := \sigma(gh)(p) - \sigma(g)(p) - g \cdot \sigma(h)(p)$. For arbitrary $p \in \mathcal{P}_{\mathfrak{g},k}$ we have

$$
\begin{aligned}
\sigma_k(gh)(p) &= f_k(p) - \Phi^*_{(gh)^{-1}}(f_k \mathrm{Ad}_{(gh)^{-1}}p) \\
&= f_k(p) - \Phi^*_{g^{-1}}\Phi^*_{h^{-1}}(f_k(\mathrm{Ad}_{h^{-1}}\mathrm{Ad}_{g^{-1}}p)) \\
&= f_k(p) - \Phi^*_{g^{-1}}(f_k(\mathrm{Ad}_{g^{-1}}p)) + \Phi^*_{g^{-1}}(f_k(\mathrm{Ad}_{g^{-1}}p)) - \Phi^*_{g^{-1}}(\Phi^*_{h^{-1}}(f_k(\mathrm{Ad}_{h^{-1}}\mathrm{Ad}_{g^{-1}}p))) \\
&= \sigma_k(g)(p) + \Phi^*_{g^{-1}}(\sigma_k(h)(\mathrm{Ad}_{g^{-1}}p)) \\
&= \sigma_k(g)(p) + g \cdot \sigma_k(h)(p).
\end{aligned}
$$

□

Definition 4.9. Let

$$\mathfrak{C} = \bigoplus_{k=1}^{n} \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$$

Let $\sigma = \sigma_1 + \sigma_2 + \cdots$. We call the map $\sigma \in \mathfrak{C}$ the cocycle corresponding to (f).

Since the components of a weak moment map do not interact, as a corollary to Proposition 4.8 we obtain

Proposition 4.10. *The map σ is a cocycle in the complex*

$$\mathfrak{C} \to C^1(G, \mathfrak{C}) \to C^2(G, \mathfrak{C}) \to \cdots$$

The next theorem shows that multisymplectic Lie algebra actions admitting weak moment maps give a well defined cohomology class, generalizing Proposition 4.4.

Theorem 4.11. *Let G act multisymplectically on (M, ω). To any weak moment map, there is a well defined cohomology class $[\sigma]$ in $H^1(G, \mathfrak{C})$. More precisely if (f) and (g) are two weak moment maps with cocycles σ and τ, then $\sigma - \tau$ is a coboundary.*

Proof. We need to show that $\sigma_k - \tau_k$ is a coboundary for each k. We have that

$$\sigma_k(g)(p) - \tau_k(g)(p) = f_k(p) - g_k(p) - \Phi^*_{g^{-1}}(f_k(\mathrm{Ad}_{g^{-1}}p) - g_k(\mathrm{Ad}_{g^{-1}}(\xi))).$$

However, (f) and (g) are both moment maps and so $d(f_k(p) - g_k(p)) = 0$. Thus $f_k - g_k$ is in $\mathfrak{C}$. Moreover, by equation (2.15), we see that $\sigma_k - \tau_k = \partial(f_k - g_k)$.

□

If (f) is not equivariant but its cocycle vanishes, then we can define a new equivariant moment map from (f), in analogy to Proposition 4.5.

Proposition 4.12. *Let (f) be a weak moment map with cocycle satisfying $[\sigma] = 0$. Then $\sigma = \partial\theta$ for some $\theta \in \mathfrak{C}$ and the map $(f) + \theta$ is an equivariant weak moment map.*

Proof. We have that $(f) + \theta$ is a moment map since $\theta(p)$ is closed for all $p \in \mathcal{P}_{\mathfrak{g},k}$. Let $\widetilde{\sigma}$ denote the corresponding cocycle. Note that by equation (2.15) we have $(\partial(\theta)(g))(p) =$

$\theta(\mathrm{Ad}_{g^{-1}}p) - \Phi_g^*\theta(p)$. By the injectivity of Φ_g^*, to show that $\widetilde{\sigma} = 0$, it is sufficient to show that $\Phi_g^*(\widetilde{\sigma}(g)(p)) = 0$ for all $g \in G$ and $p \in \mathcal{P}_{g,k}$. Indeed,

$$\begin{aligned}
\Phi_g^*(\widetilde{\sigma}_k(g)(p)) &= \Phi_g^* f(p) + \Phi_g^*\theta(p) - f(\mathrm{Ad}_{g^{-1}}p) - \theta(\mathrm{Ad}_{g^{-1}}p) \\
&= \sigma(g)(\xi) - \partial\theta(g)(\xi) \\
&= \sigma(g)(\xi) - \sigma(g)(\xi) && \text{since } \partial\theta = \sigma \\
&= 0.
\end{aligned}$$

□

4.2 Infinitesimal Equivariance in Multisymplectic Geometry

We now consider infinitesimal equivariance. We start by recalling the notions from symplectic geometry. That is, we differentiate equation (4.2) to obtain the map $\Sigma : \mathfrak{g} \times \mathfrak{g} \to C^\infty(M)$ defined by $\Sigma(\xi, \eta) := \frac{d}{dt}\big|_{t=0} \psi_{\exp(t\eta),\xi}$. A straightforward computation, which we generalize in Proposition 4.15, gives that

$$\Sigma(\xi, \eta) = f([\xi, \eta]) - \{f(\xi), f(\eta)\}.$$

Another quick computation shows that $df([\xi, \eta]) = d\{f(\xi), f(\eta)\}$, showing $\Sigma(\xi, \eta)$ is a constant function for every $\xi, \eta \in \mathfrak{g}$. That is, Σ is a function from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{R}$.

Proposition 4.13. *The map $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a Lie algebra 2-cocycle in the chain complex*

$$\mathbb{R} \to C^1(\mathfrak{g}, \mathbb{R}) \to C^2(\mathfrak{g}, \mathbb{R}) \to \cdots .$$

Definition 4.14. A moment map $f : \mathfrak{g} \to C^\infty(M)$ is infinitesimally equivariant if $\Sigma = 0$, i.e. if

$$f([\xi, \eta]) = \{f(\xi), f(\eta)\} \tag{4.5}$$

for all $\xi, \eta \in \mathfrak{g}$.

Notice that since Σ is just the derivative of σ, it follows that for a connected Lie group, infinitesimal equivariance and equivariance are equivalent. Since we will always be working with connected Lie groups, we will abuse terminology and call a moment map equivariant if it satisfies equation (4.1) or (4.5).

Now we turn our attention towards the multisymplectic setting. As in symplectic geometry, the infinitesimal equivariance of a weak moment map comes from differentiating $\psi_{\exp(t\xi),p}$

for fixed $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$.

Proposition 4.15. *Let Σ_k denote $\frac{d}{dt}\big|_{t=0} \psi^k_{\exp(t\xi),p}$. Then we have that Σ_k is a map from $\mathfrak{g}$ to $\mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$ and is given by*

$$\Sigma_k(\xi, p) = f_k([\xi, p]) + \mathcal{L}_{V_\xi} f_k(p),$$

for $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$.

Proof. We have that

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \psi^k_{\exp(t\xi),p} &= \frac{d}{dt}\bigg|_{t=0} f_k(p) - \frac{d}{dt}\bigg|_{t=0} \Phi^*_{\exp(-t\xi)}(f_k(\mathrm{Ad}_{\exp(-t\xi)}(p))) \\
&= -\frac{d}{dt}\bigg|_{t=0} \Phi^*_{\exp(-t\xi)}(f_k(\mathrm{Ad}_{\exp(-t\xi)} p)) \\
&= -f_k(\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(-t\xi)} p) - (\frac{d}{dt}\bigg|_{t=0} \Phi^*_{\exp(-t\xi)})(f_k(p)) \\
&= -f_k(-[\xi, p]) + \mathcal{L}_{\xi_M} f_k(p)
\end{aligned}$$

□

Let $R_k = \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}$. Then R_k is a $\mathfrak{g}$-module under the induced action from the tensor product of the adjoint and Lie derivative actions. Concretely, for $\alpha \in R_k$, $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$,

$$(\xi \cdot \alpha)(p) = \alpha([\xi, p]) + \mathcal{L}_{V_\xi} \alpha.$$

Consider the chain complex

$$R_k \to C^1(\mathfrak{g}, R_k) \to C^2(\mathfrak{g}, R_k) \to \cdots,$$

where the differential is given in equation (2.15).

The following is a generalization of Proposition 4.13.

Proposition 4.16. *The map Σ_k is in the kernel of ∂_k. That is, Σ_k is a cocycle.*

Proof. We need to show that $\partial\Sigma_k = 0$. Indeed, for $\xi, \eta \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$, we have that

$$\partial\Sigma_k(\xi, \eta)(p) = \xi \cdot (\Sigma_k(\eta)(p)) - \eta \cdot (\Sigma_k(\xi)(p)) + \Sigma_k([\xi, \eta])(p)$$

$$
\begin{aligned}
&= \Sigma_k(\eta)(\mathrm{ad}_\xi(p)) + \mathcal{L}_{V_\xi}(\Sigma_k(\eta)(p)) - \Sigma_k(\xi)(\mathrm{ad}_\eta(p)) \\
&\quad - \mathcal{L}_{V_\eta}(\Sigma_k(\xi)(p)) + \Sigma_k([\xi,\eta])(p).
\end{aligned}
$$

By definition of the ad map, this is equal to

$$
\Sigma_k(\eta)([\xi,p]) + \mathcal{L}_{V_\xi}(\Sigma_k(\eta)(p)) - \Sigma_k(\xi)([\eta,p]) - \mathcal{L}_{V_\eta}(\Sigma_k(\xi)(p)) + \Sigma_k([\xi,\eta])(p)
$$

and using the definition of Σ this becomes

$$
\begin{aligned}
f_k([\eta,[\xi,p]]) &+ \mathcal{L}_{V_\eta} f_k([\xi,p]) + \mathcal{L}_{V_\xi} f_k([\eta,p]) + \mathcal{L}_{V_\xi}\mathcal{L}_{V_\eta} f_k(p) \\
&- f_k([\xi,[\eta,p]]) - \mathcal{L}_{V_\xi} f_k([\eta,p]) - \mathcal{L}_{V_\eta} f_k([\xi,p]) - \mathcal{L}_{V_\eta}\mathcal{L}_{V_\xi} f_k(p) \\
&+ f_k([[\xi,\eta],p]) - \mathcal{L}_{V_{[\xi,\eta]}} f_k(p) \\
&= f_k([\eta,[\xi,p]]) - f_k([\xi,[\eta,p]]) + f_k([[\xi,\eta],p]) \\
&+ \mathcal{L}_{V_\xi}\mathcal{L}_{V_\eta} f_k(p) - \mathcal{L}_{V_\eta}\mathcal{L}_{V_\xi} f_k(p) - \mathcal{L}_{V_{[\xi,\eta]}} f_k(p).
\end{aligned}
$$

By the Jacobi identity this is equal to

$$
\mathcal{L}_{V_\xi}\mathcal{L}_{V_\eta} f_k(p) - \mathcal{L}_{V_\eta}\mathcal{L}_{V_\xi} f_k(p) - \mathcal{L}_{V_{[\xi,\eta]}} f_k(p),
$$

and this vanishes by equation (2.8).

□

As in symplectic geometry, we have that for a connected Lie group, a weak homotopy moment map is equivariant if and only if it is infinitesimally equivariant. That is, the weak homotopy k-moment map is equivariant if and only if $\sigma_k = 0$ or $\Sigma_k = 0$. A weak homotopy moment map is equivariant if $\sigma_k = 0$ or $\Sigma_k = 0$ for all $1 \leq k \leq n$.

Now that we have generalized the notions of equivariance from symplectic to multisymplectic geometry, we move on to study the existence and uniqueness of these weak homotopy moment maps.

4.3 Existence of Not Necessarily Equivariant Weak Moment Maps

For a connected Lie group G acting symplectically on a symplectic manifold (M,ω), recall the following standard results from symplectic geometry. We refer the reader to [4] for proofs and note that we give more general proofs later on in this section.

Proposition 4.17. *For any $\xi, \eta \in \mathfrak{g}$ we have*

$$[V_\xi, V_\eta] \lrcorner\, \omega = d(V_\xi \lrcorner\, V_\eta \lrcorner\, \omega).$$

Proposition 4.18. *We have that $H^1(\mathfrak{g}) = 0$ if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Combining these two propositions, we obtain:

Proposition 4.19. *If $H^1(\mathfrak{g}) = 0$, then any symplectic action admits a moment map, which is not necessarily equivariant.*

We now show how these results generalize to multisymplectic geometry. Let a connected Lie group act multisymplectically on an n-plectic manifold (M, ω).

Proposition 4.20. *For arbitrary q in $\mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$ we have that*

$$[V_q, V_\xi] \lrcorner\, \omega = -(-1)^k d(V_q \lrcorner\, V_\xi \lrcorner\, \omega).$$

Proof. By linearity it suffices to consider decomposable $q = \eta_1 \wedge \cdots \wedge \eta_k$. A quick computation shows that $[V_q, V_\xi] \lrcorner\, \omega = -V_{[q,\xi]} \lrcorner\, \omega$. Using Lemmas 2.13 and 2.16 we obtain:

$$\begin{aligned} V_{[q,\xi]} \lrcorner\, \omega &= V_{\partial(q\wedge\xi)} \lrcorner\, \omega \\ &= (-1)^k d(V_{q\wedge\xi} \lrcorner\, \omega) - \sum_{i=1}^{k} (-1)^i \eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_k \wedge \xi \lrcorner\, \mathcal{L}_{\eta_i}\omega - V_{q\wedge\xi} \lrcorner\, d\omega \\ &= (-1)^k d(V_q \lrcorner\, V_\xi \lrcorner\, \omega). \end{aligned}$$

The claim now follows. □

The next proposition is a generalization of Proposition 4.18.

Proposition 4.21. *If $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = 0$ then $\mathcal{P}_{\mathfrak{g},k} = [\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]$.*

Proof. By equation (2.15), an element $c \in H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k})$ satisfies $c([\xi, p]) = 0$ for all $\xi \in \mathfrak{g}$. That is,

$$H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k}) = [\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]^0,$$

where $[\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]^0$ is the annihilator of $[\mathcal{P}_{\mathfrak{g},k}, \mathfrak{g}]$ in $\mathcal{P}^*_{\mathfrak{g},k}$. □

We now arrive at our main theorem on the existence of not necessarily equivariant weak moment maps. The following is a generalization of Proposition 4.19.

Theorem 4.22. *Let G act multisymplectically on (M,ω). If $H^0(\mathfrak{g},\mathcal{P}^*_{\mathfrak{g},k}) = 0$ then the k-th component of a not necessarily equivariant weak moment map exists. Moreover, if $H^0(\mathfrak{g},\mathcal{P}^*_{\mathfrak{g},k}) = 0$ for all $1 \leq k \leq n$, then a not necessarily equivariant weak moment map exists. The same result holds if $H^0(\mathfrak{g},\mathcal{P}^*_{\mathfrak{g},k}\otimes\Omega^{n-k}_{\mathrm{cl}}) = 0$, and $H^0(\mathfrak{g},\Omega^{n-k}_{\mathrm{cl}}) \neq 0$.*

Proof. Note that when $H^0(\mathfrak{g},\Omega^{n-k}_{\mathrm{cl}}) \neq 0$, the space $H^0(\mathfrak{g},\mathcal{P}^*_{\mathfrak{g},k}\otimes\Omega^{n-k}_{\mathrm{cl}}) = 0$ if and only if $H^0(\mathfrak{g},\mathcal{P}^*_{\mathfrak{g},k}) = 0$ by the Kunneth formula (see for example Theorem 3.6.3 of [21]). The claim now follows from Proposition 4.21 and Proposition 4.20. Indeed, Proposition 4.20 says we may define a weak moment map on elements of the form $[p,\xi]$ by $(-1)^k V_p \lrcorner V_\xi \lrcorner \omega$, where $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$, and Proposition 4.21 says every element in $\mathcal{P}_{\mathfrak{g},k}$ is a sum of elements of this form. $\square$

Remark 4.23. Notice that for the case $n = k$, it is always true that $H^0(\mathfrak{g},\Omega^{n-k}_{\mathrm{cl}}) \neq 0$ since any constant function is closed. Hence Theorem 4.22 gives a generalization of Theorems 3.5 and 3.14 of [15] and [16] respectively. It also agrees with Lemma 4.4 and Corollary 4.2 of [19]. Moreover, by taking $n = k = 1$, we see that we are obtaining a generalization from symplectic geometry.

Remark 4.24. In [19], Corollary 4.2 gives an existence result for a full homotopy moment map under a weakly Hamiltonian action (see Definition 2.1 of [19]) on an n-plectic manifold (M,ω). The result guarantees existence of a full homotopy moment map under the assumption of vanishing de Rham cohomology, i.e. $H^k_{\mathrm{dR}}(M) = 0$ for all $1 \leq k \leq n$ and the vanishing of a specific cocycle $[c] \in H^{n+1}(\mathfrak{g})$. It follows that this corollary provides an existence result for weak moment maps, since any full homotopy moment map reduces to a weak moment map when restricted to the Lie kernel. Corollary 4.2 of [19] appears to be independent from our result as it deals with de Rham cohomology, whereas we are concerned with Lie algebra cohomology.

Remark 4.25. Further to the results of [19], the results on the existence of full homotopy moment maps provided in [7] also apply to weak moment maps, as any full moment map reduces to a weak moment map. See for example Proposition 2.5 of [7]. It is not clear if the results in [19] and [7] are related to ours, since in those papers their existence theory is derived from the double complex $(\Lambda^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M), d_{tot})$, with $d_{tot} = \delta\otimes 1 + 1\otimes d$, where δ is the Chevalley-Eilenberg differential and d is the de Rham differential. In our work, we are considering a different cohomology complex.

Example 4.26. Consider the multisympletic manifold $(\mathbb{R}^4, \omega)$ where $\omega = \mathrm{vol}$ is the standard volume form. That is, we are working in the case $n = 3$. Let $x_1, \cdots, x_4$ denote the standard coordinates. Let $G = SU(2)$ act on $\mathbb{R}^4$ by rotations. The corresponding Lie algebra action is generated by the vector fields

$$E_0 = x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_4},$$

$$E_1 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4},$$

and

$$E_2 = x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4}.$$

For the case $k = 2$, consider the distance function $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. It is clear that the distance function is invariant under rotations and hence $\mathcal{L}_{E_i} dr = 0$ for $i = 0, 1, 2, 3$. Since dr is a closed 1-form, it follows that dr is a non-zero element of $H^0(\mathfrak{g}, \Omega^1_{\mathrm{cl}}(M))$. That is, $H^0(\mathfrak{g}, \Omega^1_{\mathrm{cl}}(M)) \neq 0$.

For the case $k = 1$, consider $\alpha := dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. A quick calculation shows that $E_i \lrcorner \alpha =$ for $i = 0, 1, 2, 3$ so that α is invariant under the $\mathfrak{su}(2)$ action. Since $d\alpha = 0$, it follows that $H^0(\mathfrak{g}, \Omega^2_{\mathrm{cl}}(M)) \neq 0$ as well.

Hence, by Theorem 4.22, it follows that a weak moment map exists.

The next example gives a scenario for which Theorem 4.22 can only be applied to specific components of a weak moment map.

Example 4.27. Take the setup of Example 4.26 but instead consider the action of $SO(4)$. As in Example 4.26, dr is a non-zero closed 1-form which is invariant under the action. That is, $H^0(\mathfrak{g}, \Omega^1_{\mathrm{cl}}(M)) \neq 0$. However, in this setup, $H^0(\mathfrak{g}, \Omega^2_{\mathrm{cl}}(M)) = 0$. Indeed, the infinitesimal generators of $\mathfrak{so}(4)$ are of the form $x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ where $1 \leq i, j, \leq 4$. An arbitrary 2-form may be written as $\beta = \sum_{i,j} a_{ij} dx_i \wedge dx_j$. A computation shows that the condition $\mathcal{L}_{V_\xi}\beta = 0$ for all $\xi \in \mathfrak{so}(4)$ implies that necessarily $\beta = 0$. Hence $H^0(\mathfrak{g}, \Omega^2_{\mathrm{cl}}(M)) = 0$.

It follows that, in this case, Theorem 4.22 guarantees the existence of the second component of a weak moment map, but does not guarantee the existence of the first.

Another generalization of Proposition 4.19 to multisymplectic geometry is given by:

Proposition 4.28. *If $H^k(\mathfrak{g}) = 0$, then the k-th component of a not necessarily equivariant weak moment map exists.*

Proof. If $H^k(\mathfrak{g}) = 0$ then $\mathcal{P}_{\mathfrak{g},k} = \text{Image}(\partial_{k+1})$, since $\mathcal{P}_{\mathfrak{g},k} = \ker(\partial_k)$. But for $p \in \text{Image}(\partial_{k+1})$ we have that $p = \partial q$ for some $q \in \Lambda^{k+1}\mathfrak{g}$. Then by Lemma 2.16 we have

$$V_p \lrcorner\, \omega = (-1)^k d(V_q \lrcorner\, \omega).$$

Hence we may define $f_k(p)$ to be $(-1)^k V_q \lrcorner\, \omega$. □

Remark 4.29. Proposition 4.28 gives another generalization of the results of Madsen and Swann. Indeed, by taking $n = k$ we again arrive at Theorems 3.5 and 3.14 of [15] and [16] respectively. This also agrees with Lemma 4.4 of [7].

4.4 Obtaining an Equivariant Weak Moment Map from a Non-Equivariant Weak Moment Map

In this section we show that the theory involved in obtaining an equivariant moment map from a non-equivariant moment map extends from symplectic to multisymplectic geometry. We first recall the results from symplectic geometry. A proof of the results can be found in [4]. We give more general proofs later on in this section.

Proposition 4.13 shows that the map Σ corresponding to a moment map f is a Lie algebra 2-cocycle. The next proposition says that if the cocycle is exact then f can be made equivariant.

Proposition 4.30. *Let f be a moment map and Σ its corresponding cocycle. If $\Sigma = \partial(l)$ for some l, then $f + l$ is equivariant.*

It follows from this that

Proposition 4.31. *If $H^2(\mathfrak{g}) = 0$ then one can obtain an equivariant moment map from a non-equivariant moment map.*

Now let G be a connected Lie group acting on an n-plectic manifold (M, ω). The following proposition generalizes Proposition 4.30 to multisymplectic geometry.

Proposition 4.32. *Let f_k be the weak homotopy k-moment map, and let Σ_k denote its corresponding cocycle. If $\Sigma_k = \partial(l_k)$ for some $l_k \in H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\text{cl}})$, then $f_k + l_k$ is equivariant.*

Proof. Fix $p \in \mathcal{P}_{\mathfrak{g},k}$ and $\xi \in \mathfrak{g}$. Then

$$\begin{aligned}
(f_k + l_k)([\xi, p]) &= f_k([\xi, p]) + l_k([\xi, p]) \\
&= f_k([\xi, p]) - ((\partial l_k)(\xi))(p) + \mathcal{L}_{V_\xi} l_k(p) && \text{by equation (2.15)} \\
&= f_k([\xi, p]) - \Sigma_k([\xi, p]) + \mathcal{L}_{V_\xi} l_k(p) \\
&= \mathcal{L}_{V_\xi} f_k(p) + \mathcal{L}_{V_\xi}(l_k(p)) && \text{by definition of } \Sigma_k \\
&= \mathcal{L}_{V_\xi}((f_k + l_k)(p)).
\end{aligned}$$

□

We now arrive at our generalization of Proposition 4.31:

Theorem 4.33. *If $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) = 0$ then any weak k-moment map can be made equivariant. In particular, if $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) = 0$ for all $1 \leq k \leq n$, then any weak moment map (f) can be made equivariant.*

Proof. Let $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\mathrm{Ham}}$ be a weak k-moment map. If $H^1(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) = 0$ then the corresponding cocycle Σ_k is exact, i.e. $\Sigma_k = \partial(l_k)$ for some $l_k \in H^0(\mathfrak{g}, \mathcal{P}_{\mathfrak{g},k})$. It follows from Proposition 4.32 that $f_k + l_k$ is equivariant. □

4.5 Uniqueness of Equivariant Weak Moment Maps

We first recall the results from symplectic geometry without explicit proof. A proof can be found by setting $n = 1$ (i.e. the symplectic case) in our more general Theorem 4.37. A proof can also be found in [4]. Let $\mathfrak{g}$ be a Lie algebra acting on a symplectic manifold (M, ω).

Proposition 4.34. *If f and g are two equivariant moment maps, then $f - g$ is in $H^1(\mathfrak{g})$.*

Proof. For $\xi, \eta \in \mathfrak{g}$ we have that $(f - g)([\xi, \eta]) = \{(f - g)(\xi), (f - g)(\eta)\}$ since f and g are equivariant. However, $(f - g)(\xi)$ is a constant function since both f and g are moment maps. The claim now follows since the Poisson bracket with a constant function vanishes. □

From Proposition 4.34 it immediately follows that

Proposition 4.35. *If $H^1(\mathfrak{g}) = 0$ then equivariant moment moments are unique.*

The following is a generalization of Proposition 4.34.

Proposition 4.36. *If f_k and g_k are k-th components of two equivariant weak moment maps, then $f_k - g_k$ is in $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}})$.*

Proof. If f_k and g_k are equivariant then $(f_k - g_k)([\xi, p]) = \mathcal{L}_{V_\xi}((f_k - g_k)(p))$. Moreover, $(f_k - g_k)(p)$ is closed since both f_k and g_k are moment maps. □

We now arrive at our generalization of Proposition 4.35. Let $\mathfrak{g}$ be a Lie algebra acting on an n-plectic manifold (M, ω).

Theorem 4.37. *If $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) = 0$, then equivariant weak k-moment maps are unique. In particular, if $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}}) = 0$ for all $1 \leq k \leq n$ then equivariant weak moment maps are unique.*

Proof. If f_k and g_k are two equivariant weak k-moment maps, then Proposition 4.36 shows that $f_k - g_k$ is in $H^0(\mathfrak{g}, \mathcal{P}^*_{\mathfrak{g},k} \otimes \Omega^{n-k}_{\mathrm{cl}})$.

□

Remark 4.38. This theorem gives a generalization of the results of Madsen and Swann. Indeed, by taking $n = k$ we again arrive at Theorems 3.5 and 3.14 of [15] and [16] respectively.

5 Multisymplectic Symmetries and Conserved Quantities

In this section we give a definition of conserved quantities and continuous symmetries on multisymplectic manifolds. In symplectic geometry, the Poisson bracket plays a large role in the discussion of conserved quantities. To that end, we first generalize the Poisson bracket to multisymplectic geometry.

5.1 A Generalized Poisson Bracket

We first extend the notion of a Hamiltonian $(n-1)$-form to arbitrary forms of degree $\leq n-1$.

Definition 5.1. We call

$$\Omega^{n-k}_{\mathrm{Ham}}(M) := \{\alpha \in \Omega^{n-k}(M);\ \text{there exists } X_\alpha \in \Gamma(\Lambda^k(TM)) \text{ with } d\alpha = -X_\alpha \lrcorner\, \omega\ \}$$

the set of Hamiltonian $(n-k)$-forms. For a Hamiltonian $(n-k)$-form α, we call X_α a corresponding Hamiltonian k-vector field (or multivector field if k is not explicit).

We call

$$\mathfrak{X}^k_{\mathrm{Ham}}(M) := \{X \in \Gamma(\Lambda^k(TM)); X \lrcorner\, \omega \text{ is exact}\}$$

the set of Hamiltonian k-vector fields. We will refer to a primitive of $X \lrcorner\, \omega$ as a corresponding Hamiltonian $(n-k)$-form.

Of course, given a Hamiltonian $(n-k)$-form, it does not necessarily have a unique associated Hamiltonian multivector field. Moreover, a Hamiltonian k-vector field doesn't necessarily have a unique corresponding Hamiltonian $(n-k)$-form. However, the following is clear:

Proposition 5.2. *For $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$, any two of its Hamiltonian k-vector fields differ by an element in the kernel of ω. Conversely, for $X \in \mathfrak{X}^k_{\mathrm{Ham}}(M)$, any two of its Hamiltonian forms differ by a closed form.*

Proposition 5.2 motivates consideration of the following spaces. Let $\widetilde{\mathfrak{X}}^k_{\mathrm{Ham}}(M)$ denote the quotient space of $\mathfrak{X}^k_{\mathrm{Ham}}(M)$ by elements in the kernel of ω. Let $\widetilde{\Omega}^{n-k}_{\mathrm{Ham}}(M)$ denote the quotient of $\Omega^{n-k}_{\mathrm{Ham}}$ by closed forms. We let

$$\Omega_{\mathrm{Ham}}(M) = \oplus_{k=0}^{n-1} \Omega^k_{\mathrm{Ham}}(M)$$

and

$$\widetilde{\Omega}_{\mathrm{Ham}}(M) = \oplus_{k=0}^{n-1}\widetilde{\Omega}^k_{\mathrm{Ham}}(M).$$

Similarly, we let

$$\mathfrak{X}_{\mathrm{Ham}}(M) = \oplus_{k=0}^{n-1}\mathfrak{X}^k_{\mathrm{Ham}}(M)$$

and

$$\widetilde{\mathfrak{X}}_{\mathrm{Ham}}(M) = \oplus_{k=0}^{n-1}\widetilde{\mathfrak{X}}^k_{\mathrm{Ham}}(M).$$

It is clear that the map from $\widetilde{\Omega}^{n-k}_{\mathrm{Ham}}(M)$ to $\widetilde{\mathfrak{X}}^k_{\mathrm{Ham}}(M)$ given by $[\alpha] \mapsto [X_\alpha]$ is a bijection.

Proposition 5.3. *The vector spaces $\widetilde{\Omega}^{n-k}_{\mathrm{Ham}}(M)$ and $\widetilde{\mathfrak{X}}^k_{\mathrm{Ham}}(M)$ are isomorphic.*

Later on, we will see that there are graded Lie brackets on the vector spaces $\widetilde{\Omega}_{\mathrm{Ham}}(M)$ and $\widetilde{\mathfrak{X}}_{\mathrm{Ham}}(M)$ making them isomorphic as graded Lie algebras.

The next proposition will be used to show that certain statements about a Hamiltonian form are independent of the choice of the corresponding Hamiltonian multivector field.

Proposition 5.4. *If κ is in the kernel of ω, then for any $X \in \mathfrak{X}^k_{\mathrm{Ham}}(M)$ we have $[X,\kappa] \lrcorner\, \omega = 0$.*

Proof. Since $X \in \mathfrak{X}^k_{\mathrm{Ham}}(M)$ by definition $\mathcal{L}_X\omega = 0$. Using equation (2.7) together with the fact that $\kappa \lrcorner\, \omega = 0$ we have

$$[X,\kappa]\lrcorner\, \omega = (-1)^{k(k+1)}\mathcal{L}_X(\kappa\lrcorner\, \omega) - \kappa\lrcorner\, \mathcal{L}_X\omega = 0.$$

□

The next proposition shows that, as in symplectic geometry, any Hamiltonian multivector field preserves ω.

Proposition 5.5. *For $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ we have that $\mathcal{L}_{X_\alpha}\omega = 0$ for all Hamiltonian multivector fields X_α of α.*

Proof. Let X_α be a Hamiltonian multivector field. We have that

$$\begin{aligned}
\mathcal{L}_{X_\alpha}\omega &= \mathcal{L}_{X_\alpha}\omega \\
&= d(X_\alpha\lrcorner\, \omega) - (-1)^k X_\alpha\lrcorner\, d\omega && \text{by equation (2.5)} \\
&= d(X_\alpha\lrcorner\, \omega) && \text{since } d\omega = 0 \\
&= -d(d\alpha) && \text{by definition}
\end{aligned}$$

$$= 0.$$

□

We now put in a structure analogous to the Poisson bracket in Hamiltonian mechanics, which has analogous graded properties.

Given $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$ and $\beta \in \Omega^{n-l}_{\text{Ham}}(M)$, a first attempt would be to define their generalized bracket to be

$$\{\alpha, \beta\} := X_\beta \lrcorner X_\alpha \lrcorner \omega,$$

mimicking the Poisson bracket in symplectic geometry. However, we can see right away that this bracket is not graded anti-commutative since $\{\alpha, \beta\} = (-1)^{kl}\{\beta, \alpha\}$. Hence, we modify our grading of the Hamiltonian forms, following the work done in [5].

Definition 5.6. Let $\mathcal{H}^p(M) = \Omega^{n-p+1}_{\text{Ham}}(M)$. That is, we are assigning the grading of $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$ to be $|\alpha| = k + 1$. For $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$ and $\beta \in \Omega^{n-l}_{\text{Ham}}(M)$ (i.e. $\alpha \in \mathcal{H}^{k+1}(M)$ and $\beta \in \mathcal{H}^{l+1}(M)$) we define their (generalized) Poisson bracket to be

$$\begin{aligned}\{\alpha, \beta\} &:= (-1)^{|\beta|} X_\beta \lrcorner X_\alpha \lrcorner \omega \\ &= (-1)^{l+1} X_\beta \lrcorner X_\alpha \lrcorner \omega.\end{aligned}$$

Notice that this bracket is well defined follows directly from Proposition 5.2.

With this new grading, the generalized Poisson bracket is graded commutative.

Proposition 5.7. *Let α be a form of grading $|\alpha| = k+1$ and β a form of grading $|\beta| = l+1$. That is, $\alpha \in \Omega^{n-k}_{Ham}(M)$ and $\beta \in \Omega^{n-l}_{Ham}(M)$. Then we have that*

$$\{\alpha, \beta\} = -(-1)^{|\alpha||\beta|}\{\beta, \alpha\}.$$

Proof. By definition,

$$\begin{aligned}\{\alpha, \beta\} &= (-1)^{l+1} X_\beta \lrcorner X_\alpha \lrcorner \omega \\ &= (-1)^{l+1}(-1)^{kl} X_\alpha \lrcorner X_\beta \lrcorner \omega \\ &= -(-1)^{l(k+1)} X_\alpha \lrcorner X_\beta \lrcorner \omega \\ &= -(-1)^{(l+1)(k+1)+k+1} X_\alpha \lrcorner X_\beta \lrcorner \omega \\ &= -(-1)^{|\alpha||\beta|}(-1)^{k+1} X_\alpha \lrcorner X_\beta \lrcorner \omega \\ &= -(-1)^{|\alpha||\beta|}\{\beta, \alpha\}.\end{aligned}$$

□

The next lemma shows that the bracket of two Hamiltonian forms is Hamiltonian. In symplectic geometry, we have $X_{\{f,g\}} = [X_f, X_g]$ (or $X_{\{f,g\}} = -[X_f, X_g]$ if the defining equation for a Hamiltonian vector field is $X_\alpha \lrcorner\, \omega = d\alpha$). In multisymplectic geometry we have

Lemma 5.8. For $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ and $\beta \in \Omega^{n-l}_{\mathrm{Ham}}(M)$ their bracket $\{\alpha, \beta\}$ is in $\Omega^{n+1-k-l}_{\mathrm{Ham}}(M)$. That is, $\{\alpha, \beta\}$ is a Hamiltonian form with grading $|\{\alpha, \beta\}| = k + l - 2$. More precisely, we have that $[X_\alpha, X_\beta]$ is a Hamiltonian vector field for $\{\alpha, \beta\}$.

Proof. We have that

$$\begin{aligned}
[X_\alpha, X_\beta] \lrcorner\, \omega &= -X_\beta \lrcorner\, d(X_\alpha \lrcorner\, \omega) + (-1)^l d(X_\beta \lrcorner\, X_\alpha \lrcorner\, \omega) \\
&\quad + (-1)^{kl+k} X_\alpha \lrcorner\, X_\beta \lrcorner\, dw - (-1)^{kl+k+l} X_\alpha \lrcorner\, d(X_\beta \lrcorner\, \omega) \quad \text{by equation (2.10)} \\
&= (-1)^l d(X_\beta \lrcorner\, X_\alpha \lrcorner\, \omega) \\
&= -d(\{\alpha, \beta\}).
\end{aligned}$$

□

We now investigate the Jacobi identity for this bracket. In [5] it was mentioned that the graded Jacobi identity holds up to a closed form. We now show that the graded Jacobi identity holds up to an exact term.

Proposition 5.9. (Graded Jacobi.) Fix $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$, $\beta \in \Omega^{n-l}_{\mathrm{Ham}}(M)$ and $\gamma \in \Omega^{n-m}_{\mathrm{Ham}}(M)$. Let X_α, X_β and X_γ denote arbitrary Hamiltonian multivector fields for α, β and γ respectively. Then we have that

$$\sum_{\text{cyclic}} (-1)^{|\alpha||\gamma|} \{\alpha, \{\beta, \gamma\}\} = (-1)^{|\beta||\gamma| + |\beta||\alpha| + |\beta|} d(X_\alpha \lrcorner\, X_\beta \lrcorner\, X_\gamma \lrcorner\, \omega).$$

Proof. By definition, we have that

$$\{\alpha, \beta\} = (-1)^{|\beta|} X_\beta \lrcorner\, X_\alpha \lrcorner\, \omega = (-1)^{|\beta|+1} X_\beta \lrcorner\, d\alpha.$$

Since X_β is in $\Lambda^{|\beta|+1}(TM)$, by (2.5) it follows that

$$\begin{aligned}
\{\alpha, \beta\} &= (-1)^{|\beta|+1} (-1)^{|\beta|+1} (d(X_\beta \lrcorner\, \alpha) - \mathcal{L}_{X_\beta} \alpha) \\
&= d(X_\beta \lrcorner\, \alpha) - \mathcal{L}_{X_\beta} \alpha. \qquad (5.1)
\end{aligned}$$

Thus,

$$\begin{aligned}
\{\alpha,\{\beta,\gamma\}\} &= (-1)^{|\beta||\gamma|+1}\{\alpha,\{\gamma,\beta\}\} \\
&= (-1)^{|\beta||\gamma|+1+(|\beta|+|\gamma|)|\alpha|+1}\{\{\gamma,\beta\},\alpha\} \\
&= (-1)^{|\beta||\gamma|+1+(|\beta|+|\gamma|)|\alpha|+1}(d(X_\alpha \lrcorner \{\gamma,\beta\}) - \mathcal{L}_{X_\alpha}\{\gamma,\beta\}) && \text{by (5.1)} \\
&= (-1)^{|\beta||\gamma|+|\beta||\alpha|+|\gamma||\alpha|}(d(X_\alpha \lrcorner \{\gamma,\beta\}) - \mathcal{L}_{X_\alpha}(d(X_\beta \lrcorner \gamma)) + \mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}\gamma).
\end{aligned}$$

Hence,

$$(-1)^{|\alpha||\gamma|}\{\alpha,\{\beta,\gamma\}\} = (-1)^{|\beta||\gamma|+|\beta||\alpha|}\left(d(X_\alpha \lrcorner \{\gamma,\beta\}) - \mathcal{L}_{X_\alpha}d(X_\beta \lrcorner \gamma) + \mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}\gamma\right). \tag{5.2}$$

Similarly, since $|\{\gamma,\alpha\}| = |\gamma| + |\alpha| - 2$, we have that

$$\begin{aligned}
\{\beta,\{\gamma,\alpha\}\} &= (-1)^{(|\gamma|+|\alpha|)|\beta|+1}\{\{\gamma,\alpha\},\beta\} \\
&= (-1)^{|\gamma||\beta|+|\alpha||\beta|+1}(d(X_\beta \lrcorner \{\gamma,\alpha\}) - \mathcal{L}_{X_\beta}d(X_\alpha \lrcorner \gamma) + \mathcal{L}_{X_\beta}\mathcal{L}_{X_\alpha}\gamma). && \text{by (5.1)}
\end{aligned}$$

Hence,

$$(-1)^{|\beta||\alpha|}\{\beta,\{\gamma,\alpha\}\} = (-1)^{|\gamma||\beta|+1}\left(d(X_\beta \lrcorner \{\gamma,\alpha\}) - \mathcal{L}_{X_\beta}d(X_\alpha \lrcorner \gamma) + \mathcal{L}_{X_\beta}\mathcal{L}_{X_\alpha}\gamma\right). \tag{5.3}$$

Lastly, using Lemma 5.8 and (5.1), we have that

$$(-1)^{|\gamma||\beta|}\{\gamma,\{\alpha,\beta\}\} = (-1)^{|\gamma||\beta|}\left(d([X_\alpha,X_\beta] \lrcorner \gamma) - \mathcal{L}_{[X_\alpha,X_\beta]}\gamma\right). \tag{5.4}$$

Now we notice that by (2.8) the terms involving $\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}\gamma$ from (5.2) , $\mathcal{L}_{X_\beta}\mathcal{L}_{X_\alpha}\gamma$ from (5.3) and $\mathcal{L}_{[X_\alpha,X_\beta]}\gamma$ from (5.4) add to zero. Hence we now consider the term $(-1)^{|\gamma||\beta|}d([X_\alpha,X_\beta]\lrcorner\gamma)$ from (5.4). Using equations (2.7), (2.5), and (2.6) it follows

$$\begin{aligned}
d([X_\alpha,X_\beta] \lrcorner \gamma) &= d\left((-1)^{|\alpha|(|\beta|+1)}\mathcal{L}_{X_\alpha}(X_\beta \lrcorner \gamma) - X_\beta \lrcorner \mathcal{L}_{X_\alpha}\gamma\right) \\
&= (-1)^{|\alpha|(|\beta|+1)}d\left(\mathcal{L}_{X_\alpha}(X_\beta \lrcorner \gamma)\right) - d\left(X_\beta \lrcorner d(X_\alpha \lrcorner \gamma)\right) + (-1)^{|\alpha|+1}d\left(X_\beta \lrcorner X_\alpha \lrcorner d\gamma\right) \\
&= (-1)^{|\alpha||\beta|}\mathcal{L}_{X_\alpha}d(X_\beta \lrcorner \gamma) - \mathcal{L}_{X_\beta}d(X_\alpha \lrcorner \gamma) + (-1)^{|\alpha|}d(X_\beta \lrcorner X_\alpha \lrcorner X_\gamma \lrcorner \omega).
\end{aligned}$$

Thus

$$(-1)^{|\gamma||\beta|}d([X_\alpha, X_\beta]\lrcorner\gamma) = (-1)^{|\alpha||\beta|+|\gamma||\beta|}\mathcal{L}_{X_\alpha}d(X_\beta\lrcorner\gamma) - (-1)^{|\gamma||\beta|}\mathcal{L}_{X_\beta}d(X_\alpha\lrcorner\gamma) \\ + (-1)^{|\gamma||\beta|+|\alpha|}d(X_\beta\lrcorner X_\alpha\lrcorner X_\gamma\lrcorner\omega). \tag{5.5}$$

Thus, upon adding (5.2), (5.3) and (5.5) we are left with

$$\begin{aligned}
&(-1)^{|\alpha||\gamma|}\{\{\alpha,\beta\},\gamma\} + (-1)^{|\beta||\alpha|}\{\{\beta,\gamma\},\alpha\} + (-1)^{|\gamma||\beta|}\{\{\gamma,\alpha\},\beta\} \\
&\quad = (-1)^{|\beta||\gamma|+|\beta||\alpha|}d(X_\alpha\lrcorner\{\gamma,\beta\}) + (-1)^{|\gamma||\beta|+1}d(X_\beta\lrcorner\{\gamma,\alpha\}) \\
&\qquad + (-1)^{|\gamma||\beta|+|\alpha|}(X_\beta\lrcorner X_\alpha\lrcorner X_\gamma\lrcorner\omega) \\
&\quad = (-1)^{|\beta||\gamma|+|\beta||\alpha|+|\beta|}d(X_\alpha\lrcorner X_\beta\lrcorner X_\gamma\lrcorner\omega) + (-1)^{|\gamma||\beta|+1+|\alpha|}d(X_\beta\lrcorner X_\alpha\lrcorner X_\gamma\lrcorner\omega) \\
&\qquad + (-1)^{|\gamma||\beta|+|\alpha|}d(X_\beta\lrcorner X_\alpha\lrcorner X_\gamma\lrcorner\omega) \\
&\quad = (-1)^{|\beta||\gamma|+|\beta||\alpha|+|\beta|}d(X_\alpha\lrcorner X_\beta\lrcorner X_\gamma\lrcorner\omega).
\end{aligned}$$

□

Summing up the results of this section we have confirmed Theorem 4.1 of [5]:

Proposition 5.10. *With the above grading,* $(\widetilde{\Omega}_{Ham}(M), \{\cdot,\cdot\})$ *is a graded Lie algebra.*

Proof. The bracket is well defined on $\widetilde{\Omega}_{\mathrm{Ham}}(M)$ since if γ is closed then $\{\gamma,\alpha\} = (-1)^k X_\alpha\lrcorner d\gamma = 0$. Clearly the bracket is bilinear. Proposition 5.7 shows that the bracket is skew graded and Proposition 5.9 shows that it satisfies the Jacobi identity. □

5.2 Conserved Quantities and their Algebraic Structure

We now turn our attention towards conserved quantities. In symplectic geometry, a conserved quantity is a 0-form α that is preserved by the Hamiltonian, i.e. satisfying $\mathcal{L}_{X_H}\alpha = 0$. A generalization of this definition to multisymplectic geometry was given in [20]; however, we add the requirement that a conserved quantity is also Hamiltonian. By adding in this requirement, we can now take the generalized Poisson bracket of two conserved quantities, as in symplectic geometry.

We work with a fixed multi-Hamiltonian system (M,ω,H) with $\omega\in\Omega^{n+1}(M)$ and $H\in\Omega^{n-1}_{\mathrm{Ham}}(M)$, and let X_H denote the corresponding Hamiltonian vector field.

Definition 5.11. A Hamiltonian $(n-k)$-form α in $\Omega^{n-k}_{\mathrm{Ham}}(M)$ is called

- locally conserved if $\mathcal{L}_{X_H}\alpha$ is closed,
- globally conserved if $\mathcal{L}_{X_H}\alpha$ is exact,
- strictly conserved if $\mathcal{L}_{X_H}\alpha = 0$.

As in [20], we denote the space of locally, globally, and strictly conserved forms by $\mathcal{C}_{\mathrm{loc}}(X_H)$, $\mathcal{C}(X_H)$, and $\mathcal{C}_{\mathrm{str}}(X_H)$ respectively. We will let $\widetilde{\mathcal{C}}_{\mathrm{loc}}(X_H)$, $\widetilde{\mathcal{C}}(X_H)$ and $\widetilde{\mathcal{C}}_{\mathrm{str}}(X_H)$ denote the conserved quantities modulo closed forms. Note that $\mathcal{C}_{\mathrm{str}}(X_H) \subset \mathcal{C}(X_H) \subset \mathcal{C}_{\mathrm{loc}}(X_H)$ and $\widetilde{\mathcal{C}}_{\mathrm{str}}(X_H) \subset \widetilde{\mathcal{C}}(X_H) \subset \widetilde{\mathcal{C}}_{\mathrm{loc}}(X_H)$.

The next lemma is a generalization of Lemma 1.7 in [20].

Lemma 5.12. Fix a Hamiltonian $(n-k)$-form $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$. If α is a local conserved quantity then $[X_\alpha, X_H] \lrcorner\, \omega = 0$, for some (or equivalently every) Hamiltonian multivector field X_α of α. Conversely, if $[X_\alpha, X_H] \lrcorner\, \omega = 0$ then α is locally conserved.

Proof. Let X_α be an arbitrary Hamiltonian multivector field of α. We have that

$$\begin{aligned} [X_\alpha, X_H] \lrcorner\, \omega &= -X_H \lrcorner\, d(X_\alpha \lrcorner\, \omega) - d(X_H \lrcorner\, X_\alpha \lrcorner\, \omega) \\ &\quad + X_\alpha \lrcorner\, (d(X_H \lrcorner\, \omega)) + X_H \lrcorner\, X_\alpha \lrcorner\, d\omega && \text{by Prop 2.10} \\ &= -d(X_H \lrcorner\, X_\alpha \lrcorner\, \omega) \\ &= -\mathcal{L}_{X_H}(X_\alpha \lrcorner\, \omega) && \text{by (2.5)} \\ &= d\mathcal{L}_{X_H}\alpha && \text{by (2.6).} \end{aligned}$$

□

Recall the following standard result from Hamiltonian mechanics: If H is a Hamiltonian on a symplectic manifold and f and g are two strictly conserved quantities, i.e. $\{f, H\} = 0 = \{g, H\}$, then $\{f, g\}$ is strictly conserved. This is because $\mathcal{L}_{X_H}\{f, g\} = \{\{f, g\}, H\} = 0$ by the Jacobi identity. Moreover, if f and g are local or global conserved quantities (meaning that their bracket with H is constant) then again $\{f, g\}$ is strictly conserved by the Jacobi identity together with the fact that the Poisson bracket with a constant function vanishes.

The next proposition generalizes these results to multisymplectic geometry.

Proposition 5.13. *The bracket of two conserved quantities is a strictly conserved quantity.*

Proof. Let $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ and $\beta \in \Omega^{n-l}_{\mathrm{Ham}}(M)$ be any two conserved quantities. Let X_α and X_β denote arbitrary Hamiltonian multivector fields corresponding to α and β respectively. By definition,

$$\mathcal{L}_{X_H}\{\alpha,\beta\} = (-1)^{|\beta|}\mathcal{L}_{X_H} X_\beta \lrcorner X_\alpha \lrcorner \omega.$$

By (2.7) together with Lemma 5.12 we see that we can commute the Lie derivative and interior product. Hence,

$$\mathcal{L}_{X_H}\{\alpha,\beta\} = (-1)^{|\beta|} X_\alpha \lrcorner X_\beta \lrcorner \mathcal{L}_{X_H}\omega.$$

The claim now follows since $\mathcal{L}_{X_H}\omega = 0$, by Proposition 5.5. ☐

As a consequence, we obtain:

Proposition 5.14. *The spaces* $(\widetilde{\mathcal{C}}_{loc}(X_H), \{\cdot,\cdot\})$*,* $(\widetilde{\mathcal{C}}(X_H), \{\cdot,\cdot\})$*, and* $(\widetilde{\mathcal{C}}_{str}(X_H), \{\cdot,\cdot\})$ *are graded Lie subalgebras of* $(\widetilde{\Omega}_{Ham}(M), \{\cdot,\cdot\})$*.*

Proof. Proposition 5.13 shows that each of these spaces is preserved by the bracket. The claim now follows from Proposition 5.10. ☐

We conclude this section by showing that the Hamiltonian forms constitute an L_∞-subalgebra of the Lie n-algebra of observables. Moreover, restricting a homotopy moment map to the Lie kernel gives an L_∞-morphism into this L_∞-algebra:

Let $\widehat{L}_\infty(M,\omega) = (\widehat{L}, \{l_k\})$ denote the graded vector space $\widehat{L}_k = \Omega^{n-1-k}_{\mathrm{Ham}}(M)$ for $k = 0,\dots,n-1$, together with the maps l_k from the Lie n-algebra of observables.

Theorem 5.15. *The space* $(\widehat{L}, \{l_k\})$ *is an* L_∞*-subalgebra of* $(L, \{l_k\})$*.*

Proof. We note that l_1 preserves $\widehat{L}$ since closed forms are Hamiltonian. For $k > 1$, since l_k vanishes on elements of positive degree we need only consider

$$l_k(\alpha_1,\cdots,\alpha_k) = -(-1)^{\frac{k(k+1)}{2}} X_{\alpha_k} \lrcorner \cdots \lrcorner X_{\alpha_1} \lrcorner \omega,$$

where $\alpha_1,\dots,\alpha_k$ are Hamiltonian $(n-1)$-forms. By Lemma 3.8 we see that $l_k(\alpha_1,\dots,\alpha_k)$ is a Hamiltonian $(n+1-k)$-form. ☐

Proposition 5.16. *The spaces* $\mathcal{C}(X_H)\cap\widehat{L}$ *,* $\mathcal{C}_{loc}(X_H)\cap\widehat{L}$*, and* $\mathcal{C}_{str}(X_H)\cap\widehat{L}$ *are* L_∞*-subalgebras of* $\widehat{L}_\infty(M,\omega)$*.*

Proof. The proof is analogous to the proof of Proposition 1.15 in [20]. Since the proof is short, we include it here. From Theorem 5.15 we see that each of the spaces $\mathcal{C}(X_H) \cap \widehat{L}$, $\mathcal{C}_{\text{loc}}(X_H) \cap \widehat{L}$, and $\mathcal{C}_{\text{str}}(X_H) \cap \widehat{L}$ are closed under each l_k. It remains to show that for Hamiltonian $(n-1)$-forms $\alpha_1, \cdots, \alpha_k$ which are (locally, globally, strictly) conserved, that $l_k(\alpha_1, \cdots, \alpha_k)$ is (locally, globally, strictly) conserved. Indeed,

$$\mathcal{L}_{X_H} l_k(\alpha_1, \cdots, \alpha_k) = \mathcal{L}_{X_H} X_{\alpha_k} \lrcorner \cdots \lrcorner X_{\alpha_1} \lrcorner \omega.$$

Using equation (2.7) together with Lemma 5.12 we see that we can commute the Lie derivative and interior product. The claim then follows since $\mathcal{L}_{X_H}\omega = 0$. □

5.3 Continuous Symmetries and their Algebraic Structure

Fix a multi-Hamiltonian system (M, ω, H). Our motivation for the definition of a continuous symmetry comes from Hamiltonian mechanics; we directly generalize the definition. As is the case with conserved quantities, we define three types of continuous symmetry.

Definition 5.17. We say that a Hamiltonian multivector field $X \in \mathfrak{X}_{\text{Ham}}(M)$ is

- a local continuous symmetry if $\mathcal{L}_X H$ is closed,
- a global continuous symmetry if $\mathcal{L}_X H$ is exact,
- a strict continuous symmetry if $\mathcal{L}_X H = 0$.

Note that a continuous symmetry automatically preserves ω by Proposition 5.5. We denote the space of local, global, and strict continuous symmetries by $\mathcal{S}_{\text{loc}}(H)$, $\mathcal{S}(H)$, and $\mathcal{S}_{\text{str}}(H)$ respectively. Moreover, we let $\widetilde{\mathcal{S}}_{\text{loc}}(H)$, $\widetilde{\mathcal{S}}(H)$, and $\widetilde{\mathcal{S}}_{\text{str}}(H)$ denote the quotient by the kernel of ω.

We will say that a multivector field X is a weak (local, global, strict) continuous symmetry if $\mathcal{L}_X\omega = 0$ and $\mathcal{L}_X H$ is closed, exact, or zero respectively. That is, a weak continuous symmetry is not necessarily Hamiltonian.

Proposition 5.18. *We have $(\mathfrak{X}_{Ham}(M), [\cdot, \cdot])$ is a graded Lie subalgebra of $(\Gamma(\Lambda^\bullet(TM)), [\cdot, \cdot])$.*

Proof. By equation (2.10) we see that $[X, Y] \lrcorner \omega = (-1)^l d(X \lrcorner Y \lrcorner \omega)$. Hence the space of Hamiltonian multivector fields is closed under the Schouten bracket. □

Proposition 5.19. *The spaces $\mathcal{S}_{loc}(H)$, $\mathcal{S}(H)$, and $\mathcal{S}_{str}(H)$ are graded Lie subalgebras of $(\mathfrak{X}_{Ham}(M), [\cdot, \cdot])$.*

Proof. We see that each of $\mathcal{S}_{\mathrm{loc}}(H)$, $\mathcal{S}(H)$, and $\mathcal{S}_{\mathrm{str}}(H)$ are closed under the Schouten bracket directly from equations (2.6) and (2.8). $\square$

The next lemma generalizes Lemma 2.9 (ii) of [20].

Lemma 5.20. Let $Y \in \Gamma(\Lambda^k(TM))$. If Y is a local continuous symmetry, then $[Y, X_H] \lrcorner\, \omega = 0$. Conversely, if $[Y, X_H] \lrcorner\, \omega = 0$ and $\mathcal{L}_Y \omega = 0$, then Y is a local continuous symmetry.

Proof. We have that

$$\begin{aligned} [Y, X_H] \lrcorner\, \omega &= (-1)^{k+1} \mathcal{L}_Y (X_H \lrcorner\, \omega) - X_H \lrcorner\, \mathcal{L}_Y \omega && \text{by (2.7)} \\ &= (-1)^k \mathcal{L}_Y dH && \text{since } \mathcal{L}_Y \omega = 0 \text{ and } X_H \lrcorner\, \omega = -dH \\ &= -d\mathcal{L}_Y H && \text{by (2.6).} \end{aligned}$$

$\square$

Recall that for a group G acting on a manifold M we had defined in equation (2.11) the set $S_k := \{V_p \;:\; p \in \mathcal{P}_{\mathfrak{g},k}\}$. Proposition 2.15 showed that $S = \oplus S_k$ was a graded Lie algebra. We now get the following.

Proposition 5.21. *The spaces $\mathcal{S}_{loc}(H) \cap S$, $\mathcal{S}(H) \cap S$, and $\mathcal{S}_{str}(H) \cap S$ are graded Lie subalgebras of S.*

Proof. By Proposition 5.19 we have that the spaces of symmetries are preserved by the Schouten bracket. The claim now follows by Proposition 2.15. $\square$

6 Noether's Theorem in Multisymplectic Geometry

In this section we show how Noether's theorem extends from symplectic to multisymplectic geometry. To see this generalization explicitly, we first recall how Noether's theorem works in symplectic geometry.

6.1 Noether's Theorem in Symplectic Geometry

In this section we briefly recall the notions from symplectic geometry. More information can be found in [9], for example. Let (M, ω, H) be a Hamiltonian system. That is (M, ω) is symplectic and H is in $C^\infty(M)$. Noether's theorem gives a correspondence between symmetries and conserved quantities. If $f \in C^\infty(M)$ is a (local, global) conserved quantity then X_f is a (local, global) continuous symmetry. Conversely, if a vector field X_f is a (local, global) continuous symmetry, then f is a (local, global) conserved quantity. Note that in the symplectic case, local and strict symmetries and conserved quantities are the same thing.

If X is only a weak (local, global) continuous symmetry, then $\mathcal{L}_X \omega = 0$ so that by the Cartan formula around each point there is a neighbourhood U and a function $f \in C^\infty(U)$ such that $X = X_f$ on U. This function f is a (local, global) conserved quantity in the Hamiltonian system $(U, \omega|_U, H|_U)$.

If we only consider the symmetries and conserved quantities coming from a moment map $\mu : \mathfrak{g} \to C^\infty(M)$ then, under the assumption of an H-preserving group action, each symmetry ξ has corresponding global conserved quantity $\mu(\xi)$ and vice versa.

The rest of this subsection formalizes this, and the following sections will generalize it to multi-symplectic geometry.

Recall that an equivariant moment map gives a Lie algebra morphism between $(\mathfrak{g}, [\cdot, \cdot])$ and $(C^\infty(M), \{\cdot, \cdot\})$.

Proposition 6.1. *Let $\mu : \mathfrak{g} \to C^\infty(M)$ be a momentum map. For $\xi, \eta \in \mathfrak{g}$ we have that $\mu([\xi, \eta]) = \{\mu(\xi), \mu(\eta)\} +$ constant. If the moment map is equivariant then $\mu([\xi, \eta]) = \{\mu(\xi), \mu(\eta)\}$.*

Proof. See Theorem 4.2.8 of [1]. □

As stated above, it is clear that in the symplectic case $\mathcal{C}_{loc}(X_H) = \mathcal{C}_{str}(X_H)$ and $\mathcal{S}_{loc}(H) = \mathcal{S}_{str}(H)$. It is easily verified that the map $\alpha \mapsto X_\alpha$ is a Lie algebra morphism from

$(\mathcal{C}(X_H), \{\cdot,\cdot\})$ to $(\mathcal{S}(H), [\cdot,\cdot])$ and from $(\mathcal{C}_{loc}(X_H), \{\cdot,\cdot\})$ to $(\mathcal{S}_{loc}(H), [\cdot,\cdot])$. However, under the quotients this map turns into a Lie algebra isomorphism.

Proposition 6.2. *The map $\alpha \mapsto X_\alpha$ is a Lie algebra isomorphism from $(\widetilde{C}(X_H), \{\cdot,\cdot\})$ to $(\widetilde{S}(H), [\cdot,\cdot])$ and $(\widetilde{\mathcal{C}}_{loc}(X_H), \{\cdot,\cdot\})$ to $(\widetilde{\mathcal{S}}_{loc}(H), [\cdot,\cdot])$.*

As a consequence of this proposition, we can now see how a momentum map sets up a Lie algebra isomorphism between the symmetries and conserved quantities it generates. Let $C = \{\mu(\xi); \xi \in \mathfrak{g}\}$ and $S = \{V_\xi; \xi \in \mathfrak{g}\}$. Let $\widetilde{C}$ be the quotient of C by constant functions. Let $\widetilde{S}$ denote the quotient of S by the kernel of ω. Since the kernel of ω is trivial, $S = \widetilde{S}$. Then we get an induced well defined Poisson bracket on $\widetilde{C}$ and an induced well defined Lie bracket on $\widetilde{S}$. We thus get a Lie algebra isomorphism:

Proposition 6.3. *The map between $(\widetilde{C}, \{\cdot,\cdot\})$ and $(\widetilde{S}, [\cdot,\cdot])$ that sends $[V_\xi]$ to $[\mu(\xi)]$ is a Lie-algebra isomorphism.*

With our newly defined notions of symmetry and conserved quantity on a multisympletic manifold, we now exhibit how these concepts generalize to the setup of multisymplectic geometry.

6.2 The Correspondence between Mutlisymplectic Conserved Quantities and Continuous Symmetries

We first examine the correspondence between symmetries and conserved quantities on multi-Hamiltonian systems. We will make repeated use of the following equations. Fix $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$. By definition we have that

$$\{\alpha, H\} = -X_H \lrcorner X_\alpha \lrcorner \omega = X_H \lrcorner d\alpha = \mathcal{L}_{X_H}\alpha - d(X_H \lrcorner \alpha).$$

But we also know that $\{\alpha, H\} = -\{H, \alpha\}$, since $|H| = 2$. Thus, by definition of the Poisson bracket and equation (2.5) we have that

$$-\{H, \alpha\} = (-1)^k X_\alpha \lrcorner X_H \lrcorner \omega = (-1)^{k+1} X_\alpha \lrcorner dH = \mathcal{L}_{X_\alpha} H - d(X_\alpha \lrcorner H).$$

Putting these together we obtain

$$\mathcal{L}_{X_\alpha} H = d(X_\alpha \lrcorner H) + \mathcal{L}_{X_H}\alpha - d(X_H \lrcorner \alpha) \tag{6.1}$$

and

$$\mathcal{L}_{X_H}\alpha = d(X_H \lrcorner\, \alpha) + \mathcal{L}_{X_\alpha} H - d(X_\alpha \lrcorner\, H). \tag{6.2}$$

Theorem 6.4. *If $\alpha \in \Omega^{n-k}_{Ham}(M)$ is a (local, global) conserved quantity then any corresponding Hamiltonian k-vector field is a (local, global) continuous symmetry. Conversely, if $A \in \Gamma(\Lambda^k(TM))$ is a (local, global) continuous symmetry, then any corresponding Hamiltonian form is a (local, global) conserved quantity.*

Proof. Consider $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$. Let X_α be an arbitrary Hamiltonian multivector field. Then, by equation (6.1) we have that

$$\mathcal{L}_{X_\alpha} H = d(X_\alpha \lrcorner\, H) + \mathcal{L}_{X_H}\alpha - d(X_H \lrcorner\, \alpha).$$

Thus, if α is a (local or global) conserved quantity then X_α is a (local or global) continuous symmetry.

Conversely, suppose that A is a (local or global) continuous symmetry and let α be a corresponding Hamiltonian form. Following the same argument above, we have by equation (6.2)

$$\mathcal{L}_{X_H}\alpha = d(X_H \lrcorner\, \alpha) + \mathcal{L}_{X_\alpha} H - d(X_\alpha \lrcorner\, H)$$

□

The correspondence between strictly conserved quantities and strict continuous symmetries is a little bit different. We have that

Corollary 6.5. *If $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ is a strictly conserved quantity then X_α is a global continuous symmetry. Conversely, if A is a strict continuous symmetry then the corresponding Hamiltonian $(n-k)$-form α is a global conserved quantity.*

Proof. This follows from the proof of the above theorem. □

Remark 6.6. If we were to consider weak continuous symmetries in the above theorem, then by the Poincaré lemma, a continuous symmetry would still give a conserved quantity, but only in a neighbourhood around each point of the manifold.

The following simple example exhibits the correspondence.

Example 6.7. Consider $M = \mathbb{R}^3$ with volume form $\omega = dx \wedge dy \wedge dz$, $H = -xdy$ and $\alpha = zdx$. Then $dH = -dx \wedge dy$ so that $X_H = \frac{\partial}{\partial z}$. Also, $d\alpha = dz \wedge dx$ and so $X_\alpha = \frac{\partial}{\partial y}$. By the Cartan formula, we have that

$$\mathcal{L}_{X_\alpha} H = -dx + dx = 0,$$

which means that $X_\alpha \in \mathcal{S}_{\mathrm{str}}(H) \subset \mathcal{S}(H)$. We also have that

$$\mathcal{L}_{X_H}\alpha = d(X_H \lrcorner\, \alpha) + \{\alpha, H\} = d(X_H \lrcorner\, \alpha) - d(X_\alpha \lrcorner\, H) = dx.$$

That is, $\alpha \in \mathcal{C}(X_H)$. Thus α is a global conserved quantity and X_α is a global continuous symmetry.

6.3 Weak Moment Maps as Morphisms

We work with a fixed multi-Hamiltonian system (M, ω, H) with acting symmetry group G. By definition, a moment map is an L_∞-morphism between the Chevalley-Eilenberg complex and the Lie n-algebra of observables. Recall that in Section 4 we had defined the L_∞-algebra $\widehat{L}(M, \omega)$, where $\widehat{L}$ consisted entirely of Hamiltonian forms: $\widehat{L} = \oplus_{k=0}^{n-1} \Omega^{n-1-k}_{\mathrm{Ham}}(M)$.

Proposition 6.8. *A weak homotopy moment map is an L_∞-morphism from $(\mathcal{P}_{\mathfrak{g}}, \partial, [\cdot,\cdot])$ to $(\widehat{L}, \{l_k\})$.*

Proof. Equation (3.2) shows that a homotopy moment map sends each element of the Lie kernel to a Hamiltonian form. Hence the claim follows from Proposition 2.14 and Theorem 5.15. □

Next we study how a weak homotopy moment map interacts with the generalized Poisson bracket on the space of Hamiltonian forms. In particular, to make a connection with Proposition 6.1 from symplectic geometry, we compare the difference of $f_{k+l-1}([p, q])$ and $\{f_k(p), f_l(q)\}$.

Let G be a Lie group acting on a multi-Hamiltonian system (M, ω, H). Let (f) be a weak homotopy moment map. By equation (3.2) we see that under this restriction the image of the moment map is contained in the L_∞-algebra $\widehat{L}(M, \omega)$ of Hamiltonian forms. Moreover, we obtain that every element in the image is a conserved quantity. This was one of the main points of [20]. Indeed, in [20] Propositions 2.12 and 2.21 say:

Proposition 6.9. *If the group locally or globally preserves H, then $f_k(p)$ is a local conserved quantity for all $p \in \mathcal{P}_{\mathfrak{g},k}$. If the group strictly preserves H then $f_k(p)$ is a globally conserved quantity for all $p \in \mathcal{P}_{\mathfrak{g},k}$.*

Thus, by restricting a homotopy moment map to the Lie kernel, we see that, under assumptions on the group action, every element is a conserved quantity, analogous to the setup in symplectic geometry.

As a consequence of Theorem 6.4 we see that the moment map also gives a family of continuous symmetries.

Proposition 6.10. *If the group locally or globally preserves H, then V_p is a local continuous symmetry for all $p \in \mathcal{P}_{\mathfrak{g},k}$. If the group strictly preserves H then V_p is a global continuous symmetry for all $p \in \mathcal{P}_{\mathfrak{g},k}$.*

Example 6.11. (Motion in a conservative system under translation)

Recall that in Example 3.17 we considered the translation action of $\mathbb{R}^3$ on (M, ω, H) where $M = T^*\mathbb{R}^3 = \mathbb{R}^6$, $\omega = \text{vol}$ and

$$H = \frac{1}{2}\left((p_1q^2dq^3 - p_1q^3dq^2) + (p_2q^1dq^3 - p_2q^3dq^1) + (p_3q^1dq^2 - p_3q^2dq^1)\right) dp_1dp_2dp_3,$$

where q^1, q^2, q^3 are the standard coordinates on $\mathbb{R}^3$ and q^1, q^2, q^3, p_1, p_2, p_3 are the induced coordinates on $T^*\mathbb{R}^3$. It is easy to check that each of $\mathcal{L}_{\frac{\partial}{\partial q^i}} H$ are exact for $i = 1, 2, 3$. That is, the group action globally preserves H. Hence, by Proposition 6.9 each of the differential forms, computed in Example 3.17,

$$f_1(e_1) = \frac{1}{2}(q^2dq^3 - q^3dq^2)dp_1dp_2dp_3,$$

$$f_1(e_2) = \frac{1}{2}(q^1dq^3 - q^3dq^1)dp_1dp_2dp_3,$$

$$f_1(e_3) = \frac{1}{2}(q^1dq^2 - q^2dq^1)dp_1dp_2dp_3,$$

$$f_2(e_1 \wedge e_2) = q^3dp_1dp_2dp_3, \qquad f_2(e_1 \wedge e_3) = q^2dp_1dp_2dp_3, \qquad f_2(e_2 \wedge e_3) = q^1dp_1dp_2dp_3,$$

and

$$f_3(e_1 \wedge e_2 \wedge e_3) = \frac{1}{3}\left(p_1dp_2dp_3 + p_2dp_3dp_1 + p_3dp_1dp_2\right).$$

are all globally conserved. Thus, by Example 3.17, the Lie derivative of these differential forms by the geodesic spray are all exact.

Moreover, by Proposition 6.10, each of $\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3}, \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^3}, \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial q^3}$ and $\frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial q^3}$ are global continuous symmetries in this multi-Hamiltonian system.

Proposition 6.12. *Let $p \in \mathcal{P}_{\mathfrak{g},k}$ and $q \in \mathcal{P}_{\mathfrak{g},l}$ be arbitrary. Set $\alpha = f_{n-k}(p)$ and $\beta = f_{n-l}(q)$. Then we have that $-\zeta(k)\zeta(l)[V_p, V_q]$ is a Hamiltonian multivector field for $\{\alpha, \beta\}$.*

Proof. By definition of the moment map (equation (3.2)) we have that $X_\alpha - \zeta(k)V_p$ and $X_\beta - \zeta(l)V_q$ are in the kernel of ω. Hence, by Proposition 5.4 we have that

$$[X_\alpha - \zeta(k)V_p, X_\beta - \zeta(l)V_q] \lrcorner\, \omega = 0.$$

Proposition 5.4 also shows that

$$[X_\alpha - \zeta(k)V_p, X_\beta - \zeta(l)V_q] \lrcorner\, \omega = ([X_\alpha, X_\beta] + \zeta(k)\zeta(l)[V_p, V_q]) \lrcorner\, \omega.$$

Thus

$$[X_\alpha, X_\beta] \lrcorner\, \omega = -\zeta(k)\zeta(l)[V_p, V_q] \lrcorner\, \omega.$$

The claim now follows from Proposition 5.8. □

Our generalization of Proposition 6.1 to multisymplectic geometry is:

Proposition 6.13. *For $p \in \mathcal{P}_{\mathfrak{g},k}$ and $q \in \mathcal{P}_{\mathfrak{g},l}$ we have that*

$$\{f_k(p), f_l(q)\} - (-1)^{k+l+kl} f_{k+l-1}([p,q])$$

is a closed $(n+1-k-l)$-form.

Proof. By definition of a homotopy moment map (equation (3.10)) we have that

$$\begin{aligned}
d(f_{k+l-1}([p,q])) &= -f_{k+l-2}(\partial[p,q]) - \zeta(k+l-1)V_{[p,q]} \lrcorner\, \omega \\
&= -\zeta(k+l-1)V_{[p,q]} \lrcorner\, \omega && \text{by Proposition 2.14} \\
&= \zeta(k+l-1)[V_p, V_q] \lrcorner\, \omega && \text{by Proposition 2.15} \\
&= \zeta(k+l-1)\zeta(k)\zeta(l)[X_{f_k(p)}, X_{f_l(p)}] \lrcorner\, \omega && \text{by Proposition 6.12} \\
&= -(-1)^{k+l+kl} X_{\{f_k(p), f_l(q)\}} \lrcorner\, \omega && \text{by Proposition 5.8 and Remark 3.7} \\
&= (-1)^{k+l+kl} d(\{f_k(p), f_l(q)\}) && \text{by definition.}
\end{aligned}$$

□

From this proposition we see that a moment map does not necessarily preserve brackets; however, we now show that once we pass to certain cohomology groups then it will. Moreover, the moment map will give an isomorphism of graded Lie algebras, generalizing Proposition 6.3. Recall that we had defined $\widetilde{\mathfrak{X}}^k_{\text{Ham}}(M)$ to be the quotient of $\mathfrak{X}^k_{\text{Ham}}(M)$ by the kernel of ω restricted to $\Lambda^k(TM)$. We set $\widetilde{\mathfrak{X}}_{\text{Ham}}(M) = \oplus \widetilde{\mathfrak{X}}^k_{\text{Ham}}(M)$.

Proposition 6.14. *The Schouten bracket on $\mathfrak{X}_{\text{Ham}}(M)$ descends to a well defined bracket on $\widetilde{\mathfrak{X}}_{\text{Ham}}(M)$.*

Proof. This follows directly from Proposition 5.4. $\square$

Similarily, we let $\widetilde{\Omega}^{n-k}_{\text{Ham}}(M)$ denote the quotient of $\Omega^{n-k}_{\text{Ham}}(M)$ by the closed forms of degree $n-k$ and set $\widetilde{\Omega}_{\text{Ham}}(M) = \oplus_{k=1}^n \widetilde{\Omega}^{n-k}_{\text{Ham}}(M)$. Recall that Proposition 5.10 showed that $(\widetilde{\Omega}_{\text{Ham}}(M), \{\cdot,\cdot\})$ was a well defined graded Lie algebra.

Theorem 6.15. *The map $\alpha \mapsto X_\alpha$ is a graded Lie algebra isomorphism from $(\widetilde{\Omega}_{\text{Ham}}(M), \{\cdot,\cdot\})$ to $(\widetilde{\mathfrak{X}}_{\text{Ham}}(M), [\cdot,\cdot])$.*

Proof. The map is well defined since the Hamiltonian multivector field of a closed form is the zero vector field. The map is clearly surjective. It is injective since if $X_\alpha = X_\beta$ then $d\alpha = d\beta$. Lastly, by Lemma 5.8, we have that $X_{\{\alpha,\beta\}} = [X_\alpha, X_\beta]$ in the quotient space. $\square$

We have now obtained a generalization of Proposition 6.2 from symplectic geometry.

Corollary 6.16. *The map $\alpha \mapsto X_\alpha$ is a graded Lie algebra isomorphism from $(\widetilde{\mathcal{C}}(X_H), \{\cdot,\cdot\})$ to $(\widetilde{\mathcal{S}}(H), [\cdot,\cdot])$ and from $(\widetilde{\mathcal{C}}_{loc}(X_H), \{\cdot,\cdot\})$ to $(\widetilde{\mathcal{S}}_{loc}(H), [\cdot,\cdot])$.*

Proof. We know from Proposition 5.13 that each of the spaces of conserved quantities are closed under the Poisson bracket. Similarly, by Proposition 5.19, the spaces of continuous symmetries are all closed under the Schouten bracket. The claim now follows from Theorem 6.15. $\square$

We now give a generalization of Proposition 6.3 to multisymplectic geometry: We let C_k denote the image of the Lie kernel under the moment map. That is, let $C_k = f_{n-k}(\mathcal{P}_{\mathfrak{g},n-k})$. Let $\widetilde{C}_k$ denote the quotient of C_k by closed forms and set $\widetilde{C} = \oplus_{k=1}^n \widetilde{C}_k$. Recall that we had defined S_k to be the set $\{V_p; p \in \mathcal{P}_{\mathfrak{g},k}\}$. Let $\widetilde{S}_k$ denote the quotient of S_k by elements in the Lie kernel and set $\widetilde{S} = \oplus_{k=1}^n \widetilde{S}_k$. Our generalization of Proposition 6.3 is given by the following corollaries.

Corollary 6.17. *For an H-preserving group action, a momentum map induces an L_∞-algebra morphism from $\widetilde{S}$ to $\widetilde{C} \cap \widehat{L}$ given by $V_p \mapsto f_k(p)$.*

Proof. We see by Proposition 6.13 that the Poisson bracket preserves $\widetilde{C}$. The claim follows since by definition a homotopy moment map is an L_∞-morphism. □

Corollary 6.18. *For an H-preserving group action, an equivariant homotopy moment map induces an isomorphism of graded Lie algebras between $(\widetilde{S}, [\cdot,\cdot])$ and $(\widetilde{C}, \{\cdot,\cdot\})$. Explicitly, the map is given by $[V_p] \mapsto [f_k(p)]$.*

Proof. The Lie algebra isomorphism given in Theorem 6.15 is precisely the moment map. Indeed, if $\alpha = f_k(p)$ for $p \in \mathcal{P}_{\mathfrak{g},k}$, then $X_{f_k(p)} = V_p$ in $\widetilde{\mathfrak{X}}_{\mathrm{Ham}}(M)$, since both are Hamiltonian vector fields for α. Proposition 6.13 now shows that the moment map preserves the Lie brackets on these quotient spaces.

□

The morphism properties of a weak moment map are also related to its equivariance. Recall that for a symplectic action of a connected Lie group G acting on a symplectic manifold (M,ω) and a moment map, $f : \mathfrak{g} \to C^\infty(M)$, then by Definition 4.14, f is equivariant if and only if f is a Lie algebra morphism from $(\mathfrak{g}, [\cdot,\cdot])$ to $(C^\infty(M), \{\cdot,\cdot\})$. That is, if and only if

$$f([\xi,\eta]) = \{f(\xi), f(\eta)\}.$$

Taking d of both sides of this equation yields:

Proposition 6.19. *A moment map f induces a morphism from $\mathfrak{g}$ onto the quotient of $C^\infty(M)$ by constant functions. That is, a moment map induces a Lie algebra morphism from $(\mathfrak{g}, [\cdot,\cdot])$ to $(C^\infty(M)/closed, \{\cdot,\cdot\})$, regardless of equivariance. Moreover, the moment map f is equivariant if and only if f is a morphism from $(\mathfrak{g}, [\cdot,\cdot])$ to $(C^\infty(M), \{\cdot,\cdot\})$.*

We now restate Proposition 6.19 in an equivalent way, but which will allow for a direct generalization to multisymplectic geometry: Notice that $\mathfrak{g}$ is a $\mathfrak{g}$-module under the Lie bracket action and $C^\infty(M)$ is $\mathfrak{g}$-module under the action $\xi \cdot \mathfrak{g} = L_{V_\xi} g$, where $\xi \in \mathfrak{g}$ and $g \in C^\infty(M)$. Proposition 6.19 is equivalent to:

Proposition 6.20. *A moment map f always induces a $\mathfrak{g}$-module morphism from $\mathfrak{g}$ to $C^\infty(M)/closed$. Moreover, the moment map f is equivariant if and only if it is a $\mathfrak{g}$-module morphism from $\mathfrak{g}$ to $C^\infty(M)$.*

Now let a connected Lie group G act multisymplectically on an n-plectic manifold (M,ω).

Proposition 6.21. *For any $1 \leq k \leq n$, we have that $\mathcal{P}_{\mathfrak{g},k}$ is a $\mathfrak{g}$-module under the action $\xi \cdot p = [p,\xi]$, where $p \in \mathcal{P}_{\mathfrak{g},k}$, $\xi \in \mathfrak{g}$, and $[\cdot,\cdot]$ is the Schouten bracket.*

Proof. This follows since Lemma 2.13 shows that $[p,\xi]$ is in the Lie kernel. □

Proposition 6.22. *For any $1 \leq k \leq n$, we have that $\Omega^{n-k}_{\mathrm{Ham}}(M)$ is a $\mathfrak{g}$-module under the action $\xi \cdot \alpha = \mathcal{L}_{V_\xi}\alpha$, where $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ and $\xi \in \mathfrak{g}$.*

Proof. Suppose that $\alpha \in \Omega^{n-k}_{\mathrm{Ham}}(M)$ is a Hamiltonian $(n-k)$-form. Then $d\alpha = -X_\alpha \lrcorner\, \omega$ for some $X_\alpha \in \Gamma(\Lambda^k(TM))$. Then, for $\xi \in \mathfrak{g}$,

$$\begin{aligned} d\mathcal{L}_{V_\xi}\alpha &= -\mathcal{L}_{V_\xi}(X_\alpha \lrcorner\, \omega) \\ &= -\mathcal{L}_{V_\xi}(X_\alpha \lrcorner\, \omega) + X_\alpha \lrcorner\, \mathcal{L}_{V_\xi}\omega && \text{since } \mathcal{L}_{V_\xi}\omega = 0 \\ &= [V_\xi, X_\alpha] \lrcorner\, \omega && \text{by the product rule} \end{aligned}$$

Hence $\mathcal{L}_{V_\xi}\alpha$ is in $\Omega^{n-k}_{\mathrm{Ham}}(M)$.

□

Our generalization of Proposition 6.20 to multisymplectic geometry is:

Theorem 6.23. *For any $1 \leq k \leq n$, the k-th component of a moment map f_k is a $\mathfrak{g}$-module morphism from $\mathcal{P}_{\mathfrak{g},k}$ to $\Omega^{n-k}_{\mathrm{Ham}}(M)/$closed. Moreover, a weak k-moment map f_k is equivariant if and only if it is a $\mathfrak{g}$-module morphism from $\mathcal{P}_{\mathfrak{g},k}$ to $\Omega^{n-k}_{\mathrm{Ham}}(M)$.*

Proof. Suppose that (f) is a weak moment map. Then, by definition

$$\begin{aligned} df_k([\xi,p]) &= -\zeta(k)V_{[\xi,p]} \lrcorner\, \omega \\ &= -\zeta(k)[V_\xi, V_p] \lrcorner\, \omega \\ &= -\zeta(k)\mathcal{L}_{V_\xi}(V_p \lrcorner\, \omega) \\ &= \zeta(k)\zeta(k)d\mathcal{L}_{V_\xi}f_k(p) \\ &= d\mathcal{L}_{V_\xi}f_k(p). \end{aligned}$$

This proves the first statement of the theorem. Now suppose f_k is equivariant. It follows that $\Sigma_k = 0$. Thus, by Proposition 4.15 we have $f_k([\xi,p]) = \mathcal{L}_{V_\xi}f_k(p)$. Conversely, if f_k

is a $\mathfrak{g}$-module morphism, that $f_k([\xi, p]) = \mathcal{L}_{V_\xi} f_k(p)$ for every $\xi \in \mathfrak{g}$ and $p \in \mathcal{P}_{\mathfrak{g},k}$. That is, $\Sigma_k = 0$. $\square$

7 Applications

We first apply the generalized Poisson bracket to extend the theory of the classical momentum and position functions on the phase space of a manifold to the multisymplectic phase space.

7.1 Classical Multisymplectic Momentum and Position Forms

Recall the following notions from Hamiltonian mechanics:

Let N be a manifold and $(T^*N, \omega = -d\theta)$ the canonical phase space. Given a group action on N we can extend this to a group action on T^*N that preserves both tautological forms θ and ω. It is easy to check that a moment map for this action is given by $f : \mathfrak{g} \to C^\infty(N)$, $\xi \mapsto V_\xi \lrcorner\, \theta$. From this moment map we can introduce the classical momentum and position functions, as discussed in Propositions 4.2.12 and 5.4.4 of [1]. Given $X \in \Gamma(TN)$, its classical momentum function is $P(X) \in C^\infty(T^*N)$ defined by $P(X)(\alpha_q) := \alpha_q(X_q)$. Corollary 4.2.11 of [1] then shows that

$$P(V_\xi) = f(\xi). \tag{7.1}$$

Next, given $h \in C^\infty(N)$ define $\widetilde{h} \in C^\infty(T^*N)$ by $\widetilde{h} = h \circ \pi$. The function $\widetilde{h}$ is referred to as the corresponding position function. The following Poisson bracket relations between the momentum and position functions are then obtained in Proposition 4.2.12 of [1]:

$$\{P(X), P(Y)\} = P([X, Y]) \tag{7.2}$$

$$\{\widetilde{h}, \widetilde{g}\} = 0 \tag{7.3}$$

and

$$\{\widetilde{h}, P(X)\} = \widetilde{X(h)}. \tag{7.4}$$

These bracket relations are the starting point for obtaining a quantum system from a classical system.

Remark 7.1. In [1] their first bracket relation actually reads $\{P(X), P(Y)\} = -P([X, Y])$. This is because their defining equation for a Hamiltonian vector field is $dh = X_h \lrcorner\, \omega$, as compared to our $dh = -X_h \lrcorner\, \omega$.

The goal of this subsection is to show how these concepts generalize to the multisymplectic phase space. The results we obtain are all new. As in Example 3.2, let N be a manifold and (M,ω) the multisymplectic phase space. That is, $M=\Lambda^k(T^*N)$ and $\omega=-d\theta$ is the canonical $(k+1)$-form on M. Let $\pi: M\to N$ denote the projection map. In Example 3.13 we showed that

$$f_l:\mathcal{P}_{\mathfrak{g},l}\to\Omega^{k-l}_{\mathrm{Ham}}(M)\qquad\qquad p\mapsto-\zeta(l+1)V_p\lrcorner\,\theta$$

was a weak homotopy moment map for the action on M induced from the action on N.

Definition 7.2. Given decomposable $X=X_1\wedge\cdots\wedge X_l$ in $\Gamma(\Lambda^l(TN))$ we define its momentum form $P(X)\in\Omega^{k-l}(M)$ by

$$P(X)(\mu_x)(Z_1,\ldots,Z_{k-l}):=-\zeta(l+1)\mu_x(X_1,\cdots,X_l,\pi_*Z_1,\cdots,\pi_*Z_{k-l}),$$

where μ_x is in M, and then extend by linearity to non-decomposables. Moreover, given $\alpha\in\Omega^{k-l}(N)$, we define the corresponding position form to be $\pi^*\alpha$, a $(k-l)$-form on M. Notice that in symplectic case $(k=1)$ these definitions coincide with the classical momentum and position functions.

Definition 7.3. Given a vector field $Y\in\Gamma(TN)$ with flow θ_t, the complete lift of Y is the vector field $Y^\sharp\in\Gamma(TM)$ whose flow is $(\theta_t^*)^{-1}$. For a decomposable multivector field $Y=Y_1\wedge\cdots\wedge Y_l\in\Gamma(\Lambda^l(TN))$ we define its complete lift $Y^\sharp$ to be $Y_1^\sharp\wedge\cdots\wedge Y_l^\sharp$ and then extend by linearity.

For $\xi\in\mathfrak{g}$, let V_ξ denote its infinitesimal generator on N and let $V_\xi^\sharp$ denote its infinitesimal generator on M. Similarly, for $p=\xi_1\wedge\cdots\wedge\xi_l$ in $\Lambda^l\mathfrak{g}$ let V_p denote $V_{\xi_1}\wedge\cdots\wedge V_{\xi_l}$ and $V_p^\sharp$ denote $V_{\xi_1}^\sharp\wedge\cdots\wedge V_{\xi_l}^\sharp$. Notice that by definition, we are not abusing notation by letting $V_p^\sharp$ denote both the complete lift of V_p and the infinitesimal generator of p under the induced action on M.

Lastly, note that by the equivariance of $\pi: M\to N$, we have

$$\pi_*(V_p^\sharp)=V_p\circ\pi.$$

We now examine the bracket relations between our momentum and position forms. We first rewrite the momentum form in a different way:

Proposition 7.4. *For $Y\in\Gamma(\Lambda^l(TN))$ we have that $P(Y)=-\zeta(l+1)Y^\sharp\lrcorner\,\theta$.*

Proof. Let $Y=Y_1\wedge\cdots\wedge Y_l$ be an arbitrary decomposable element of $\Gamma(\Lambda^l(TN))$. Let $Z_1,\cdots,Z_{k-l}$ be arbitrary vector fields on M. Fix $\mu_x\in M$. Then

$$\begin{aligned}(Y^\sharp \lrcorner\, \theta)_{\mu_x}(Z_1, \cdots, Z_{k-l}) &= \theta_{\mu_x}(Y_1^\sharp, \cdots, Y_l^\sharp, Z_1, \cdots, Z_{k-l}) \\ &= \mu_x(\pi_* Y_1^\sharp, \cdots, \pi_* Y_l^\sharp, \pi_* Z_1, \cdots, \pi_* Z_{k-l}) \\ &= \mu_x(Y_1, \cdots, Y_l, \pi_* Z_1, \cdots, \pi_* Z_{k-l}) \\ &= -\zeta(l+1) P(Y)_{\mu_x}(Z_1, \cdots, Z_{k-l}).\end{aligned}$$

□

As a corollary to the above proposition, we obtain a generalization of (7.1) to multisymplectic geometry:

Corollary 7.5. *For $p = \xi_1 \wedge \cdots \wedge \xi_l$ in $\mathcal{P}_{\mathfrak{g},l}$ we have*

$$P(V_p) = f_l(p).$$

Proof. This follows immediately from Proposition 7.4 since $V_p^\sharp$ is the infinitesimal generator of p on M. □

In the symplectic case, given $Y \in \Gamma(TN)$ the complete lift $Y^\sharp \in \Gamma(T(T^*N))$ preserves the tautological forms θ and ω. Hence $d(Y^\sharp \lrcorner\, \theta) = Y^\sharp \lrcorner\, \omega$, showing that each momentum function is Hamiltonian with Hamiltonian vector field the complete lift of the base vector field.

In the multisymplectic case, it is no longer true that $\mathcal{L}_{Y^\sharp}\theta = 0$ for a multivector field Y. Instead, we need to restrict our attention to multivector fields in the Lie kernel, which we defined in Definition 2.12. We quickly recall this definition and some terminology and notation introduced in [16].

Any degree l-multivector field is a sum of multivectors of the form $Y = Y_1 \wedge \cdots \wedge Y_l$. We consider the differential graded Lie algebra $(\Gamma(\Lambda^\bullet(TN)), \partial)$ where $\partial_l : \Gamma(\Lambda^l(TN)) \to \Gamma(\Lambda^{l-1}(TN))$ is given by

$$\partial_l(Y_1 \wedge \cdots \wedge Y_l) = \sum_{1 \le i \le j \le l} [Y_i, Y_j] \wedge Y_1 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_l.$$

As in [16], for a differential form τ, let

$$(\lrcorner\, \mathcal{L})_Y \tau = \sum_{i=1}^{l} Y_1 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge Y_l \lrcorner\, \mathcal{L}_{Y_i} \tau.$$

A more general version of Lemma 2.16 is given by Lemma 3.4 of [16]:

Lemma 7.6. For a differential form τ and $Y = Y_1 \wedge \cdots \wedge Y_l \in \Gamma(\Lambda^l(TN))$ we have that

$$Y \lrcorner\, d\tau - (-1)^l d(Y \lrcorner\, \tau) = (\lrcorner\, \mathcal{L})_Y \tau - \partial_l(Y) \lrcorner\, \tau.$$

Definition 7.7. As in Definition 2.12, we call $\mathcal{P}_l = \ker\, \partial_l$ the l-th Lie kernel.

Proposition 7.8. *For an l-multivector field in the Lie kernel, $Y \in \mathcal{P}_l$, we have that $P(Y)$ is in $\Omega^{k-l}_{\mathrm{Ham}}(M)$. More precisely, $\zeta(l)Y^\sharp$ is a Hamiltonian multivector field for $P(Y)$.*

Proof. Abusing notation, let ∂_l denote the differential on both $\Gamma(\Lambda^\bullet(TN))$ and $\Gamma(\Lambda^\bullet(TM))$. By definition, we have $\partial_l(Y) = 0$. It follows that $\partial_l(Y^\sharp) = 0$. Now, since the action on M preserves θ, we have that $(\lrcorner\, \mathcal{L})_{Y^\sharp}\theta = 0$. Thus, by Proposition 7.4 and Lemma 7.6, we have that

$$\begin{aligned}
d(P(Y)) &= -\zeta(l+1) d(Y^\sharp \lrcorner\, \theta) \\
&= -\zeta(l+1)(-1)^l (Y^\sharp \lrcorner\, d\theta) \\
&= \zeta(l+1)(-1)^l (Y^\sharp \lrcorner\, \omega) \\
&= -\zeta(l) Y^\sharp \lrcorner\, \omega,
\end{aligned}$$

where in the last equality we used Remark 3.7. $\square$

Remark 7.9. In the setup of classical Hamiltonian mechanics, the phase space of N is just T^*N, and so $k = l = 1$. Since $\mathcal{P}_1 = \Gamma(TN)$ we see that we are obtaining a generalization from Hamiltonian mechanics.

We now arrive at our generalization of equation (7.2):

Proposition 7.10. *For $Y_1 \in \mathcal{P}_s$ and $Y_2 \in \mathcal{P}_t$ we have that*

$$\{P(Y_1), P(Y_2)\} = -(-1)^{ts+s+t} P([Y_1, Y_2]) - \zeta(s+1)\zeta(t+1) d(Y_1^\sharp \lrcorner\, Y_2^\sharp \lrcorner\, \theta).$$

Proof. Using Proposition 7.8, Remark 3.7 and the definition of the bracket, we have

$$\begin{aligned}
\{P(Y_1), P(Y_2)\} &= (-1)^{t+1}\zeta(t)\zeta(s) Y_2^\sharp \lrcorner\, Y_1^\sharp \lrcorner\, \omega \\
&= (-1)^{ts+t}\zeta(s+t) Y_2^\sharp \lrcorner\, Y_1^\sharp \lrcorner\, \omega.
\end{aligned} \tag{7.5}$$

On the other hand, by Proposition 7.4 and Remark 3.7 we have

$$P([Y_1, Y_2]) = -\zeta(s+t)[Y_1^\sharp, Y_2^\sharp] \lrcorner\, \theta.$$

By Proposition 7.8 and Remark 3.7 we have that

$$d(Y_1^\sharp \lrcorner \theta) = (-1)^{s+1} Y_1^\sharp \lrcorner \omega$$

and

$$d(Y_2^\sharp \lrcorner \theta) = (-1)^{t+1} Y_2^\sharp \lrcorner \omega.$$

By equation (2.10) we have that

$$[Y_1^\sharp, Y_2^\sharp] \lrcorner \theta = -Y_2^\sharp \lrcorner (d(Y_1^\sharp \lrcorner \theta)) + (-1)^t d(Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \theta) - (-1)^{st+s} Y_1^\sharp \lrcorner Y_2^\sharp \lrcorner \omega - (-1)^{st+s+t} Y_1^\sharp \lrcorner (d(Y_2^\sharp \lrcorner \theta)),$$

and using the two equations above, this is equal to

$$(-1)^s (Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \omega) + (-1)^t d(Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \theta) - (-1)^{st+s} Y_1^\sharp \lrcorner Y_2^\sharp \lrcorner \omega - (-1)^{st+s+t} (-1)^{t+1} Y_1^\sharp \lrcorner Y_2^\sharp \lrcorner \omega.$$

Simplifying this equation we obtain that

$$[Y_1^\sharp, Y_2^\sharp] \lrcorner \theta = (-1)^s (Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \omega) + (-1)^t d(Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \theta).$$

Thus,

$$P([Y_1^\sharp, Y_2^\sharp]) = -\zeta(s+t)(-1)^s (Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \omega) - \zeta(s+t)(-1)^t d(Y_2^\sharp \lrcorner Y_1^\sharp \lrcorner \theta). \tag{7.6}$$

Equating equations (7.5) and (7.6) and using Remark 3.7 gives the result. □

To generalize (7.3) and (7.4) to the multisymplectic phase space, we need the following lemma:

Lemma 7.11. Let α be an arbitrary $(k-l)$-form on N and let $\pi^*\alpha$ be the corresponding classical position form in $\Omega^{k-l}(M)$. Then $\pi^*\alpha$ is Hamiltonian and $\pi_*(X_{\pi^*\alpha}) = 0$.

Proof. Let $q^1, \cdots, q^n$ denote coordinates on N, and let $\{p_{i_1 \cdots i_k}; 1 \leq i_1 < \cdots < i_k \leq n\}$ denote the induced fibre coordinates on M. In these coordinates we have that

$$\theta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} p_{i_1 \cdots i_k} dq^{i_1} \wedge \cdots \wedge dq^{i_k}$$

so that

$$\omega = -d\theta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} -dp_{i_1 \cdots i_k} \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_k}.$$

An arbitrary $(k-l)$-form α on N is given by

$$\alpha = \alpha_{i_1 \cdots i_{k-l}} dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.$$

Abusing notation, it follows that

$$\pi^*\alpha = \alpha_{i_1 \cdots i_{k-l}} dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.$$

Thus,

$$d\pi^*\alpha = \frac{\partial \alpha_{i_1 \cdots i_{k-l}}}{\partial q^j} dq^j \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.$$

An arbitrary l-vector field on M is of the form

$$X = a^{i_1 \cdots i_l} \frac{\partial}{\partial q^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial q^{i_l}} + a_{J_1}^{i_1 \cdots i_{l-1}} \frac{\partial}{\partial q^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial q^{i_{l-1}}} \wedge \frac{\partial}{\partial p^{J_1}} + \cdots + a_{J_1 \cdots J_l} \frac{\partial}{\partial p^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial p^{J_l}}.$$

Now, the multivector field $X_{\pi^*\alpha}$ we are looking for satisfies $X_{\pi^*\alpha} \lrcorner\, \omega = d\pi^*\alpha$. An exercise in combinatorics shows that there always exists an l-vector field X satisfying $X \lrcorner\, \omega = d\pi^*\alpha$, proving that $\pi^*\alpha$ is Hamiltonian. Note we can see directly from the equality $X \lrcorner\, \omega = d\pi^*\alpha$ that necessarily

$$a^{i_1 \cdots i_l} = 0.$$

Thus $\pi_*(X_{\pi^*\alpha}) = 0$ as desired.

□

Our generalization of (7.3) is:

Proposition 7.12. *For $\alpha \in \Omega^{k-i}(N)$ and $\beta \in \Omega^{k-j}(N)$ we have that*

$$\{\pi^*\alpha, \pi^*\beta\} = 0.$$

Proof. Let $Z_1, \cdots, Z_{k+1-i-j} \in \Gamma(TM)$ be arbitrary. Then,

$$\begin{aligned} \{\pi^*\alpha, \pi^*\beta\}(Z_1, \cdots, Z_{k+1-i-j}) &= (-1)^{j+1} X_{\pi^*\beta} \lrcorner\, X_{\pi^*\alpha} \lrcorner\, \omega(Z_1, \cdots, Z_{k+1-i-j}) \\ &= (-1)^j X_{\pi^*\beta} \lrcorner\, (\pi^* d\alpha)(Z_1, \cdots, Z_{k+1-i-j}) \\ &= (-1)^j \pi^* d\alpha(X_{\pi^*\beta}, Z_1, \cdots, Z_{k+1-i-j}) \\ &= (-1)^j d\alpha(0, \pi_* Z_1, \cdots, \pi_* X_{k+1-i-j}) && \text{by Lemma 7.11} \\ &= 0. \end{aligned}$$

$\square$

Our generalization of (7.4) is:

Proposition 7.13. *For $\alpha \in \Omega^{k-i}(N)$ and $Y \in \mathcal{P}_j$, we have that*

$$\{\pi^*\alpha, P(Y)\} = -\zeta(j)\pi^*(Y \lrcorner d\alpha).$$

Proof. Let $Z_1, \cdots, Z_{k+1-i-j} \in \Gamma(TM)$ be arbitrary. Then,

$$\begin{aligned}
\{\pi^*\alpha, P(Y)\}(Z_1, \cdots, Z_{k+1-i-j}) &= (-1)^{j+1} X_{P(Y)} \lrcorner X_{\pi^*\alpha} \lrcorner \omega(Z_1, \cdots, Z_{k+1-i-j}) \\
&= (-1)^{j+1}\zeta(j) Y^\sharp \lrcorner X_{\pi^*\alpha} \lrcorner \omega(Z_1, \cdots, Z_{k+1-i-j}) \quad \text{by Lemma 7.8} \\
&= (-1)^{j}\zeta(j+1) Y^\sharp \lrcorner \pi^* d\alpha(Z_1, \cdots, Z_{k+1-i-j}) \\
&= -\zeta(j) Y^\sharp \lrcorner \pi^* d\alpha(Z_1, \cdots, Z_{k+1-i-j}) \quad \text{by Remark 3.7} \\
&= -\zeta(j) d\alpha(\pi_* Y^\sharp, \pi_* Z_1, \cdots, \pi_* Z_{k+1-i-j}) \\
&= -\zeta(j) d\alpha(Y, \pi_* Z_1, \cdots, \pi_* Z_{k+1-i-j}) \\
&= -\zeta(j)\pi^*(Y \lrcorner d\alpha)(Z_1, \cdots, Z_{k+1-i-j}).
\end{aligned}$$

$\square$

7.2 Closed G_2 Structures

We first recall the standard G_2 structure on $\mathbb{R}^7$. More details for the material in this section can be found in [12]. Let $x^1, \cdots, x^7$ denote the standard coordinates on $\mathbb{R}^7$ and consider the 3-form φ_0 defined by

$$\varphi_0 = dx^{123} + dx^1(dx^{45} - dx^{67}) + dx^2(dx^{46} - dx^{75}) - dx^3(dx^{47} - dx^{56})$$

where we have omitted the wedge product signs. The stabilizer of this 3-form is given by the Lie group G_2. For an arbitrary 7-manifold we define a G_2 structure to be a 3-form φ which has around every point $p \in M$ local coordinates with $\varphi = \varphi_0$, at the point p.

The 3-form induces a unique metric g and volume form, vol, determined by the equation

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6g(X, Y)\text{vol}.$$

From the volume form we get the Hodge star operator and hence a 4-form $\psi := *\varphi$. We will refer to the data (M^7, φ, ψ, g) as a manifold with G_2 structure. We remark that the G_2 form φ is more than just non-degenerate:

Proposition 7.14. *The G_2 form φ is fully nondegenerate. This means that $\varphi(X, Y, \cdot)$ is non-zero whenever X and Y are linearly independent.*

Proof. See Theorem 2.2 of [16]. □

For the rest of this section we assume that $d\varphi = 0$. We now briefly recall some first order differential operators on a G_2 manifold, while refering the reader to section 4 of [14] for more details. Given $X \in \Gamma(TM)$ we will let $X^\flat$ denote the metric dual one form $X^\flat = X \lrcorner\, g$. Conversely, given $\alpha \in \Omega^1(M)$, let $\alpha^\sharp$ denote the metric dual vector field. Recall that given $f \in C^\infty(M)$ its gradient is defined by

$$\mathrm{grad}(f) = (df)^\sharp.$$

From the metric and the three form we can define the cross product of two vector fields. Given $X, Y, Z \in \Gamma(TM)$ the cross product $X \times Y$ is defined by the equation

$$\varphi(X, Y, Z) = g(X \times Y, Z).$$

Equivalently, the cross product is defined by

$$(X \times Y) = (Y \lrcorner\, X \lrcorner\, \varphi)^\sharp.$$

In coordinates, this says that

$$(X \times Y)^l = X^i Y^i \varphi_{ijk} g^{kl}. \tag{7.7}$$

The last differential operator we will consider is the curl of a vector field. We first need to recall the following decomposition of two forms on a G_2 manifold.

Proposition 7.15. *The space of 2-forms on a G_2 manifold has the G_2 irreducible decomposition*

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M),$$

where

$$\Omega^2_7(M) = \{X \lrcorner\, \varphi; X \in \Gamma(TM)\}$$

and

$$\Omega^2_{14}(M) = \{\alpha \in \Omega^2(M); \psi \wedge \alpha = 0\}.$$

The projection maps: $\pi_7 : \Omega^2(M) \to \Omega^2_7(M)$ *and* $\pi_{14} : \Omega^2(M) \to \Omega^2_{14}(M)$ *are given by*

$$\pi_7(\alpha) = \frac{\alpha - *(\varphi \wedge \alpha)}{3} \tag{7.8}$$

and

$$\pi_{14}(\alpha) = \frac{2\alpha + *(\varphi \wedge \alpha)}{3}. \tag{7.9}$$

Proof. See Section 2.2 of [13]. □

We can now define the curl of a vector field. Given $X \in \Gamma(TM)$ its curl is defined by

$$(\mathrm{curl}(X))^\flat = *(dX^\flat \wedge \psi). \tag{7.10}$$

This is equivalent to saying that

$$\pi_7(dX^\flat) = \mathrm{curl}(X) \lrcorner \varphi. \tag{7.11}$$

In coordinates,

$$\mathrm{curl}(X)^l = (\nabla_a X_b) g^{ai} g^{bj} \varphi_{ijk} g^{kl}, \tag{7.12}$$

where ∇ is the Levi-Civita connection corresponding to g. This is reminiscent of the fact that in $\mathbb{R}^3$ the curl is given by the cross product of ∇ with X. Again, we refer the reader to Section 4.1 of [14] for more details.

We now translate our definition of Hamiltonian forms and vector fields into the language of G_2 geometry. By definition, we see that a 1-form is Hamiltonian if and only if its differential is in $\Omega^2_7(M)$. That is,

$$\Omega^1_{\mathrm{Ham}}(M) = \{\alpha \in \Omega^1(M); \pi_{14}(d\alpha) = 0\}.$$

Similarly,

$$\mathfrak{X}^1_{\mathrm{Ham}}(M) = \{X \in \Gamma(TM); X = \mathrm{curl}(\alpha^\sharp) \text{ and } \pi_{14}(d\alpha) = 0 \text{ for some } \alpha \in \Omega^1(M)\}.$$

Note that if M is compact, then it follows from (7.8) and Hodge theory that there are no non-zero Hamiltonian 1-forms.

Proposition 7.16. *If α is a Hamiltonian* 1*-form then its corresponding Hamiltonian vector field is* $\mathrm{curl}(\alpha^\sharp)$.

Proof. Since a Hamiltonian 1-form satisfies $\pi_{14}(\alpha) = 0$, this follows immediately from equation (7.11). ☐

From Proposition 7.16 and equation (7.7) we see that the generalized Poisson bracket is given by the cross product:

$$\{\alpha, \beta\} = \mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp), \tag{7.13}$$

for $\alpha, \beta \in \Omega^1_{\mathrm{Ham}}(M)$.

Proposition 5.5 showed that a Hamiltonian vector field preserves the n-plectic form. In the language of G_2 geometry this gives:

Proposition 7.17. *Given $\alpha \in \Omega^1(M)$ with $\pi_{14}(d\alpha) = 0$, the curl of $\alpha^\sharp$ preserves the G_2 structure. That is,*

$$\mathcal{L}_{\mathrm{curl}(\alpha^\sharp)}\varphi = 0.$$

Proof. This follows immediately from Propositions 7.16 and 5.5. ☐

As a consequence of Proposition 5.8, we get the following:

Proposition 7.18. *Let α and β be in $\Omega^1(M)$ with $\pi_{14}(d\alpha) = 0 = \pi_{14}(d\beta)$. Then*

$$\pi_{14}(\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)) = 0.$$

Moreover,

$$\mathrm{curl}(\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)) = [\mathrm{curl}(\alpha^\sharp), \mathrm{curl}(\beta^\sharp)].$$

Proof. By equation (7.13) and Lemma 5.8 we see that $d(\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)) = [X_\alpha, X_\beta] \lrcorner\, \omega$. Thus, $\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)$ is in $\Omega^2_7(M)$, showing that $\pi_{14}(\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)) = 0$. Moreover, we have that

$$\begin{aligned}
\mathrm{curl}(\mathrm{curl}(\alpha^\sharp) \times \mathrm{curl}(\beta^\sharp)) \lrcorner\, \varphi &= \mathrm{curl}(\{\alpha, \beta\}) \lrcorner\, \varphi && \text{by (7.13)} \\
&= d(\{\alpha, \beta\}) \\
&= [X_\alpha, X_\beta] \lrcorner\, \varphi && \text{by Lemma 5.8}
\end{aligned}$$

$$= [\mathrm{curl}(\alpha^\sharp), \mathrm{curl}(\beta^\sharp)] \lrcorner \varphi \qquad \text{by Proposition 7.16.}$$

The proposition now follows since φ is non-degenerate. □

We now consider the definition of a homotopy moment map in the setting of a G_2 manifold. The equations defining the components of a homotopy moment map, i.e. (3.2), reduce to finding functions $f_1 : \mathfrak{g} \to \Omega^1(M)$ and $f_2 : \mathcal{P}_{\mathfrak{g},2} \to C^\infty(M)$ satisfying

$$\pi_{14}(d(f_1(\xi))) = 0 \text{ and } \mathrm{curl}((f_1(\xi))^\sharp) = V_\xi. \tag{7.14}$$

$$V_\xi \times V_\eta = -(d(f_2(\xi \wedge \eta)))^\flat. \tag{7.15}$$

We finish this section by computing a homotopy moment map in the following set up, extending Example 6.7 of [15].

Consider $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ with standard 3-form given by

$$\varphi = \frac{1}{2}(dz_1 \wedge dz_2 \wedge dz_3 + d\overline{z}_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3) - \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_2 \wedge d\overline{z}_3) \wedge dt.$$

In terms of $t, x_1, x_2, x_3, y_1, y_2, y_3$ this is

$$\varphi = dx_1dx_2dx_3 - dx_1dy_2dy_3 - dy_1dx_2dy_3 - dy_1dy_2dx_3 - dtdx_1dy_1 - dtdx_2dy_2 - dtdx_3dy_3,$$

where we have omitted the wedge signs. Equivalently,

$$\varphi = \mathrm{Re}(\Omega_3) - dt \wedge \omega_3$$

where Ω_3 is the standard holomorphic volume and ω_3 is the standard Kahler form on $\mathbb{C}^3$. That is, $\Omega_3 = dz_1 \wedge dz_2 \wedge dz_3$ and $\omega_3 = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3)$.

As in Examples 3.15 and 3.16 we consider the standard action by the diagonal maximal torus $T^2 \subset SU(3)$ given by $(e^{i\theta}, e^{i\eta}) \cdot (t, z_1, z_2, z_3) = (t, e^{i\theta}z_1, e^{i\eta}z_2, e^{-i(\theta+\eta)}z_3)$. We have $\mathfrak{t}^2 = \mathbb{R}^2$ and that the infinitesimal generators of $(1,0)$ and $(0,1)$ are

$$A = \frac{i}{2}\left(z_1\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_3} - \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + \overline{z}_3\frac{\partial}{\partial \overline{z}_3}\right)$$

and

$$B = \frac{i}{2}\left(z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - \overline{z}_2\frac{\partial}{\partial \overline{z}_2} + \overline{z}_3\frac{\partial}{\partial \overline{z}_3}\right)$$

respectively.

By Example 3.15 it follows that

$$\begin{aligned}
A \lrcorner \varphi &= A \lrcorner (\Omega_3 - dt \wedge \omega_3) \\
&= \frac{1}{2} d(\mathrm{Im} z_1 z_3 dz_2) - \frac{1}{4} dt \wedge d(|z_1|^2 - |z_3|^2) \\
&= \frac{1}{2} d\left(\mathrm{Im}(z_1 z_3 dz_2) - \frac{1}{2}(|z_1|^2 - |z_3|^2) dt \right).
\end{aligned}$$

Similarly,

$$B \lrcorner \varphi = \frac{1}{2} d\left(\mathrm{Im}(z_1 z_2 dz_3) - \frac{1}{2}(|z_1|^2 - |z_2|^2) dt \right).$$

It follows that

$$f_1((1,0)) = \frac{1}{2}\mathrm{Im}(z_1 z_3 dz_2) - \frac{1}{4}(|z_1|^2 - |z_3|^2) dt$$

and

$$f_1((0,1)) = \frac{1}{2}\mathrm{Im}(z_1 z_2 dz_3) - \frac{1}{4}(|z_1|^2 - |z_2|^2) dt$$

give the first component of a homotopy moment map. Plugging in $f_1((1,0))$ and $f_1((0,1))$ into (7.9) shows that

$$\pi_{14}(f_1((1,0))) = 0 = \pi_{14}(f_1((0,1))).$$

Moreover, using (7.12) one can directly verify that

$$\mathrm{curl}(f_1((1,0)))^\sharp = A$$

and

$$\mathrm{curl}(f_1((0,1)))^\sharp = B,$$

confirming (7.14).

Using Example 3.16 it follows that

$$\begin{aligned}
B \lrcorner A \lrcorner \varphi &= B \lrcorner A \lrcorner (\Omega_3 - dt \wedge \omega_3) \\
&= B \lrcorner A \lrcorner \Omega_3 \\
&= \frac{1}{4} d(\mathrm{Re}(z_1 z_2 z_3)).
\end{aligned}$$

Thus the second component of the homotopy moment map is given by

$$f_2(A \wedge B) = \frac{1}{4}\mathrm{Re}(z_1 z_2 z_3),$$

in accordance with Example 6.7 of [15]. Written out in the coordinates t, x_1, x_2, x_3, y_1, y_2, y_3, the infinitesimal vector fields coming from the torus action are

$$A = \frac{1}{2}\left(-y_1\frac{\partial}{\partial x_1} + y_3\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial y_1} - x_3\frac{\partial}{\partial y_3}\right),$$

$$B = \frac{1}{2}\left(-y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial y_2} - x_3\frac{\partial}{\partial y_3}\right).$$

Using the metric to identify 1-forms and vector fields, equation (7.7) gives the cross product of A and B to be

$$\begin{aligned} 4(A \times B) &= (y_2y_3 - x_2x_3)\frac{\partial}{\partial x_1} + (y_1y_3 - x_1x_3)\frac{\partial}{\partial x_2} + (y_1y_2 - x_1x_2)\frac{\partial}{\partial x_3} + \\ &\quad + (x_2y_3 + x_3y_2)\frac{\partial}{\partial y_1} + (x_3y_1 + x_1y_3)\frac{\partial}{\partial y_2} + (x_1y_2 + x_2y_1)\frac{\partial}{\partial y_3} \\ &= d(x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3) \\ &= d(\mathrm{Re}(z_1z_2z_3)) \end{aligned}$$

confirming equation (7.15). We thus have extended Example 6.7 of [15] by obtaining a full homotopy moment map for the diagonal torus action on $\mathbb{R}^7$ with the standard torsion-free G_2 structure.

8 Concluding Remarks

This work poses many natural questions for future research. The following are just a few ideas:

1. In our work, we provided a few examples of multi-Hamiltonian systems. What are some examples of other physical or interesting multi-Hamiltonian systems to which this work could be applied?

2. In Section 6.1 we generalized the classical momentum and position functions on the phase space of a manifold to momentum and position forms on the multisymplectic phase space. Since, as discussed in [1], the classical momentum and position functions play an important role in connecting classical and quantum mechanics, a natural question is if there is an analogous application of our more general theory to quantum mechanics?

3. In symplectic geometry, Proposition 4.13 shows that a moment map $f : \mathfrak{g} \to C^\infty(M)$ induces a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to the quotient space $(C^\infty(M)/\text{closed}, \{\cdot, \cdot\})$, and if f is equivariant then it is a Lie algebra morphism from $(\mathfrak{g}, [\cdot, \cdot])$ to $(C^\infty(M), \{\cdot, \cdot\})$. Moreover, Proposition 4.10 showed that both $\Omega^\bullet_{\text{Ham}}(M)/\text{closed}$ and $\Omega^\bullet_{\text{Ham}}(M)/\text{exact}$ are graded Lie algebras while Proposition 5.9 showed that a weak homotopy moment map is always a graded Lie algebra morphism from $\mathcal{P}_\mathfrak{g}$ to $\Omega^\bullet_{\text{Ham}}(M)/\text{closed}$.

   Hence, a natural question is:

   If (f) is an equivariant weak moment map, does it induce a graded Lie algebra morphism from $(\mathcal{P}_\mathfrak{g}, [\cdot, \cdot])$ to $(\Omega^\bullet_{\text{Ham}}(M)/\text{exact}, \{\cdot, \cdot\})$? Does the converse hold?

4. In our work, we provided a couple of examples of n-plectic group actions to which our theory of the existence and uniqueness of moment maps could be applied. There are many other interesting n-plectic geometries; see for example [3], [9] and [20]. What does the work done in this thesis say about the existence and uniqueness of moment maps in these setups?

5. Given a weak moment map (f) with $f_k : \mathcal{P}_{\mathfrak{g},k} \to \Omega^{n-k}_{\text{Ham}}(M)$, does there exists a full homotopy moment map (h) whose restriction to the Lie kernel is (f)? Moreover, in [7] an interpretation of homotopy moment maps was given in terms of equivariant cohomology. Is there an analogous interpretation for weak moment maps?

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