MODULI SPACE AND DEFORMATIONS OF SPECIAL LAGRANGIAN SUBMANIFOLDS WITH EDGE SINGULARITIES

(Thesis format: Monograph)

by

Josue Rosario-Ortega

Graduate Program in Mathematics

A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

The School of Graduate and Postdoctoral Studies
The University of Western Ontario
London, Ontario, Canada

Abstract

Special Lagrangian submanifolds are submanifolds of a Calabi-Yau manifold calibrated by the real part of the holomorphic volume form. In this thesis we use elliptic theory for edge-degenerate differential operators on singular manifolds to study general deformations of special Lagrangian submanifolds with edge singularities. We obtain a general theorem describing the local structure of the moduli space. When the obstruction space vanishes the moduli space is a smooth, finite dimensional manifold.

Keywords: singular manifolds, special Lagrangian submanifolds, edge-degenerate differential operators, boundary value problems, moduli spaces.

To all the people who have supported me during my life

Acknowledgments

There are a lot of people I have to thank for their support, help and contribution to make possible for me to reach this point. I want to thank my parents Nabor and Concepción for all their love and support during these years. My father has always had the strong conviction that the best inheritance he can give me is an education and he has worked tirelessly to make sure I got it. Many thanks for that.

Thanks to my brother and sister for all their love and support. Thanks to my lifelong friends Ayocuan Topitlzin, Israel Cueto and Mauro Garcia for their support and friendship.

Mathematically speaking there are a lot o people to thank too. I will do it chronologically. My real appreciation and enjoyment of mathematics started very late in my life when I was around 20 years old. I want to thank Prof. Tiburcio Fernández whose lectures on fluid dynamics changed completely my way of thinking about mathematics and the direction of my life.

I want to thank all the people at the Universidad Autonoma Metropolitana campus Iztapalapa. I spent 4 wonderful years there learning and thinking about mathematics. Thanks to Prof. Ernesto Lacomba (deceased), Prof. Antoni Wawrzyñczyk and Prof. Lourdes Palacios for the support they gave me to undertake graduate studies abroad.

During my years at the Universidad Autonoma Metropolitana I made long lasting friends Jhonatan Castro and Adrian "Dinobot". Their friendship has always been there to support me during all these years in Canada. Thank you my friends.

A special thank is for Prof. Nikolai Vasilevski at CINVESTAV for the yearly seminar "Análisis: Norte-Sur" that he organizes every Fall. Every year I used to wait eagerly to attend this seminar from which I obtained a lot of mathematical inspiration that encouraged me to study abroad.

I have relied on countless of friends in the math department at UWO during the last four years. Many thanks to Chandra Rajamani for many mathematical and non-mathematical conversations, Ali Fathi, Asghar Ghorbanpour, Javad Rastegari, Marina Palasti, Masoud Ataei, Brett Bridges, Dinesh Valluri, Mitsuru Wilson, Octavian Mitrea, Nick Meadows, Nadia Alluhaibi and Ahmed Ashraf. Thanks to my office mate Shahab Azarfar for all the conversations, friendship and companion during this years of extremely hard work.

Undoubtedly I am in debt with Chris Whalen. He has shown me the kindness, generosity and how big-hearted the Canadian people are. Definitely he is the best landlord in the world.

I think there are still some friends missing in this list but you can be sure that I appreciate your friendship even if your name is not explicitly written here.

I am infinitely grateful to Prof. Tatyana Barron and Prof. Spiro Karigiannis. Thanks to Prof. Tatyana Barron for her infinitely patience and support during the last 4 years. Thanks for your understanding during the process of finding myself mathematically speaking. Thanks to my co-supervisor Prof. Spiro Karigiannis for his support during the last 3 years, for having suggested the research topic that inspired this thesis and for all the mathematical conversations we have had. Thanks to all the organizing committee of the Geometric Analysis Colloquium at Fields Institute where this work started with a series of informal conversations with Prof. Spiro Karigiannis.

Thanks to my supervisory committee Prof. Masoud Khalkhali, Prof. Martin Pinsonnault and Prof. Xingfu Zou for reading this thesis and their comments. Thanks to Prof. Jason Lotay for accepting being my external examiner and all his comments and feedback to improve this thesis.

Thanks to Prof. McKenzie Wang at McMaster University for the invitation to give a talk in the geometry and topology seminar. Thanks to Prof. Frédéric Rochon at UQAM for the conversations we had and suggestions he made during my visit.

I want to thank the Department of Mathematics at UWO for the financial support, the excellent library services and the opportunity to perform graduate studies at this wonderful university.

Last but not least, I want to say thank you to my lovely wife Elena. Thanks for your understanding and patient during the last five years. I know that it has not been easy for you at all. Thanks for your support despite of countless of hours I had to spend at the office and absent from home.

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Introduction

The study of moduli spaces of deformations of a special Lagrangian submanifold in a Calabi-Yau manifold started with the work of McLean [McL98], where he studied the deformation of compact special Lagrangian submanifolds (without boundary). He proved that the moduli space is a finite dimensional, smooth manifold with dimension equal to the dimension of its space of harmonic 1-forms. The role of special Lagrangian fibrations of Calabi-Yau manifolds in mirror symmetry and especially the presence of singular fibers, motivated the study of special Lagrangian submanifolds with conical singularities/ends [Mar, Pac04, Pac13, Joy04b]. Moreover in the simplest example of a Calabi-Yau manifold, $\mathbb{C}^n$, the fact that special Lagrangian submanifolds are minimal implies the nonexistence of compact special Lagrangian submanifolds. Hence the search for special Lagrangian submanifolds in $\mathbb{C}^n$ must be done in the category of non-compact or singular spaces.

Broadly speaking, the study of moduli spaces of special Lagrangian deformations is performed by identifying nearby special Lagrangian submanifolds with elements in the zero set of a non-linear elliptic partial differential operator that governs the deformations. By means of the Implicit Function theorem for Banach spaces, this is reduced to the analysis of the linearised equation. Hence the study of moduli spaces of deformations of a special Lagrangian submanifold requires a good understanding of elliptic equations on the base space.

The theory of linear elliptic partial differential equations on smooth, compact manifolds without boundary is well-developed: the construction of parametrices of inverse order, a complete calculus of elliptic ΨDOs, elliptic regularity, the equivalence between ellipticity and the existence of an a priori estimate, the equivalence of ellipticity and Fredholmness and finally the celebrated Atiyah-Singer index formula. All these elements are already classical tools when studying elliptic equations on compact manifolds.

In contrast, in non-compact or singular spaces there is no canonical approach or methods to study elliptic equations. Even the concept of ellipticity on a non-compact or singular manifold is not canonical as it is in the compact case. The basic model of singularity in the theory of PDEs on singular manifolds is the conical singularity. Near a vertex, a manifold with conical singularities looks like $\frac{\overline{\mathbb{R}}^+ \times \mathcal{X}}{\{0\} \times \mathcal{X}}$, where $\mathcal{X}$ is a compact manifold without boundary. The usual approach in this case is to blow-up the vertices to obtain a compact manifold with boundary, the stretched manifold, with collar neighborhood $\overline{\mathbb{R}}^+ \times \mathcal{X}$.

The most common type of degenerate differential operator studied on the collar neighborhood is the Fuchs type operator

$$\mathrm{P} = r^{-m} \sum_{j \leq m} a_j(r) \, (-r\partial_r)^j$$

with coefficients $a_j \in C^\infty(\overline{\mathbb{R}}^+, \mathrm{Diff}^{m-j}(\mathcal{X}))$, where $\mathrm{Diff}^{m-j}(\mathcal{X})$ is the set of classical differential operators of order $m-j$ on the compact manifold $\mathcal{X}$.

This class of differential operators arises naturally when changing into polar coordinates a differential operator on $\mathbb{R}^n$. In particular the Laplace-Beltrami operator defined by a conical metric $r^2 g_{\mathcal{X}} + dr^2$ is a Fuchs type operator. Many authors have studied this type of equation with different approaches: Lockhart and McOwen [LM85, Loc87], Melrose [Mel93], Schulze [Sch91, ES97, Sch98], Kozlov, Mazya and Rossmann [KMR97], among possibly others. The b-calculus of Melrose [Mel93] and the cone algebra of Schulze [Sch91] are robust and systematic approaches to Fuchs operators with the goal of constructing a calculus or an algebra of ΨDOs that contains parametrices of Fusch type operators (see [LS01] for a comparison of both approaches).

Higher order singularities arise by means of conification or edgification of a manifold with conical singularities. The conification of a manifold with conical singularities produces a manifold with corners, where locally the singularities look like $\mathbb{R}^+ \times \left(\frac{\overline{\mathbb{R}}^+ \times \mathcal{X}}{\{0\} \times \mathcal{X}}\right)$. Similarly to the conical singularity case, there is a natural class of degenerate corner differential operators associated with this type of singularity. An example of such an operator is the Laplace-Beltrami operator associated to a corner metric $t^2(r^2 g_{\mathcal{X}} + dr^2) + dt^2$.

On the other hand, the edgification of a manifold with conical singularities produces a manifold with edge singularities, where locally near the singularity it looks like $\mathbb{R}^n \times \left(\frac{\overline{\mathbb{R}}^+ \times \mathcal{X}}{\{0\} \times \mathcal{X}}\right)$. Here we also have a class of edge-degenerate differential operators. A typical example is the Laplace-Beltrami operator associated to an edge metric $r^2 g_{\mathcal{X}} + dr^2 + g_{\mathcal{E}}$ where $g_{\mathcal{E}}$ is a Riemannian metric on a smooth manifold $\mathcal{E}$ (the edge) without boundary (see section 2.2.1).

This thesis is concerned with deformations of special Lagrangian submanifolds with edge singularity (see section 2.2.1 for the precise definition of a manifold with edge singularity). The motivations to study special Lagrangian submanifolds with edge singularities are the following: it is a natural next step in the category of singularities where there is a well-developed elliptic theory [Sch91, ES97, Sch98], hence the analysis of the linearised equation that governs the deformations is accessible. An alternative approach to study edge-degenerate operators developed by R. Mazzeo and B. Vertman can be found in [Maz91], [MV14].

On the other hand we are interested in the deformation of calibrated vector bundles, specially, special Lagrangian submanifolds obtained as a conormal bundle $\mathcal{N}^*(M) \subset T^*(\mathbb{R}^n) \cong \mathbb{C}^n$ of an austere submanifold M in $\mathbb{R}^n$. In this direction, Karigiannis and Leung [KL12] obtained special Lagrangian deformations of $\mathcal{N}^*(M)$ by affinely translating the fibers, see section 2.4. One of the main examples of austere submanifolds in $\mathbb{R}^n$ is the class of austere cones [Bry91]. These cones are of the form $\mathbb{R}^+ \times \mathcal{X} \subset \mathbb{R}^n$ where $\mathcal{X} \subset S^{n-1}$ is an austere submanifold of the sphere. If we assume that the conormal bundle is a trivial bundle near the vertex of the cone (for example if $\mathbb{R}^+ \times \mathcal{X}$ is an orientable hypersurface in $\mathbb{R}^n$ then the conormal bundle is trivial) then it is diffeomorphic to $\mathbb{R}^+ \times \mathcal{X} \times \mathbb{R}^q$ where q is the codimension of $\mathbb{R}^+ \times \mathcal{X}$ in $\mathbb{R}^n$. This implies that we can consider $\mathcal{N}^*(M)$ as a manifold with an edge singularity. Here we have a non-compact edge $\mathbb{R}^q$. This will restrict us in the type of deformations that we study. When the edge is a compact manifold $\mathcal{E}$ more complete results are obtained.

In [Sch91, ES97, Sch98], B.-W. Schulze and coauthors have developed a comprehensive elliptic theory of edge-degenerate differential operators:

$$\mathrm{P} = r^{-m} \sum_{j+|\alpha|\leq m} a_{j\alpha}(r,y)\,(-r\partial_r)^j\,(rD_s)^\alpha\,.$$

with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}^+ \times \Omega, , \mathrm{Diff}^{m-(j+|\alpha|)}(\mathcal{X}))$.

In this thesis we use Schulze's approach to study deformation of special Lagrangian submanifolds with edge singularities. We use Schulze's approach to analyze the Hodge-Laplace Δ and Hodge-deRham $d+d^*$ operators acting on sections of differential forms induced by edge-degenerate vector fields on M.

In previous works of Joyce, McLean, Marshall and Pacini [Joy04b],[McL98], [Mar], [Pac04],[Pac13], finite dimension of the moduli space follows from the Fredholmness of the Hodge-Laplace operator acting on (weighted) Sobolev spaces. As was mentioned above, there is no canonical notion of ellipticity in non-compact or singular manifolds, however in most approaches, once suitable Banach spaces have been defined, the ellipticity of an operator is defined in such a way that it implies the Fredholm property of the operator acting between those Banach spaces. In manifolds with conical singularities with local model $\mathbb{R}^+ \times \mathcal{X}$, the concept of ellipticity is based on the symbolic structure of the Fuchs operator. This is given by two symbols $(\sigma_b^m(\mathrm{P}), \sigma_M^m(\mathrm{P})(z))$. The first symbol $\sigma_b^m(\mathrm{P})$ is the homogeneous boundary principal symbol and $\sigma_M^m(\mathrm{P})(z)$ is the Mellin conormal symbol. The symbol $\sigma_M^m(\mathrm{P})(z)$ is an operator-valued symbol given by a holomorphic family of continuous operators parametrized by $z \in \mathbb{C}$ and acting on the base of the cone $\mathcal{X}$, $\sigma_M^m(\mathrm{P})(z) : H^s(\mathcal{X}) \to H^{s-m}(\mathcal{X})$. The ellipticity of P on $\mathcal{X}^\wedge := \mathbb{R}^+ \times \mathcal{X}$ implies that $\sigma_M^m(\mathrm{P})(z)$ is a family of isomorphisms for all $z \in \Gamma_{\frac{n+1}{2}-\gamma} = \{z \in \mathbb{C} : \mathrm{Re}(z) = \frac{n+1}{2} - \gamma\}$ for some weight $\gamma \in \mathbb{R}$. In the approaches of Melrose and Lockhart-McOwen similar symbolic structures are used to define ellipticity. See section 2.3.2 for a complete discussion of the symbolic structure.

In the analysis of edge-degenerate operators, the symbolic structure for an adequate notion of ellipticity involves the edge symbol $\sigma_\wedge^m(u,\eta)$. This is an operator-valued symbol given by a family of continuous operators acting on cone-Sobolev spaces $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ (see definition 13) and parametrized by the cotangent bundle of the edge. For $(u,\eta) \in T^*\mathcal{E} \setminus 0$ we have a continuous operator

$$\sigma_\wedge^m(u,\eta) : \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(\mathcal{X}^\wedge).$$

Analogous to the conical case, a necessary condition for the ellipticity of P is that $\sigma_\wedge^m(u,\eta)$ is an isomorphism for every $(u,\eta) \in T^*\mathcal{E} \setminus \{0\}$. However, this is rarely the case (for example, in general, the Laplace-Beltrami operator induced by an edge metric does not satisfy this condition). It is more natural to expect the family $\sigma_\wedge^m(u,\eta)$ to be only Fredholm for every $(u,\eta) \in T^*\mathcal{E} \setminus 0$.

In this case, in order to have a family of isomorphisms, we need to complete the edge symbol with boundary and coboundary conditions. This completion is achieved by adding trace and potential operators and an operator acting on the edge that represent a reduction of the elliptic problem to the boundary:

$$\begin{pmatrix} \sigma_\wedge^m(s,\eta) & \mathrm{k}(s,\eta) \\ \mathrm{t}(s,\eta) & \mathrm{r}(s,\eta) \end{pmatrix},$$

see section 3.2.3 for further details.

The need for completing the symbols means that P is not Fredholm unless we impose complementary edge boundary conditions. Moreover boundary and coboundary conditions are an essential part of the regularity of solutions of elliptic edge-degenerate equations (see section 2.3.5). Therefore, if we are interested in studying moduli spaces of deformations of a special Lagrangian submanifold with edge singularities, we need to consider deformations with boundary conditions in the edge in order to obtain regular enough deformations that allow the existence of a smooth, finite dimensional moduli space of deformations. These boundary conditions are given by the trace pseudo-differential operator that appears in the completion of the symbol. Moreover solutions of elliptic equations near singularities have a well-known conormal asymptotic expansion. This occurs even in the simplest case when solving an elliptic equation in $\mathbb{R}^+$. Our case is not an exception and our deformations have conormal asymptotic expansions near the edge, see section 2.3.3.

Once the symbol is completed (this is possible because, in our case, the topological obstruction vanishes, see 3.2.3) we obtain a Fredholm operator in the edge algebra with a parametrix (with asymptotics) of inverse order. At this point we want to use the Implicit Function theorem for Banach spaces to obtain finite dimensionality and smoothness of the moduli space of deformations. However the possible non-surjectivity of the linearised deformation map produces an obstruction space. The presence of an obstruction space is not unexpected because even in the case of a compact manifold with isolated conical singularities each of the singular cones contributes to the obstruction space. This was studied in detail by Joyce [Joy04b].

Given a special Lagrangian submanifold in $\mathbb{C}^n$ with edge singularity, $\Phi : M \longrightarrow \mathbb{C}^n$, our moduli space has as parameters an admissible weight $\gamma > \frac{\dim \mathcal{X}+3}{2}$ and a trace pseudo-differential operator, $\mathcal{T}$, such that it belongs to a set of boundary condition for an elliptic edge boundary value problem for the Hodge-deRham operator on M.

Our main result, theorem 5.2, is a theorem describing the local structure of the moduli space $\mathfrak{M}(M,\Phi,\mathcal{T},\gamma)$ considering the possible obstructions (see chapter 4 for the precise definition of the moduli space and further details).

Theorem 0.1. Locally near M the moduli space $\mathfrak{M}(M,\Phi,\mathcal{T},\gamma)$ is homeomorphic to the zero set of a smooth map $\mathfrak{G}$ between smooth manifolds $\mathcal{M}_1$, $\mathcal{M}_2$ given as neighborhoods of zero in finite dimensional Banach spaces. The map $\mathfrak{G} : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfies $\mathfrak{G}(0) = 0$ and $\mathfrak{M}(M,\Phi,\mathcal{T},\gamma)$ near M is a smooth manifold of finite dimension when $\mathfrak{G}$ is the zero map.

Chapter 1

Special Lagrangian submanifolds and calibrations

1.1 Special Lagrangian submanifolds

In this section we explain the basics of special Lagrangian submanifolds both in $\mathbb{C}^n$ and in general Calabi-Yau manifolds. This section is based in material from [Joy07] and [HL82].

1.1.1 Special Lagrangian submanifolds in $\mathbb{C}^n$

Let $\mathbb{C}^n := \{(z_1, \cdots, z_n) : z_k \in \mathbb{C} \text{ for all } 1 \leq k \leq n\}$ be the complex n-dimensional space. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n} = \mathbb{R}^n_x \oplus \mathbb{R}^n_y$ in the following, specific way

$$(x_1, \cdots, x_n, y_1, \cdots y_n) \longrightarrow (x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n), \tag{1.1}$$

hence with this identification $z_k = x_k + \sqrt{-1}y_k$. Now let's consider the automorphism $J : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ given by $J(z) = \sqrt{-1}z$. Then, under the identification $\mathbb{C}^n \cong \mathbb{R}^n_x \oplus \mathbb{R}^n_y$ we have

$$J = \begin{bmatrix} 0 & -Id_{\mathbb{R}^n} \\ Id_{\mathbb{R}^n} & 0 \end{bmatrix} : \mathbb{R}^n_x \oplus \mathbb{R}^n_y \longrightarrow \mathbb{R}^n_x \oplus \mathbb{R}^n_y. \tag{1.2}$$

Definition 1. Let $\xi = \xi_1 \wedge \cdots \wedge \xi_n$ be an oriented, real n-plane in $\mathbb{C}^n$, where $\xi_1, \cdots, \xi_n$ is an oriented, orthonormal basis of ξ. We say that ξ is a Lagrangian n-plane if $J(\xi) = \xi^\perp$ where

$$\xi^\perp = \{\eta \in \mathbb{C}^n : \langle \eta, v \rangle_{g_{\mathbb{R}^{2n}}} = 0 \text{ for all } v \in \xi\}. \tag{1.3}$$

Observe that in (1.3) the inner product is taken in $\mathbb{R}^{2n}$ (the standard flat Riemannian metric) under the identification (1.1).

Remark 1.1. We can also describe Lagrangian planes in terms of the vanishing of the Kähler form

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \tag{1.4}$$

i.e. ξ is Lagrangian if and only if $\omega|_\xi = 0$. This follows immediately from the fact that $\langle Ju, v \rangle_{g_{\mathbb{R}^{2n}}} = \omega_{g_{\mathbb{C}^n}}(u, v)$ for every $u, v \in \mathbb{C}^n$.

Now, let's consider the standard volume form in $\mathbb{R}^{2n} = \mathbb{R}^n_x \oplus \mathbb{R}^n_y$ given by

$$dV_{\mathbb{R}^{2n}} = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots dy_n, \tag{1.5}$$

then under the identification (1.1) we can rewrite it as

$$dV_{\mathbb{R}^{2n}} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2} \right)^n \Omega \wedge \overline{\Omega} \tag{1.6}$$

where

$$\Omega = dz_1 \wedge \cdots \wedge dz_n. \tag{1.7}$$

Definition 2. The complex n-form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ is called the holomorphic volume form of $\mathbb{C}^n$.

Definition 3. An oriented n-submanifold $\psi : M \longrightarrow \mathbb{C}^n$ is a Lagrangian submanifold of $\mathbb{C}^n$ if each tangent plane $\psi_*(T_pM) \subset T_{\psi(p)}\mathbb{C}^n \cong \mathbb{C}^n$ is a Lagrangian n-plane in $\mathbb{C}^n$ for every $p \in M$, where ψ_* denotes the push-forward.

Let M be a Lagrangian submanifold in $\mathbb{C}^n$ and $p \in M$. Let's take an oriented, orthonormal basis $\{e_1(p), \cdots, e_n(p)\}$ of T_pM. Then $\Omega|_M(e_1(p), \cdots, e_n(p)) \in \mathbb{C}$ and the fact that T_pM is a Lagrangian plane implies that

$$\big|\Omega(e_1(p), \cdots, e_n(p))\big| = 1 \tag{1.8}$$

because $\big|\Omega(e_1(p), \cdots, e_n(p))\big| = |\xi \wedge J\xi|_{g_{\mathbb{C}^n}}$. See theorem 1.7 in [HL82] for details.

Given any other oriented, orthonormal basis $\{\tilde{e}_i(p)\}$ the matrix that sends $\{e_i(p)\}$ to $\{\tilde{e}_i(p)\}$ is an element in $\mathrm{SO(n,\mathbb{R})}$ hence $\Omega|_M(e_1(p), \cdots, e_n(p))$ is independent of the choice of oriented, orthonormal basis and by (1.8) there exists a function

$$\theta : M \longrightarrow \mathbb{R}/2\pi\mathbb{Z} \cong \ S^1 \tag{1.9}$$

such that $\Omega|_M(e_1(p), \cdots, e_n(p)) = e^{\sqrt{-1}\theta(p)}$ for all $p \in M$.

Definition 4. The function $\theta : M \longrightarrow S^1$ in (1.9) is called the Lagrangian angle or phase function of M.

Now, let's suppose that M is a Lagrangian submanifold in $\mathbb{C}^n$ such that its phase function is constant i.e. $\theta(p) = \theta_0$ for all $p \in M$. Consequently

$$e^{-\sqrt{-1}\theta_0}\Omega|_M(e_1(p), \cdots, e_n(p)) = 1 \tag{1.10}$$

for every $p \in M$ and any oriented, orthonormal basis. Therefore

$$\begin{cases} \mathrm{Re}(e^{-\sqrt{-1}\theta_0}\Omega)\big|_M = dV_M \\ \mathrm{Im}(e^{-\sqrt{-1}\theta_0}\Omega)\big|_M = 0 \end{cases}. \tag{1.11}$$

Definition 5. An oriented Langrangian submanifold M in $\mathbb{C}^n$ is called a special Lagrangian submanifold with phase θ_0 if its phase function is constant with value θ_0. Equivalently M is special Lagrangian with phase θ_0 if (1.11) are satisfied.

Observe that $\mathrm{Re}(e^{-\sqrt{-1}\theta_0}\Omega)\big|_M$ is a real-valued n-form on $\mathbb{C}^n$ and the fact that

$$\mathrm{Re}(e^{-\sqrt{-1}\theta_0}\Omega)\big|_M = dV_M$$

implies that that M is calibrated by $\mathrm{Re}(e^{-\sqrt{-1}\theta_0}\Omega)\big|_M$. More precisely we have the following definition.

Definition 6. Let (M, g_M) be a Riemannian manifold and $\varphi \in C^\infty(M, \bigwedge^k T^*M)$ a closed k-form. We say that φ is a calibration on M if for every oriented k-subspace $V \subset T_pM$ we have

$$\varphi|_V \leq \mathrm{Vol}_V \tag{1.12}$$

i.e. $\varphi|_V = \lambda\,\mathrm{Vol}_V$ for some $\lambda \leq 1$.

Definition 7. Let (M, g_M) be a Riemannian manifold with a calibration φ. An oriented submanifold N of dimension k is said to be calibrated by φ if

$$\varphi = dV_N.$$

Remark 1.2. Harvey and Lawson proved in [HL82] that compact calibrated submanifolds are minimal and volume-minimizing submanifolds in their homology class. In the non-compact case they proved that calibrated submanifolds are locally volume-minimizing. These results follows easily from definition 6 as for any submanifold N' in the same homology class as N i.e. $[N] = [N']$ we have

$$\mathrm{Vol}(N) = \langle[\varphi], [N]\rangle = \langle[\varphi], [N']\rangle = \int_{N'} \varphi \leq \int_{N'} dV_{N'} = \mathrm{Vol}(N').$$

Harvey and Lawson proved in [HL82] that $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$ is a calibration on $\mathbb{C}^n$. Hence we have that special Lagrangian submanifolds in $\mathbb{C}^n$ with any phase θ_0 are calibrated submanifolds. In fact we have a family of special Lagrangian calibrations parametrized by S^1 and given by $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$ with $0 \leq \theta < 2\pi$. However, observe that given a special Lagrangian submanifold $\Phi : M \longrightarrow \mathbb{C}^n$ with phase θ i.e. a Lagrangian submanifold calibrated by $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$, the submanifold given by

$$e^{-\sqrt{-1}\frac{\theta}{n}}\Phi : M \longrightarrow \mathbb{C}^n$$

is a special Lagrangian submanifold with phase $\theta = 0$. This can be easily seen as $\Omega|_{e^{-\sqrt{-1}\frac{\theta}{n}}M} = (e^{-\sqrt{-1}\frac{\theta}{n}}\Phi)^*(\Omega) = e^{-\sqrt{-1}\theta}\Phi^*(\Omega) = e^{-\sqrt{-1}\theta}\Omega|_M$. Therefore by rotating a special Lagrangian submanifold with phase θ we transform it into a special Lagrangian submanifold with phase zero. Henceforth, when we consider special Lagrangian submanifolds in $\mathbb{C}^n$, we shall focus and discuss only the case with phase zero.

However we want to remark that in the general case when the ambient manifold is a Calabi-Yau manifold $\mathfrak{X}$ different to $\mathbb{C}^n$, it is not possible to rotate the submanifold in order to change the phase as we did in the case $\mathbb{C}^n$.

1.1.2 Special Lagrangian submanifolds in Calabi-Yau manifolds

Let $(\mathfrak{X}, \omega, J, g_{\mathfrak{x}})$ be a Kähler manifold of complex dimension n with Kähler form ω, complex structure J and Kähler metric $g_{\mathfrak{x}}$. Recall that $\mathfrak{X}$ is called a Calabi-Yau manifold if the holonomy group of $g_{\mathfrak{x}}$ is a subgroup of $\mathrm{SU}(n)$, i.e.

$$\mathrm{Hol}(g_{\mathfrak{x}}) \subseteq \mathrm{SU}(n).$$

This implies that the Ricci curvature vanishes $\mathrm{Ric}(g) = 0$ and the canonical line bundle $\bigwedge^{n,0} T^*\mathfrak{X}$ is trivial. From the set of non-vanishing holomorphic sections of $\bigwedge^{n,0} T^*\mathfrak{X}$ we choose one of those sections $\Omega_{\mathfrak{X}}$ normalized by the condition

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega_{\mathfrak{X}} \wedge \overline{\Omega}_{\mathfrak{X}}. \tag{1.13}$$

Definition 8. A non-vanishing, holomorphic section $\Omega_{\mathfrak{X}}$ normalized by (1.13) is called a normalized holomorphic volume form of the Calabi-Yau manifold $(\mathfrak{X}, \omega, J, g_{\mathfrak{x}})$.

Observe that given a normalized holomorphic volume form $\Omega_{\mathfrak{X}}$ we have that $e^{\sqrt{-1}\theta}\Omega_{\mathfrak{X}}$ is also a normalized holomorphic volume form for every $0 \leq \theta < 2\pi$. Here we will focus on the case $\theta = 0$.

Note that $\mathbb{C}^n$ is a Calabi-Yau manifold with the structure

$$(\mathbb{C}^n, g_{\mathbb{C}^n}, \omega_{\mathbb{C}^n}, \Omega)$$

where $g_{\mathbb{C}^n} = |dz_1|^2 + \cdots + |dz_N|^2$, $\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ and $\Omega = dz_1 \wedge \cdots \wedge dz_N$. Analogously to the case $\mathbb{C}^n$ we have the following result for general Calabi-Yau manifolds.

Proposition 1.3. The real part of a normalized holomorphic volume form $\mathrm{Re}(\Omega_{\mathfrak{X}})$ is a calibration on $\mathfrak{X}$. Moreover, for any $0 \leq \theta < 2\pi$, $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega_{\mathfrak{X}})$ is also a calibration.

Harvey and Lawson in [HL82] characterized special Lagrangian submanifolds in a way that has been extremely useful to study the deformation theory.

Proposition 1.4. Let $(\mathfrak{X}, \omega, J, g_{\mathfrak{x}}, \Omega_{\mathfrak{X}})$ be a Calabi-Yau manifold and M a n-dimensional real submanifold. Then M admits an orientation making it into a special Lagrangian submanifold if and only if

$$\begin{cases} \omega\big|_M \equiv 0 \\ \mathrm{Im}\,\Omega_{\mathfrak{X}}\big|_M \equiv 0 \end{cases}. \tag{1.14}$$

1.2 Deformation of special Lagrangian submanifolds

Given a Calabi-Yau manifold $(\mathfrak{X}, \omega, J, g_{\mathfrak{x}}, \Omega_{\mathfrak{X}})$ and a special Lagrangian submanifold

$$\Phi : M \longrightarrow \mathfrak{X},$$

we are interested in deformations of M, as a submanifold of $\mathfrak{X}$, such that the deformed submanifold is special Lagrangian. More precisely we are looking for submanifolds $\Psi : M \longrightarrow \mathfrak{X}$ such that Φ is isotopic to Ψ and $\Psi(M) = M_\Psi$ is special Lagrangian. If we are able to find special Lagrangian deformations of M then we want to investigate the structure of the space containing those special Lagrangian deformations i.e. the moduli space of special Lagrangian deformations $\mathfrak{M}(M, \Phi)$. In general the moduli space will be the space of special Lagrangian embeddings $\Psi : M \longrightarrow \mathfrak{X}$ (equivalent up to diffeomorphism) isotopic to our original Φ. If we require the isotopy through special Lagrangian submanifolds then we are considering only the connected component in the moduli space containing M. If we do not require the intermediate submanifolds to be special Lagrangian then we are considering all the connected components of the moduli space.

If we consider nearby enough submanifolds then it is possible to obtained deformations of M by moving it in a normal direction $\mathcal{V}$ given by a section of the normal bundle $\mathcal{V} \in C^\infty(M, \mathcal{N}(M))$. This is possible thanks to the tubular neighborhood theorem (see [Lan95] Ch 4, theorem 5.1).

Theorem 1.5. Let $(\mathfrak{X}, g_\mathfrak{x})$ be a Riemannian manifold and M an embedded submanifold. Then there exists an open neighborhood $\mathfrak{A}$ of the zero section in $\mathcal{N}(M)$ and an open neighborhood U of M in $\mathfrak{X}$ such that the exponential map $\exp_{g_\mathfrak{x}} : \mathfrak{A} \subset \mathcal{N}(M) \longrightarrow U \subset \mathfrak{X}$ is a diffeomorphism.

Observe that the submanifold M is not required to be closed, see [Lee03] theorem 10.19.

Therefore any normal section $\mathcal{V}$ lying in $\mathfrak{A}$ will produce an embedded submanifold given by

$$\exp_{g_\mathfrak{x}}(\mathcal{V}) \circ \Phi : M \longrightarrow \mathfrak{X}$$

such that $(\exp_{g_\mathfrak{x}}(\mathcal{V}) \circ \Phi)(M) = M_\mathcal{V} \subset U$. Once we have the deformed submanifold $M_\mathcal{V}$ we want to investigate if it is special Lagrangian. Equations (1.14) imply that $M_\mathcal{V}$ is special Lagrangian if and only if

$$\begin{cases} (\exp_{g_\mathfrak{x}}(\mathcal{V}) \circ \Phi)^*(\omega) \equiv 0 \\ (\exp_{g_\mathfrak{x}}(\mathcal{V}) \circ \Phi)^*(\operatorname{Im}\Omega_\mathfrak{X}) \equiv 0 \end{cases}. \tag{1.15}$$

This provides us with two explicit equations that $\mathcal{V}$ must satisfy in order to produce a special Lagrangian deformation of M. Taking advantage of the fact that M is a Lagrangian submanifold we can use the bundle isomorphisms

$$T^*M \overset{g_\mathfrak{x}^{-1}}{\cong} TM \overset{J}{\cong} \mathcal{N}(M)$$

so that we obtain that each differential form $\Xi \in \mathcal{C}^\infty(M, T^*M)$ defines a unique section of the normal bundle

$$\mathcal{V}_\Xi = J(g_\mathfrak{x}^{-1}(\Xi)) \in \mathcal{C}^\infty(M, \mathcal{N}(M)).$$

Therefore, we can express the deformation problem as a non-linear operator P acting on differential forms on M:

$$\mathrm{P}:\mathcal{C}^{\infty}(M,T^{*}M)\big|_{\mathfrak{A}}\longrightarrow\mathcal{C}^{\infty}(M,\bigwedge^{2}T^{*}M)\oplus\mathcal{C}^{\infty}(M,\bigwedge^{n}T^{*}M) \tag{1.16}$$

given by

$$\mathrm{P}(\Xi)=\Big((\exp_{\mathrm{g}_{\mathfrak{X}}}(\mathcal{V}_{\Xi})\circ\Phi)^{*}(\omega),(\exp_{\mathrm{g}_{\mathfrak{X}}}(\mathcal{V}_{\Xi})\circ\Phi)^{*}(\operatorname{Im}\Omega)\Big), \tag{1.17}$$

where $\mathcal{C}^{\infty}(M,T^{*}M)\big|_{\mathfrak{A}}$ is the space of differential forms Ξ such that their image under the bundle map $J\circ g_{\mathfrak{X}}^{-1}$ belongs to $\mathfrak{A}$.

The zero set of this operator contains those special Lagrangian deformations of M lying in U, that is,

$$\mathrm{P}^{-1}(0)=$$
$$\Big\{\Xi\in C^{\infty}(M,T^{*}M)|_{\mathfrak{A}}:\exp_{g_{\mathfrak{X}}}(\mathcal{V}_{\Xi})\circ\Phi:M\longrightarrow\mathfrak{X}\quad\text{is a special Lagrangian embedding}\Big\}.$$

Hence, the local structure of the moduli space $\mathfrak{M}(M,\Phi)$ near M is given by $\mathrm{P}^{-1}(0)$, the zero set of a non-linear operator. A classical result in non-linear functional analysis that has been used to describe the zero set of non-linear operators in deformation of calibrated submanifolds is the Implicit Function Theorem for Banach spaces (see, for example, [Lan93] chapter 14, theorem 2.1).

Theorem 1.6. Let X and Y be Banach spaces, $\mathfrak{A}\subset X$ an open neighborhood of zero and $\mathrm{P}:\mathfrak{A}\subset X\longrightarrow Y$ a $\mathcal{C}^{k}$-map such that

i) $\mathrm{P}(0)=0$

ii) $\mathrm{DP}[0]:X\longrightarrow Y$ is surjective

iii) $\mathrm{DP}[0]$ splits X i.e. $X=\operatorname{Ker}\mathrm{DP}[0]\oplus Z$ for some closed subspace Z

then there exist open subsets $W_1\subset\operatorname{Ker}\mathrm{DP}[0]$, $W_2\subset Z$ and a unique $\mathcal{C}^{k}$-map $G:W_1\subset\operatorname{Ker}\mathrm{DP}[0]\longrightarrow W_2\subset Z$ such that

i) $0\in W_1\cap W_2$

ii) $W_1\oplus W_2\subset\mathcal{U}$

iii) $\mathrm{P}^{-1}(0)\cap(W_1\oplus W_2)=\{(x,G(x)):x\in W_1\}$.

Ideally, in order to apply theorem 1.6, we expect to define Banach spaces of differential forms X and Y such that the deformation operator P acts smoothly, adapt the tubular neighborhood given by theorem 1.5 such that an open neighborhood of zero $\mathfrak{A}\subset X$ fits into it. Moreover we would like that with this choice of Banach spaces the linearisation of the deformation operator at zero $\mathrm{DP}[0]$ is a Fredholm operator. Thus its kernel is a finite dimensional space and it splits X. Moreover if the cokernel of $\mathrm{DP}[0]$ vanishes, then theorem 1.6 applies immediately and gives us that the moduli space $\mathfrak{M}(M,\Phi)$, locally around M, is a finite dimensional, smooth manifold with dimension equal to $\dim\operatorname{Ker}DP[0]$. Moreover any infinitesimal deformation, i.e. $x\in W_1\subset\operatorname{Ker}\mathrm{DP}[0]$, can be lifted to an authentic deformation given by $(x, G(x))$ with $\mathrm{P}(x + G(x)) = 0$ i.e. there are no obstructions.

This ideal situation turned out to be true in the compact case. In [McL98], McLean studied the deformation of a compact special Lagrangian submanifold inside a Calabi-Yau manifold $\mathfrak{X}$. McLean used the classical and well-developed elliptic theory on compact manifolds to analyze the deformation operation and its linearisation. He obtained the following very complete result.

Theorem 1.7. If $\Phi : M \longrightarrow \mathfrak{X}$ is an immersed compact special Lagrangian submanifold, the moduli space of special Lagrangian deformations

$$\mathfrak{M}(M, \Phi) := \{\psi : M \longrightarrow \mathfrak{X} : \psi \text{ is a special Lagrangian immersion isotopic to } \Phi\}$$

is a smooth, finite dimensional manifold with tangent space at M isomorphic to $\mathcal{H}^1(M)$, the space of harmonic 1-forms, therefore $\dim \mathfrak{M}(M, \Phi) = b^1(M)$. Acting on $C^\infty(M, T^*M)$, the linearisation of the deformation operator P at zero is given by the Hodge-deRham operator i.e.

$$\mathrm{DP}[0] = d + d^*.$$

Moreover there are no obstructions to extend infinitesimal deformations to authentic special Lagrangian deformations.

Now let's consider the case $\mathfrak{X} = \mathbb{C}^n$. As any special Lagrangian submanifold is a minimal submanifold (see remark 1.2), we have that any non-trivial special Lagrangian submanifold in $\mathbb{C}^n$ must be non-compact (in particular singular) or with boundary. This is due to the fact that any isometrically immersed submanifold in an Euclidean space is minimal if and only if their components are harmonic, see [Xin03] corollary 1.3.2 for details.

Singular special Lagrangian submanifolds attracted a lot of attention due to the SYZ conjecture in mirror symmetry. We will not get into details here but we only mention that special Lagrangian fibrations over a 3-manifold B play a fundamental role. Over the singular locus $\Delta \subset B$ of this fibration the fibers are singular special Lagrangian submanifolds. We refer the reader to [Joy03] for a comprehensive review.

In this direction, special Lagrangian submanifolds with conical singularities and conical ends have been studied intensively, see section 2.4 for an introduction to this kind of submanifold. For the last sixteen years, moduli spaces, obstructions, gluing and desingularization constructions have been studied by several authors, see [Joy04a],[Joy04b], [Joy04c],[Joy04d],[Mar],[Pac04],[Pac13]. Most of these results rely on the elliptic and Hodge theory of Lockhart and McOwen [LM85], [Loc87]. An exception is [Pac04] where Pacini used the b-calculus of R. Melrose [Mel93] to analyze the linearisation of the deformation operator.

Chapter 2

Manifolds with Singularities

2.1 Preliminaries

We are interested in the deformation of special Lagrangian submanifolds with singularities. In this section we provide the definitions and concepts related to singular manifolds that we use throughout this thesis. Most of this section and section 2.2 is based on chapter 1 and 2 of [NSSS06], we refer the reader to that book for further details.

First let's recall the definition of the algebra of classical differential operators on a manifold M. For any $\mathbb{C}$-linear map $\mathrm{P} : C^\infty(M) \longrightarrow C^\infty(M)$ and any $f \in C^\infty(M)$ we can define the $\mathbb{C}$-linear map $[\mathrm{P}, f] : C^\infty(M) \longrightarrow C^\infty(M)$ given by

$$[\mathrm{P}, f](g) := \mathrm{P}(fg) - f\,\mathrm{P}(g).$$

A differential operator of order zero $\mathrm{P} \in \mathrm{Diff}^0(M)$ is a $\mathbb{C}$-linear map $\mathrm{P} : C^\infty(M) \longrightarrow C^\infty(M)$ such that $[\mathrm{P}, f] \equiv 0$ for every $f \in C^\infty(M)$. Inductively we define $\mathrm{Diff}^l(M)$ as the set of $\mathbb{C}$-linear maps P such that $[\mathrm{P}, f] \in \mathrm{Diff}^{l-1}(M)$ for every $f \in C^\infty(M)$. It can be proven that a $\mathbb{C}$-linear map $\mathrm{P} : C^\infty(M) \longrightarrow C^\infty(M)$ belongs to $\mathrm{Diff}^l(M)$ if and only if at any patch of local coordinates (x_i) on M we have

$$\mathrm{P} = \sum_{|\alpha| \leq l} a_\alpha(x) \partial_x^\alpha,$$

where $a_\alpha(x)$ is a $\mathbb{C}$-valued smooth function on the patch of coordinates for every multi-index α. See [Nic07] chapter 10, section 10.1 for the proof and further details.

We denote by

$$\mathrm{Diff}(M) := \bigcup_{l \geq 0} \mathrm{Diff}^l(M)$$

the algebra of classical differential operators on M.

Definition 9. A singular manifold is a pair $(M, \mathfrak{D})$ where M is a smooth manifold possibly non-compact and $\mathfrak{D} \subset \mathrm{Diff}(M)$ is a subalgebra of differential operators such that its restriction $\mathfrak{D}\big|_U$ at every open subset U with compact closure $\overline{U} \subset M$ is equal to the restriction of the algebra of all differential operators $\mathrm{Diff}(M)\big|_U$.

The most relevant situation in this definition is when M is non-compact. In this case the subalgebra $\mathfrak{D}$ will consist of differential operators that degenerate at the limit in the non-compact part of M. The degeneration of the differential operators in $\mathfrak{D}$ will reflect the geometric singularities of M. Away from the limit i.e. on an open subset $U \Subset M$, as in definition 9, the operators in $\mathfrak{D}$ have no degeneration and coincide with the classical operators in $\mathrm{Diff}(M)$. In order to visualize the geometric singularities on M reflected by the degeneration of the operators in $\mathfrak{D}$, a topological space $\hat{M}$, called the singular space associated with the singular manifold M, is defined in such a way that M is an open dense subset of $\hat{M}$. See section 2.2.1 for specific examples of singular manifolds.

2.2 Construction of singular manifolds

We start with a specific way of constructing the algebra $\mathfrak{D}$ that is general enough to produce the type of singular manifolds that we are interested in, namely manifolds with conical or edge singularities.

The algebra $\mathfrak{D}$ is generated by a function space $\mathcal{F}$ such that $C_0^\infty(M) \subset \mathcal{F} \subset C^\infty(M)$ and a space of vector fields $\mathbf{V}$ on M such that $C_0^\infty(M, TM) \subset \mathbf{V} \subset C^\infty(M, TM)$.

We obtain the function space $\mathcal{F}$ by embedding M into a compact manifold with boundary $\mathbb{M}$ and defining $\mathcal{F}$ as the restriction of the space of smooth functions on $\mathbb{M}$ i.e. $\mathcal{F} := C^\infty(\mathbb{M})\big|_M$. In order to define the space $\mathbf{V}$ we need a Riemannian metric $g_{_M}$ on M such that it extends to a smooth symmetric 2-tensor on $\mathbb{M}$ but possibly degenerate on $\mathbb{M} \setminus M$. The space of vector fields $\mathbf{V}$ is obtained by means of duality with respect to g_M and $\mathcal{F}$. More precisely we have the following definition.

Definition 10. A linear space of vector fields $V \subset C^\infty(M, TM)$ is self-dual with respect to the $\mathcal{F}$-valued pairing defined by $g_{_M}$ if $V = V'$ where

$$V' = \{v \in C^\infty(M, TM) : g_{_M}(v, u) \in \mathcal{F} \quad \forall u \in V\}.$$

We obtain $\mathbf{V}$ by requiring that $\mathbf{V}$ is self-dual with respect to the $\mathcal{F}$-pairing and

$$C^\infty(\mathbb{M}, T\mathbb{M}) \subset \mathbf{V} \subset C^\infty(M, TM). \tag{2.1}$$

The degeneration of $g_{_M}$ on $\mathbb{M} \setminus M$ will define the geometric singularity of M in the following way.

Definition 11. The singular space $\hat{M}$ associated with the pair $(M, \mathfrak{D})$ is the quotient space

$$\hat{M} = \mathbb{M}/\sim$$

where $p \sim q$ if $d_{g_{_M}}(p, q) = 0$ with $d_{g_{_M}}$ the distance function on $\mathbb{M}$ induced by $g_{_M}$.

2.2.1 Examples

i) Let's consider a smooth manifold $\mathbb{M}$ with smooth boundary $\partial\mathbb{M} = \mathcal{X}$, $\dim \mathcal{X} = m$ and define $M := \mathbb{M} \setminus \partial\mathbb{M}$. Let $g_{_M}$ be a conical metric on M i.e. $g_{_M}$ is a Riemannian metric on M such that on a collar neighborhood of $\partial\mathbb{M}$ given by $(0,1)\times\mathcal{X}\subset M$ we have

$$g_{_M} = r^2 g_{\mathcal{X}} + dr^2$$

where $g_{\mathcal{X}}$ is a Riemannian metric on $\mathcal{X}$. Then $g_{_M}$ extends to a smooth, symmetric 2-tensor on $\mathbb{M}$ that degenerates in each tangent direction to $\mathcal{X}$. Following the scheme to construct singular manifolds explained in the previous subsection, we set $\mathcal{F}$ to be the restriction of $C^\infty(\mathbb{M})$ to M.

Now, let $\mathcal{V}\in C^\infty(\mathbb{M},T\mathbb{M})$ be a vector field with length of the order of unity with respect to $g_{_M}$, i.e.

$$\left|\mathcal{V}(p)\right|_{g_{_M}} \leq C$$

for any $p\in\mathbb{M}$ and $C>0$ independent of p.

On a neighborhood $[0,1)\times\mathcal{U}\subset[0,1)\times\mathcal{X}$ it is easy to see that

$$\mathcal{V} = \mathcal{A}\partial_r + \sum_{k=1}^{m}\mathcal{B}_k\frac{1}{r}\partial_k \tag{2.2}$$

where ∂_k are the local coordinate vector fields on $\mathcal{U}\subset\mathcal{X}$ and $\mathcal{A},\mathcal{B}_k\in C^\infty([0,1)\times\mathcal{U})$.

Observe that the linear space of vector fields of the form (2.2) is self-dual with respect to the $C^\infty(\mathbb{M})\big|_M$-pairing induced by $g_{_M}$, therefore any such $\mathcal{V}$ must belong to $\mathbf{V}$. Moreover, it is not difficult to prove (see [NSSS06] proposition 1.17) that the space of vector fields on $\mathbb{M}$ with length of the order of unity with respect to $g_{_M}$ is the unique linear space that is self-dual and satisfies the expression (2.1), hence this is the choice for $\mathbf{V}$.

From the discussion above we conclude that the algebra of degenerate operators $\mathfrak{D}$ is generated by functions on M smooth up to $r=0$ i.e. $C^\infty(\mathbb{M})\big|_M$ and vector fields $\mathcal{V}$ such that on the collar neighborhood $[0,1)\times\mathcal{X}$ are given by

$$\mathcal{V} = \mathcal{A}\partial_r + \Theta$$

where $\mathcal{A}\in C^\infty([0,1)\times\mathcal{X})$ and $r\Theta\in C^\infty([0,1),T\mathcal{X})$.

This algebra is called the algebra of cone-degenerate operators $\mathrm{Diff}_{\mathrm{cone}}(M)$. From the local expressions above we have that every cone-degenerate operator P of order l can be written in the collar neighborhood as

$$\mathrm{P} = r^{-l}\sum_{i\leq l} a_i(r)(-r\partial_r)^i \tag{2.3}$$

where $a_i\in C^\infty\Big([0,1),\mathrm{Diff}^{l-i}(\mathcal{X})\Big)$, with $\mathrm{Diff}^{l-i}(\mathcal{X})$ denoting the space of classical differential operators of order $l-i$ on $\mathcal{X}$. Cone-degenerate operators are also called Fuchs-type operators.

The singular space $\hat{M}$ associated to $(M,\mathrm{Diff}_{\mathrm{cone}}(M))$ is the quotient space $\mathbb{M}/\sim$ where $p\sim q$ if and only if $p=q$ or $p,q\in\partial\mathbb{M}$. This is a consequence of the degeneration of the cone metric. Hence the singular space $\hat{M}$ is obtained by collapsing the boundary to a point $\mathfrak{v}$, the vertex of the cone. The natural projection $\pi : \mathbb{M} \longrightarrow \hat{M}$ defines a diffeomorphism between $\mathbb{M} \setminus \partial\mathbb{M}$ and $\hat{M} \setminus \{\mathfrak{v}\}$. The vertex $\mathfrak{v}$ has a neighborhood homeomorphic to a cone with base $\mathcal{X}$:

$$\mathcal{X}^{\triangle} := \big([0,1) \times \mathcal{X}\big) \Big/ \big(\{0\} \times \mathcal{X}\big)\,. \tag{2.4}$$

We say that $(M, \mathrm{Diff}_{\mathrm{cone}}(M))$ is a manifold with conical singularity.

ii) Let $\mathbb{M}$ be a smooth compact manifold with boundary $\partial\mathbb{M}$. This time we will assume that the boundary has a fiber bundle structure in order to produce more elaborate singularities. More precisely, let $\mathcal{X}$ and $\mathcal{E}$ be smooth, compact manifolds without boundary such that $\partial\mathbb{M}$ is the total space of a smooth $\mathcal{X}$-fibration over $\mathcal{E}$

$$\pi : \partial\mathbb{M} \longrightarrow \mathcal{E}.$$

Observe that any collar neighborhood of the boundary $[0,1) \times \partial\mathbb{M}$ has the structure of a $\mathcal{X}$-fibration over $\mathcal{E} \times [0,1)$. By fixing a collar neighborhood, we use the bundle coordinates on $[0,1) \times \partial\mathbb{M}$ as admissible coordinates i.e. coordinates of the form (r, σ_k, u_l) where (u_l, r) are coordinates on $\mathcal{E} \times [0,1)$ and (σ_k) local coordinates on the fiber $\mathcal{X}$.

Now, we apply the scheme to construct singular manifolds. Let $M = \mathbb{M} \setminus \partial\mathbb{M}$ and equip M with an edge metric

$$g_{_M} = r^2 g_{_\mathcal{X}} + dr^2 + g_{_\mathcal{E}}$$

where $g_{_\mathcal{E}}$ is a smooth Riemannian metric on $\mathcal{E}$.

Observe that the edge metric $g_{_M}$ extends to a smooth symmetric 2-tensor on $\mathbb{M}$ that degenerates on each $\mathcal{X}$-fiber over $\partial\mathbb{M}$. In order to define the algebra of degenerate differential operators on M we set $\mathcal{F} = C^\infty(\mathbb{M})\big|_M$. Analogous to the conical case it can be proven (see [NSSS06] sec. 1.3.1) that $\mathbf{V}$ is defined uniquely as the set of vector fields with length of the order of unity with respect to the edge metric $g_{_M}$. In admissible coordinates on the collar neighborhood, $[0,1) \times \mathcal{U} \times \Omega \subset [0,1) \times \partial\mathbb{M}$, these vectors fields are given by

$$V = \mathcal{A}\partial_r + \sum_{k=1}^{m} \mathcal{B}_k \frac{1}{r}\partial_k + \sum_{l=1}^{q} \mathcal{C}_l \partial_{u_l} \tag{2.5}$$

where $\mathcal{A}, \mathcal{B}_k, \mathcal{C}_l \in C^\infty([0,1) \times \mathcal{U} \times \Omega)$, ∂_k are local coordinate vector fields on $\mathcal{U} \subset \mathcal{X}$ and ∂_{u_l} are local coordinate vector fields on $\Omega \subset \mathcal{E}$.

The algebra generated by $\mathcal{F}$ and $\mathbf{V}$ is called the algebra of edge-degenerate operators $\mathrm{Diff}_{\mathrm{edge}}(M)$. In admissible coordinates, every edge-degenerate operator of order l is given by

$$\mathrm{P} = r^{-l} \sum_{i \leq l} a_{i,\alpha}(r,u)(-r\partial_r)^i (rD_u)^\alpha \tag{2.6}$$

where $a_{i,\alpha} \in C^\infty\left([0,1) \times \Omega, \mathrm{Diff}^{l-i-|\alpha|}(\mathcal{X})\right)$ and $D_{u_l} = -\sqrt{-1}\partial_{u_l}$.

Similarly we define edge-degenerate differential operators acting on sections of an admissible vector bundle E over M (see definition 23). In this case the coefficients $a_{j\alpha}$ belongs to $C^\infty(\overline{\mathbb{R}}^+ \times \Omega, \mathrm{Diff}^{m-(j+|\alpha|)}(\mathcal{X}, E_{\mathcal{X}}))$.

The singular space $\hat{M}$ associated to $(M, \mathrm{Diff}_{\mathrm{edge}}(M))$ is the quotient space $\mathbb{M}/\sim$ where $x \sim y$ if and only if $x = y$ or $x, y \in \partial\mathbb{M}$ and $\pi(x) = \pi(y)$ i.e. x and y belong to the same fiber over $\mathcal{E}$. The singular space $\hat{M}$ is obtained by collapsing each fiber of $\partial\mathbb{M}$ to a point. Observe that the collar neighborhood $[0,1) \times \partial\mathbb{M}$ under the relation $\sim$ becomes a fiber bundle over $\mathcal{E}$ with fiber the singular conical space $\mathcal{X}^\triangle$. We say that $(M, \mathrm{Diff}_{\mathrm{edge}}(M))$ is a manifold with edge singularity.

2.3 Analysis on Manifolds with edges.

In this section we describe the necessary elements to study partial differential equations on manifolds with conical or edge singularities. In particular we introduce the relevant concepts and definitions needed for the analysis of deformations of singular special Lagrangian submanifolds carried out in chapter 3. This section is based on [Sch98] chapters 2 and 3. We refer the reader interested in full details and explanations to that book.

2.3.1 Sobolev spaces on singular manifolds

We start by introducing suitable Banach spaces on which cone-degenerate operators act. In the same way as the algebras of degenerate operators $\mathfrak{D}$ coincide with the classical differential operators away from the singular set, the norm of Sobolev spaces defined on manifolds with cone or edge singularities will be equivalent to the norm of classical Sobolev spaces for functions supported away from the singularities. Near the singular set these Sobolev spaces are defined by means of the Mellin transformation as it plays a similar role in the symbolic structure of cone-degenerate operators as the Fourier transformation for classical operators.

Recall that the Mellin transformation $\mathcal{M}$ is a continuous operator $\mathcal{M} : C_0^\infty(\mathbb{R}^+) \longrightarrow \mathcal{A}(\mathbb{C})$ given by the integral formula

$$(\mathcal{M}f)(z) = \int_0^\infty r^{z-1} f(r) dr, \tag{2.7}$$

where $\mathcal{A}(\mathbb{C})$ is the space of holomorphic functions on $\mathbb{C}$. Some of the elementary properties of the Mellin transformation are the following:

i) $\mathcal{M}\left(-r\frac{d}{dr}f\right)(z) = z\left(\mathcal{M}f\right)(z)$;

ii) $\mathcal{M}\left(r^\gamma f\right)(z) = \left(\mathcal{M}f\right)(z+\gamma)$ for any $\gamma \in \mathbb{R}$;

iii) $\mathcal{M}\left(\log(r)f\right)(z) = \frac{d}{dz}\left(\mathcal{M}f\right)(z)$;

iv) $\mathcal{M}\left(f(r^\gamma)\right)(z) = \gamma^{-1}\left(\mathcal{M}f\right)(\gamma^{-1}z)$ for any $\gamma \in \mathbb{R}$.

Very often we need the restriction of the holomorphic function $\mathcal{M}f$ to subsets isomorphic to $\mathbb{R}$ given by

$$\Gamma_\beta = \{z \in \mathbb{C} : Re(z) = \beta\}. \tag{2.8}$$

The restricted Mellin transformation, denoted by $\mathcal{M}_\gamma$, maps $C_0^\infty(\mathbb{R}^+)$ into the Schwartz space $\mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$. This map extends continuously to an isomorphism of Banach spaces $\mathcal{M}_\gamma : r^\gamma L^2(\mathbb{R}^+) \longrightarrow L^2(\Gamma_{\frac{1}{2}-\gamma})$, where $r^\gamma L^2(\mathbb{R}^+)$ is endowed with the weighted L^2-norm i.e. $f \in r^\gamma L^2(\mathbb{R}^+)$ if and only if $r^{-\gamma} f \in L^2(\mathbb{R}^+)$. It can be easily computed that the inverse Mellin transformation $\mathcal{M}_\gamma^{-1} : \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}) \longrightarrow r^\gamma L^2(\mathbb{R}^+)$ is given by

$$\left(\mathcal{M}_\gamma^{-1} g\right)(r) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} g(z) dz.$$

The role of the Mellin transformation in cone-degenerate operators is given by the following basic fact: $(-r\partial_r) f = \mathcal{M}^{-1} z \mathcal{M} f$ for any $f(r,\sigma) \in C_0^\infty(\mathbb{R}_r^+ \times \mathbb{R}_\sigma^m)$. This follows immediately from the first property numbered above. Therefore any cone-degenerate operator $\mathrm{P} = r^{-l} \sum\limits_{i \le l} a_i(r)(-r\partial_r)^i$ is given in terms of the Mellin transformation as follows

$$\mathrm{P} = r^{-l} \mathcal{M}^{-1} h(r,z) \mathcal{M},$$

where $h(r,z) = \sum\limits_{i \le l} a_i(r) z^i$.

Observe that $h(0,z) : \mathbb{C} \to \mathcal{L}\left(H^s(\mathcal{X}), H^{s-l}(\mathcal{X})\right)$ defines an operator-valued polynomial of degree l on $\mathbb{C}$ where $H^s(\mathcal{X})$ denotes the classical Sobolev space of order s on a closed manifold. It is easy to check that the Fréchet derivative with respect to z satisfies the usual differentiation rule as in the scalar case. Therefore $h(0,z)$ defines a holomorphic family of operators in $\mathcal{L}\left(H^s(\mathcal{X}), H^{s-l}(\mathcal{X})\right)$ for every $s \in \mathbb{R}$. The family of operators $h(0,z)$ is called the Mellin symbol of P. In an analogous way, when trying to invert the Mellin symbol we obtain a meromorphic family of operators $h(0,z)^{-1}$ whose poles induce asymptotics to solutions of elliptic cone-degenerate equations, see section 3.5 below.

Definition 12. The local cone-Sobolev space of order s and weight γ, $\mathcal{H}^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^m)$, with $s, \gamma \in \mathbb{R}$, is defined as the closure of $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^m)$ with respect to the norm

$$\|f\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)} := \left(\frac{1}{2\pi\sqrt{-1}} \int\limits_{\Gamma_{\frac{m+1}{2}-\gamma}} \int\limits_{\mathbb{R}^m} (1+|z|^2+|\xi|^2)^s \left| \mathcal{M}_{\gamma-\frac{m}{2}, r\to z} \mathcal{F}_{\sigma\to\xi} f \right|^2 d\xi dz \right)^{\frac{1}{2}},$$

where $\mathcal{F}_{\sigma\to\xi}$ denotes the Fourier transformation in the variable $\sigma \in \mathbb{R}^m$ and the symbol $\mathcal{M}_{\gamma-\frac{m}{2}, r\to z}$ denotes the restricted Mellin transformation acting on the variable $r \in \mathbb{R}^+$.

The local cone-Sobolev spaces are Hilbert spaces with inner product given by

$$\frac{1}{2\pi\sqrt{-1}} \left\langle (1+|z|^2+|\xi|^2)^{\frac{s}{2}} \mathcal{M}_{\gamma-\frac{m}{2}, r\to z} \mathcal{F}_{\sigma\to\xi} f, (1+|z|^2+|\xi|^2)^{\frac{s}{2}} \mathcal{M}_{\gamma-\frac{m}{2}, r\to z} \mathcal{F}_{\sigma\to\xi} g \right\rangle_{L^2(\mathbb{R}\times\mathbb{R}^m)}. \tag{2.9}$$

The relation between the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^m)$ and the standard Sobolev spaces $H^s(\mathbb{R}^{m+1})$ is given by the following transformation. First consider the transformation

$$S_{\gamma-\frac{m}{2}} : C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^m) \longrightarrow C_0^\infty(\mathbb{R}_t \times \mathbb{R}^m_x) \tag{2.10}$$

such that $S_{\gamma-\frac{m}{2}}(f)(t,x) := e^{-(\frac{1}{2}-(\gamma-\frac{m}{2}))t} f(e^{-t},x)$. This transformation extends to a Banach space isomorphism between $\mathcal{H}^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^m)$ and the standard Sobolev spaces $H^s(\mathbb{R}^{m+1})$. Therefore the norms $\|f\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)}$ and $\left\|S_{\gamma-\frac{m}{2}}(f)\right\|_{H^s(\mathbb{R}^{m+1})}$ are equivalent.

In order to define the global cone-Sobolev space on a manifold with conical singularities M, we choose a finite open covering $\{\mathcal{U}_\lambda, \chi_\lambda\}$ of $\mathcal{X}$ given by coordinate neighborhoods such that $\chi_\lambda : \mathcal{U}_\lambda \longrightarrow \mathbb{R}^m$ and $I \times \chi_\lambda : \mathbb{R}^+ \times \mathcal{U}_\lambda \longrightarrow \mathbb{R}^+ \times \mathbb{R}^m$ with $(I\times\chi_\lambda)(r,p) = (r, \chi_\lambda(p))$ are diffeomorphisms for every λ. Let $\{\varphi_\lambda\}$ be a partition of unity subordinate to $\{\mathcal{U}_\lambda\}$. The global cone-Sobolev space near the conical singularities is modelled on the space $\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ defined on the open cone $\mathcal{X}^\wedge := \mathbb{R}^+ \times \mathcal{X}$ as the closure of $C_0^\infty(\mathcal{X}^\wedge)$ with respect to the norm

$$\|f\|_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} := \left(\sum_\lambda \left\| (I \times \chi_\lambda^*)^{-1} \varphi_\lambda f \right\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)} \right)^{\frac{1}{2}} \tag{2.11}$$

where $m = \dim \mathcal{X}$. This definition is independent of the open covering and partition of unity up to norm equivalence. For $s \in \mathbb{N}$ the space $\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ can be characterized as the space of $f \in r^{\gamma-\frac{m}{2}} L^2(\mathcal{X}^\wedge, drd\sigma)$ such that $(r\partial_r)^{\alpha_0} \mathcal{V}_1^{\alpha_1} \cdots \mathcal{V}_1^{\alpha_k} f \in r^{\gamma-\frac{m}{2}} L^2(\mathcal{X}^\wedge, drd\sigma)$ for any $\mathcal{V}_i \in C^\infty(\mathcal{X}, T\mathcal{X})$ with $\alpha_0 + \cdots + \alpha_k \le s$. See [Sch98] proposition 2.1.45.

In order to glue together the space $\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ with the classical Sobolev space away from the edge we use a cut-off function $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}^+)$ such that $\omega(r) = 1$ for $0 \le r < \varepsilon_1$ and $\omega(r) = 0$ for $r \ge \varepsilon_2$ for some $0 < \varepsilon_1 < \varepsilon_2$. Given a Banach space H such that $\omega f \in H$ for every $f \in H$ we denote by $[\omega]H$ the closure of the set $\{\omega f : f \in H\}$ with respect to the norm in H. Recall that if $\mathbb{M}$ is a compact manifold with boundary the double manifold

$$2\mathbb{M} := (\mathbb{M} \coprod \mathbb{M})/\sim$$

denotes the compact manifold without boundary obtained as the quotient of the disjoint union of 2 copies of $\mathbb{M}$ with the relation that identifies their boundaries.

Definition 13. Given a compact manifold M with conical singularity, the cone-Sobolev space of order s and weight γ is defined as follows

$$\mathcal{H}^{s,\gamma}(M) := [\omega]\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge) + [1-\omega]H^s(2\mathbb{M}).$$

The cone-Sobolev space on the open cone $\mathcal{X}^\wedge$ is defined in an analogous manner

$$\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge) := [\omega]\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge) + [1-\omega]H^s_{\text{cone}}(\mathcal{X}^\wedge).$$

Both of these spaces are endowed with the topology of the non-direct sum.

Remark 2.1. Here we provide some explanations about the notation in the previous definition.

i) The cone space $H^s_{\text{cone}}(\mathcal{X}^\wedge)$ controls the behavior of the functions away from the vertex. Roughly speaking, it makes the functions behave like functions on standard Sobolev space $H^s(\mathbb{R}^{m+1})$ away from the origin. It is defined in the following way: let $\{\phi_k\}$ be a finite partition of unity of $\mathcal{X}$ subordinate to $\{U_k\}$ and consider the conical charts $\varphi_k : \mathbb{R}^+ \times U_k \longrightarrow \hat{V}_k \subset \mathbb{R}^{m+1}$ where $\hat{V}_k$ are conical subsets. Then

$$\left\|(1-\omega)u\right\|_{H^s_{\text{cone}}(\mathcal{X}^\wedge)} := \left(\sum_k \left\|(\varphi^*)^{-1}(1-\omega)\phi_k u\right\|^2_{H^s(\mathbb{R}^{m+1})}\right)^{1/2}$$

ii) Given two Banach spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ which are subspaces of a Hausdorff topological vector space F, the non-direct sum of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ is defined in the following way: let's consider the direct sum $\mathcal{H} \oplus \tilde{\mathcal{H}}$ and the space $\Delta = \{(h, -h) : h \in \mathcal{H} \cap \tilde{\mathcal{H}}\}$. Both of them are Banach spaces and we can consider Δ as a closed subspace of the direct sum. The non-direct sum is defined as the quotient space $(\mathcal{H} \oplus \tilde{\mathcal{H}})/\Delta$ with the quotient Banach space structure. In the first part of the previous definition we consider $[\omega]\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ and $[1-\omega]H^s(2\mathbb{M})$ as subspaces of the Hausdorff topological vector space $H^s_{loc}(M)$ and for the second part $[\omega]\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ and $[1-\omega]H^s(\mathcal{X}^\wedge)$ as subspaces of the Hausdorff topological vector space $H^s_{loc}(\mathcal{X}^\wedge)$.

Cone-degenerate differential operators act continuously on cone-Sobolev spaces. The next proposition can be found in [Sch98] theorem 2.1.14.

Proposition 2.2. Let $\mathrm{P} \in \mathrm{Diff}^l_{\text{cone}}(M)$, then

$$\mathrm{P} : \mathcal{H}^{s,\gamma}(M) \longrightarrow \mathcal{H}^{s-l,\gamma-l}(M)$$

is a continuous operator for every l and any s, γ.

Now, let's define the edge-Sobolev spaces. As a motivation for this definition consider the so-called anisotropic description of the standard Sobolev space $H^s(\mathbb{R}^{1+m} \times \mathbb{R}^q)$ with respect to $\mathbb{R}^q$. Let's take $f(w,u) \in H^s(\mathbb{R}^{1+m} \times \mathbb{R}^q)$ and $g(w) \in H^s(\mathbb{R}^{1+m})$ with norms

$$\|f\|_{H^s(\mathbb{R}^{1+m}\times\mathbb{R}^q)} = \left(\int\limits_{\mathbb{R}^q}\int\limits_{\mathbb{R}^{1+m}} \left(|\xi|^2 + [\eta]^2\right)^s \left|(\mathcal{F}f)(\xi,\eta)\right|^2 d\xi d\eta\right)^{\frac{1}{2}}$$

where $[\eta] \in C^\infty(\mathbb{R}^q)$ is a fixed strictly positive function such that $[\eta] = |\eta|$ for $|\eta| > c$ for some $c > \mathbb{R}$ and

$$\|g\|_{H^s(\mathbb{R}^{1+m})} = \left(\int\limits_{\mathbb{R}^{1+m}} \left(1 + |\xi|^2\right)^s \left|(\mathcal{F}g)(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Now consider the following $\mathbb{R}^+$-action on the spaces $H^s(\mathbb{R}^{1+m})$: given $\lambda \in \mathbb{R}^+$ and $g \in H^s(\mathbb{R}^{1+m})$ we define $(\kappa_\lambda g)(w) := \lambda^{\frac{m+1}{2}} g(\lambda w)$. This defines a continuous one-parameter group of invertible operators on $H^s(\mathbb{R}^{1+m})$, $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}^+} \in \mathcal{C}\left(\mathbb{R}^+, \mathcal{L}(H^s(\mathbb{R}^{1+m})\right)$ where $\mathcal{L}\left(H^s(\mathbb{R}^{1+m})\right)$ is equipped with the strong operator topology. Then the following proposition is the anisotropic description of the standard Sobolev space $H^s(\mathbb{R}^{1+m}\times\mathbb{R}^q)$ with respect to $\mathbb{R}^q$. A proof of this proposition can be found in [Sch98] lemma 3.1.12.

Proposition 2.3.

$$\|f\|_{H^s(\mathbb{R}^{1+m}\times\mathbb{R}^q)} = \left(\int_{\mathbb{R}^q} [\eta]^{2s} \left\| \kappa^{-1}_{[\eta]}(\mathcal{F}_{u\to\eta} f)(w,\eta)\right\|^2_{H^s(\mathbb{R}^{1+m})} d\eta\right)^{\frac{1}{2}}$$

for every $s\in\mathbb{R}$.

The definition of edge-Sobolev spaces is inspired by this anisotropic description. In the edge case the Sobolev space will be anisotropic with respect to the edge $\mathcal{E}$. The $\mathbb{R}^+$-action on the cone-Sobolev space $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ is given by $(\kappa_\lambda f)(r,\sigma) := \lambda^{\frac{m+1}{2}} f(\lambda r,\sigma)$. Again this defines a continuous one-parameter group of invertible operators with the strong operator topology.

Definition 14. Edge-Sobolev spaces.

i) We define the edge-Sobolev space on the open edge $\mathcal{X}^\wedge\times\mathbb{R}^q$ as the the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge))$ with respect to the norm

$$\|f\|_{\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge\times\mathbb{R}^q)} = \left(\int [\eta]^{2s} \left\| \kappa^{-1}_{[\eta]}(\mathcal{F}_{u\to\eta} f(\eta))\right\|^2_{\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)} d\eta\right)^{\frac{1}{2}}_{.} \tag{2.12}$$

ii) Given a compact manifold M with edge singularity $\mathcal{E}$ the edge-Sobolev space $\mathcal{W}^{s,\gamma}(M)$ is defined as the closure of $C_0^\infty(M)$ with respect to the norm

$$\|f\|_{\mathcal{W}^{s,\gamma}(M)} = \left(\sum_j \left\|\omega\phi_j f\right\|^2_{\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge\times\mathbb{R}^q)} + \left\|(1-\omega)f\right\|^2_{H^s(2\mathbb{M})}\right)^{\frac{1}{2}}_{.} \tag{2.13}$$

where ϕ_j is a partition of unity associated to a finite open cover $\{\Omega_j\}$ of $\mathcal{E}$ and ω is the cut-off function supported near the edge.

Similarly we can define $\mathcal{W}^{s,\gamma}(M,E)$ with E an admissible vector bundle over M (see definition 23).

The next proposition establishes the continuity of edge-degenerate operators on edge-Sobolev spaces. A proof of this proposition can be found in [Sch91] section 3.1, proposition 5.

Proposition 2.4. Let $\mathrm{P}\in\mathrm{Diff}^l_{\mathrm{edge}}(M)$, then

$$\mathrm{P} : \mathcal{W}^{s,\gamma}(M) \longrightarrow \mathcal{W}^{s-l,\gamma-l}(M)$$

is a continuous operator for every l and any s,γ.

Before concluding this section we want to make some important remarks on continuous one-parameter group of operators. In general, if $\mathfrak{B}$ is a Banach space and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}^+} \in \mathcal{C}\left(\mathbb{R}^+, \mathcal{L}(\mathfrak{B})\right)$ is a continuous one-parameter group of invertible operators we have that there exist positive constants $K, \mathfrak{c}$ such that

$$\|\kappa_\lambda\|_{\mathcal{L}(\mathfrak{B})} \leq \begin{cases} K\lambda^{\mathfrak{c}} \text{ for } \lambda \geq 1 \\ K\lambda^{-\mathfrak{c}} \text{ for } 0 < \lambda \leq 1 \end{cases} . \tag{2.14}$$

See [Sch98] proposition 1.3.1 for details.

When $\mathfrak{B} = \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ and $(\kappa_\lambda f)(r,\sigma) = \lambda^{\frac{m+1}{2}} f(\lambda r, \sigma)$ we can use 2.10 to compute $\|\kappa_\lambda\|_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} = \lambda^\gamma$ (see [Sch91] section 1.1). By the proof of proposition 1.3.1 in [Sch98] it is easy to see that the constant $\mathfrak{c}$ in (2.14) depends only on the weight γ. When $\mathfrak{B} = \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ we denote this constant by $\mathfrak{c}_\gamma$.

As a consequence of (2.14), we have the following continuous embeddings

$$\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge \times \mathbb{R}^q) \longrightarrow H^{s-\mathfrak{c}_\gamma}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)) \tag{2.15}$$

$$H^{s}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)) \longrightarrow \mathcal{W}^{s+\mathfrak{c}_\gamma,\gamma}(\mathcal{X}^\wedge \times \mathbb{R}^q) \tag{2.16}$$

for all $s \in \mathbb{R}$ where $H^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge))$ is the standard vector-valued Sobolev space with norm given by

$$\|f\|_{H^s(\mathbb{R}^q,\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge))} := \left(\int_{\mathbb{R}^q} [\eta]^{2s} \left\|\mathcal{F}_{u\to\eta} f(\eta)\right\|^2_{\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)} d\eta \right)^{\frac{1}{2}} .$$

The reader is refer to [Sch98] proposition 1.3.1 and remark 1.3.21 for details.

2.3.2 Symbolic structure

Consider a cone-degenerate differential operator (see (2.3) above)

$$\mathrm{P} = r^{-l} \sum_{i\leq l} a_i(r)(-r\partial_r)^i.$$

As an element in $\mathrm{Diff}(M)$ it has its classical principal homogeneous symbol given by

$$\sigma^l(\mathrm{P})(r,x,\rho,\xi) = r^{-l} \sum_{i+|\alpha|=l} a_{i,\alpha}(r,x)(-\sqrt{-1}r\rho)^i \xi^\alpha$$

with smooth coefficients $a_{i,\alpha}(r,x)$ up to $r=0$. The classical symbol $\sigma^l(\mathrm{P})$ is a function acting on the cotangent bundle T^*M, homogeneous of degree l on the fibers and singular on the singular set $\mathcal{E}$ (corresponding to $r=0$). In order to reflect the singular behavior on the symbolic structure two additional symbols are introduced. First we have the homogeneous boundary symbol

$$\sigma^l_b(\mathrm{P})(r,x,\tilde{\rho},\xi) = \sum_{i+|\alpha|=l} a_{i,\alpha}(r,x)(-\sqrt{-1}\tilde{\rho})^i \xi^\alpha$$

defined on the cotangent bundle of the stretched manifold $T^*\mathbb{M}$ and smooth up to $r=0$. Observe that we have the relation

$$\sigma_b^l(\mathrm{P})(r,x,\rho,\xi) = r^l\sigma^l(\mathrm{P})(r,x,r^{-1}\rho,\xi).$$

The second symbol is the Mellin symbol $\sigma_M^l(\mathrm{P})$. Recall that any cone-degenerate operator $\mathrm{P} = r^{-l}\sum_{i\leq l} a_i(r)(-r\partial_r)^i$ is given in terms of the Mellin transformation as follows

$$\mathrm{P} = r^{-l}\mathcal{M}^{-1}h(r,z)\mathcal{M},$$

where $h(r,z) = \sum_{i\leq l} a_i(r)z^i$. At the singular set i.e. when $r=0$ we have a holomorphic family of operators

$$h(0,z) : \mathbb{C} \longrightarrow \mathcal{L}\left(H^s(\mathcal{X}), H^{s-l}(\mathcal{X})\right). \tag{2.17}$$

This holomorphic operator-valued function is the Mellin symbol of P i.e. $\sigma_M^l(\mathrm{P})(z) := h(0,z)$. As we shall see below the ellipticity of P will be given by the invertibility of the symbolic structure $(\sigma_b^l(\mathrm{P}), \sigma_M^l(\mathrm{P}))$, the first one as a bundle map on $T^*\mathbb{M}\setminus\{0\}$ and the second as a family of invertible continuous operators.

Now consider an edge-degenerate differential operator (see (2.6) above)

$$\mathrm{P} = r^{-l}\sum_{i+|\alpha|\leq l} a_{i,\alpha}(r,u)(-r\partial_r)^i(rD_u)^\alpha.$$

In the same way as the cone-degenerate case it has a classical homogeneous principal symbol and a homogeneous boundary symbol

$$\sigma^l(\mathrm{P})(r,x,u,\rho,\xi,\eta) = r^{-l}\sum_{i+|\alpha|+|\beta|=l} a_{i,\alpha,\beta}(r,x,u)(-\sqrt{-1}r\rho)^i(r\eta)^\alpha\xi^\beta$$

$$\sigma_b^l(\mathrm{P})(r,x,u,\tilde{\rho},\xi,\tilde{\eta}) = \sum_{i+|\alpha|+|\beta|=l} a_{i,\alpha,\beta}(r,x,u)(-\sqrt{-1}\tilde{\rho})^i\tilde{\eta}^\alpha\xi^\beta \tag{2.18}$$

satisfying the relation

$$\sigma_b^l(\mathrm{P})(r,x,u,\rho,\xi,\eta) = r^l\sigma^l(\mathrm{P})(r,x,r^{-1}\rho,\xi,r^{-1}\eta).$$

In the conical case the Mellin symbol is an operator-valued function such that those operators act on Sobolev spaces defined on the link $\mathcal{X}$ i.e. one level below in the singular hierarchy. Analogously, in the edge case we have the edge symbol $\sigma_\wedge^l(\mathrm{P})$ that as we expected is defined as an operator-valued function acting on spaces one level below in the singular hierarchy, in this case those operators act on the cone-Sobolev spaces $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ as cone-degenerate operators. More precisely we have

$$\sigma_\wedge^l(\mathrm{P}) : T^*\mathcal{E}\setminus\{0\} \longrightarrow \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge), \mathcal{K}^{s-l,\gamma-l}(\mathcal{X}^\wedge)\right)$$

given by

$$\sigma_\wedge^l(\mathrm{P})(u,\eta) = r^{-l}\sum_{i+|\alpha|\leq l} a_{i,\alpha}(0,u)(-r\partial_r)^i(r\eta)^\alpha. \tag{2.19}$$

This is a family of cone-degenerate operators parametrized by the cotangent bundle of the edge $\mathcal{E}$. Then, as in the conical case, the edge symbol has a Mellin symbol $\sigma^l_M(\sigma^l_\wedge(\mathrm{P}))$ associated to it. The ellipticity of P requires the invertibility of the symbolic structure $(\sigma^l_b(\mathrm{P}), \sigma^l_\wedge(\mathrm{P}))$. In general, the invertibily of the homogeneous boundary symbol only implies that the edge symbol defines a family of Fredholm operators in $\mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge), \mathcal{K}^{s-l,\gamma-l}(\mathcal{X}^\wedge)\right)$. This imposes the need for including boundary and coboundary conditions to complete the edge symbol. Once the symbol is complete we obtain a 2 by 2 matrix of operators with the original operator P in the upper left corner (see (2.30) below).

2.3.3 Conormal asymptotics

Given a manifold with conical singularities M and an elliptic cone-degenerate operator $\mathrm{P} \in \mathrm{Diff}_{\mathrm{cone}}(M)$, we are interested in the solutions of the equation $\mathrm{P}\, f = 0$ on cone-Sobolev spaces. It was proved by Kondratev in the 60's that solutions of such equations always have conormal asymptotic expansions near the singular points, see [Kon67]. Since then, such asymptotics have been an important part in the formulation of several calculus of (pseudo) differential operators on manifolds with conical or edge singularities. In this subsection we recall the basic facts of such asymptotics. For a complete presentation see [Sch98] section 2.3.

A sequence $O = \{(p_j, m_j)\}_{j\in\mathbb{N}}$ in $\mathbb{C} \times \mathbb{Z}^+$ is called an asymptotic type for the weight data $\gamma \in \mathbb{R}$ if

$$\operatorname{Re} p_j < \frac{\dim \mathcal{X} + 1}{2} - \gamma$$

and $\operatorname{Re} p_j \to -\infty$ when $j \to \infty$.

Definition 15. Let $O = \{(p_j, m_j)\}_{j\in\mathbb{N}}$ be an asymptotic type for the weight $\gamma \in \mathbb{R}$. The cone-Sobolev space with conormal asymptotics O, denoted by $\mathcal{K}^{s,\gamma}_O(\mathcal{X}^\wedge)$, is defined as the set of all $f \in \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ such that for every $l \in \mathbb{N}$ there is $N(l) \in \mathbb{N}$ such that

$$f(r,\sigma,y) - \omega(r) \sum_{j=0}^{N(l)} \sum_{k=0}^{m_j} c_{j,k}(\sigma) r^{-p_j} \log^k(r) \in \mathcal{K}^{s,\gamma+l}(\mathcal{X}^\wedge) \tag{2.20}$$

with $c_{j,k}(\sigma) \in C^\infty(\mathcal{X})$.

The asymptotics (2.20) arise naturally from the symbolic structure of the cone-degenerate operators, more precisely from the Mellin symbol (2.17). Given a cone-degenerate operator $\mathrm{P} \in \mathrm{Diff}^l_{\mathrm{cone}}(M)$, its Mellin symbol $\sigma^l_M(\mathrm{P}) = h(0,z)$ defines a holomorphic operator-valued function (2.17). Thus, the inverse $h^{-1}(0,z)$ defines a meromorphic operator-valued function. The set of poles $p_j \in \mathbb{C}$ with their respective multiplicities $n_j \in \mathbb{N}$ will define an associated asymptotic type $\{(p_j, m_j)\}_{j\in\mathbb{N}}$ that will produce the asymptotic expansion (2.20). See section 2.3.4 for an explicit example.

The space $\mathcal{K}^{s,\gamma}_O(\mathcal{X}^\wedge)$ has the structure of a Fréchet space given as an inductive limit of spaces with asymptotics of finite type $\mathcal{K}^{s,\gamma}_{O_k}(\mathcal{X}^\wedge)$ where $O_k = \{(p_j, m_j) \in O : \frac{\dim \mathcal{X}+1}{2} - \gamma - k < \operatorname{Re} p_j < \frac{\dim \mathcal{X}+1}{2} - \gamma\}$, see [ES97] sec. 8.1.1 for details. By using this inductive limit structure we define the edge-Sobolev space with conormal asymptotics O as the inductive limit of Fréchet spaces

$$\mathcal{W}^{s,\gamma}\left(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{O}(\mathcal{X}^\wedge)\right) := \varinjlim_{k} \mathcal{W}^{s,\gamma}\left(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{O_k}(\mathcal{X}^\wedge)\right).$$

In particular, if $f \in \mathcal{W}^{\infty,\gamma}_{O}(M)$ then for every $l \in \mathbb{N}$ there is $N(l) \in \mathbb{N}$ such that

$$f(r,\sigma,y) - \omega(r) \sum_{j=0}^{N(l)} \sum_{k=0}^{m_j} c_{j,k}(\sigma) v_{j,k}(y) r^{-p_j} \log^k(r) \in \mathcal{W}^{\infty,\gamma+l}(M)$$

with $c_{j,k}(\sigma) \in C^\infty(\mathcal{X})$ and $v_{j,k}(y) \in H^\infty(\mathcal{E})$, see [Sch98] proposition 3.1.33.

2.3.4 Examples

In order to illustrate the role of the conormal asymptotics we consider a simple example. Here we follow [ES97] section 9.4.2.

Let's consider the half-axis $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$ and the Laplace operator $\Delta_{\mathbb{R}^+} = \left(\frac{d}{dr}\right)^2$ induced by the flat Euclidean metric $g_{\mathbb{R}}$. We can rewrite this operator as an explicit cone-degenerate operator:

$$\Delta_{\mathbb{R}^+} = \frac{1}{r^2}\left(\left(-r\frac{d}{dr}\right) + \left(-r\frac{d}{dr}\right)^2\right).$$

Thus, it defines a continuous linear operator acting between the following spaces

$$\Delta_{\mathbb{R}^+} : \mathcal{K}^{s,\gamma}(\mathbb{R}^+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^+).$$

Its Mellin symbol (2.17) is given by

$$\sigma^2_M(\Delta_{\mathbb{R}^+}) = z + z^2, \tag{2.21}$$

hence we can write the Laplacian in terms of the Mellin transformation as

$$\Delta_{\mathbb{R}^+} = \frac{1}{r^2}\mathcal{M}^{-1}(z+z^2)\mathcal{M}. \tag{2.22}$$

Let $\varphi(r) \in \mathcal{S}(\mathbb{R}^+)$ and let's consider a solution $f(r)$ of the equation $\Delta_{\mathbb{R}^+} f = \varphi$. By (2.22) and the elementary properties of the Mellin transformation in section 2.3.1 we have that $(z+z^2)(\mathcal{M}f)(z) = \mathcal{M}(\varphi)(z+2)$. Now, by using the definition of the Mellin transformation (2.7) we can prove that

$$\mathcal{M}(\varphi)(z) = \int_0^\infty r^{z-1}\varphi(r)dr$$

is meromorphic with simple poles at $z \in \mathbb{Z}^- \cup \{0\}$. Therefore, $\mathcal{M}(\varphi)(z+2)$ has simple poles at $z \in \mathbb{C}$ such that $z = -2, -3, -4, \cdots$.

On the other hand the polynomial $h(z) = z + z^2$ has simple roots at $z = 0, -1$. Then $h^{-1}(z)$ is a meromorphic function with simple poles at $z = 0, -1$. Hence, the Mellin transformation of the solution of the equation is given by

$$(\mathcal{M}f)(z) = \frac{1}{(z + z^2)} \mathcal{M}(\varphi)(z + 2) \tag{2.23}$$

with simple poles at $z \in \mathbb{Z}^- \cup \{0\}$.

Let's consider one of these poles, let's say $z = -k$. Then there exits an open neighborhood $\mathcal{U}_k \subset \mathbb{C}$ around $z = -k$, a complex number $a^{(k)} \in \mathbb{C}$ and a function $G_{(k)}(z)$ holomorphic on $\mathcal{U}_k$ such that

$$(\mathcal{M}f)(z) = \frac{a^{(k)}}{z + k} + G_{(k)}(z) \tag{2.24}$$

on $\mathcal{U}_k \subset \mathbb{C}$.

Now, by using the fact that the Mellin transformation satisfies

$$\mathcal{M}(r^{-p} \log^j(r)) = \frac{(-1)^j j!}{(z - p)^{j+1}}$$

with $p \in \mathbb{C}$ and $j \in \mathbb{N}$, we obtain that

$$\mathcal{M}\left(f(r) - a^{(k)} r^k \log^0(r)\right)(z) \tag{2.25}$$

is holomorphic on $\mathcal{U}_k$. Observe that as the poles have multiplicity 1, the log terms do not play a role, $\log^0(r) = 1$. By subtracting more terms in (2.25) we have that for every $N \in \mathbb{N}$ the Mellin transformation

$$\mathcal{M}\left(f(r) - \sum_{k=0}^{N} a^{(k)} r^k \log^0(r)\right)(z)$$

defines a holomorphic function on the strip $\{z \in \mathbb{C} : N + 1 < \operatorname{Re} z\}$.

Based in the previous discussion, it is possible to prove that near the vertex $r = 0$ we have

$$\omega(r) f(r) - \omega(r) \sum_{k=0}^{N} a^{(k)} r^k \log^0(r) \in \mathcal{K}^{\infty, N}(\mathbb{R}^+) \tag{2.26}$$

for every $N \in \mathbb{N}$. The reader is referred to [ES97] section 9.4.2 for complete details.

2.3.5 Ellipticity on manifolds with conical and edge singularities

In this subsection we present the definition and principal implications of ellipticity for cone and edge-degenerate differential operators.

Definition 16. A cone-degenerate differential operator of order l, $\mathrm{P} \in \mathrm{Diff}^l_{\mathrm{cone}}(M)$, is called elliptic with respect to the weight γ if

i) $\sigma_b^l(\mathrm{P})(\tilde{\rho}, \xi) \neq 0$ on $T^*\mathbb{M} \setminus \{0\}$

ii) the Mellin symbol $\sigma_M^l(\mathrm{P})(z) = h(0, z)$ defines a family of Banach space isomorphisms in $\mathcal{L}\left(H^s(\mathcal{X}), H^{s-l}(\mathcal{X})\right)$ for some $s \in \mathbb{R}$ and all $z \in \Gamma_{\frac{\dim \mathcal{X}+1}{2}-\gamma}$.

As we mentioned before, the concept of ellipticity in the conical case requires the invertibility of the symbolic structure $(\sigma_b^l(\mathrm{P}), \sigma_M^l(\mathrm{P}))$. Observe that the invertibility of the Mellin symbol is required along the set $\Gamma_{\frac{\dim \mathcal{X}+1}{2}-\gamma} \cong \mathbb{R}$ (see (2.8)). These sets are completely determined by γ hence the invertibilty of the Mellin symbol imposes conditions on the weights γ and therefore on the spaces $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ on which the operator P is elliptic.

Definition 17. If $\gamma \in \mathbb{R}$ such that the Mellin symbol is invertible along $\Gamma_{\frac{\dim \mathcal{X}+1}{2}-\gamma}$ we say that γ is an admissible weight.

Now we have some important implications of being elliptic.

Theorem 2.5. Let P be a cone-degenerate operator of order l.

i) (Fredholm property) P is elliptic with respect to the weight γ if and only if

$$\mathrm{P} : \mathcal{H}^{s,\gamma}(M) \longrightarrow \mathcal{H}^{s-l,\gamma-l}(M)$$

is Fredholm for every $s \in \mathbb{R}$.

ii) (Parametrix) If P is elliptic then there exists a parametrix of order $-l$ with asymptotics i.e.

$$\mathfrak{P} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma-l}(M), \mathcal{H}^{s+l,\gamma}(M))$$

such that

$$\mathfrak{P}\,\mathrm{P} - I \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma}(M), \mathcal{H}_O^{\infty,\gamma}(M)) \tag{2.27}$$

$$\mathrm{P}\,\mathfrak{P} - I \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma-l}(M), \mathcal{H}_{O'}^{\infty,\gamma-l}(M)) \tag{2.28}$$

with some asymptotic types O, O' associated to γ and $\gamma - l$ respectively.

iii) (Elliptic regularity) If P is elliptic with respect to γ and $\mathrm{P}\,f = g$ with $g \in \mathcal{H}_O^{s-l,\gamma-l}(M)$ and $f \in \mathcal{H}^{-\infty,\gamma}(M)$ for some $s \in \mathbb{R}$ and some asymptotic type O associated with $\gamma - l$ then $f \in \mathcal{H}_Q^{s,\gamma}(M)$ with an asymptotic type Q associated with γ.

Remark 2.6. Theorem 2.5 also holds for spaces on the open cone space $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$, see [Sch98] thm. 2.4.40. Here we need an extra condition on the non-compact end of the open cone $\mathcal{X}^\wedge$, see [Sch98] definition 2.4.35. Moreover, elliptic regularity is also valid for spaces with asymptotics $\mathcal{K}_O^{s,\gamma}(\mathcal{X}^\wedge)$, [Sch98] thm. 2.4.42.

Remark 2.7. The resulting asymptotic type Q in the elliptic regularity statement in theorem 2.5 is related to the initial asymptotic type O by means of the poles of the Mellin symbol $\sigma_M^l(\mathrm{P})(z) = h(0, z)$. See [ES97], section 8.1 theorem 3 for details.

Remark 2.8. Consider a cone-degenerate operator P and $f \in \mathcal{H}^{s,\gamma}(M)$ such that $\mathrm{P} f = 0$. The parametrix with asymptotics, and specifically (2.27) implies that $(\mathfrak{P}\,\mathrm{P} - I)(f) = f \in \mathcal{H}^{\infty,\gamma}_{O}(M)$. Therefore solutions of cone-degenerate equations are smooth on the regular part of M and have conormal asymptotics expansions near the vertex of the cone.

In order to introduce the notion of ellipticity in the edge singular setting we need to make some assumptions. Assume that there exist vector bundles J^+ and J^- over $\mathcal{E}$ and operator families parametrized by $T^*\mathcal{E} \setminus \{0\}$ acting as follows

$$\sigma^l_\wedge(\mathrm{T})(u,\eta) : \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge) \longrightarrow J^+_u,$$

$$\sigma^l_\wedge(\mathrm{C})(u,\eta) : J^-_u \longrightarrow \mathcal{K}^{s-l,\gamma-l}(\mathcal{X}^\wedge),$$

$$\sigma^l_\wedge(\mathrm{B})(u,\eta) : J^-_u \longrightarrow J^+_u;$$

such that

$$\begin{bmatrix} \sigma^1_\wedge(\mathrm{P})(u,\eta) & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge) \\ \oplus \\ J^-_u \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-l,\gamma-l}(\mathcal{X}^\wedge) \\ \oplus \\ J^+_u \end{matrix} \tag{2.29}$$

is a family of continuous operators for every $(u,\eta) \in T^*\mathcal{E} \setminus \{0\}$.

The existence of the vector bundles $J^\pm$ and operators acting between the fibers and cone-Sobolev spaces will be discussed in section 3.2.3. Here we only mention that there is a topological obstruction (see theorem 3.12) that must be satisfied in order to guarantee the existence of $J^\pm$ and the operators.

Definition 18. An edge-degenerate differential operator $\mathrm{P} \in \mathrm{Diff}^l_{\mathrm{edge}}(M)$ of order l for which (2.29) exists, is called elliptic with respect to the weight γ if

i) $\sigma^l_b(\mathrm{P}) \neq 0$ on $T^*\mathbb{M} \setminus \{0\}$

ii) the operator matrix (2.29) defines an invertible operator for some $s \in \mathbb{R}$ and each $(u,\eta) \in T^*\mathcal{E} \setminus \{0\}$.

Theorem 2.9. Let P be an edge-degenerate operator of order l.

i) (Fredholm property) P is elliptic with respect to the weight γ if and only if the operator

$$\mathcal{A}_{\mathrm{P}} := \begin{bmatrix} \mathrm{P} & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} = \mathcal{F}^{-1}_{\eta\to u} \begin{bmatrix} \sigma^1_\wedge(\mathrm{P})(u,\eta) & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix} \mathcal{F}_{u'\to\eta} \tag{2.30}$$

acting on the spaces

$$\mathcal{A}_{\mathrm{P}} : \begin{matrix} \mathcal{W}^{s,\gamma}(M) \\ \oplus \\ H^s(\mathcal{E}, J^-) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-l,\gamma-l}(M) \\ \oplus \\ H^{s-l}(\mathcal{E}, J^+) \end{matrix}$$

is Fredholm for every $s \in \mathbb{R}$.

ii) (Parametrix) If P is elliptic then there exists a parametrix of order $-l$ with asymptotics for $\mathcal{A}_{\mathrm{P}}$ i.e.

$$\mathfrak{P} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}^{s,\gamma-l}(M) & \mathcal{W}^{s+l,\gamma}(M) \\ \oplus & , \quad \oplus \\ H^s(\mathcal{E},J^+) & H^{s+l}(\mathcal{E},J^-) \end{pmatrix}$$

such that

$$\mathfrak{P}\mathcal{A}_{\mathrm{P}} - I \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}^{s,\gamma}(M) & \mathcal{W}^{\infty,\gamma}_{O}(M) \\ \oplus & , \quad \oplus \\ H^s(\mathcal{E},J^-) & H^{\infty}(\mathcal{E},J^-) \end{pmatrix}$$

and

$$\mathcal{A}_{\mathrm{P}}\mathfrak{P} - I \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}^{s,\gamma-l}(M) & \mathcal{W}^{\infty,\gamma-l}_{O'}(M) \\ \oplus & , \quad \oplus \\ H^s(\mathcal{E},J^+) & H^{\infty}(\mathcal{E},J^+) \end{pmatrix}$$

with some asymptotic type O, O' associated to γ and $\gamma - l$ respectively.

iii) (Elliptic regularity) If P is elliptic with respect to γ and $\mathcal{A}_{\mathrm{P}} f = g$ with

$$g \in \begin{matrix} \mathcal{W}^{s-l,\gamma-l}_{O}(M) \\ \oplus \\ H^{s-l}(\mathcal{E},J^+) \end{matrix} \quad \text{and} \quad f \in \begin{matrix} \mathcal{W}^{-\infty,\gamma}(M) \\ \oplus \\ H^{-\infty}(\mathcal{E},J^-) \end{matrix}$$

for some $s \in \mathbb{R}$ and some asymptotic type O associated with $\gamma - l$ then

$$f \in \begin{matrix} \mathcal{W}^{s,\gamma}_{Q}(M) \\ \oplus \\ H^{s}(\mathcal{E},J^-) \end{matrix}$$

with an asymptotic type Q associated with γ.

Remark 2.10. Analogously to remark 2.8 we have that solutions to the edge-degenerate equation $\mathcal{A}_{\mathrm{P}} f = 0$ belong to $\mathcal{W}^{\infty,\gamma}_{O}(M)$, hence they are smooth and have conormal asymptotics near the edge $\mathcal{E}$.

2.3.6 Examples

In this subsection we illustrate the concept of ellipticity discussed in the previous section. This example is based on [NSSS06] section 6.1.1.

Consider the upper half-space $\mathbb{R}^+ \times \mathbb{R}^n = \{(x_0, \cdots, x_n) : x_0 > 0\}$. This space can be considered as a manifold with edge singularities where the link of the cone is a point i.e. $\mathcal{X} = \{*\}$ and $\mathcal{E} = \mathbb{R}^n$. Let $\Delta_{\mathbb{R}^+\times\mathbb{R}^n}$ be the Laplacian induced by the restriction of the flat metric $g_{\mathbb{R}^{n+1}}$ to $\mathbb{R}^+ \times \mathbb{R}^n$. Then

$$\Delta_{\mathbb{R}^+\times\mathbb{R}^n} = \sum_{k=0}^{n} D^2_{x_k},$$

where $D_{x_k} = \frac{1}{\sqrt{-1}}\partial_{x_k}$. We can rewrite $\Delta_{\mathbb{R}^+\times\mathbb{R}^n}$ as an explicit edge-degenerate differential operator (2.6) in the following way:

$$\Delta_{\mathbb{R}^+\times\mathbb{R}^n} = \frac{1}{r^2}\left(-(-r\partial_r)^2 - (-r\partial_r) + (rD_{x_1})^2 + \cdots + (rD_{x_n})^2\right),$$

where $r = x_0$.

Let's analyze the symbolic structure and ellipticity of the edge-degenerate operator $\Delta_{\mathbb{R}^+\times\mathbb{R}^n}$.

i) The homogeneous boundary symbol (2.18):

$$\sigma_b^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(r,u,\tilde{\rho},\tilde{\eta}) = \tilde{\rho}^2 + |\tilde{\eta}|^2 .$$

Hence $\sigma_b^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(r,u,\tilde{\rho},\tilde{\eta}) \neq 0$ for every $(\tilde{\rho},\tilde{\eta}) \neq (0,0)$.

ii) The edge symbol (2.19):

$$\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n}) : T^*\mathbb{R}^n \setminus \{0\} \longrightarrow \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathbb{R}^+), \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^+)\right)$$

is given by

$$\begin{aligned}\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta) &= \frac{1}{r^2}\left(-(-r\partial_r)^2 - (-r\partial_r) + (r\eta_1)^2 + \cdots + (r\eta_n)^2\right)\\ &= |\eta|^2 - \partial_r^2.\end{aligned}$$

This is a family of cone-degenerate differential operators parametrized by $(u,\eta) \in T^*\mathbb{R}^n \setminus \{0\}$.

iii) The Mellin symbol (2.17):

$$\sigma_M^2(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)) : \mathbb{C} \longrightarrow \mathbb{C}$$

is given by

$$\sigma_M^2(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta))(z) = -(z^2+z).$$

The Mellin symbol is not invertible when $z = 0, -1$, hence, by definition 16, we have that the exceptional weights are $\gamma = \frac{1}{2}, \frac{3}{2}$. Therefore by theorem 2.5 we have that the edge symbol

$$\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta) = |\eta|^2 - \partial_r^2 : \mathcal{K}^{s,\gamma}(\mathbb{R}^+) \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^+),$$

is a Fredholm operator for every $(u,\eta) \in T^*\mathbb{R}^n \setminus \{0\}$ and $\gamma \neq \frac{1}{2}, \frac{3}{2}$. Now, let's compute $\mathrm{Ker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right)$.

We can easily check that a basis for the space of solutions of the ordinary differential equation

$$\left(|\eta|^2 - \partial_r^2\right) f(r) = 0$$

is given by $\{e^{|\eta|r}, e^{-|\eta|r}\}$. The solution $e^{|\eta|r}$ is not bounded and grows very fast when $r \to +\infty$. We can easily check that $e^{|\eta|r}$ does not belong to $\mathcal{K}^{s,\gamma}(\mathbb{R}^+)$. On the other hand we can check that $e^{-|\eta|r} \in \mathcal{K}^{s,\gamma}(\mathbb{R}^+)$ only if $\gamma < \frac{1}{2}$. Let's consider all the possible cases.

i) Case $\gamma < \frac{1}{2}$.

In this case $e^{-|\eta| r} \in \mathcal{K}^{s,\gamma}(\mathbb{R}^+)$, therefore $\dim \operatorname{Ker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = 1$. In order to find $\dim \operatorname{Coker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right)$ let's consider the formal adjoint operator. The operator $|\eta|^2 - \partial_r^2$ is formally self-adjoint, thus the dual operator is given by

$$|\eta|^2 - \partial_r^2 : \mathcal{K}^{-s+2,-\gamma+2}(\mathbb{R}^+) \longrightarrow \mathcal{K}^{-s,-\gamma}(\mathbb{R}^+). \tag{2.31}$$

Observe that $\gamma < \frac{1}{2}$ implies $-\gamma + 2 > \frac{3}{2}$. Thus, $\operatorname{Coker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = \{0\}$ and $\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)$ is surjective. On the cosphere bundle $S^*(\mathbb{R}^n)$, the kernel $\operatorname{Ker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right)$ defines the fiber $J^-_{(u,\eta)}$ of a complex line bundle J^- over $(u,\eta) \in S^*(\mathbb{R}^n)$. Moreover, as $\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta) = 1 - \partial_r^2$ for $(u,\eta) \in S^*\mathbb{R}^n$, we have that the operators defined by $\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)$ do not depend on (u,η). Therefore the bundle J^- is a trivial line bundle over $S^*\mathbb{R}^n$.

Let

$$\mathrm{t}(\mathrm{u},\eta) : \mathcal{K}^{s,\gamma}(\mathbb{R}^+) \longrightarrow \operatorname{Ker}\left(\sigma_\wedge^2(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = J^-_{(u,\eta)} \cong \mathbb{C}$$

be the projection onto the subspace generated by $\{e^{-|\eta| r}\}$. Then

$$\mathrm{t}(\mathrm{u},\eta)(f) = \int_0^\infty \phi(u,\eta)(r) f(r) dr$$

where $\phi(u,\eta)(r)$ is an unitary function in the subspace spanned by $\{e^{-|\eta| r}\}$.

Therefore, it follows immediately that the family of operators

$$\begin{bmatrix} |\eta|^2 - \partial_r^2 \\ \mathrm{t}(u,\eta) \end{bmatrix} : \mathcal{K}^{s,\gamma}(\mathbb{R}^+) \longrightarrow \begin{array}{c} \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^+) \\ \oplus \\ \mathbb{C} \end{array}$$

is a family of isomorphisms of Banach spaces for every $(u,\eta) \in T^*\mathbb{R}^n \setminus \{0\}$. Now, observe that

$$\mathcal{F}^{-1}_{\eta\to u}\left(|\eta|^2 - \partial_r^2\right)\mathcal{F}_{u'\to\eta} = \Delta_{\mathbb{R}^+\times\mathbb{R}^n}$$

and the trace pseudo-differential operator

$$\mathcal{T} := \mathcal{F}^{-1}_{\eta\to u}\,\mathrm{t}(u,\eta)\mathcal{F}_{u'\to\eta} : \mathcal{W}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^n) \longrightarrow H^{s-2}(\mathbb{R}^n)$$

is given explicitly by

$$\mathcal{T}(G)(r,u) = \int_{\mathbb{R}^n}\int_0^\infty e^{\sqrt{-1}\eta\cdot u}\phi(u,\eta)(r)\mathcal{F}_{u\to\eta}(G)(r,\eta)drd\eta.$$

Hence by theorem 2.9 the edge boundary value problem

$$\begin{bmatrix} \Delta_{\mathbb{R}^+\times\mathbb{R}^n} \\ \mathcal{T} \end{bmatrix} : \mathcal{W}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^n) \longrightarrow \begin{array}{c} \mathcal{W}^{s-2,\gamma-2}(\mathbb{R}^+\times\mathbb{R}^n) \\ \oplus \\ H^{s-2}(\mathbb{R}^n) \end{array}$$

is elliptic and therefore it is a Fredholm operator.

ii) Case $\frac{1}{2} < \gamma < \frac{3}{2}$.

In this case $e^{-|\eta|r} \notin \mathcal{K}^{s,\gamma}(\mathbb{R}^+)$, therefore $\dim \operatorname{Ker}\left(\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = 0$. Moreover, the fact that $\gamma < \frac{3}{2}$, implies $-\gamma + 2 > \frac{1}{2}$. Thus the kernel of the formal adjoint operator (2.31) is trivial and $\dim \operatorname{Coker}\left(\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = 0$. Hence the edge symbol $\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)$ defines a family of isomorphism of Banach spaces for every $(u,\eta) \in T^*\mathbb{R}^n \setminus \{0\}$. By theorem 2.9 we have that

$$\Delta_{\mathbb{R}^+\times\mathbb{R}^n} : \mathcal{W}^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^n) \longrightarrow \mathcal{W}^{s-2,\gamma-2}(\mathbb{R}^+ \times \mathbb{R}^n)$$

is elliptic and therefore a Fredholm operator.

iii) Case $\frac{3}{2} < \gamma$.

In this case $e^{-|\eta|r} \notin \mathcal{K}^{s,\gamma}(\mathbb{R}^+)$, therefore $\dim \operatorname{Ker}\left(\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = 0$. However, the fact that $\gamma > \frac{3}{2}$, implies $-\gamma + 2 < \frac{1}{2}$. Thus the kernel of the formal adjoint operator (2.31) is spanned by $\{e^{-|\eta|r}\}$ and $\dim \operatorname{Coker}\left(\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)\right) = 1$ for every $(u,\eta) \in T^*\mathbb{R}^n \setminus \{0\}$. At each point in the cosphere bundle $(u,\eta) \in S^*\mathbb{R}^n$ we can fix an isomorphism $\mathrm{c}(u,\eta) : \mathbb{C} \longrightarrow \operatorname{Coker}(\sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta)) \cong \mathbb{C}$ such that

$$\begin{bmatrix} \sigma^2_\wedge(\Delta_{\mathbb{R}^+\times\mathbb{R}^n})(u,\eta) & \mathrm{c}(u,\eta) \end{bmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(\mathbb{R}^+) \\ \oplus \\ \mathbb{C} \end{matrix} \longrightarrow \mathcal{K}^{s-2,\gamma-2}(\mathbb{R}^+)$$

is a family of isomorphisms of Banach spaces.

Therefore, by considering the co-boundary pseudo-differential operator

$$\mathrm{C} := \mathcal{F}^{-1}_{\eta\to u}\, \mathrm{c}(u,\eta)\mathcal{F}_{u'\to\eta} : H^s(\mathbb{R}^n) \longrightarrow \mathcal{W}^{s-2,\gamma-2}(\mathbb{R}^+ \times \mathbb{R}^n)$$

given explicitly by

$$\mathrm{C}(\phi)(r,u) = \int\limits_{\mathbb{R}^n} e^{\sqrt{-1}\eta\cdot u}\, \mathrm{c}(u,\eta)\mathcal{F}_{u\to\eta}(\phi)(\eta)d\eta,$$

theorem 2.9 implies that the edge co-boundary value problem

$$\begin{bmatrix} \Delta_{\mathbb{R}^+\times\mathbb{R}^n} & \mathrm{C} \end{bmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{R}^+ \times \mathbb{R}^n) \\ \oplus \\ H^s(\mathbb{R}^n) \end{matrix} \longrightarrow \mathcal{W}^{s-2,\gamma-2}(\mathbb{R}^+ \times \mathbb{R}^n)$$

is elliptic and therefore a Fredholm operator.

2.3.7 Green operators in the edge algebra

In this subsection we collect some basic facts about Green operators and their relation with edge-degenerate differential operators. We will use these concepts in section 4.2. Consider an elliptic edge-degenerate differential operator $\mathrm{P} \in \operatorname{Diff}^l_{\text{edge}}(M)$ with associated elliptic edge boundary value problem given by

$$\mathcal{A}_{\mathrm{P}} := \begin{bmatrix} \mathrm{P} & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} = \mathcal{F}^{-1}_{\eta\to u} \begin{bmatrix} \sigma^1_\wedge(\mathrm{P})(u,\eta) & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix} \mathcal{F}_{u'\to\eta} \tag{2.32}$$

acting on the spaces

$$\mathcal{A}_{\mathrm{P}} = \begin{bmatrix} \mathrm{P} & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(M) \\ \oplus \\ H^s(\mathcal{E}, J^-) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-l,\gamma-l}(M) \\ \oplus \\ H^{s-l}(\mathcal{E}, J^+) \end{matrix}$$

with γ an admissible weight.

Definition 19. Let $\Omega \subset \mathbb{R}^q$ be an open set and $\gamma \in \mathbb{R}$. An operator function

$$g(u,\eta) \in \bigcap_{s\in\mathbb{R}} C^\infty\left(\Omega\times\mathbb{R}^q, \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge), \mathcal{K}^{\infty,\gamma-l}(\mathcal{X}^\wedge)\right)\right) \tag{2.33}$$

is called a Green symbol of order $\mu \in \mathbb{R}$ with asymptotics if there are asymptotic types O and O' associated with $\gamma - l$ and $-\gamma$ respectively such that

$$g(u,\eta) \in \bigcap_{s\in\mathbb{R}} S^\mu_{\mathrm{cl}}\left(\Omega\times\mathbb{R}^q, \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge), \mathcal{S}^{\gamma-l}_O(\mathcal{X}^\wedge)\right)\right), \tag{2.34}$$

$$g^*(u,\eta) \in \bigcap_{s\in\mathbb{R}} S^\mu_{\mathrm{cl}}\left(\Omega\times\mathbb{R}^q, \mathcal{L}\left(\mathcal{K}^{s,-\gamma}(\mathcal{X}^\wedge), \mathcal{S}^{-\gamma}_{O'}(\mathcal{X}^\wedge)\right)\right) \tag{2.35}$$

where

$$\mathcal{S}^{\gamma-l}_O(\mathcal{X}^\wedge) := [\omega]\mathcal{K}^{\infty,\gamma-l}_O(\mathcal{X}^\wedge) + [1-\omega]\mathcal{S}(\overline{\mathbb{R}}^+ \times \mathcal{X})$$

and $S^\mu_{\mathrm{cl}}\left(\Omega\times\mathbb{R}^q, \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge), \mathcal{S}^{\gamma-l}_O(\mathcal{X}^\wedge)\right)\right)$ denotes the space of operator-valued classical symbols (see [Sch98] section 3.3.1 for details).

When P is elliptic we are able to complete the edge symbol $\sigma^l_\wedge(\mathrm{P})$ by means of boundary and coboundary conditions given by

$$\begin{bmatrix} 0 & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix}. \tag{2.36}$$

As we explain in section 3.2.3, for each $(u,\eta) \in T^*\mathcal{E}\setminus\{0\}$ these operators are basically projections onto the kernel of Fredholm cone-degenerate operators acting on the fibers of the $\mathcal{X}^\wedge$-fibration over $\mathcal{E}$. By elliptic regularity properties of cone-degenerate operators (see remark 2.8) these operators project onto cone-Sobolev spaces with asymptotics. Furthermore it is easy to prove that

$$\begin{bmatrix} 0 & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix} \tag{2.37}$$

is a Green symbol of order l with asymptotics as in definition 19.

Therefore

$$\mathcal{F}^{-1}_{\eta \to u} \begin{bmatrix} 0 & \sigma^l_\wedge(\mathrm{C})(u,\eta) \\ \sigma^l_\wedge(\mathrm{T})(u,\eta) & \sigma^l_\wedge(\mathrm{B})(u,\eta) \end{bmatrix} \mathcal{F}_{u' \to \eta} = \begin{bmatrix} 0 & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} \tag{2.38}$$

is a Green operator and it acts on edge-Sobolev spaces as follows

$$\begin{bmatrix} 0 & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(M) \\ \oplus \\ H^s(\mathcal{E}, J^-) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-l,\gamma-l}_O(M) \\ \oplus \\ H^{s-l}(\mathcal{E}, J^+) \end{matrix} . \tag{2.39}$$

A proof of this fact can be found in [Sch98] theorem 3.4.3.

2.4 Special Lagrangian submanifolds with singularities

In this section we summarize definitions and main results about singular special Lagrangian submanifolds. In particular we recall the results obtained by various authors about deformations of conical singular and asymptotically conical special Lagrangian submanifolds in $\mathbb{C}^n$. Moreover we explain the construction of special Lagrangian submanifolds given as conormal bundles, some results about deformations by twisting the fibers, [KL12], and their interpretation as manifolds with edge singularity.

2.4.1 Asymptotically conical special Lagrangian submanifolds

Let $\Phi : M \longrightarrow \mathbb{C}^n$ be a non-compact embedded submanifold in $\mathbb{C}^n$, $M_0 \subset M$ a compact subset such that there exists a diffeomorphism $\Upsilon : M \setminus M_0 \longrightarrow \mathbb{R}^+ \times \mathcal{X}$ with $\mathcal{X}$ a compact submanifold of S^{2n-1} without boundary. Let $\psi : \mathbb{R}^+ \times \mathcal{X} \longrightarrow \mathbb{C}^n$ be given by $\psi(r,\theta) = r\theta$ such that ψ defines a diffeormorphism with the cone $\mathcal{C} \subset \mathbb{C}^n$ with link $\mathcal{C} \cap S^{2n-1} = \mathcal{X}$. Observe that $\psi^*(g_{\mathbb{R}^{2n}}) = g_{\text{cone}} = r^2 g_{\mathcal{X}} + dr^2$.

Definition 20. We say M is an asymptotically conical submanifold with rate $\gamma < 2$ if there is a constant $R > 0$ such that for every $k \geq 0$

$$\left| \nabla^k \left(\Phi \circ \Upsilon^{-1}(r,\theta) - \psi(r,\theta) \right) \right|_{g_{\text{cone}}} = O(r^{\gamma-1-k}) \quad \forall (r,\theta) \in (R,\infty) \times \mathcal{X}. \tag{2.40}$$

Definition 20 implies that the induced metric $\Phi^*(g_{\mathbb{C}^n})$ converges to the conical metric g_{cone} as $r \to \infty$. In [Mar], [Pac04] and [Pac13] Marshall and Pacini studied moduli spaces of asymptotically conical special Lagrangian submanifolds in $\mathbb{C}^n$. Here we recall theorem 8.5 in the extended version of [Pac13] that can be found in [Pac12].

Theorem 2.11. Let M be an asymptotically conical special Lagrangian submanifold with rate $\gamma < 2$ Then

i) If $\gamma \in (0,2)$ is an admissible weight for the Laplace operator Δ_M on M with the induced asymptotically conical metric $g_{_M} = \Phi^*(g_{\mathbb{C}^n})$ and acting on weighted Sobolev spaces, then the moduli space of asymptotically conical special Lagrangian deformations of M is a smooth manifold of dimension equal to $b^1(M) + \dim \mathrm{Ker}(\Delta_M) - 1$.

ii) If $\gamma \in (2-n, 0)$ then the moduli space is a smooth manifold of dimension $b^1_c(M)$ i.e. the dimension of the first cohomology group with compact support of M.

2.4.2 Conically singular special Lagrangian submanifolds

These submanifolds are defined in a similar way to our previous example. For simplicity we consider only the case with one singular point i.e. $\mathcal{X}$ is connected. Let M be a manifold with conical singularity (see section 2.2.1) with an open neighborhood of the vertex $\mathfrak{v}$ given by $M \setminus M_0$ and a diffeomorphism $\Upsilon : M \setminus M_0 \longrightarrow \mathbb{R}^+ \times \mathcal{X}$ as in the previous example.

Definition 21. We say M is a conically singular submanifold with rate $\gamma > 2$ if there is a constant $\varepsilon > 0$ such that for every $k \geq 0$

$$\left| \nabla^k \left(\Phi \circ \Upsilon^{-1}(r,\theta) - \psi(r,\theta) \right) \right|_{g_{\mathrm{cone}}} = O(r^{\gamma-1-k}) \quad \forall (r,\theta) \in (0,\varepsilon) \times \mathcal{X}. \tag{2.41}$$

In [Joy04b] Joyce studied the moduli space of conically singular special Lagrangian submanifolds. Here we recall one of the main results of [Joy04b] as is presented in [Pac12] theorem 8.7.

Theorem 2.12. Let M be a conically singular special Lagrangian submanifold with rate $\gamma > 2$. Then if γ is an admissible weight for the Laplace operator Δ_M on M with the induced asymptotically conical metric $g_M = \Phi^*(g_{\mathbb{C}^n})$ and acting on weighted Sobolev spaces, then the moduli space is locally homeomorphic to the zero set of a smooth map Ψ between finite dimensional spaces $\Psi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$. When $\mathcal{M}_2 = \{0\}$, the moduli space is a smooth manifold of dimension equal to $\dim \mathcal{M}_1 = b^1_c(M)$.

2.4.3 Conormal bundles as special Lagrangian submanifolds with edge singularities

In [HL82] Harvey and Lawson constructed special Lagrangian submanifolds in $\mathbb{C}^n$ by considering $\mathbb{C}^n \cong \mathbb{R}^n_x \oplus \mathbb{R}^n_y$ as the cotangent bundle $T^*\mathbb{R}_x$. The construction is as follows. Given an immersed submanifold $M \subset \mathbb{R}_x$, the conormal bundle $\mathcal{N}^*(M)$ is a subbundle of $T^*\mathbb{R}_x|_M$. Moreover $\mathcal{N}^*(M)$ is a real n-dimensional Lagrangian submanifold of the Calabi-Yau manifold $T^*\mathbb{R}_x \cong \mathbb{C}^n$. Hence it is natural to look for conditions on M such that $\mathcal{N}^*(M)$ is special Lagrangian.

Definition 22. Let (X, g) be a Riemannian manifold and M be an immersed submanifold in X. Let's denote by $A : TM \otimes \mathcal{N}(M) \longrightarrow TM$ the second fundamental form of M given by $A(X, \mathcal{V}) := (\nabla^g_X \mathcal{V})^\top$. Then we say that M is an austere submanifold if for every $\mathcal{V} \in \mathcal{N}_p(M)$ the odd symmetric polynomial in the eigenvalues of $A(\cdot, \mathcal{V}) : T_pM \longrightarrow T_pM$ vanish.

A direct computation shows that at each $p \in M$ the operator $A(\cdot, \mathcal{V})$ is symmetric, hence diagonalizable and their eigenvalues are real.

Theorem 2.13. Let M be an immersed submanifold in $\mathbb{R}^n_x$ with codimension q. The conormal bundle $\mathcal{N}^*(M)$ is a special Lagrangian submanifold of $\mathbb{C}^n \cong T^*\mathbb{R}^n_x$ with respect to the calibration $\mathrm{Re}((\sqrt{-1})^{-q} dz_1 \wedge \cdots \wedge dz_n)$ if and only if M is an austere submanifold.

Karigiannis and Leung [KL12] obtained special Lagrangian deformations of $\mathcal{N}^*(M)$ by affinely translating each fiber $\mathcal{N}^*_p(M)$ by a cotangent vector $\xi(p) \in T^*_p(M)$, i.e. passing from $\mathcal{N}^*_p(M)$ to $\mathcal{N}^*_p(M) + \xi(p)$. They obtained conditions on $\xi \in C^\infty(T^*M)$ such that $\mathcal{N}^*(M) + \xi$ is a special Lagrangian submanifold of $T^*\mathbb{R}^n_x \cong \mathbb{C}^n$. This generalizes previous results of Borisenko [Bor93]. These twisted spaces are no longer vector bundles therefore the moduli space of these calibrated submanifolds includes deformations through non-vector bundles.

One of the main examples of austere submanifolds in $\mathbb{R}^n$ is the class of austere cones. Austere cones in $\mathbb{R}^n$ are in one to one relation with austere submanifolds in S^{n-1} i.e. a cone $\mathcal{C} \subset \mathbb{R}^n$ is austere if and only if its link $\mathcal{C} \cap S^{n-1} = \mathcal{X}$ is an austere submanifold of S^{n-1}, see [Xin03] theorem 5.2.22. Further investigations on austere cones were done by Bryant in [Bry91].

Consider an austere cone $\mathcal{C} \subset \mathbb{R}^n$. Then by theorem 2.13 the conormal bundle $\mathcal{N}^*\mathcal{C}$ is a special Lagrangian submanifold of $\mathbb{C}^n \cong T^*\mathbb{R}^n_x$ with respect to the calibration $\mathrm{Re}(i^{-q} dz_1 \wedge \cdots \wedge dz_n)$ where q is the codimension of the cone. Now let's suppose that the conormal bundle is a trivial bundle over $\mathcal{C}$. Then $\mathcal{N}^*\mathcal{C}$ is diffeomorphic to $\mathbb{R}^+ \times \mathcal{X} \times \mathbb{R}^q$ and can be consider as a manifold with edge singularity (see section 2.2.1). Observe that the stretched manifold is obtained by stretching (blowing-up) the tip of the cone to obtain $\mathbb{M} = \overline{\mathbb{R}}^+ \times \mathcal{X} \times \mathbb{R}^q$ such that $\partial\mathbb{M} = \{0\} \times \mathcal{X} \times \mathbb{R}^q$ is a trivial $\mathcal{X}$-bundle over the edge $\mathbb{R}^q$. Moreover the singular space associated to $\mathcal{N}^*\mathcal{C}$ is given by collapsing the fibers of $\partial\mathbb{M}$ to points and this space is homeomorphic to $\overline{\mathcal{C}} \times \mathbb{R}^q$ where $\overline{\mathcal{C}}$ is the closure of the cone i.e. $\frac{[0,\infty)\times\mathcal{X}}{\{0\}\times\mathcal{X}}$. Thus to get the singular space associated to $\mathcal{N}^*\mathcal{C}$ basically we attached the fictitious fiber over the tip of the cone and the resulting space is singular along that fiber. In particular observe that if $\mathcal{X} \subset S^{n-1}$ is an orientable hypersurface in S^{n-1} then the normal bundle $\mathcal{N}(\mathcal{C})$ is a trivial bundle (and therefore the conormal bundle too).

More generally, we can consider special Lagrangian submanifolds M with edge singularity $\mathcal{E}$ in a Calabi-Yau manifold $\mathfrak{X}$. For example if $\mathfrak{X}$ is a Calabi-Yau manifold, $\mathcal{E}$ a compact special Lagrangian submanifold in $\mathfrak{X}$ and M a special Lagrangian submanifold with conical singularity in $\mathbb{C}^n$ then $M \times \mathcal{E}$ is a special Lagrangian submanifold in $\mathbb{C}^n \times \mathfrak{X}$ singular along $\mathcal{E}$.

In order to study the deformation theory we need to use the elliptic theory on manifolds with edge singularities summarized in the previous section. Observe that we have assumed that the edge $\mathcal{E}$ is compact in order to have the Fredholm property. In case of non-compact edge, like $\mathbb{R}^q$, it is necessary to impose certain regularity conditions at infinity and a different elliptic theory would be needed. However, in the case of non-compact edge, if we restrict to deformations compactly supported in the fiber direction the elliptic theory of the previous section applies as we have compact embeddings of edge-Sobolev spaces for functions compactly supported in the fiber direction (see [Sch98] theorem 3.1.23).

Chapter 3

Deformation theory

3.1 Preliminaries

In order to have a deformation operator on a singular manifold M compatible with the edge singularity we shall use edge-degenerate differential forms. These forms are dual to the edge-degenerate vector fields (2.5) with respect to the edge metric $g_{_M} = r^2 g_{\mathcal{X}} + dr^2 + g_{\mathcal{E}}$. More precisely, consider the following space of differential forms γ on the stretched manifold $\mathbb{M}$ such that they vanish on all tangent directions to the fibers on $\partial\mathbb{M}$:

$$C^\infty(T^*_\wedge \mathbb{M}) := \{\gamma \in C^\infty(T^*\mathbb{M}) : \gamma|_{T\mathcal{X}_y} = 0 \ \forall y \in \mathcal{E}\}.$$

The space $C^\infty(T^*_\wedge \mathbb{M})$ is a locally free $C^\infty(\mathbb{M})$-module. By the Swan theorem [Swa62], this is the space of sections of a vector bundle $T^*_\wedge \mathbb{M}$ over $\mathbb{M}$. The vector bundle $T^*_\wedge \mathbb{M}$ is called the stretched cotangent bundle of the manifold with edges M (see [NSSS06] section 1.3.1). In local coordinates (r, σ_k, u_l) we have

$$\gamma = \mathcal{A} dr + \sum_{k=1}^{m} \mathcal{B}_k r d\sigma_k + \mathcal{C}_l du_l$$

with $\mathcal{A}, \mathcal{B}_k, \mathcal{C}_l$ in $C^\infty(\mathbb{M})$. Observe that these are differential forms that degenerate at each direction tangent to $\mathcal{X}$. We will denote by $T^*_\wedge M$ the restriction of the stretched cotangent bundle to M.

At this point we have to make an assumption on the vector bundles we consider on a manifold with edge (or conical) singularity.

Definition 23. Let M be a manifold with edge singularity. We say that a vector bundle E over M is admissible if on a collar neighborhood $(0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$ the restriction $E|_{(0,\varepsilon)\times\mathcal{X}\times\mathcal{E}}$ is the pull-back of a vector bundle $E_{\mathcal{X}}$ over $\mathcal{X}$.

Now let's consider the stretched cotangent bundle $T^*_\wedge M$ as an admissible vector bundle. In order to do that let's define

$$E_{\mathcal{X}} := \wedge^0 \mathcal{X} \oplus T^*\mathcal{X} \oplus \underbrace{\wedge^0 \mathcal{X} \oplus \wedge^0 \mathcal{X} \oplus ... \oplus \wedge^0 \mathcal{X}}_{\text{q-times}}$$

where $q = \dim \mathcal{E}$.

We shall assume that on the collar neighborhood

$$[0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$$

the stretched cotangent bundle $T^*_\wedge M$ is isomorphic to the pull-back vector bundle $\pi^*_{\mathbb{R}^+\times\mathcal{E}} E_{\mathcal{X}}$ where $\pi_{\mathbb{R}^+\times\mathcal{E}}$ is the projection

$$\pi_{\mathbb{R}^+\times\mathcal{E}} : \mathcal{X}^\wedge \times \mathcal{E} \longrightarrow \mathcal{X}. \tag{3.1}$$

where $\mathcal{X}^\wedge = \mathbb{R}^+ \times \mathcal{X}$. We shall also define the bundle $E_{\mathcal{X}^\wedge} := \pi^*_{\mathbb{R}^+} E_{\mathcal{X}}$ as the pull-back of the bundle $E_{\mathcal{X}}$ by the projection

$$\pi_{\mathbb{R}^+} : \mathcal{X}^\wedge \longrightarrow \mathcal{X}. \tag{3.2}$$

In particular if the edge $\mathcal{E}$ is a parallelizable manifold then the stretched cotangent bundle $T^*_\wedge M$ is an admissible bundle.

Throughout this section we consider $\mathbb{C}^n$ with its standard Calabi-Yau structure

$$(\mathbb{C}^n, g_{\mathbb{C}^n}, \omega_{\mathbb{C}^n}, \Omega)$$

where $g_{\mathbb{C}^n} = |dz_1|^2 + \cdots + |dz_n|^2$, $\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ and $\Omega = dz_1 \wedge \cdots \wedge dz_n$. We consider $\mathbb{C}^n$ with a fictitious edge structure as follows:

$$\mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n \cong \frac{\overline{\mathbb{R}}^+ \times S^{n-1}}{\{0\} \times S^{n-1}} \times \mathbb{R}^n.$$

Associated with this edge structure we have the stretched space

$$\mathbb{C}^n_{\mathrm{Str}} := \left(\overline{\mathbb{R}}^+ \times S^{n-1}\right) \times \mathbb{R}^n$$

such that

$$\mathbb{C}^n_{\mathrm{Str}} \setminus \partial \mathbb{C}^n_{\mathrm{Str}} = \left(\mathbb{R}^n \setminus \{0\}\right) \times \mathbb{R}^n \cong \mathbb{C}^n \setminus \left(\{0\} \times \mathbb{R}^n\right).$$

3.1.1 Submanifolds with edge singularities in $\mathbb{C}^n$

Let M be a compact manifold with edge singularity $\mathcal{E}$ (see section 2.2.1). Then the boundary of the stretched manifold $\mathbb{M}$ has a $\mathcal{X}$-fibration structure over $\mathcal{E}$, $\pi : \partial\mathbb{M} \to \mathcal{E}$, where $\mathcal{X}$ and $\mathcal{E}$ are compact smooth manifolds with $q = \dim \mathcal{E}$ and $m = \dim \mathcal{X}$. We assume that $\mathcal{X}$ is diffeomorphic to an embedded submanifold of the sphere S^{n-1} with diffeomorphism given by $\theta : \mathcal{X} \longrightarrow S^{n-1} \subset \mathbb{R}^n$. Consider the cone $\mathcal{X}^\wedge$ with cross section $\mathcal{X}$ i.e. $\mathcal{X}^\wedge = \mathcal{X} \times \mathbb{R}^+$ and let's define a diffeomorphism of $\mathcal{X}^\wedge$ with a cone $\mathcal{C} \subset \mathbb{R}^n$ by $\psi : \mathcal{X}^\wedge \longrightarrow \mathcal{C} \subset \mathbb{R}^n$ where $\psi(r, p) := (r\theta_1(p), \ldots, r\theta_n(p)) \in \mathbb{R}^n$. We shall also assume that $\mathcal{E}$ is embedded in $\mathbb{R}^n$ by $\tau : \mathcal{E} \longrightarrow \mathbb{R}^n$.

Definition 24. Let M be a compact manifold with edge singularity $\mathcal{E}$.

i) A smooth embedding $\Phi : M \longrightarrow \mathbb{C}^n$ is called an edge embedding if on a collar neighborhood $(0,\varepsilon)\times\partial\mathbb{M}$, which has the structure of a $\mathcal{X}^\wedge$-bundle over $\mathcal{E}$, the embedding Φ splits as $\Phi(r,p,v) = (\psi(r,p),\tau(v))$ with respect to the identification $\mathbb{C}^n \cong \mathbb{R}^n_x \oplus \mathbb{R}^n_y$.

ii) If $\Phi : M \longrightarrow \mathbb{C}^n$ is an edge embedding such that $\Phi(M)$ is a special Lagrangian submanifold of $\mathbb{C}^n$, we say that (M,Φ) is a special Lagrangian submanifold with edge singularity.

Observe that the definition 24 implies that the $\mathcal{X}^\wedge$-bundle structure of the collar neighborhood is trivial and the embedding Φ extends to a bundle map between the $\mathcal{X}^\triangle$-bundle over the singular space $\hat{M}$, see (2.4), and the fiber bundle over $\mathbb{R}^n_y$ with conical fibers given by $\frac{[0,1)\times S^{n-1}}{\{0\}\times S^{n-1}} \times \mathbb{R}^n_y$.

3.2 The Deformation Operator

Let $\Phi : M \longrightarrow \mathbb{C}^n$ be a compact special Lagrangian submanifold with edge singularity. In order to study the moduli space of special Lagrangian deformations of M as a manifold with edge singularities, we have to study small deformations of M inside $\mathbb{C}^n$. These deformations are produced by sections of the normal bundle $\varphi \in \mathcal{N}(M)$ via the exponential map $\exp_{g_{\mathbb{C}^n}}$. The equations

$$\begin{cases} \omega_{\mathbb{C}^n}\big|_M \equiv 0 \\ \operatorname{Im}\Omega\big|_M \equiv 0 \end{cases} \tag{3.3}$$

define a first order non-linear partial differential operator P such that φ must satisfy the equation $\mathrm{P}(\varphi) = 0$ in order to produce a special Lagrangian deformation (see (1.17)). Because we are interested in small deformations we can use the Implicit Function Theorem for Banach spaces (if applicable) to describe small solutions of the equation $\mathrm{P}(\varphi) = 0$ in terms of solutions of the linearised equation at zero i.e. $\mathrm{DP}[0](\varphi) = 0$. In particular, on a collar neighborhood $(0,\varepsilon)\times\partial\mathbb{M}$, equipped with the edge metric $g_M = r^2 g_{\mathcal{X}} + dr^2 + g_{\mathcal{E}}$, we want to solve the equation $\mathrm{DP}[0](\varphi) = 0$. This is a problem of analysis of PDEs on singular spaces and this observation suggests the approach to follow. First, we expect the operators P and DP[0] to be edge-degenerate on $(0,\varepsilon)\times\partial\mathbb{M}$. This is achieved by using sections of the stretched cotangent bundle $T^*_\wedge M$ to produce small deformations. This is natural as differential forms in $T^*_\wedge M$ have a degenerate behavior compatible with the edge singularity of M in the sense that their degenerations are induced by the pairing of the edge metric g_M with edge-degenerate vector fields. Then, in order to invoke the Implicit Function Theorem for Banach spaces we need that DP[0] is an elliptic operator in the edge calculus (hence a Fredholm operator). This is achieved by completing the edge symbol $\sigma^1_\wedge(\mathrm{DP}[0])$ with boundary and coboundary conditions as the Atiyah-Bott obstruction vanishes, see section 3.2.2 below.

3.2.1 The non-linear deformation operator

Given a compact special Lagrangian submanifold with edge singularity $\Phi : M \longrightarrow \mathbb{C}^n$, let $\mathcal{N}(M) \subset T(\mathbb{C}^n)$ be the normal bundle. By using the identification $\mathbb{C}^n \cong \mathbb{R}^n_x \oplus \mathbb{R}^n_y$ we have $T^*(\mathbb{C}^n) \cong T^*(\mathbb{R}^n_x \oplus \mathbb{R}^n_y)$. Now, the complex structure J induces and isomorphism of vector bundles $J : T(M) \longrightarrow \mathcal{N}(M)$, hence we have a bundle isomorphism $J \circ \Phi_* \circ g^*_{_M} : T^*(M) \longrightarrow \mathcal{N}(M) \subset T(\mathbb{R}^n_x \oplus \mathbb{R}^n_y)|_M$ where $g^*_{_M}$ is the dual metric on the cotangent bundle T^*M inducing a bundle map $g^*_{_M} : T^*M \longrightarrow TM$.

Lemma 3.1. Let $\Xi \in C^\infty(M, T^*M)$ with local expression in an edge neighborhood $(0,\varepsilon) \times \mathcal{U} \times \Omega \subset \mathcal{X}^\wedge \times \mathcal{E}$, in local coordinates (r, σ, u), be given by

$$\Xi(r,\sigma,u) = \mathcal{A}(r,\sigma,u)dr + \sum_{k=1}^{m} \mathcal{B}_k(r,\sigma,u)d\sigma_k + \sum_{l=1}^{q} \mathcal{C}_l(r,\sigma,u)du_l,$$

then, its image under the map $J \circ \Phi_* \circ g^*_M$ is given by the following expression in the restriction of the tangent bundle $T(\mathbb{R}^n_x \oplus \mathbb{R}^n_y)\big|_{(0,\varepsilon)\times\mathcal{U}\times\Omega}$

$$\mathcal{V}_\Xi := J(\Phi_*(g^*_M(\Xi))) = \sum_{i=1}^{n} -\tilde{\mathcal{C}}_i(r,\sigma,u)\partial_{x_i} + (\mathcal{A}(r,\sigma,u)\theta^\circ_i + \frac{1}{r}\tilde{\mathcal{B}}_i(r,\sigma,u))\partial_{y_i}$$

where $\tilde{\mathcal{B}}_i$ and $\tilde{\mathcal{C}}_l$ are the components of the corresponding pushforwards

$$\theta_* \left(g^*_{\mathcal{X}} \left(\sum_{k=1}^{m} \mathcal{B}_k(r,\sigma,u)d\sigma_k \right) \right),$$

$$\tau_* \left(g^*_{\mathcal{E}} \left(\sum_{l=1}^{q} \mathcal{C}_l(r,\sigma,u)du_l \right) \right)$$

and $(\theta^\circ_1, \ldots \theta^\circ_n) = \theta(\sigma_1, \ldots \sigma_m)$.

Proof. It follows from the expression of the dual edge metric $g^*_{_M} = \frac{1}{r^2}g^*_{\mathcal{X}} + \partial_r \otimes \partial_r + g^*_{\mathcal{E}}$ that

$$g^*_{_M}(\Xi) = \mathcal{A}\partial_r + \frac{1}{r^2}\sum_{k=1}^{m} \hat{\mathcal{B}}_k \partial_k + \sum_{l=1}^{q} \hat{\mathcal{C}}_l \partial_{u_l}$$

where $\sum\limits_{k=1}^{m} \hat{\mathcal{B}}_k \partial_k = g^*_{\mathcal{X}}(\sum\limits_{k=1}^{m} \mathcal{B}_k d\sigma_k)$ and $\sum\limits_{l=1}^{q} \hat{\mathcal{C}}_l \partial_{u_l} = g^*_{\mathcal{E}}(\sum\limits_{l=1}^{q} \mathcal{C}_l du_l)$. Let $p \in (0,\varepsilon) \times \mathcal{U} \times \Omega$ and take a curve $\mathcal{J} : I \subset \mathbb{R} \longrightarrow M$ given by $\mathcal{J}(t) = (r(t), \sigma(t), u(t))$, such that $\mathcal{J}(0) = p$ and $\mathcal{J}'(0) = g^*_{_M}(\Xi(p))$. Then

$$\Phi \circ \mathcal{J} : I \subset \mathbb{R} \longrightarrow \mathbb{C}^n \cong \mathbb{R}^n_x \oplus \mathbb{R}^n_y$$

defines a curve given by

$$\begin{aligned}(\Phi \circ \mathcal{J})(t) &= \Phi(r(t), \sigma(t), u(t)) \\ &= (r(t)\theta_1(\sigma(t)), \ldots, r(t)\theta_n(\sigma(t)), \tau_1(u(t)), \ldots, \tau_n(u(t))),\end{aligned}$$

therefore

$$(\Phi \circ \mathcal{J})'(0) =$$

$$(r'(0)\theta_1(\sigma(0)) + r(0)\theta_1'(\sigma(0)), \ldots, r'(0)\theta_n(\sigma(0)) + r(0)\theta_n'(\sigma(0)), \tau_1'(u(0)), \ldots, \tau_n'(u(0))) =$$

$$(\mathcal{A}\theta_1^\circ + \frac{1}{r}\tilde{\mathcal{B}}_1, \ldots, \mathcal{A}\theta_n^\circ + \frac{1}{r}\tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_n) \in \mathbb{R}_x^n \oplus \mathbb{R}_y^n \cong T_p(\mathbb{R}_x^n \oplus \mathbb{R}_y^n).$$

By applying the standard complex structure J on $\mathbb{C}^n$ under the identification (1.2) we obtain the result. ■

Proposition 3.2. If $\Xi \in \mathcal{W}^{s,\gamma}(M, T_\wedge^* M)$ then $\mathcal{V}_\Xi$ belongs to $\mathcal{W}^{s,\gamma}(M, \mathcal{N}(M))$ where $\mathcal{N}(M)$ is endowed with the restriction of the standard flat metric $g_{\mathbb{C}^n} = g_{\mathbb{R}^{2n}}$.

Proof. We have to prove that $\|\mathcal{V}_\Xi\|_{\mathcal{W}^{s,\gamma}(M,\mathcal{N}(M))} < \infty$. By (2.13) we have to estimate near the edge with the edge-Sobolev norm

$$\|\omega \mathcal{V}_\Xi\|_{\mathcal{W}^{s,\gamma}(M,\mathcal{N}(M))}$$

and away from the edge with the classical Sobolev norm

$$\left\|(1-\omega)\mathcal{V}_\Xi\right\|_{H^s(2\mathbb{M},\mathcal{N}(M))}.$$

Let $\{\Omega_j, \beta_j\}$ and $\{\mathcal{U}_\lambda, \chi_\lambda\}$ be finite coverings of $\mathcal{E}$ and $\mathcal{X}$ respectively, given by coordinate neighborhoods such that

$$\beta_j : \Omega_j \to \mathbb{R}^q$$

and

$$I \times \chi_\lambda : \mathbb{R}^+ \times \mathcal{U}_\lambda \longrightarrow \mathbb{R}^+ \times \mathbb{R}^m$$

are diffeomorphism and let $\{\phi_j\}$ and $\{\varphi_\lambda\}$ be corresponding subordinate partitions of unity. Let $\omega(r)$ be the cut-off function defining the edge-Sobolev space (see definition 13). In an edge neighborhood $\mathbb{R}^+ \times \mathcal{U}_\lambda \times \Omega_j$ we have

$$\Xi(r, \sigma, u) = \mathcal{A}(r, \sigma, u)dr + \sum_{k=1}^{m} \mathcal{B}_k(r, \sigma, u) r d\sigma_k + \sum_{l=1}^{q} \mathcal{C}_l(r, \sigma, u) du_l,$$

and the fact that $\Xi \in \mathcal{W}^{s,\gamma}(M, T_\wedge^* M)$ implies that $\omega\phi_j\varphi_\lambda\mathcal{A}$, $\omega\phi_j\varphi_\lambda\mathcal{B}_k$ and $\omega\phi_j\varphi_\lambda\mathcal{C}_l$ belong to $\mathcal{W}^{s,\gamma}(M)$. This follows from (2.11) and (2.13).

By lemma 3.1 we have

$$\mathcal{V}_\Xi(r, \sigma, u) = \sum_{i=1}^{n} -\tilde{\mathcal{C}}_i(r, \sigma, u)\partial_{x_i} + (\mathcal{A}(r, \sigma, u)\theta_i^\circ + \tilde{\mathcal{B}}_i(r, \sigma, u))\partial_{y_i}. \tag{3.4}$$

By 2.12, near the edge we want to estimate the terms

$$\left\|\omega\phi_j\varphi_\lambda\tilde{\mathcal{B}}_k\right\|^2_{\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge \times \mathbb{R}^q)} = \int_{\mathbb{R}^q_\eta} [\eta]^{2s} \left\|\kappa^{-1}_{[\eta]}\mathcal{F}_{u\to\eta}(\omega\phi_j\varphi_\lambda\tilde{\mathcal{B}}_k)\right\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} d\eta$$

and

$$\left\|\omega\phi_j\varphi_\lambda\tilde{\mathcal{C}}_l\right\|^2_{\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge\times\mathbb{R}^q)} = \int\limits_{\mathbb{R}^q_\eta}[\eta]^{2s}\left\|\kappa^{-1}_{[\eta]}\mathcal{F}_{u\to\eta}(\omega\phi_j\varphi_\lambda\tilde{\mathcal{C}}_l)\right\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)}\,d\eta.$$

First, we observe that

$$\left\|\kappa^{-1}_{[\eta]}\mathcal{F}_{u\to\eta}(\omega\phi_j\varphi_\lambda\tilde{\mathcal{B}}_k)\right\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} \approx \sum_\lambda\left\|((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]})\varphi_\lambda\omega\mathcal{F}_{u\to\eta}(\phi_j\tilde{\mathcal{B}}_k)(\eta)\right\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)} \tag{3.5}$$

by (2.11). By lemma 3.1 $\tilde{\mathcal{B}}_k$ is obtained by applying $\theta_* g^*_{\mathcal{X}}$. This pull-back and push-forward acts locally on the components $\mathcal{B}_k$ by multiplying by $g^{ij}_{\mathcal{X}}$ and partial derivatives of the component functions of $\theta : \mathcal{X} \to S^{n-1} \subset \mathbb{R}^n$. We claim that both of these operations preserve membership in $\mathcal{W}^{s,\gamma}(M)$. Indeed, in local coordinates $\mathcal{U}_\lambda$ we have $g^*_{\mathcal{X}} = \sum\limits_{i,j=1}^m g^{ij}_{\mathcal{X}}\partial_i\otimes\partial_j$ and

$$g^*_{\mathcal{X}}(\sum_{k=1}^m \mathcal{B}_k(r,\sigma,u)d\sigma_k) = \sum_{j=1}^m(\sum_{k=1}^m g^{kj}_{\mathcal{X}}\mathcal{B}_k(r,\sigma,u))\partial_j.$$

The norm

$$\left\|((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]}\varphi_\lambda\omega\mathcal{F}_{u\to\eta}(\phi_j g^{kj}_{\mathcal{X}}\mathcal{B}_k)(\eta)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)}$$

is equivalent to

$$\left\|\varphi_\lambda g^{kj}_{\mathcal{X}} S_{\gamma-\frac{m}{2}}\left(((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]}\omega\mathcal{F}_{u\to\eta}(\phi_j\mathcal{B}_k)(\eta)\right)\right\|_{H^s(\mathbb{R}^{m+1})}$$

by (2.10). The functions $g^{kj}_{\mathcal{X}}$ are bounded on the support of φ_λ, hence they are bounded functions on $\mathbb{R}^{m+1}$ under the coordinate map $I\times\chi_\lambda$. By the general theory of multipliers on Sobolev spaces $H^s(\mathbb{R}^{m+1})$, ([Agr15] theorem 1.9.1 and 1.9.2), multiplication by any bounded function defines a bounded operator, therefore there exists a constant C_{kj} depending only on $g^{kj}_{\mathcal{X}}$ such that

$$\begin{aligned}&\left\|((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]}\varphi_\lambda\omega\mathcal{F}_{u\to\eta}(\phi_j g^{kj}_{\mathcal{X}}\mathcal{B}_k)(\eta)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)}\\ &\le C_{kj}\left\|((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]}\varphi_\lambda\omega\mathcal{F}_{u\to\eta}(\phi_j\mathcal{B}_k)(\eta)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)}.\end{aligned}$$

By hypothesis

$$\int\limits_{\mathbb{R}^q_\eta}[\eta]^{2s}\left\|((I\times\chi_\lambda)^*)^{-1}\kappa^{-1}_{[\eta]}\varphi_\lambda\omega\mathcal{F}_{u\to\eta}(\phi_j\mathcal{B}_k)(\eta)\right\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}^+\times\mathbb{R}^m)}\,d\eta < \infty$$

then, by (3.5), g^* preserves $\mathcal{W}^{s,\gamma}(M)$ near the edge.

The push-forward θ_* is induced by a diffeomorphism $\theta : \mathcal{X} \longrightarrow S^{n-1} \subset \mathbb{R}^n$. Locally this push-forward acts on the components of vector fields by multiplications by the derivatives of the component functions θ_k, hence the same argument as above applies and we conclude that $\omega\phi_j\tilde{\mathcal{B}}_k \in \mathcal{W}^{s,\gamma}(M)$.

Now, the components $\tilde{\mathcal{C}}_l$ are obtained by applying $\tau_* g_\varepsilon^*$ to the components $\mathcal{C}_l$. Given $g_\varepsilon^* = \sum g_\varepsilon^{ij}(u)\partial_{u_i} \otimes \partial_{u_j}$, it acts on the components $\mathcal{C}_l$ by multiplication by $g_\varepsilon^{ij}(u)$. In the same way the push-forward τ_* acts through multiplication by derivatives of its components $\frac{\partial \tau^i}{\partial u_l}$. When composed with the coordinate function β_k, the maps

$$(\phi_k g_\varepsilon^{ij}) \circ \beta_k^{-1} : \mathbb{R}^q \longrightarrow \mathbb{R}$$

and

$$(\phi_k \frac{\partial \tau^i}{\partial u_l}) \circ \beta_k^{-1} : \mathbb{R}^q \longrightarrow \mathbb{R}$$

belong to $\mathcal{S}(\mathbb{R}^q)$ as they have compact support. By [Sch98] theorem 1.3.34, multiplication by an element in $\mathcal{S}(\mathbb{R}^q)$ defines a continuous operator on $\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge \times \mathbb{R}^q)$. Hence, by the same argument as in the first part of the proof, $\tau_* g_\varepsilon^*$ preserves $\mathcal{W}^{s,\gamma}(M)$ near the edge and $\omega\phi_k\varphi_\lambda\tilde{\mathcal{C}}_l \in \mathcal{W}^{s,\gamma}(M)$.

Away from the edge on the compact manifold $M \setminus \big((0,\varepsilon) \times \partial\mathbb{M}\big)$ take a finite covering of coordinate neighborhoods $\{W_i\}$ with a subordinate partition of unity $\{\mu_i\}$. Then

$$\big\|(1-\omega)\mathcal{V}_\Xi\big\|^2_{H^s(2\mathbb{M},\mathcal{N}(M))} = \sum_i \|\mu_i \mathcal{V}_\Xi\|^2_{H^s(\mathbb{R}^n,\mathbb{R}^q)} . \tag{3.6}$$

As on each of those patches of local coordinates the support of $\mu_i\mathcal{V}_\Xi$ is compact, clearly $\big\|(1-\omega)\mathcal{V}_\Xi\big\|^2_{H^s(2\mathbb{M},\mathcal{N}(M))} < \infty$. ■

Proposition 3.3. Let $\Xi \in C_0^\infty(T^*_\wedge M)$ and $\mathcal{V}_\Xi = J(\Phi_*(g_M^*(\Xi))) \in C^\infty(\mathcal{N}(M))$. Then the pull-back of the standard Kähler form $\omega_{\mathbb{C}^n}$ by the map $\exp(\mathcal{V}_\Xi)\circ\Phi$ is given in terms of edge-degenerate differential operators of order 1 by the following expression in a neighborhood $(0,\varepsilon) \times \mathcal{U}_\lambda \times \Omega_j$ near the edge:

$$\begin{aligned}
&\big(\exp(\mathcal{V}_\Xi)\circ\Phi\big)^*(\omega_{\mathbb{C}^n}) = \\
&\sum_{k=1}^m \Bigg(\sum_{i=1}^n \mathrm{P}_{k,i}(\mathcal{A}) + \mathrm{Q}_{k,i}(\tilde{\mathcal{B}}_i) + (\mathrm{S}_{k,i}(\mathcal{A}) + \mathrm{T}_{k,i}(\tilde{\mathcal{B}}_i))\,\mathrm{R}_{k,i}(\tilde{\mathcal{C}}_i) \\
&+ (\mathrm{L}_{k,i}(\mathcal{A}) + \mathrm{M}_{k,i}(\tilde{\mathcal{B}}_i))\,\mathrm{O}_{k,i}(\tilde{\mathcal{C}}_i) \Bigg) r dr \wedge d\sigma_k \\
&+ \sum_{l=1}^q \Bigg(\sum_{i=1}^n \mathrm{P}_{l,i}(\mathcal{A}) + \mathrm{Q}_{l,i}(\tilde{\mathcal{B}}_i) + (\mathrm{S}_{l,i}(\mathcal{A}) + \mathrm{T}_{l,i}(\tilde{\mathcal{B}}_i))\,\mathrm{R}_{l,i}(\tilde{\mathcal{C}}_i) \\
&+ (\mathrm{L}_{l,i}(\mathcal{A}) + \mathrm{M}_{l,i}(\tilde{\mathcal{B}}_i))\,\mathrm{O}_{l,i}(\tilde{\mathcal{C}}_i) \Bigg) dr \wedge du_l \\
&+ \sum_{k=1}^m \sum_{l=1}^q \Bigg(\sum_{i=1}^n \mathrm{P}_{k,l,i}(\mathcal{A}) + \mathrm{Q}_{k,l,i}(\tilde{\mathcal{B}}_i) + \mathrm{U}_{k,l,i}(\tilde{\mathcal{C}}_i) + (\mathrm{S}_{k,l,i}(\mathcal{A}) + \mathrm{T}_{k,l,i}(\tilde{\mathcal{B}}_i))\,\mathrm{R}_{k,l,i}(\tilde{\mathcal{C}}_i) \\
&+ (\mathrm{L}_{k,l,i}(\mathcal{A}) + \mathrm{M}_{k,l,i}(\tilde{\mathcal{B}}_i))\,\mathrm{O}_{k,l,i}(\tilde{\mathcal{C}}_i) \Bigg) r d\sigma_k \wedge du_l
\end{aligned}$$

$$+\sum_{k=1}^{m}\sum_{j=1}^{m}\Bigg(\sum_{i=1}^{n}\mathrm{P}_{k,j,i}(\mathcal{A})+\mathrm{Q}_{k,j,i}(\tilde{\mathcal{B}}_i)+(\mathrm{S}_{k,j,i}(\mathcal{A})+\mathrm{T}_{k,j,i}(\tilde{\mathcal{B}}_i))\,\mathrm{R}_{k,j,i}(\tilde{\mathcal{C}}_i)$$

$$+(\mathrm{L}_{k,j,i}(\mathcal{A})+\mathrm{M}_{k,j,i}(\tilde{\mathcal{B}}_i))\,\mathrm{O}_{k,j,i}(\tilde{\mathcal{C}}_i)\Bigg)r^2 d\sigma_k\wedge d\sigma_j$$

$$+\sum_{s=1}^{q}\sum_{l=1}^{q}\Bigg(\sum_{i=1}^{n}\mathrm{P}_{s,j,i}(\tilde{\mathcal{C}}_i)+(\mathrm{S}_{s,j,i}(\mathcal{A})+\mathrm{T}_{s,j,i}(\tilde{\mathcal{B}}_i))\,\mathrm{R}_{s,j,i}(\tilde{\mathcal{C}}_i)$$

$$+(\mathrm{L}_{s,j,i}(\mathcal{A})+\mathrm{M}_{s,j,i}(\tilde{\mathcal{B}}_i))\,\mathrm{O}_{s,j,i}(\tilde{\mathcal{C}}_i)\Bigg)du_s\wedge du_l$$

where $\mathcal{A},\tilde{\mathcal{B}}_i,\tilde{\mathcal{C}}_i,\ i=1,\dots,n$ are the components of $\mathcal{V}_\Xi$ in a neighborhood of the edge as in (3.4) and the operators $\mathrm{L}_\bullet,\mathrm{M}_\bullet,\mathrm{O}_\bullet,\mathrm{P}_\bullet,\mathrm{Q}_\bullet,\mathrm{R}_\bullet,\mathrm{S}_\bullet,\mathrm{T}_\bullet,\mathrm{U}_\bullet$ belong to $\mathrm{Diff}^1_{\mathrm{edge}}(M)$.

Proof. Let $\Xi=\mathcal{A}dr+\sum_{k=1}^{m}\mathcal{B}_k r d\sigma_k+\sum_{l=1}^{q}\mathcal{C}_l du_l$ in local coordinates near the edge, then

$$\mathcal{V}_\Xi=J\Phi_*(g_M^*(\Xi))=\sum_{i=1}^{n}-\tilde{\mathcal{C}}_i(r,\sigma,u)\partial_{x_i}+(\mathcal{A}(r,\sigma,u)\theta_i^\circ+\tilde{\mathcal{B}}_i(r,\sigma,u))\partial_{y_i}$$

and

$$(\exp(\mathcal{V}_\Xi)\circ\Phi)(r,\sigma,u)=(r\theta_1-\tilde{\mathcal{C}}_1,\dots,r\theta_n-\tilde{\mathcal{C}}_n,\tau_1+\mathcal{A}\theta_1+\tilde{\mathcal{B}}_1,\dots,\tau_n+\mathcal{A}\theta_n+\tilde{\mathcal{B}}_n).$$

The standard Kähler form in $\mathbb{C}^n$ is given by

$$\omega_{\mathbb{C}^n}=\sum_{i=1}^{n}dx_i\wedge dy_i,$$

then a direct computation shows that

$$(\exp(\mathcal{V}_\Xi)\circ\Phi)^*(\omega_{\mathbb{C}^n})=\sum_{i=1}^{n}(\exp(\mathcal{V}_\Xi)\circ\Phi)^*(dx_i\wedge dy_i)$$

$$=\sum_{i=1}^{n}d\left(r\theta_i(\sigma)-\tilde{\mathcal{C}}_i(r,\sigma,u)\right)\wedge d\left(\tau_i(u)+\mathcal{A}\theta_i(\sigma)+\tilde{\mathcal{B}}_i\right)$$

$$=\sum_{k=1}^{m}\Bigg(\sum_{i=1}^{n}\frac{1}{r}\left(\theta_i\partial_k(\theta_i)+\theta_i\partial_k+\theta_i\partial_k(\theta_i)(-r\partial_r)\right)(\mathcal{A})+\frac{1}{r}\left(\theta_i\partial_k+\partial_k(\theta_i)(-r\partial_r)(\tilde{\mathcal{B}}_i)\right)$$

$$-\frac{1}{r}\left(\left(\partial_k(\theta_i)+\theta_i\partial_k\right)(\mathcal{A})+\partial_k(\tilde{\mathcal{B}}_i)\right)\partial_r(\tilde{\mathcal{C}}_i)+\frac{1}{r}\partial_k(\tilde{\mathcal{C}})\big(\theta_i\partial_r(\mathcal{A})$$

$$+\partial_r(\tilde{\mathcal{B}}_i)\big)\Bigg)rdr\wedge d\sigma_k$$

$$
\begin{aligned}
&+\sum_{l=1}^{q}\Bigg(\sum_{i=1}^{n}\theta_i^2\partial_{u_l}(\mathcal{A})+\theta_i\partial_{u_l}(\tilde{\mathcal{B}}_i)-\partial_{u_l}(\tau_i)\partial_r(\tilde{\mathcal{C}}_i)-\partial_r(\tilde{\mathcal{C}}_i)\Big(\theta_i\partial_{u_l}(\mathcal{A})+\partial_{u_l}(\tilde{\mathcal{B}}_i)\Big)\\
&+\partial_{u_l}(\tilde{\mathcal{C}}_i)\Big(\theta_i\partial_r(\mathcal{A})+\partial_r\tilde{\mathcal{B}}_i\Big)\Bigg)dr\wedge du_l\\
&+\sum_{k=1}^{m}\sum_{l=1}^{q}\Bigg(\sum_{i=1}^{n}\frac{1}{r}\left(\partial_k(\theta_i)\theta_i(r\partial_{u_l})\right)(\mathcal{A})+\frac{1}{r}\left(\partial_k(\theta_i)(r\partial_{u_l})\right)(\tilde{\mathcal{B}}_i)-\frac{1}{r}\partial_{u_l}(\tau_i)\partial_k(\tilde{\mathcal{C}}_i)\\
&-\frac{1}{r}\partial_k(\tilde{\mathcal{C}}_i)\Big(\theta_i\partial_{u_l}(\mathcal{A})+\partial_{u_l}(\tilde{\mathcal{B}}_i)\Big)+\partial_{u_l}(\tilde{\mathcal{C}}_i)\Big((\partial_k(\theta_i)+\theta_i\partial_k)(\mathcal{A})+\partial_k(\tilde{\mathcal{B}}_i)\Big)\Bigg)rd\sigma_k\wedge du_l+\\
&+\sum_{j=1}^{m}\sum_{k=1}^{m}\Bigg(\sum_{i=1}^{n}\frac{1}{r}\left(\theta_i\partial_j(\theta_i)\partial_k-\theta_i\partial_k(\theta_i)\partial_j\right)(\mathcal{A})+\frac{1}{r}\left(\partial_j(\theta_i)\partial_k-\partial_k(\theta_i)\partial_j\right)(\tilde{\mathcal{B}}_i)\\
&-\frac{1}{r}\partial_j(\tilde{\mathcal{C}}_i)\left(\frac{1}{r}\left(\partial_k(\theta_i)+\theta_i\partial_k\right)(\mathcal{A})+\frac{1}{r}\partial_k(\tilde{\mathcal{B}}_i)\right)\\
&+\frac{1}{r}\partial_k(\tilde{\mathcal{C}}_i)\left(\frac{1}{r}\left(\partial_j(\theta_i)+\theta_i\partial_j\right)(\mathcal{A})+\frac{1}{r}\partial_j(\tilde{\mathcal{B}}_i)\right)\Bigg)r^2d\sigma_j\wedge d\sigma_k\\
&+\sum_{\lambda=1}^{q}\sum_{l=1}^{q}\Bigg(\sum_{i=1}^{n}\left(-\partial_{u_l}(\tau_i)\partial_{u_\lambda}+\partial_{u_\lambda}(\tau_i)\partial_{u_l}\right)(\tilde{\mathcal{C}}_i)-\partial_{u_\lambda}(\tilde{\mathcal{C}}_i)\Big(\theta_i\partial_{u_l}(\mathcal{A})+\partial_{u_l}(\tilde{\mathcal{B}}_i)\Big)\\
&+\partial_{u_l}(\tilde{\mathcal{C}}_i)\Big(\theta_i\partial_{u_\lambda}(\mathcal{A})+\partial_{u_\lambda}(\tilde{\mathcal{B}}_i)\Big)\Bigg)du_\lambda\wedge du_l.
\end{aligned}
$$

Each of these terms are edge-degenerate differential operators acting on the components of Ξ and products of these as it is claimed in the proposition. Note that we have used the fact that

$$\Big(\exp(\mathcal{V}_0)\circ\Phi\Big)^*(\omega_{\mathbb{C}^n})=0$$

to remove products in each term that do not contain any of the component functions $\mathcal{A}$, $\tilde{\mathcal{B}}_i$ and $\tilde{\mathcal{C}}_i$. ■

Corollary 3.4. The map

$$\mathrm{P}_{\omega_{\mathbb{C}^n}}:C_0^\infty(T_\wedge^*M)\longrightarrow C_0^\infty(M,\bigwedge\nolimits^2 T_\wedge^*M)$$

defined by $\mathrm{P}_{\omega_{\mathbb{C}^n}}(\Xi):=\Big(\exp(\mathcal{V}_\Xi)\circ\Phi\Big)^*(\omega_{\mathbb{C}^n})$, extends to a continuous non-linear operator

$$\mathrm{P}_{\omega_{\mathbb{C}^n}}:\mathcal{W}^{s,\gamma}(M,T_\wedge^*M)\longrightarrow\mathcal{W}^{s-1,\gamma-1}(M,\bigwedge\nolimits^2 T_\wedge^*M)$$

for $s>\frac{\dim\mathcal{X}+\dim\mathcal{E}+3}{2}$ and $\gamma>\frac{\dim\mathcal{X}+1}{2}$.

Proof. Let's consider a sequence $\{\Xi_i\}_{i\in\mathbb{N}}\subset C_0^\infty(T_\wedge^*M)$ such that it is a Cauchy sequence in the edge-Sobolev space $\mathcal{W}^{s,\gamma}(M,T_\wedge^*M)$. Then, in a neighborhood near the edge, the components of the elements of the sequence $\{\Xi_i\}_{i\in\mathbb{N}}$ define Cauchy sequences in $\mathcal{W}^{s,\gamma}(M)$ i.e.

$$\left\|\omega\phi_j\varphi_\lambda\mathcal{A}^i - \omega\phi_j\varphi_\lambda\mathcal{A}^j\right\|_{\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)} < \varepsilon,$$

$$\left\|\omega\phi_j\varphi_\lambda\tilde{\mathcal{B}}^i_k - \omega\phi_j\varphi_\lambda\tilde{\mathcal{B}}^j_k\right\|_{\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)} < \varepsilon \text{ for all } k = 1, 2, \ldots, m$$

and

$$\left\|\omega\phi_j\varphi_\lambda\tilde{\mathcal{C}}^i_l - \omega\phi_j\varphi_\lambda\tilde{\mathcal{C}}^j_l\right\|_{\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)} < \varepsilon \text{ for all } l = 1, 2, \ldots, q$$

for all $i, j > N(\varepsilon)$. Away from the edge we have

$$\left\|(1-\omega)(\Xi_i - \Xi_j)\right\|_{H^s(2\mathbb{M},T^*M)} < \varepsilon.$$

We want to estimate

$$\left\|\mathrm{P}_{\omega_{\mathbb{C}^n}}(\Xi_i) - \mathrm{P}_{\omega_{\mathbb{C}^n}}(\Xi_j)\right\|_{\mathcal{W}^{s-1,\gamma-1}(M,T^*_\wedge M)}.$$

Observe that the conditions on s and γ imply that the edge-Sobolev spaces are Banach algebras (A.42), hence multiplication is well-defined and we have the estimate $\|fg\|_{s,\gamma} \leq C_{s,\gamma}\|f\|_{s,\gamma}\|g\|_{s,\gamma}$ with a constant $C_{s,\gamma}$ depending only on s and γ. To simplify the notation we will use $\|\cdot\|_{s,\gamma}$ to denote $\|\cdot\|_{\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)}$.

Now, near the edge, the components of the operator $\mathrm{P}_{\omega_{\mathbb{C}^n}}$ are given by expressions of the form $\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}$ where $\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S},\mathrm{T} \in \mathrm{Diff}^1_{\mathrm{edge}}(M)$. By estimating one of these expressions we can apply the same argument to all components. Given $\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S},\mathrm{T}$ in $\mathrm{Diff}^1_{\mathrm{edge}}(M)$ and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{A}',\mathcal{B}',\mathcal{C}'$ in $\mathcal{W}^{s,\gamma}(M)$ by the continuity these operators (proposition 2.4) and the elementary identity $ab - a'b' = \frac{1}{2}(a+a')(b-b') + \frac{1}{2}(a-a')(b+b')$ we have

$$\begin{aligned}
&\Big\|\mathrm{P}(\mathcal{A}) + \mathrm{Q}(\mathcal{B}) + (\mathrm{S}(\mathcal{A}) + \mathrm{T}(\mathcal{B}))\,\mathrm{R}(\mathcal{C}) - \big(\mathrm{P}(\mathcal{A}') + \mathrm{Q}(\mathcal{B}') + (\mathrm{S}(\mathcal{A}') + \mathrm{T}(\mathcal{B}'))\,\mathrm{R}(\mathcal{C}')\big)\Big\|_{s-1,\gamma-1}\\
&\leq \|\mathrm{P}\|\left\|\mathcal{A}-\mathcal{A}'\right\|_{s\gamma} + \|\mathrm{Q}\|\left\|\mathcal{B}-\mathcal{B}'\right\|_{s,\gamma}\\
&+ \left\|(\mathrm{S}(\mathcal{A}) + \mathrm{T}(\mathcal{B}))\,\mathrm{R}(\mathcal{C}) - (\mathrm{S}(\mathcal{A}') + \mathrm{T}(\mathcal{B}'))\,\mathrm{R}(\mathcal{C}')\right\|_{s-1,\gamma-1}\\
&\leq \|\mathrm{P}\|\left\|\mathcal{A}-\mathcal{A}'\right\|_{s,\gamma} + \|\mathrm{Q}\|\left\|\mathcal{B}-\mathcal{B}'\right\|_{s,\gamma}\\
&+ \frac{1}{2}\Big(\|\mathrm{R}\|\|\mathrm{S}\|\left\|\mathcal{A}-\mathcal{A}'\right\|_{s,\gamma} + \|\mathrm{R}\|\|\mathrm{T}\|\left\|\mathcal{B}-\mathcal{B}'\right\|_{s,\gamma}\Big)\left\|\mathcal{C}+\mathcal{C}'\right\|_{s,\gamma}\\
&+ \frac{1}{2}\Big(\|\mathrm{R}\|\|\mathrm{S}\|\left\|\mathcal{A}+\mathcal{A}'\right\|_{s,\gamma} + \|\mathrm{R}\|\|\mathrm{T}\|\left\|\mathcal{B}+\mathcal{B}'\right\|_{s,\gamma}\Big)\left\|\mathcal{C}-\mathcal{C}'\right\|_{s,\gamma}.
\end{aligned}$$

Therefore, if $\{\Xi_i\}$ is a Cauchy sequence in $\mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$ then

$$\{\mathrm{P}(\mathcal{A}_i) + \mathrm{Q}(\mathcal{B}_i) + (\mathrm{S}(\mathcal{A}_i) + \mathrm{T}(\mathcal{B}_i))\,\mathrm{R}(\mathcal{C}_i)\}_{i\in\mathbb{N}}$$

is a Cauchy sequence in $\mathcal{W}^{s-1,\gamma-1}(M)$ which implies that $\{\omega(r)\,\mathrm{P}_{\omega_{\mathbb{C}^n}}(\Xi_i)\}_i$ is Cauchy too.

Away from the edge we are in the setting of standard Sobolev spaces $H^s(2\mathbb{M}, T^*_\wedge M)$ and the operator $\mathrm{P}_{\omega_{\mathbb{C}^n}}$ is given by products of two differential operators of order 1. By using their continuity on $H^s(2\mathbb{M}, T^*M)$, the fact that standard Sobolev spaces are Banach algebras for $s > \frac{\dim M}{2}$ and a similar argument give us that $\{(1-\omega)\,\mathrm{P}_{\omega_{\mathbb{C}^n}}(\Xi_i)\}_i$ is Cauchy in $H^s(2\mathbb{M}, T^*M)$. Therefore the corollary follows immediately. ∎

Proposition 3.5. Let $\Xi \in C_0^\infty(T^*_\wedge M)$ and $\mathcal{V}_\Xi = J\Phi_*(g^*_M(\Xi)) \in C^\infty(\mathcal{N}(M))$. Then, on a neighborhood $(0,\varepsilon) \times \mathcal{U}_\lambda \times \Omega_j$ near the edge, the pull-back of the imaginary part of the holomorphic volume form in $\mathbb{C}^n$

$$\operatorname{Im}\Omega = \operatorname{Im}(dz_1 \wedge \cdots \wedge dz_n),$$

by the map $\exp(\mathcal{V}_\Xi) \circ \Phi$ is given as a sum of n products of the form

$$\mathrm{P}_{i_1}(F_{i_1})\,\mathrm{P}_{i_2}(F_{i_2}) \cdot \cdots \cdot \mathrm{P}_{i_n}(F_{i_n})$$

where $\mathrm{P}_{i_j} \in \operatorname{Diff}^1_{\text{edge}}(M)$ and $F_{i_j} \in \mathcal{W}^{s,\gamma}(M)$.

Proof. The holomorphic volume form in $\mathbb{C}^n$ is given by

$$\begin{aligned}\operatorname{Im}\Omega = \operatorname{Im}(dz_1 \wedge \cdots \wedge dz_n) &= \operatorname{Im} d(x_1 + iy_1) \wedge \cdots \wedge d(x_n + iy_n)\\ &= \sum_{|I|=odd} c_I dy_I \wedge dx_1 \wedge \cdots \wedge \widehat{dx_I} \wedge \dots dx_n\end{aligned}$$

where the sum is taken over all increasingly ordered multi-indexes $I = (i_1, \dots, i_k)$ of odd length k, the hat means that we omit the corresponding terms and $c_I = \pm 1$. Then

$$(\exp(\mathcal{V}_\Xi) \circ \Phi)^* \left(\operatorname{Im}\Omega\right) = \sum_{|I|=odd} c_I (\exp(\mathcal{V}_\Xi) \circ \Phi)^* \left(dy_I \wedge dx_1 \wedge \cdots \wedge \widehat{dx_I} \wedge \dots dx_n\right)$$

$$= \sum_{|I|=odd} c_I d(\tau_{i_1} + \mathcal{A}\theta_{i_1} + \tilde{\mathcal{B}}_{i_1}) \wedge \cdots \wedge d(\tau_{i_k} + \mathcal{A}\theta_{i_k} + \tilde{\mathcal{B}}_{i_k}) \wedge d(r\theta_1 - \tilde{\mathcal{C}}_1) \wedge \cdots \wedge d(\widehat{r\theta_I - \tilde{\mathcal{C}}_I}) \wedge \cdots \wedge d(r\theta_n - \tilde{\mathcal{C}}_n).$$

Each of the terms in the sum is a n-form on $(0,\varepsilon) \times \mathcal{U}_\lambda \times \Omega_j$, hence

$$c_I (\exp(\mathcal{V}_\Xi) \circ \Phi)^* \left(dy_I \wedge dx_1 \wedge \cdots \wedge \widehat{dx_I} \wedge \dots dx_n\right) = F_I(r,\sigma,u) r^m dr \wedge d\sigma_1 \wedge \cdots \wedge d\sigma_m \wedge du_1 \wedge \cdots \wedge du_q,$$

where $F_I(r,\sigma,u)$ is the determinant of the following matrix

$$\begin{bmatrix} \partial_r(\tau_{i_1} + \mathcal{A}\theta_{i_1} + \tilde{\mathcal{B}}_{i_1}) & \dots & \partial_r(r\theta_1 - \tilde{\mathcal{C}}_1) & \dots & \partial_r(\widehat{r\theta_I - \tilde{\mathcal{C}}_I}) & \dots & \partial_r(r\theta_n - \tilde{\mathcal{C}}_n) \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \frac{1}{r}\partial_j(\tau_{i_1} + \mathcal{A}\theta_{i_1} + \tilde{\mathcal{B}}_{i_1}) & \dots & \frac{1}{r}\partial_j(r\theta_1 - \tilde{\mathcal{C}}_1) & \dots & \frac{1}{r}\partial_j(\widehat{r\theta_I - \tilde{\mathcal{C}}_I}) & \dots & \frac{1}{r}\partial_j(r\theta_n - \tilde{\mathcal{C}}_n) \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \partial_{u_q}(\tau_{i_1} + \mathcal{A}\theta_{i_1} + \tilde{\mathcal{B}}_{i_1}) & \dots & \partial_{u_q}(r\theta_1 - \tilde{\mathcal{C}}_1) & \dots & \partial_{u_q}(\widehat{r\theta_I - \tilde{\mathcal{C}}_I}) & \dots & \partial_{u_q}(r\theta_n - \tilde{\mathcal{C}}_n) \end{bmatrix}.$$

Therefore $F_I(r,\sigma,u)$ is the sum of products of the form

$$\mathrm{P}_1(\tau_{i_1} + \mathcal{A}\theta_{i_1} + \tilde{\mathcal{B}}_{i_1}) \dots \mathrm{P}_n(r\theta_n - \tilde{\mathcal{C}}_n)$$

with all the operators P_k, $k = 1, \dots, n$, in $\operatorname{Diff}^1_{\text{edge}}(M)$. By expanding these products and observing that the sum of all terms of the form

$$\mathrm{P}_1(\tau_{i_1}) \dots \mathrm{P}_n(r\theta_n),$$

i.e. all those products not containing any of the functions $\mathcal{A}, \tilde{\mathcal{B}}_\bullet$ or $\tilde{\mathcal{C}}_\bullet$, is equal to zero as

$$\left(\exp(\mathcal{V}_0) \circ \Phi\right)^* (\operatorname{Im} \Omega) = 0;$$

we obtain that $F_I(r, \sigma, u)$ is the sum of products of the form

$$\mathrm{P}_{i_1}(F_{i_1})\, \mathrm{P}_{i_2}(F_{i_2}) \cdot \dots \cdot \mathrm{P}_{i_n}(F_{i_n}) \tag{3.7}$$

where $\mathrm{P}_{i_j} \in \mathrm{Diff}^1_{\mathrm{edge}}(M)$ and $F_{i_j} \in \mathcal{W}^{s,\gamma}(M)$ as claimed. ■

Corollary 3.6. The map

$$\mathrm{P}_{\operatorname{Im}\Omega} : C_0^\infty(T^*_\wedge M) \longrightarrow C_0^\infty(M, \bigwedge\nolimits^n T^*_\wedge M)$$

given by $\mathrm{P}_{\operatorname{Im}\Omega}(\Xi) := \left(\exp(\mathcal{V}_\Xi) \circ \Phi\right)^* (\operatorname{Im}(\Omega))$ extends to a continuous non-linear operator

$$\mathrm{P}_{\operatorname{Im}\Omega} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge\nolimits^n T^*_\wedge M)$$

for $s > \frac{\dim \mathcal{X} + \dim \mathcal{E} + 3}{2}$ and $\gamma > \frac{\dim \mathcal{X} + 1}{2}$.

Proof. Given Ξ and Ξ' in $C_0^\infty(T^*_\wedge M)$, near the edge, we have

$$\mathrm{P}_{\operatorname{Im}\Omega}(\Xi) - \mathrm{P}_{\operatorname{Im}\Omega}(\Xi') =$$

$$\left(\sum_I \mathrm{P}_{i_1}(F_{i_1}) \cdot \dots \cdot \mathrm{P}_{i_n}(F_{i_n}) - \mathrm{P}_{i_1}(F'_{i_1}) \cdot \dots \cdot \mathrm{P}_{i_n}(F'_{i_n})\right) r^m dr \wedge d\sigma_1 \cdots \wedge d\sigma_m \wedge du_1 \wedge \cdots \wedge du_q.$$

By applying the elementary identity $ab - a'b' = \frac{1}{2}(a + a')(b - b') + \frac{1}{2}(a - a')(b + b')$ we can decompose each term of the sum in factors, each of them containing one of the subtractions

$$P_{i_{n-k}}(F_{i_{n-k}}) - \mathrm{P}_{i_{n-k}}(F_{i_{n-k}}) \tag{3.8}$$

for $k = 0, \dots, n$.

Now, if $\{\Xi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(T^*_\wedge M)$ is a Cauchy sequence in $\mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$ as in corollary 3.4, the sequences $\mathrm{P}_{i_j}(F^{(k)}_{i_j})$ are Cauchy in $\mathcal{W}^{s-1,\gamma-1}(M)$ (by the continuity of edge-degenerate operators on edge-Sobolev spaces 2.4) and

$$\left\| \mathrm{P}_{i_{n-k}}(F^{(k)}_{i_{n-k}}) - \mathrm{P}_{i_{n-k}}(F^{(k')}_{i_{n-k}}) \right\|_{s-1,\gamma-1} < \varepsilon$$

for all $k, k' > N(\varepsilon)$. Therefore, by the Banach algebra property (A.42) and (3.8), we have that $\left\{\omega(r)\, \mathrm{P}_{\operatorname{Im}\Omega}(\Xi_k)\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^n T^*_\wedge M)$. Away from the edge a completely similar argument using the spaces $H^s(2\mathbb{M}, T^*M)$ implies that $\left\{(1-\omega)\, \mathrm{P}_{\operatorname{Im}\Omega}(\Xi_k)\right\}_{k\in\mathbb{N}}$ is Cauchy in $H^s(2\mathbb{M}, T^*M)$ and the corollary follows immediately. ■

Corollary 3.4 and 3.6 tell us that we have a continuous non-linear deformation operator:

$$\mathrm{P} := \begin{array}{c} \mathrm{P}_{\omega_{\mathbb{C}^n}} \\ \oplus \\ \mathrm{P}_{\operatorname{Im}\Omega} \end{array} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow \begin{array}{c} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^2 T^*_\wedge M) \\ \oplus \\ \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^n T^*_\wedge M) \end{array}$$

for $s > \frac{\dim \mathcal{X} + \dim \mathcal{E} + 3}{2}$ and $\gamma > \frac{\dim \mathcal{X}+1}{2}$.

In fact, this operator is smooth as the following corollary shows.

Corollary 3.7. The non-linear deformation operator

$$\begin{array}{c} \mathrm{P}_{\omega_{\mathbb{C}^n}} \\ \oplus \\ \mathrm{P}_{\operatorname{Im}\Omega} \end{array} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow \begin{array}{c} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^2 T^*_\wedge M) \\ \oplus \\ \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^n T^*_\wedge M). \end{array}$$

is smooth.

Proof. Recall that from proposition 3.3 the components of the operator $\mathrm{P}_{\omega_{\mathbb{C}^n}}$ are given by expressions of the form $\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}$ where $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}, \mathrm{T} \in \mathrm{Diff}^1_{\mathrm{edge}}(M)$. Therefore, given $f, \nu \in \mathcal{W}^{s,\gamma}(M)$ we have

$$\begin{aligned} &(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R})(f + \nu) - (\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R})(f) \\ &= \mathrm{P}(\nu) + \mathrm{Q}(\nu) + \mathrm{R}(f)(\mathrm{S}(\nu) + \mathrm{T}(\nu)) + \mathrm{S}(f)\,\mathrm{R}(\nu) + \mathrm{T}(f)\,\mathrm{R}(\nu) + \mathrm{S}(\nu)\,\mathrm{R}(\nu) + \mathrm{T}(\nu)\,\mathrm{R}(\nu). \end{aligned}$$

Observe that by the continuity of S

$$\left\| \frac{\mathrm{S}(\nu)\,\mathrm{R}(\nu)}{\|\nu\|} \right\|_{s-1,\gamma-1} \leq \|\mathrm{S}\|_{s,\gamma} \left\|\mathrm{R}(\nu)\right\|_{s-1,\gamma-1},$$

hence by the continuity of R we have $\lim_{\|\nu\|\to 0} \frac{\mathrm{S}(\nu)\,\mathrm{R}(\nu)}{\|\nu\|} = 0$ and the same holds for $\mathrm{T}(\nu)\,\mathrm{R}(\nu)$. This implies that the Fréchet derivative of $\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}$ at f is given by

$$\mathrm{D}\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[f] = \mathrm{P} + \mathrm{Q} + \mathrm{R}(f)(\mathrm{S} + \mathrm{T}) + \mathrm{S}(f)\,\mathrm{R} + \mathrm{T}(f)\,\mathrm{R}$$

and from this expression is clear that

$$\mathrm{D}\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[\bullet] : \mathcal{W}^{s,\gamma}(M) \longrightarrow \mathcal{L}\left(\mathcal{W}^{s,\gamma}(M), \mathcal{W}^{s-1,\gamma-1}(M)\right)$$

is a continuous operator. Moreover, observe that the second derivative is constant. More precisely, the second derivative of $\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}$ at $f \in \mathcal{W}^{s,\gamma}(M)$ is the linear, continuous map

$$\mathrm{D}^2\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[f] \in \mathcal{L}\left(\mathcal{W}^{s,\gamma}(M), \mathcal{L}\left(\mathcal{W}^{s,\gamma}(M), \mathcal{W}^{s-1,\gamma-1}(M)\right)\right)$$

such that

$$\frac{\mathrm{D}\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[f + \nu] - \mathrm{D}\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[f] - \mathrm{D}^2\left(\mathrm{P} + \mathrm{Q} + (\mathrm{S} + \mathrm{T}) \cdot \mathrm{R}\right)[f](\nu)}{\|\nu\|}$$

goes to zero in $\mathcal{L}\left(\mathcal{W}^{s,\gamma}(M), \mathcal{W}^{s-1,\gamma-1}(M)\right)$ when $\|\nu\| \to 0$.

A direct computation shows that

$$\mathrm{D}\left(\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}\right)[f+\nu]-\mathrm{D}\left(\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}\right)[f] = \mathrm{R}(\nu)(\mathrm{S}+\mathrm{T})+\mathrm{S}(\nu)\,\mathrm{R}+\mathrm{T}(\nu)\,\mathrm{R}$$

hence by uniqueness of derivatives we have

$$\mathrm{D}^2\left(\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}\right)[f](\nu) = \mathrm{R}(\nu)(\mathrm{S}+\mathrm{T})+\mathrm{S}(\nu)\,\mathrm{R}+\mathrm{T}(\nu)\,\mathrm{R}$$

for any $\nu \in \mathcal{W}^{s,\gamma}(M)$ and all $f \in \mathcal{W}^{s,\gamma}(M)$. Thus the second derivative

$$\mathrm{D}^2\left(\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}\right)[\bullet] : \mathcal{W}^{s,\gamma}(M) \longrightarrow \mathcal{L}\Big(\mathcal{W}^{s,\gamma}(M), \mathcal{L}\left(\mathcal{W}^{s,\gamma}(M), \mathcal{W}^{s-1,\gamma-1}(M)\right)\Big)$$

is constant and given by

$$\mathrm{D}^2\left(\mathrm{P}+\mathrm{Q}+(\mathrm{S}+\mathrm{T})\cdot\mathrm{R}\right)[f](g,h) = \mathrm{R}(g)(\mathrm{S}+\mathrm{T})(h)+(\mathrm{S}+\mathrm{T})(g)\,\mathrm{R}(h)$$

for any $f \in \mathcal{W}^{s,\gamma}(M)$. We conclude that $\mathrm{P}_{\omega_{\mathbb{C}^n}}$ is smooth.

Now, the operator $\mathrm{P}_{\mathrm{Im}\,\Omega}$ is given as a sum of products $\mathrm{P}_1(F_1)\,\mathrm{P}_2(F_2)\cdot\cdots\cdot\mathrm{P}_n(F_n)$ where $\mathrm{P}_i \in \mathrm{Diff}^1_{\mathrm{edge}}(M)$ and $F_i \in \mathcal{W}^{s,\gamma}(M)$. Then we have that the expression

$$\mathrm{P}_1(F_1+\nu)\,\mathrm{P}_2(F_2+\nu)\cdot\cdots\cdot\mathrm{P}_n(F_n+\nu) - \mathrm{P}_1(F_1)\,\mathrm{P}_2(F_2)\cdot\cdots\cdot\mathrm{P}_n(F_n)$$

is a sum of products of the operators P_i evaluated at F_i or at ν with at least one ν in each product. Those products containing 2 or more terms with ν does not contribute to the Fréchet derivative as in the first part of the proof. Hence the Fréchet derivative is computed only with products having one term with ν which produce continuous linear operators of the form $\mathrm{P}_{i_1}(F_{i_1})\,\mathrm{P}_{i_2}(F_{i_2})\cdot\cdots\cdot\mathrm{P}_{i_{n-1}}(F_{i_{n-1}})\,\mathrm{P}_{i_n}$. Furthermore, it is easily seen that the nth derivative of each of the products $\mathrm{P}_1(F_1)\,\mathrm{P}_2(F_2)\cdot\cdots\cdot\mathrm{P}_n(F_n)$ is constant and equal to

$$\sum_{(i_1,\ldots,i_n)\in S_n} \mathrm{P}_{i_1}\,\mathrm{P}_{i_2}\cdot\cdots\cdot\mathrm{P}_{i_n}$$

where the sum is taken over the group of permutations with n elements. From this it follows that $\mathrm{P}_{\mathrm{Im}\,\Omega}$ is smooth. ■

Observe that any $\alpha \in \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^n T^*_\wedge M)$ is given as $\alpha = f\,\mathrm{Vol}_M$ where $f \in \mathcal{W}^{s-1,\gamma-1}(M)$ and $f_k\,\mathrm{Vol}_M \longrightarrow f\,\mathrm{Vol}_M$ in $\mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^n T^*_\wedge M)$ if and only if $f_k \longrightarrow f$ in $\mathcal{W}^{s-1,\gamma-1}(M)$. Hence we can consider the operator $\mathrm{P}_{\mathrm{Im}\,\Omega}$ as an operator acting between the following spaces:

$$\mathrm{P}_{\mathrm{Im}\,\Omega} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow \mathcal{W}^{s-1,\gamma-1}(M).$$

3.2.2 The linear operator DP[0]

In this subsection we consider the operator DP[0], the linearisation at zero of the deformation operator $\mathrm{P} = \mathrm{P}_{\omega_{\mathbb{C}^n}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega}$. A careful analysis of this operator is necessary as we want to apply the Implicit Function Theorem for Banach spaces to this linear operator in order to describe solutions (nearby zero) of the non-linear equation $\mathrm{P}(f) = 0$ in term of $\operatorname{Ker} \mathrm{DP}[0]$. McLean's results in [McL98] implies that the linearisation of the deformation operator at zero acting on $C_0^\infty(M, T^*_\wedge M)$ is given by the Hodge-deRham operator i.e.

$$\mathrm{DP}[0]\Big|_{C_0^\infty(M, T^*_\wedge M)} = d + d^*.$$

In this section we analyse the extension of this operator to edge-Sobolev spaces, its ellipticity and the Fredholm property.

Observe that on a collar neighborhood any $\Xi \in C^\infty\left(\bigwedge^k T^*_\wedge M\right)$ can be written as

$$\Xi = dr \wedge r^{k-1}\Theta_{\mathcal{X}} + dr \wedge \Theta_{\mathcal{E}} + dr \wedge \sum_{i=1}^{k-2} r^i \Lambda^i_{\mathcal{X},\mathcal{E}} + \sum_{j=1}^{k-1} r^j \tilde{\Lambda}^j_{\mathcal{X},\mathcal{E}} + r^k \tilde{\Theta}_{\mathcal{X}} + \tilde{\Theta}_{\mathcal{E}} \tag{3.9}$$

where:

i) $\Theta_{\mathcal{X}}$ is a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{E}}(\bigwedge^{k-1} T^*\mathcal{X})$;

ii) $\Theta_{\mathcal{E}}$ is a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{X}}(\bigwedge^{k-1} T^*\mathcal{E})$;

iii) $\Lambda^i_{\mathcal{X},\mathcal{E}}$ is the wedge product of a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{E}}(\bigwedge^{i} T^*\mathcal{X})$ with a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{X}}(\bigwedge^{k-1-i} T^*\mathcal{E})$;

iv) $\tilde{\Lambda}^j_{\mathcal{X},\mathcal{E}}$ is the wedge product of a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{E}}(\bigwedge^{j} T^*\mathcal{X})$ with a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{X}}(\bigwedge^{k-j} T^*\mathcal{E})$;

v) $\tilde{\Theta}_{\mathcal{X}}$ is a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{E}}(\bigwedge^{k} T^*\mathcal{X})$;

vi) $\tilde{\Theta}_{\mathcal{E}}$ is a smooth section of the bundle $\pi^*_{\mathbb{R}^+\times\mathcal{X}}(\bigwedge^{k} T^*\mathcal{E})$.

Recall that $\pi_{\mathbb{R}^+\times\mathcal{X}}$ and $\pi_{\mathbb{R}^+\times\mathcal{E}}$ are the corresponding projections

$$\pi_{\mathbb{R}^+\times\mathcal{X}} : \mathcal{X}^\wedge \times \mathcal{E} \longrightarrow \mathcal{E}$$
$$\pi_{\mathbb{R}^+\times\mathcal{E}} : \mathcal{X}^\wedge \times \mathcal{E} \longrightarrow \mathcal{X}.$$

More generally, the bundle $\bigwedge^\bullet T^*_\wedge M$ can be decomposed in the following way:

$$\begin{aligned}
\bigwedge^\bullet T^*_\wedge M = {} & dr \wedge \sum_{k=0}^{m} r^k \pi^*_{\mathbb{R}^+\times\mathcal{E}}\left(\bigwedge^k T^*\mathcal{X}\right) \oplus dr \wedge \pi^*_{\mathbb{R}^+\times\mathcal{X}}\left(\bigwedge^\bullet T^*\mathcal{E}\right) \\
& \oplus dr \wedge \sum_{k=1}^{m} r^k \pi^*_{\mathbb{R}^+\times\mathcal{E}}\left(\bigwedge^k T^*\mathcal{X}\right) \wedge \pi^*_{\mathbb{R}^+\times\mathcal{X}}\left(\bigwedge^\bullet T^*\mathcal{E}\right) \\
& \oplus \sum_{k=1}^{m} r^k \pi^*_{\mathbb{R}^+\times\mathcal{E}}\left(\bigwedge^k T^*\mathcal{X}\right) \wedge \pi^*_{\mathbb{R}^+\times\mathcal{X}}\left(\bigwedge^\bullet T^*\mathcal{E}\right) \\
& \oplus \sum_{k=1}^{m} r^k \pi^*_{\mathbb{R}^+\times\mathcal{E}}(\bigwedge^k T^*\mathcal{X}) \oplus \pi^*_{\mathbb{R}^+\times\mathcal{X}}(\bigwedge^\bullet T^*\mathcal{E}).
\end{aligned}$$

With respect to this splitting we can compute an explicit expression for $d + d^*$ acting on $C_0^\infty(M, \bigwedge^\bullet T^*_\wedge M)$ to prove the following proposition.

Proposition 3.8. The operator $\mathrm{DP}[0]\Big|_{C_0^\infty(M,T^*_\wedge M)} = d+d^*$ extends to a continuous linear operator acting between edge-Sobolev spaces

$$\mathrm{DP}[0]: \ \ \mathcal{W}^{s,\gamma}(M,{\textstyle\bigwedge^\bullet} T^*_\wedge M) \ \longrightarrow \ \mathcal{W}^{s-1,\gamma-1}(M,{\textstyle\bigwedge^\bullet} T^*_\wedge M) \ .$$

Proof. By using the splitting (3.9) and arranging in a vector the components of $\Xi \in C_0^\infty(M,\bigwedge^k T^*_\wedge M)$ in the following way

$$\begin{pmatrix} \Theta_{\mathcal{X}}, \Theta_{\mathcal{E}}, \Lambda^1_{\mathcal{X},\mathcal{E}}, \ldots, \Lambda^{k-2}_{\mathcal{X},\mathcal{E}} \\ \tilde{\Lambda}^1_{\mathcal{X},\mathcal{E}}, \ldots, \tilde{\Lambda}^{k-1}_{\mathcal{X},\mathcal{E}}, \tilde{\Theta}_{\mathcal{X}}, \tilde{\Theta}_{\mathcal{E}} \end{pmatrix},$$

a direct computation shows that the Hodge-deRham operator acting on Ξ is given by

$$d + d^* = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right],$$

where the operator matrices are given by

$$\mathbf{A} = \begin{bmatrix} \frac{1}{r}(d_{\mathcal{X}}+d^*_{\mathcal{X}}) & 0 & 0 & 0 & \cdots & \cdots & -d^*_{\mathcal{E}} \\ 0 & -\frac{1}{r}(d_{\mathcal{E}}+d^*_{\mathcal{E}}) & -\frac{1}{r}d^*_{\mathcal{X}} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{1}{r}d^*_{\mathcal{X}} & -(d_{\mathcal{E}}+\frac{1}{r}d^*_{\mathcal{X}}) & 0 & 0 & \cdots & \vdots \\ \vdots & 0 & -(d^*_{\mathcal{E}}+\frac{1}{r}d_{\mathcal{X}}) & -(d_{\mathcal{E}}+\frac{1}{r}d^*_{\mathcal{X}}) & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & -(d^*_{\mathcal{E}}+\frac{1}{r}d_{\mathcal{X}}) & -(d_{\mathcal{E}}+\frac{1}{r}d^*_{\mathcal{X}}) & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & -\frac{1}{r}d_{\mathcal{X}} & -d_{\mathcal{E}} \\ -d_{\mathcal{E}} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{r}d_{\mathcal{X}} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \frac{k}{r}+\partial_r & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \partial_r \\ \frac{1}{r}+\partial_r & 0 & \cdots & \vdots & 0 & 0 \\ 0 & \ddots & \ddots & 0 & \vdots & 0 \\ \vdots & \ddots & & 0 & \vdots & 0 \\ 0 & \cdots & 0 & \frac{k-1}{r}+\partial_r & 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & \frac{1}{r}-\partial_r & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \cdots & 0 & \frac{k-2}{r}-\partial_r \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{k}{r}-\partial_r & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -\partial_r & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{\mathcal{E}}+d^*_{\mathcal{E}} & \frac{1}{r}d^*_{\mathcal{X}} & 0 & \cdots & \cdots & 0 & \frac{1}{r}d_{\mathcal{X}} \\ \frac{1}{r}d_{\mathcal{X}} & d_{\mathcal{E}}+d^*_{\mathcal{E}} & \frac{1}{r}d^*_{\mathcal{X}} & 0 & \cdots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 \\ \vdots & & & & & \vdots & \vdots \\ \vdots & \cdots & \frac{1}{r}d_{\mathcal{X}} & d_{\mathcal{E}}+d^*_{\mathcal{E}} & \frac{1}{r}d^*_{\mathcal{X}} & 0 & \vdots \\ \vdots & \cdots & 0 & \frac{1}{r}d_{\mathcal{X}} & d_{\mathcal{E}} & 0 & \vdots \\ \vdots & \cdots & \cdots & 0 & \frac{1}{r}d_{\mathcal{X}} & d_{\mathcal{E}} & \vdots \\ 0 & \cdots & \cdots & 0 & d^*_{\mathcal{E}} & \frac{1}{r}(d_{\mathcal{X}}+d^*_{\mathcal{X}}) & 0 \\ \frac{1}{r}d^*_{\mathcal{X}} & 0 & \cdots & \cdots & 0 & 0 & d_{\mathcal{E}}+d^*_{\mathcal{E}} \end{bmatrix}.$$

Observe that each of the elements in these matrices is an element of $\mathrm{Diff}^1_{\mathrm{edge}}(M, \bigwedge^\bullet T^*_\wedge M)$. From this, it follows that $d + d^* \in \mathrm{Diff}^1_{\mathrm{edge}}\left(\bigwedge^\bullet T^*_\wedge M\right)$ which implies that DP[0] extends to a continuous linear operator between the corresponding edge-Sobolev spaces. $\blacksquare$

In order to verify some properties of the symbolic structure of $d + d^*$ it will be useful to have similar explicit expressions for the Hodge-Laplacian associated with the edge metric g_M acting on $\bigwedge^\bullet T^*_\wedge M$.

Proposition 3.9. The Hodge-Laplace operator $\Delta_{g_M}\Big|_{C_0^\infty(M, T^*_\wedge M)}$ extends to a continuous linear operator acting between edge-Sobolev spaces

$$\Delta_{g_M} : \ \mathcal{W}^{s,\gamma}(M, {\textstyle\bigwedge^\bullet} T^*_\wedge M) \ \longrightarrow \ \mathcal{W}^{s-2,\gamma-2}(M, {\textstyle\bigwedge^\bullet} T^*_\wedge M).$$

Proof. By arranging the components of $\Xi \in C_0^\infty(M, \bigwedge^k T^*_\wedge M)$ in the same way as in the previous proposition, a direct computation shows that the Hodge-Laplacian acting on k-forms

$$\Delta_{g_M} : C_0^\infty(M, \bigwedge^k T^*_\wedge M) \longrightarrow C_0^\infty(M, \bigwedge^k T^*_\wedge M)$$

is given by

$$\Delta_{g_M} = \left[\begin{array}{c|c} \mathbf{A}' & \mathbf{B}' \\ \hline \mathbf{C}' & \mathbf{D}' \end{array}\right],$$

where the operator matrices are given by

$$\mathbf{A}' = \begin{bmatrix} \frac{1}{r^2}\Delta_{\mathcal{X}} + d^*_{\mathcal{E}} d_{\mathcal{E}} - \partial_r^2 + \frac{(k-1)(k-2)}{r^2} & 0 & 0 & 0 \ \cdots & \cdots & \frac{1}{r}(d^*_{\mathcal{E}} d_{\mathcal{X}} + d_{\mathcal{X}} d^*_{\mathcal{E}}) \\ 0 & \Delta_{\mathcal{E}} + \frac{1}{r^2} d^*_{\mathcal{X}} d_{\mathcal{X}} - \partial_r^2 & \frac{1}{r}(d^*_{\mathcal{X}} d_{\mathcal{E}} + d_{\mathcal{E}} d^*_{\mathcal{X}}) & 0 \ \cdots & \cdots & 0 \\ 0 & \frac{1}{r}(d^*_{\mathcal{E}} d_{\mathcal{X}} + d_{\mathcal{X}} d^*_{\mathcal{E}}) & \frac{1}{r^2}\Delta_{\mathcal{X}} + \Delta_{\mathcal{E}} - \partial_r^2 & 0 \ \cdots & \cdots & \vdots \\ \vdots & 0 & & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & \frac{1}{r}(d^*_{\mathcal{X}} d_{\mathcal{E}} + d_{\mathcal{E}} d^*_{\mathcal{X}}) \\ \frac{1}{r}(d^*_{\mathcal{X}} d_{\mathcal{E}} + d_{\mathcal{E}} d^*_{\mathcal{X}}) & 0 & \cdots & \cdots \ 0 & \frac{1}{r}(d^*_{\mathcal{E}} d_{\mathcal{X}} + d_{\mathcal{X}} d^*_{\mathcal{E}}) & \frac{1}{r^2}\Delta_{\mathcal{X}} + \Delta_{\mathcal{E}} - \partial_r^2 + \frac{(k-2)^2-(k-2)}{r^2} \end{bmatrix}$$

$$\mathbf{B}' = \begin{bmatrix} 0 & \cdots & 0 & \frac{-1}{r} d^*_{\mathcal{E}} & \frac{-2}{r^2} d^*_{\mathcal{X}} & 0 \\ \frac{-2}{r^2} d^*_{\mathcal{X}} & 0 & \cdots & 0 & 0 & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{-2}{r} d^*_{\mathcal{X}} & 0 & 0 \end{bmatrix} \quad \mathbf{C}' = \begin{bmatrix} 0 & \frac{-2}{r^2} d_{\mathcal{X}} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & & \ddots & \vdots \\ \vdots & \vdots & \ddots & & \ddots & 0 \\ \vdots & \vdots & \cdots & & 0 & \frac{-2}{r^2} d_{\mathcal{X}} \\ \frac{-2}{r^2} d_{\mathcal{X}} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$$
\mathbf{D}' = \begin{bmatrix}
\frac{1}{r^2}\Delta_{\mathcal{X}}+\Delta_{\mathcal{E}}-\partial_r^2 & \frac{1}{r}(d^*_{\mathcal{X}}d_{\mathcal{E}}+d_{\mathcal{E}}d^*_{\mathcal{X}}) & 0 & \cdots & \cdots & 0 & \frac{1}{r}(d^*_{\mathcal{E}}d_{\mathcal{X}}+d_{\mathcal{X}}d^*_{\mathcal{E}}) \\
\frac{1}{r}(d^*_{\mathcal{E}}d_{\mathcal{X}}+d_{\mathcal{X}}d^*_{\mathcal{E}}) & & & & & \vdots & 0 \\
0 & \ddots & \ddots & & \ddots & \vdots & \vdots \\
\vdots & & & & & \vdots & \vdots \\
\vdots & \ddots & & \ddots & \ddots & \vdots & \vdots \\
\vdots & & & & & 0 & \vdots \\
\vdots & \cdots & 0 & & & \frac{1}{r}(d^*_{\mathcal{X}}d_{\mathcal{E}}+d_{\mathcal{E}}d^*_{\mathcal{X}}) & \vdots \\
0 & \cdots & \cdots & 0 & \frac{1}{r}(d^*_{\mathcal{E}}d_{\mathcal{X}}+d_{\mathcal{X}}d^*_{\mathcal{E}}) & \frac{1}{r^2}\Delta_{\mathcal{X}}+d^*_{\mathcal{E}}d_{\mathcal{E}}-\partial_r^2-\frac{k(k-1)}{r^2} & 0 \\
\frac{1}{r}(d^*_{\mathcal{X}}d_{\mathcal{E}}+d_{\mathcal{E}}d^*_{\mathcal{X}}) & \cdots & \cdots & \cdots & 0 & 0 & \Delta_{\mathcal{E}}+\frac{1}{r^2}d^*_{\mathcal{X}}d_{\mathcal{X}}-\partial_r^2
\end{bmatrix}.
$$

Again each of the operators in the matrices is an element of $\mathrm{Diff}^2_{\mathrm{edge}}(M, \bigwedge^\bullet T^*_\wedge M)$, hence the result follows. ∎

3.2.3 The symbolic structure of DP[0]

Recall from section 2.3.2 that the symbolic structure of the edge-degenerate operator DP[0] is given by the pair

$$
\Big(\; \sigma_b^1\left(\mathrm{DP}[0]\right)(r,\sigma,u,\tilde{\rho},\xi,\tilde{\eta}), \quad \sigma_\wedge^1\left(\mathrm{DP}[0]\right)(u,\eta) \; \Big)
$$

where $\sigma_b^1\left(\mathrm{DP}[0]\right)$ is a bundle map on $\pi^*_{T^*\mathbb{M}}(\bigwedge^\bullet T^*_\wedge \mathbb{M})$. The edge symbol $\sigma_\wedge^1\left(\mathrm{DP}[0]\right)$ is a family of continuous linear operators acting on cone-Sobolev spaces and parametrized by the cosphere bundle over $\mathcal{E}$:

$$
\sigma_\wedge^1\left(\mathrm{DP}[0]\right) : S^*\mathcal{E} \longrightarrow \mathcal{L}\left(\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge, \bigwedge^\bullet T^*_\wedge M), \mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge, \bigwedge^\bullet T^*_\wedge M)\right).
$$

The ellipticity of the operator DP[0] requires the invertibility of its symbolic structure. From proposition 3.8 and 3.9 we obtain the first part of the desired result.

Proposition 3.10.

$$
\sigma_b^1(\mathrm{DP}[0])(r,\sigma,u,\tilde{\rho},\xi,\tilde{\eta}) : \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M} \longrightarrow \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M}
$$

is a bundle isomorphism for every non-zero $(r,\sigma,u,\tilde{\rho},\xi,\tilde{\eta}) \in T^*\mathbb{M}$ up to $r=0$.

Proof. To prove this result we shall use the symbolic structure of the Hodge-Laplace operator

$$
\Big(\; \sigma_b^2(\Delta_{g_M})(r,\sigma,u,\tilde{\rho},\xi,\tilde{\eta}), \quad \sigma_\wedge^2(\Delta_{g_M})(u,\eta) \; \Big).
$$

From [Sch98] theorem 3.4.56 we have the natural expected symbolic relation

$$
\sigma_b^2(\Delta_{g_M}) = \sigma_b^1(\mathrm{DP}[0]) \circ \sigma_b^1(\mathrm{DP}[0]) \tag{3.10}
$$

$$
\sigma_\wedge^2(\Delta_{g_M}) = \sigma_\wedge^1(\mathrm{DP}[0]) \circ \sigma_\wedge^1(\mathrm{DP}[0]) \tag{3.11}
$$

as $\mathrm{DP}[0] = d + d^*$ and $\Delta_{g_M} = (d + d^*) \circ (d + d^*)$. Now, from the matrices representing Δ_{g_M} in proposition 3.9 we have that the elements in $\mathbf{B}'$ and $\mathbf{C}'$ are operators of order 1 hence they do not intervene in the computation of $\sigma_b^2(\Delta_{g_M})$. Hence let's focus on the operators in $\mathbf{A}'$ and $\mathbf{D}'$.

Observe that for any $\alpha \in \bigwedge^\bullet T^*_\wedge M$

$$\begin{aligned}
\sigma_b^2(d^*_{\mathcal{E}} d_{\mathcal{X}} + d_{\mathcal{X}} d^*_{\mathcal{E}})(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta})(\alpha) &= \tilde{\eta}^* \lrcorner (\xi \wedge \alpha) + \xi \wedge (\tilde{\eta}^* \lrcorner \alpha) \\
&= (\tilde{\eta}^* \lrcorner \xi) \wedge \alpha - \xi \wedge (\tilde{\eta}^* \lrcorner \alpha) + \xi \wedge (\tilde{\eta}^* \lrcorner \alpha) \\
&= 0
\end{aligned}$$

as $\tilde{\eta}^* \in T\mathcal{E}$ and $\xi \in T^*\mathcal{X}$. Moreover

$$\begin{aligned}
\sigma_b^2(d^*_{\mathcal{E}} d_{\mathcal{E}})(\tilde{\eta})(\Theta_{\mathcal{X}}) &= \tilde{\eta}^* \lrcorner (\tilde{\eta} \wedge \Theta_{\mathcal{X}}) \\
&= \Theta_{\mathcal{X}} + \tilde{\eta} \wedge (\tilde{\eta}^* \lrcorner \Theta_{\mathcal{X}}) \\
&= \Theta_{\mathcal{X}}
\end{aligned}$$

and in the same way

$$\sigma_b^2(d^*_{\mathcal{X}} d_{\mathcal{X}})(\xi)(\Theta_{\mathcal{E}}) = \Theta_{\mathcal{E}}.$$

Therefore $\sigma_b^2(\Delta_{g_M})(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta})$ is a diagonal matrix with entries given by

$$\sigma_b^2(\Delta_{\mathcal{X}} + d^*_{\mathcal{E}} d_{\mathcal{E}} - \partial_r^2)(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) = |\xi|^2_{g_{\mathcal{X}}} + 1 + |\tilde{\rho}|^2$$

$$\sigma_b^2(\Delta_{\mathcal{E}} + d^*_{\mathcal{X}} d_{\mathcal{X}} - \partial_r^2)(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) = |\tilde{\eta}|^2_{g_{\mathcal{E}}} + 1 + |\tilde{\rho}|^2$$

and

$$\sigma_b^2(\Delta_{\mathcal{X}} + \Delta_{\mathcal{E}} - \partial_r^2)(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) = |\xi|^2_{g_{\mathcal{X}}} + |\tilde{\eta}|^2_{g_{\mathcal{E}}} + |\tilde{\rho}|^2.$$

Hence

$$\sigma_b^2(\Delta_{g_M})(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) : \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M} \longrightarrow \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M}$$

is an isomorphism for every non-zero $(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) \in T^*\mathbb{M}$ up to $r = 0$. By (3.10) we have that

$$\sigma_b^1(\mathrm{DP}[0])(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) : \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M} \longrightarrow \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M}$$

has the same property. ■

In order to obtain information about the invertibility of the edge symbol $\sigma_\wedge^1(\mathrm{DP}[0])$ we will use proposition 3.10 together with theorem 2.4.18 and theorem 3.5.1 in [Sch98]. These theorems state the existence of admissible weights $\gamma \in \mathbb{R}$ such that $\sigma_\wedge^1(\mathrm{DP}[0])$ is a Fredholm operator on the corresponding cone-Sobolev spaces of any order. We adapt those theorems to our setting in the following result. Its proof follows immediately from theorem 2.4.18 and theorem 3.5.1 in [Sch98].

Theorem 3.11. The condition that

$$\sigma_b^1(\mathrm{DP}[0])(r, \sigma, u, \tilde{\rho}, \xi, \tilde{\eta}) : \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M} \longrightarrow \bigwedge^\bullet T^*_{\wedge,(r,\sigma,u)}\mathbb{M}$$

is an isomorphism for every non-zero $(r,\sigma,u,\tilde{\rho},\xi,\tilde{\eta}) \in T^*_\wedge\mathbb{M}$ up to $r=0$, implies that there exists a countable set $\Lambda \subset \mathbb{C}$, where $\Lambda \cap K$ is finite for every $K \subset\subset \mathbb{C}$, such that

$$\sigma^1_M\left(\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)\right)(z) : H^s(\mathcal{X},\bigwedge\nolimits^\bullet T^*_\wedge M) \longrightarrow H^{s-1}(\mathcal{X},\bigwedge\nolimits^\bullet T^*_\wedge M)$$

is an isomorphism (invertible, linear, continuous operator) for every $z \in \mathbb{C}\setminus\Lambda$ and all $s \in \mathbb{R}$. This implies that there is a countable subset $D \subset \mathbb{R}$ given by $D = \Lambda \cap \mathbb{R}$, with the property that $D \cap \{z : a \le \operatorname{Re} z \le b\}$ is finite for every $a \le b$, such that

$$\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta) : \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge,\bigwedge\nolimits^\bullet T^*_\wedge M) \longrightarrow \mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge,\bigwedge\nolimits^\bullet T^*_\wedge M)$$

is a family of Fredholm operators for each $\gamma \in \mathbb{R}\setminus D$ and $(u,\eta) \in S^*\mathcal{E}$ with $\eta \neq 0$.

Theorem 3.11 tell us that for an admissible weight γ, the wedge symbol $\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)$ defines a Fredholm operator for each $(u,\eta) \in S^*\mathcal{E}$. However, if we require the ellipticity of DP[0] we need to have a family of invertible operators. In some cases this can be achieved by adding boundary and coboundary operators that defines an elliptic edge boundary value problem. This can be done in the following way.

For each $(u,\eta) \in S^*\mathcal{E}$ we have that $\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)$ is a Fredholm operator, then

$$\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)\right) \subset \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge,\bigwedge\nolimits^\bullet T^*_\wedge M)$$

is a finite dimensional subspace. Moreover $\operatorname{Coker}\left(\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)\right)$ is finite dimensional too.

Let $N(u,\eta) = \dim\operatorname{Coker}\left(\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)\right)$ and choose an isomorphism

$$\mathrm{k}(u,\eta) : \mathbb{C}^{N(u,\eta)} \longrightarrow \operatorname{Coker}\left(\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta)\right),$$

then

$$\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{k}\end{pmatrix}(y,\eta) : \begin{array}{c}\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge,\bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ \mathbb{C}^{N(u,\eta)}\end{array} \longrightarrow \mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge,\bigwedge\nolimits^\bullet T^*_\wedge M)$$

is a surjective operator. Now, because the set of surjective operators is an open set and the space $S^*\mathcal{E}$ is compact, there exists $N^+ \in \mathbb{N}$ and $\mathrm{c} \in \mathcal{L}\left(\mathbb{C}^{N^+},\mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge,\bigwedge^\bullet T^*_\wedge M)\right)$ such that

$$\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{c}\end{pmatrix}(y,\eta) : \begin{array}{c}\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge,\bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ \mathbb{C}^{N^+}\end{array} \longrightarrow \mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge,\bigwedge\nolimits^\bullet T^*_\wedge M) \tag{3.12}$$

is Fredholm and surjective for each $(y,\eta) \in S^*\mathcal{E}$ (see [Sch98] theorem 1.2.30 for further details). Because $\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{c}\end{pmatrix}(y,\eta)$ is Fredholm and surjective we have that the kernel of $\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{c}\end{pmatrix}(y,\eta)$ has constant dimension equal to its index for every $(y,\eta) \in S^*\mathcal{E}$:

$$\dim\operatorname{Ker}\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{c}\end{pmatrix}(y,\eta) = \operatorname{Ind}\begin{pmatrix}\sigma^1_\wedge(\mathrm{DP}[0]) & \mathrm{c}\end{pmatrix}(y,\eta) := N^- \quad \forall (u,\eta) \in S^*\mathcal{E}.$$

The finite dimensional spaces $\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)(y,\eta)$ define a smooth vector bundle over $S^*\mathcal{E}$ (see section 1.2.4 in [Sch98]).

Now consider the trivial bundle of dimension N^+ over $S^*\mathcal{E}$. Here we denote it simply as $\mathbb{C}^{N^+}$. The formal difference of these vector bundles defines an element in the K-theory of $S^*\mathcal{E}$

$$\left[\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)\right] - \left[\mathbb{C}^{N^+}\right] \in K(S^*\mathcal{E}).$$

This element in the K-group represents a topological obstruction to the existence of an elliptic edge boundary value problem for the operator DP[0]. More precisely we have the following theorem. For its proof and more details about the obstruction of ellipticity in the edge calculus see [NSSS06] section 6.2.

Theorem 3.12. A necessary and sufficient condition for the existence of an elliptic edge problem for DP[0] is given by

$$\left[\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)\right] - \left[\mathbb{C}^{N^+}\right] \in \pi^*_{S^*\mathcal{E}} K(\mathcal{E}) \tag{3.13}$$

where $\pi_{S^*\mathcal{E}} : S^*\mathcal{E} \longrightarrow \mathcal{E}$ is the natural projection and $\pi^*_{S^*\mathcal{E}}K(\mathcal{E})$ is the subgroup of $K(S^*\mathcal{E})$ generated by vector bundles lifted from $\mathcal{E}$ by means of $\pi_{S^*\mathcal{E}}$.

Now, assume for the moment that the condition in theorem 3.12 is satisfied. Then the bundle defined by $\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)(y,\eta)$ is stably equivalent to a vector bundle J^- lifted from $\mathcal{E}$. Then, by adding zeros to c if needed, we can assume that the bundle $\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)(y,\eta)$ is isomorphic to J^-. By extending this isomorphism by zero on the orthogonal complement of $\operatorname{Ker}\left(\sigma^1_\wedge(\mathrm{DP}[0]) \quad \mathrm{c}\right)(y,\eta)$ we obtain a map

$$\left(\mathrm{t}(u,\eta) \quad \mathrm{b}(u,\eta)\right) : \begin{array}{c}\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge, \bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ \mathbb{C}^{N^+}\end{array} \longrightarrow J^-_{(u,\eta)} \tag{3.14}$$

such that

$$\begin{bmatrix}\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta) & \mathrm{c}(u,\eta)\\ \mathrm{t}(u,\eta) & \mathrm{b}(u,\eta)\end{bmatrix} : \begin{array}{c}\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge, \bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ \mathbb{C}^{N^+}\end{array} \longrightarrow \begin{array}{c}\mathcal{K}^{s-1,\gamma-1}(\mathcal{X}^\wedge, \bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ J^-_{(u,\eta)}\end{array}$$

is an invertible, linear operator for every $\eta \neq 0$.

Then, the operator

$$\mathcal{A}_{\mathrm{DP}[0]} = \begin{bmatrix}\mathrm{DP}[0] & \mathrm{C}\\ \mathcal{T} & \mathrm{B}\end{bmatrix} = \mathcal{F}^{-1}_{\eta\to u}\begin{bmatrix}\sigma^1_\wedge(\mathrm{DP}[0])(u,\eta) & \mathrm{c}(u,\eta)\\ \mathrm{t}(u,\eta) & \mathrm{b}(u,\eta)\end{bmatrix}\mathcal{F}_{u'\to\eta}$$

acting on the spaces

$$\begin{bmatrix}\mathrm{DP}[0] & \mathrm{C}\\ \mathcal{T} & \mathrm{B}\end{bmatrix} : \begin{array}{c}\mathcal{W}^{s,\gamma}(M, \bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+})\end{array} \longrightarrow \begin{array}{c}\mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M)\\ \oplus \\ H^{s-1}(\mathcal{E}, J^-)\end{array}$$

is an elliptic edge operator (see definition 18) for all $s \in \mathbb{R}$ and γ the admissible weight chosen at the beginning.

In order to prove the claim that condition (3.13) is satisfied we have the following theorem which is contained in [NSSS06] theorem 6.30.

Theorem 3.13. If the Atiyah-Bott obstruction vanishes for an edge-degenerate operator A on the stretched manifold $\mathbb{M}$, then there exists an elliptic edge problem for A.

Recall that given a compact manifold with boundary $\mathbb{M}$, the principal symbol an elliptic differential operator $\mathrm{A} \in \mathrm{Diff}^l(\mathbb{M})$ defines an bundle map

$$\sigma(\mathrm{A}) : \pi_{\mathbb{M}}^* T^*\mathbb{M} \longrightarrow \pi_{\mathbb{M}}^* T^*\mathbb{M}$$

where $\pi_{\mathbb{M}} : T^*\mathbb{M} \longrightarrow \mathbb{M}$ is the natural projection. Note that $\sigma(\mathrm{A})$ is an isomorphism outside the compact set diffeomorphic to $\mathbb{M}$ given by the zero section of $T^*\mathbb{M}$. Thus the triple $(\pi_{\mathbb{M}}^* T^*\mathbb{M}, \pi_{\mathbb{M}}^* T^*\mathbb{M}, \sigma(\mathrm{A}))$ defines an element $[\sigma]$ in $K_c(T^*\mathbb{M})$, the K-group with compact support of $T^*\mathbb{M}$. See [NSSS06] section 3.6.1 for further details.

The Atiyah-Bott obstruction of A is an element in the K-theory with compact support of $\partial T^*\mathbb{M}$. It is given by

$$\left[\sigma(\mathrm{A})|_{\partial T^*\mathbb{M}}\right] \in K_c(\partial T^*\mathbb{M}).$$

This is the topological obstruction for the existence of boundary conditions given by continuous linear operators

$$\mathrm{B}_i : H^s(\mathbb{M}) \longrightarrow H^{s-\frac{1}{2}-r_j}(\partial\mathbb{M}) \quad i = 1, \ldots, k$$

where r_j is the order of B_i, such that the boundary value problem

$$\begin{bmatrix} \mathrm{A} \\ \mathrm{B}_1 \\ \vdots \\ \mathrm{B}_k \end{bmatrix} : H^s(\mathbb{M}) \longrightarrow H^{s-l}(\mathbb{M}) \bigoplus_{j=0}^{k} H^{s-\frac{1}{2}-r_j}(\partial\mathbb{M}) \tag{3.15}$$

is elliptic for $s > \frac{1}{2} + \max\{r_j\}$ and therefore (3.15) is a Fredholm operator. The reader is referred to [AB64] for details and [BB85] part III chapters 6 and 7 for a comprehensive review of this obstruction; in particular for how it is derived from the classical Shapiro-Lopatinsky condition for boundary value problems.

In our case DP[0] is the Hodge-deRham operator, the vanishing of the Atiyah-Bott obstruction for this operator was proved by Atiyah-Patodi-Singer in [APS75], hence the topological condition (3.13) is satisfied.

Chapter 4

Conormal deformations and regularity

4.1 Preliminaries

Given a special Lagrangian submanifold of $\mathbb{C}^n$ with edge singularity, (M, Φ) (see 2.2.1), in this section we define the moduli space of special Lagrangian deformations of (M, Φ). Broadly speaking, we want to have in the moduli space all nearby special Lagrangian submanifolds with edge singularity. This rough idea has two aspects that must be considered for the moduli space. First, as the manifold M is non-compact, the important aspect to consider when defining the concept of nearby submanifold is the behavior on the collar neighborhood $(0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$. Here we shall define the concept of nearby submanifold by means of its asymptotic behavior with respect to the conormal variable r and weight γ. Second, the property of being special Lagrangian is completely determined by the equations (1.14). As we mentioned in remark 2.8 and 2.10, solutions of the linearised deformation equation have conormal asymptotics (section 2.3.3). This asymptotic behavior is transferred to the induced metric of the deformed submanifold making the induced metric asymptotic to the original edge metric $g_{_M}$ in a very special way that reflects the fact it comes from the solution of an edge-degenerate PDE on a singular space. All of these considerations are formalized in the following definition.

Definition 25. Given an embedding $\Upsilon : M \longrightarrow \mathbb{C}^n$ we say that Υ is conormal asymptotic to (M, Φ) with rate γ if :

i) For every multi-index α we have

$$\left|\partial^{\alpha}_{(r,\sigma,u)} \left(\Upsilon(r, \sigma, u) - \Phi(r, \sigma, u)\right)\right| = O(r^{\gamma-|\alpha|}) \quad \forall (r, \sigma, u) \in (0, \varepsilon) \times \mathcal{X} \times \mathcal{E};$$

ii) $\Upsilon^* g_{\mathbb{C}^n} = r^2 g_{\mathcal{X}} + dr^2 + g_{\mathcal{E}} + \beta$ where β is a symmetric 2-tensor on $\Upsilon(M) = M_\Upsilon$ such that their components β_{ij} have conormal asymptotic expansions on the collar neighborhood $(0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$ with respect to some asymptotic type associated to γ.

Because we want to describe a small neighborhood of (M, Φ) in the moduli space by means of the Implicit Function Theorem 1.6 applied to a neighborhood of zero in edge-Sobolev spaces, we want to make sure that smooth elements in edge-Sobolev spaces with small norm will produce submanifolds. In order to show this, we will define a neighborhood of deformations i.e. we will define a small neighborhood of M in $\mathbb{C}^n$ such that small deformations will be inside this neighborhood. Because of the geometric singularities of the manifold M and the behavior of the elements in edge-Sobolev spaces, this neighborhood will be constructed as an edge neighborhood to guarantee that small submanifolds induced by edge-degenerate forms will fit inside it.

Proposition 4.1. There exists an open edge neighborhood of the zero section in the normal bundle $\mathcal{N}\big((0,\varepsilon)\times\mathcal{X}\times\mathcal{E}\big)$ such that it is given by $V\times W$ where $V\subset\mathcal{N}((0,\varepsilon)\times\mathcal{X})\subset T\mathbb{R}^n_x$ is an open conical set and $W\subset\mathcal{N}(\mathcal{E})\subset T\mathbb{R}^n_y$ is an open set both of them being neighborhoods of the zero section in the corresponding normal bundles and diffeomorphic to an open edge set $\tilde{V}\times\tilde{W}\subset\mathbb{R}^n_x\oplus\mathbb{R}^n_y\cong\mathbb{C}^n$ with diffeomorphism given by the exponential map $\exp_{g_{\mathbb{C}^n}}$. Moreover, for every $\gamma>\frac{m+3}{2}$ and $s>\frac{q+m+1}{2}+\mathfrak{c}_\gamma$, where $\mathfrak{c}_\gamma$ is the positive constant defined in (2.14), there exists $\vartheta>0$ depending on s and γ such that

$$\Big\{\mathcal{V}_\Xi\big|_{(0,\varepsilon)\times\mathcal{X}\times\mathcal{E}}:\Xi\in\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)\text{ and }\|\Xi\|_{s,\gamma}<\vartheta\Big\}\subset V\times W.$$

Proof. First let's define the conical open neighborhood $V\subset\mathcal{N}((0,\varepsilon)\times\mathcal{X})\subset T\mathbb{R}^n_x$. The tubular neighborhood theorem 1.5 applied to $\mathcal{X}$ as a compact submanifold in $\mathbb{R}^n$ gives us an open neighborhood of the zero section in $\mathcal{N}(\mathcal{X})$. Take $l_0>0$ to be the maximum l such that the uniform neighborhood $\{X\in\mathcal{N}(\mathcal{X}):|X|_{g_{\mathbb{R}^n}}<l\}$ is inside the tubular neighborhood. By applying the $\mathbb{R}^+$-action defined on the cone $\mathcal{X}^\wedge$ to this uniform neighborhood we can obtain the desired open conical neighborhood V of the zero section in the normal bundle $\mathcal{N}((0,\varepsilon)\times\mathcal{X})\subset T\mathbb{R}^n_x$. Now, for any section $\mathcal{V}$ of the normal bundle $\mathcal{N}(\mathcal{X})$ lying in V we have $\big|\mathcal{V}(r,\sigma,u)\big|_{g_{\mathbb{C}^n}}<C_1\cdot r$ for all $(r,\sigma,u)\in(0,\varepsilon)\times\mathcal{X}\times\mathcal{E}$, where the constant C_1 is independent of $\mathcal{V}$. The constant C_1 can be taken to be the maximum $l>0$ chosen above. Now choose a uniform tubular neighborhood of the zero section in the normal bundle $\mathcal{N}(\mathcal{E})$ given by $\{Y\in\mathcal{N}(\mathcal{E}):|Y|_{g_{\mathbb{R}^n}}<\vartheta\}$ for some $\vartheta>0$. Clearly this is possible because $\mathcal{E}$ is compact. If necessary we can choose a smaller ε such that $\vartheta>C_1\varepsilon$. Define W as this uniform neighborhood of the zero section $W=\{Y\in\mathcal{N}(\mathcal{E}):|Y|_{g_{\mathbb{R}^n}}<\vartheta\}$. Then $V\times W$ is our open edge neighborhood of the zero section in $\mathcal{N}\big((0,\varepsilon)\times\mathcal{X}\times\mathcal{E}\big)$.

To prove the second part of the proposition let $\Xi\in\mathcal{W}^{s,\gamma}(M,T^*_\wedge M)$ with $s>\frac{q+m+1}{2}+\mathfrak{c}_\gamma$ and consider its local expression in a neighborhood $(0,\varepsilon)\times\mathcal{U}\times\Omega\subset(0,\varepsilon)\times\mathcal{X}\times\mathcal{E}$ given by

$$\Xi(r,\sigma,u)=\mathcal{A}(r,\sigma,u)dr+\sum_{k=1}^{m}\mathcal{B}_k(r,\sigma,u)rd\sigma_k+\sum_{l=1}^{q}\mathcal{C}_l(r,\sigma,u)du_l,$$

where $\omega\phi_j\varphi_\lambda\mathcal{A}$, $\omega\phi_j\varphi_\lambda\mathcal{B}_k$ and $\omega\phi_j\varphi_\lambda\mathcal{C}_l$ belong to $\mathcal{W}^{s,\gamma}(M)$ as in lemma 3.1. Then by proposition A.5 there exists a constant $C>0$ depending only on s and γ such that

$$\big|\mathcal{A}(r,\sigma,u)\big|\le C\|\Xi\|_{s,\gamma}\,r^{\gamma-\frac{m+1}{2}}\tag{4.1}$$

$$\big|\mathcal{B}(r,\sigma,u)\big|\le C\|\Xi\|_{s,\gamma}\,r^{\gamma-\frac{m+1}{2}}\tag{4.2}$$

$$\big|\mathcal{C}(r,\sigma,u)\big|\le C\|\Xi\|_{s,\gamma}\,r^{\gamma-\frac{m+1}{2}}\tag{4.3}$$

for all $(r, \sigma, u) \in (0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$. Hence by lemma 3.1 there exists a constant C' depending only on s and γ such that

$$\left|\tilde{\mathcal{C}}(r, \sigma, u)\right| \leq C' \|\Xi\|_{s,\gamma} r^{\gamma - \frac{m+1}{2}} \tag{4.4}$$

$$\left|\mathcal{A}(r, \sigma, u)\theta_i + \tilde{\mathcal{B}}_i(r, \sigma, u)\right| \leq C' \|\Xi\|_{s,\gamma} r^{\gamma - \frac{m+1}{2}}. \tag{4.5}$$

Then, by (4.4) and because $0 < \varepsilon < 1$, we have that

$$\left|\tilde{\mathcal{C}}(r, \sigma, u)\right| \leq C_1 r \tag{4.6}$$

for all $(r, \sigma, u) \in (0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$ if

$$\gamma > \log\left(\frac{C_1}{C' \|\Xi\|_{s,\gamma}}\right) \frac{1}{\log(r)} + \frac{m+3}{2}. \tag{4.7}$$

Note that if $\|\Xi\|_{s,\gamma}$ is small enough, then (4.7) is satisfied and this implies (4.6). More precisely, if $\frac{C_1}{C'} \geq \|\Xi\|_{s,\gamma}$ then (4.7) is satisfied as $\log(r) < 0$ for $r \leq \varepsilon$. Therefore $\frac{C_1}{C'} \geq \|\Xi\|_{s,\gamma}$ implies

$$\left|\tilde{\mathcal{C}}(r, \sigma, u)\right| \leq C_1 r.$$

Analogously, $\frac{C_1}{C'} > \|\Xi\|_{s,\gamma}$ implies that

$$\left|\mathcal{A}(r, \sigma, u)\theta_i + \tilde{\mathcal{B}}_i(r, \sigma, u)\right| \leq C_1 r$$

for any $\gamma > \frac{m+3}{2}$. Then it follows from our chose of ϑ that

$$\left|\mathcal{A}(r, \sigma, u)\theta_i + \tilde{\mathcal{B}}_i(r, \sigma, u)\right| \leq \vartheta$$

for all $(r, \sigma, u) \in (0, \varepsilon) \times \mathcal{X} \times \mathcal{E}$. ■

Observe that the manifold $M \setminus \big((0, \varepsilon) \times \partial\mathbb{M}\big)$ is compact hence we can extend our edge neighborhood $V \times W$ to this compact space to get a open neighborhood of the zero section in $\mathcal{N}(M)$ such that near the edge this neighborhood corresponds to the edge open neighborhood constructed above. We denote this neighborhood as $\mathfrak{A}$. Moreover this proposition implies that any smooth form $\Xi \in \mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$ as above with $\|\Xi\|_{s,\gamma} < \vartheta$ produces a smooth embedded submanifold inside the neighborhood of deformations $\mathfrak{A}$. This submanifold is defined by the embedding $\exp_{g_{\mathbb{C}^n}}(\mathcal{V}_\Xi) \circ \Phi$.

4.2 Regularity of Deformations

Let's consider an elliptic edge problem (see section 3.2.3) for the operator DP[0] acting on edge-Sobolev spaces with admissible weigh $\gamma > \frac{m+1}{2}$,

$$\mathcal{A}_{\mathrm{DP}[0]} = \begin{bmatrix} \mathrm{DP}[0] & \mathrm{C} \\ \mathcal{T} & \mathrm{B} \end{bmatrix} : \begin{array}{c} \mathcal{W}^{s,\gamma}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{array}.$$

Then, by augmenting the deformation operator $\mathrm{P} = \mathrm{P}_{\omega_{\mathbb{C}^n}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega}$ with the trace operator

$$\mathcal{T} : \mathcal{W}^{s,\gamma}(M, \bigwedge^{\bullet} T^*_{\wedge} M) \longrightarrow H^{s-1}(\mathcal{E}, J^-),$$

we obtain a non-linear boundary value problem for P:

$$\begin{bmatrix} \mathrm{P}_{\omega_{\mathbb{C}^N}} \oplus P_{\mathrm{Im}\,\Omega} \\ \mathcal{T} \end{bmatrix} : \mathfrak{A} \subset \mathcal{W}^{s,\gamma}(M, T^*_{\wedge} M) \longrightarrow \begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^{\bullet} T^*_{\wedge} M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix}$$

whose linearisation at zero is given by

$$\begin{bmatrix} \mathrm{DP}[0] \\ \mathcal{T} \end{bmatrix} : \mathcal{W}^{s,\gamma}(M, T^*_{\wedge} M) \longrightarrow \begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^{\bullet} T^*_{\wedge} M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix} .$$

In this section we consider some properties of solutions of the equation

$$\begin{bmatrix} \mathrm{P}_{\omega_{\mathbb{C}^N}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega} \\ \mathcal{T} \end{bmatrix} (\Xi) = 0 \tag{4.8}$$

where $\Xi \in \mathcal{W}^{s,\gamma}(M, T^*_{\wedge} M)$. We are mainly interested in those solutions given by the Implicit Function Theorem for Banach spaces (when applicable) i.e. we assume that $\Xi = \Xi_1 + \Xi_2$ where Ξ_1 is solution of the linear boundary value problem

$$\begin{bmatrix} \mathrm{DP}[0] \\ \mathcal{T} \end{bmatrix} (\Xi_1) = 0 \tag{4.9}$$

and Ξ_2 belongs to the Banach space complement in $\mathcal{W}^{s,\gamma}(M, T^*_{\wedge} M)$ defined by a splitting (not unique) induced by the finite dimensional space $\mathrm{Ker}\,\mathcal{A}_{\mathrm{DP}[0]}$. First we have some straightforward observations. The ellipticity of the operator $\mathcal{A}_{\mathrm{DP}[0]}$ (theorem 3.13) and the fact that

$$\mathcal{A}_{\mathrm{DP}[0]} \begin{bmatrix} \Xi_1 \\ 0 \end{bmatrix} = 0$$

implies that $\Xi_1 \in \mathcal{W}^{\infty,\gamma}(M, T^*_{\wedge} M)$ by elliptic regularity (theorem 2.9). Moreover, because $\mathcal{W}^{s,\gamma}(M, T^*_{\wedge} M) \subset H^s_{loc}(M, T^*_{\wedge} M)$ for all $s \in \mathbb{R}$ ([ES97], section 9.3, proposition 5), standard Sobolev embeddings (theorem A.1) imply that Ξ_1 is smooth. The ellipticity of $\mathcal{A}_{\mathrm{DP}[0]}$ implies the existence of a parametrix $\mathrm{B}_{\mathrm{DP}[0]}$ with asymptotics O (theorem 2.9) i.e.

$$\mathrm{B}_{\mathrm{DP}[0]}\, \mathcal{A}_{\mathrm{DP}[0]} - \mathrm{I} : \quad \begin{matrix} \mathcal{W}^{s,\gamma}(M, \bigwedge^{\bullet} T^*_{\wedge} M) \\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{\infty,\gamma}_O(M, \bigwedge^{\bullet} T^*_{\wedge} M) \\ \oplus \\ H^{\infty}(\mathcal{E}, J^-) \end{matrix} .$$

Consequently, any element in the kernel of the operator $\mathcal{A}_{\mathrm{DP}[0]}$ belongs to $\mathcal{W}^{\infty,\gamma}_O(M, \bigwedge^{\bullet} T^*_{\wedge} M)$ for some asymptotic type O associated to γ. In particular

$$\Xi_1 \in \mathcal{W}^{\infty,\gamma}_O(M, T^*_{\wedge} M). \tag{4.10}$$

Now let's consider the regularity properties of Ξ_2.

Let $\Xi \in \mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$ such that

$$(\mathrm{P}_{\omega_{\mathbb{C}^n}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega})(\Xi) = 0. \tag{4.11}$$

Hence $\exp_{g_{\mathbb{C}^n}}(\mathcal{V}_\Xi) \circ \Phi : M \longrightarrow \mathbb{C}^n$ is a special Lagrangian submanifold. Harvey and Lawson pointed out in [HL82] theorem 2.7 that C^2 special Lagrangian submanifolds in $\mathbb{C}^n$ are real analytic, in particular they are smooth. Therefore, by choosing s large enough, (4.11) implies that $\Xi \in C^\infty(M, T^*_\wedge M)$ which, together with (4.10), allow us to conclude that Ξ_2 is smooth.

Even though $\Xi_1 + \Xi_2$ is solution of the non-linear edge boundary value problem (4.8) we cannot conclude immediately that Ξ has a conormal asymptotic expansion near the singular set $\mathcal{E}$. The edge calculus tells us that solutions of the linearised equation (4.9), here denoted by Ξ_1, have such asymptotics. It turns out that it is possible to prove that Ξ_2 also has a conormal expansion i.e. the whole solution of the non-linear edge boundary value problem has conormal expansion. In order to prove this we follow and adapt to our very specific setting in the next two propositions the general argument in [CMR15] theorem 5.1. The author thanks Frédéric Rochon for pointing out and explaining his work.

Observe that

$$\begin{bmatrix} \mathrm{P}_{\omega_{\mathbb{C}^N}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega} \\ \mathcal{T} \end{bmatrix} (\Xi_1 + \Xi_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies that $\mathcal{T}(\Xi_2) = 0$ because $\mathcal{T}(\Xi_1) = 0$ due to the fact that Ξ_1 is solution of the linearised equation (4.9). By writing the non-linear equation as $\mathrm{P}_{\omega_{\mathbb{C}^N}} \oplus \mathrm{P}_{\mathrm{Im}\,\Omega} = \mathrm{DP}[0] + \mathrm{Q}$ (see proposition 3.3 and (3.7)) where Q is a non-linear operator locally defined by the sum of products of 2 or more operators in $\mathrm{Diff}^1_{\mathrm{edge}}(M)$ acting on Ξ_1 or Ξ_2 we have

$$(\mathrm{DP}[0] + \mathrm{Q})(\Xi_2) = -\,\mathrm{Q}(\Xi_1) - \sum_{j \geq 2} \mathrm{Q}_{i_1}(\Xi_\bullet) \cdots \mathrm{Q}_{i_j}(\Xi_\bullet). \tag{4.12}$$

In order to avoid cumbersome notation to keep track of the specific asymptotic types, we say that an element belongs to $\mathcal{W}^{s,\gamma}_{\mathrm{As}}(M, T^*_\wedge M)$ if it belongs to the edge-Sobolev space with some asymptotic type associated to γ.

Proposition 4.2. Let $\xi_1 \in \mathcal{W}^{\infty,\gamma-1}_{\mathrm{As}}(M, \bigwedge^\bullet T^*_\wedge M)$ and $\Xi_2 \in \mathcal{W}^{\infty,\gamma}(M, T^*_\wedge M)$ such that

$$(\mathrm{DP}[0] + \mathrm{Q})(\Xi_2) = \xi_1. \tag{4.13}$$

Assume that γ is an admissible weight for DP[0] and there exists $\beta > 0$ such that $\mathrm{Q}(\Xi_2) \in \mathcal{W}^{\infty,\gamma+\beta}(M, \bigwedge^\bullet T^*_\wedge M)$ and $\gamma + \beta + 1$ is an admissible weight. Then $\Xi_2 = E_1 + E_2$ with $E_2 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_\wedge M)$ and $E_1 \in \mathcal{W}^{\infty,\gamma+\beta+1}(M, T^*_\wedge M)$.

Proof. Let's consider the Fredholm operator defined by DP[0] acting on edge-Sobolev spaces with weight $\gamma + \beta + 1$

$$\mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1} : \begin{matrix} \mathcal{W}^{s+1,\gamma+\beta+1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s+1}(\mathcal{E}, \mathbb{C}^{N^+}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s,\gamma+\beta}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s}(\mathcal{E}, J^-) \end{matrix} .$$

Because $\operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1}$ is finite dimensional and $C_0^\infty(M, \bigwedge^\bullet T_\wedge^* M)$ is a dense subset of the edge-Sobolev spaces we have

$$\begin{bmatrix} -\operatorname{Q}(\Xi_2) \\ 0 \end{bmatrix} = \mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1} \begin{bmatrix} E_1 \\ e_1 \end{bmatrix} + \begin{bmatrix} \mathfrak{F} \\ f \end{bmatrix}$$

with

$$\begin{bmatrix} E_1 \\ e_1 \end{bmatrix} \in \begin{array}{c} \mathcal{W}^{s+1,\gamma+\beta+1}(M, \bigwedge^\bullet T_\wedge^* M) \\ \oplus \\ H^{s+1}(\mathcal{E}, \mathbb{C}^{N^+}) \end{array}$$

and $\mathfrak{F} \in C_0^\infty(M, \bigwedge^\bullet T_\wedge^* M)$, $f \in C_0^\infty(\mathcal{E}, \mathbb{C}^{N^+})$. Observe that this implies

$$\mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1} \begin{bmatrix} E_1 \\ e_1 \end{bmatrix} \in \begin{array}{c} \mathcal{W}^{\infty,\gamma+\beta}(M, \bigwedge^\bullet T_\wedge^* M) \\ \oplus \\ H^{\infty}(\mathcal{E}, \mathbb{C}^{N^+}) \end{array}$$

as $\operatorname{Q}(\Xi_2) \in \mathcal{W}^{\infty,\gamma+\beta}(M, T_\wedge^* M)$.

Hence by elliptic regularity (theorem 2.9)

$$\begin{bmatrix} E_1 \\ e_1 \end{bmatrix} \in \begin{array}{c} \mathcal{W}^{\infty,\gamma+\beta+1}(M, \bigwedge^\bullet T_\wedge^* M) \\ \oplus \\ H^{\infty}(\mathcal{E}, \mathbb{C}^{N^+}) \end{array} .$$

Now define $\begin{bmatrix} E_2 \\ e_2 \end{bmatrix} := \begin{bmatrix} \Xi_2 \\ 0 \end{bmatrix} - \begin{bmatrix} E_1 \\ e_1 \end{bmatrix}$, then by (4.13)

$$\mathcal{A}_{\mathrm{DP}[0],\gamma} \begin{bmatrix} E_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} \xi_1 - \operatorname{Q}(\Xi_2) \\ 0 \end{bmatrix} - \mathcal{A}_{\mathrm{DP}[0],\gamma} \begin{bmatrix} E_1 \\ e_1 \end{bmatrix}.$$

Observe that as $\mathcal{A}_{\mathrm{DP}[0],\gamma}$ and $\mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1}$ are 2×2 operator matrices with $\mathrm{DP}[0]$ in the upper left corner they differ by Green operators with asymptotics acting on the corresponding spaces (see section 2.3.7). Hence we can write $\mathcal{A}_{\mathrm{DP}[0],\gamma} = \mathcal{A}_{\mathrm{DP}[0],\gamma+\beta+1} - G_{\mathrm{DP}[0],\gamma+\beta+1} + G_{\mathrm{DP}[0],\gamma}$, where $G_{\mathrm{DP}[0],\gamma+\beta+1}$ is the Green operator matrix with the elliptic boundary conditions for $\mathrm{DP}[0]$ acting on spaces with weight $\gamma+\beta+1$ and analogously for $G_{\mathrm{DP}[0],\gamma}$. This implies

$$\mathcal{A}_{\mathrm{DP}[0],\gamma} \begin{bmatrix} E_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{F} \\ f \end{bmatrix} - (-G_{\mathrm{DP}[0],\gamma+\beta+1} + G_{\mathrm{DP}[0],\gamma}) \begin{bmatrix} E_1 \\ e_1 \end{bmatrix}$$

therefore, by the mapping properties of Green operators (2.39),

$$\mathcal{A}_{\mathrm{DP}[0],\gamma} \begin{bmatrix} E_2 \\ e_2 \end{bmatrix} \in \mathcal{W}_{\mathrm{As}}^{\infty,\gamma-1}(M, \bigwedge^\bullet T_\wedge^* M).$$

By elliptic regularity we conclude

$$\begin{bmatrix} E_2 \\ e_2 \end{bmatrix} \in \begin{matrix} \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, \bigwedge^{\bullet} T^*_{\wedge} M) \\ \oplus \\ H^{\infty}(\mathcal{E}, \mathbb{C}^{N^+}) \end{matrix}$$

and $\Xi_2 = E_1 + E_2$ as claimed. ■

Proposition 4.3. Let $\Xi_1 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_{\wedge}M)$ and $\Xi_2 \in \mathcal{W}^{\infty,\gamma}(M, T^*_{\wedge}M)$ with an admissible weight $\gamma > \frac{m+5}{2}$ such that

$$(\mathrm{DP}[0] + \mathrm{Q})(\Xi_1 + \Xi_2) = 0 \tag{4.14}$$

and $\mathrm{DP}[0](\Xi_1) = 0$. Then $\Xi_2 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_{\wedge}M)$.

Proof. Equation (4.14) can be written as

$$(\mathrm{DP}[0] + \mathrm{Q})(\Xi_2) = -\,\mathrm{Q}(\Xi_1) - \sum_{j\geq 2} \mathrm{Q}_{i_1}(\Xi_{\bullet}) \cdots \mathrm{Q}_{i_j}(\Xi_{\bullet}), \tag{4.15}$$

(see (4.12)). The right hand side of (4.15) consists of products where at least one of the operators in each of these products is acting on Ξ_1, let's say $\mathrm{Q}_1(\Xi_1)$. Now, the term $\mathrm{Q}_1(\Xi_1)$ has asymptotics (associated to $\gamma - 1$) and the other elements $\mathrm{Q}_{i_k}(\Xi_{\bullet})$ in the products satisfy the estimate (A.7) near the edge. Hence by [Sch98] theorem 2.3.13, multiplication by elements in $C^{\infty}_0(\mathbb{M})$ induces a continuous operator on Sobolev spaces with asymptotics (with possibly different asymptotic type but associated to the same weight). Therefore we conclude that the right hand side in (4.15) belongs to $\mathcal{W}^{\infty,\gamma-1}_{\mathrm{As}}(M, T^*_{\wedge}M)$.

Now, the fact that $\gamma > \frac{m+5}{2}$ together with (A.43) implies that $\mathrm{Q}(\Xi_2) \in \mathcal{W}^{\infty,\gamma+\beta}(M, T^*_{\wedge}M)$ for some $\beta > 0$. If necessary we can choose β small enough such that $\gamma + \beta + 1$ is an admissible weight for $\mathrm{DP}[0]$. Then, proposition 4.2 implies $\Xi_2 = E_1 + E_2$ with $E_1 \in \mathcal{W}^{\infty,\gamma+\beta+1}(M, T^*_{\wedge}M)$ and $E_2 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_{\wedge}M)$.

Define $S_1 := \Xi_2 - E_2 = E_1$ and observe that (4.15) and a similar argument as above implies

$$(\mathrm{DP}[0] + \mathrm{Q})(S_1) := \xi_2 \in \mathcal{W}^{\infty,\gamma-1}_{\mathrm{As}}(M, T^*_{\wedge}M).$$

Moreover, (A.43) and $\gamma > \frac{m+5}{2}$ imply that $\mathrm{Q}(S_1)$ belongs to the space $\mathcal{W}^{\infty,\gamma+\beta+1+\beta'}(M, T^*_{\wedge}M)$ for some $\beta' > 0$. Then by following the same argument as in proposition 4.2 we have

$$S_1 = E_3 + E_4$$

with $E_3 \in \mathcal{W}^{\infty,\gamma+\beta+\beta'+2}(M, T^*_{\wedge}M)$ and $E_4 = \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_{\wedge}M)$. Hence we have found an element $E_2 + E_4 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_{\wedge}M)$ such that

$$\Xi_2 - (E_2 + E_4) = E_3 \in \mathcal{W}^{\infty,\gamma+\beta+\beta'+2}(M, T^*_{\wedge}M).$$

We continue this recursion argument by setting $S_2 = S_1 - E_4 = \Xi_2 - E_2 - E_4 = E_3$ and

$$(\mathrm{DP}[0] + \mathrm{Q})(S_2) = \xi_3.$$

Therefore by means of this iterative process we conclude that for every $l > 0$ there exists $E \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_\wedge M)$ such that $\Xi_2 - E \in \mathcal{W}^{\infty,\gamma+l}(M, T^*_\wedge M)$. Note that if

$$E(r,\sigma,y) - \omega(r) \sum_{j=0}^{N(l)} \sum_{k=0}^{m_j} c_{j,k}(\sigma) v_{j,k}(y) r^{-p_j} \log^k(r) \in \mathcal{W}^{\infty,\gamma+l}(M, T^*_\wedge M)$$

then, by adding and subtracting $\omega(r) \sum_{j=0}^{N(l)} \sum_{k=0}^{m_j} c_{j,k}(\sigma) v_{j,k}(y) r^{-p_j} \log^k(r) \in \mathcal{W}^{\infty,\gamma+l}(M, T^*_\wedge M)$ to $\Xi_2 - E$, we obtain

$$\Xi_2(r,\sigma,y) - \omega(r) \sum_{j=0}^{N(l)} \sum_{k=0}^{m_j} c_{j,k}(\sigma) v_{j,k}(y) r^{-p_j} \log^k(r) \in \mathcal{W}^{\infty,\gamma+l}(M, T^*_\wedge M).$$

Therefore $\Xi_2 \in \mathcal{W}^{\infty,\gamma}_{\mathrm{As}}(M, T^*_\wedge M)$. ∎

Chapter 5

The Moduli space

In this section we define the moduli space we are interested in and prove the main result of this thesis. Let (M, Φ) be a special Lagrangian submanifold with edge singularity i.e. M is a manifold with edge singularity (section 2.2.1) and $\Phi : M \longrightarrow \mathbb{C}^n$ is an edge special Lagrangian embedding (see definition 24).

Definition 26. Given an admissible weight $\gamma > \frac{m+5}{2}$, we define the moduli space $\mathfrak{M}(M, \Phi, \mathcal{T}, \gamma)$ of conormal asymptotic special Lagrangian deformations of (M, Φ) with rate γ and elliptic boundary trace condition $\mathcal{T}$ as the space of smooth embeddings $\Upsilon : M \longrightarrow \mathbb{C}^n$, conormal asymptotic to (M, Φ) with rate γ, such that they satisfy the boundary condition $\mathcal{T}(\Upsilon) = 0$ where $\mathcal{T}$ is a trace pseudo-differential operator:

$$\mathcal{T} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow H^{s-1}(\mathcal{E}, J^-) \tag{5.1}$$

with $s > \max\left\{\frac{m+1+q}{2} + \mathfrak{c}_\gamma, \frac{m+3+q}{2}\right\}$ such that $\mathcal{T}$ belongs to a set of boundary conditions for an elliptic edge boundary value problem for the operator $\mathrm{DP}[0]$ on the edge-Sobolev space $\mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$.

Given a special Lagrangian submanifold with edge singularity (M, Φ), the moduli space of conormal asymptotic special Lagrangian deformations depends on the parameters γ and $\mathcal{T}$. The role of the weight γ is explained in the definition of conormal asymptotic embedding (see definition 25). The existence of a trace operator $\mathcal{T}$ and its role in the elliptic theory of edge degenerate equations was discussed in section 3.2.3. Here we want to include further details about $\mathcal{T}$. The edge symbol defining $\mathcal{T}$ was defined in (3.14) as a family of continuous linear maps $\mathrm{t}(u, \eta) : \mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge) \longrightarrow J^-_{(u,\eta)}$ continuously parametrized by $T^*\mathcal{E} \setminus \{0\}$. Note that as J^- is a finite rank vector bundle, the operators $\mathrm{t}(u, \eta)$ are finite rank operators (in particular compact operators). The trace operator $\mathcal{T}$ was locally defined by

$$\mathcal{T} = \mathrm{Op}(\mathrm{t}(u, \eta)) = \mathcal{F}^{-1}_{\eta \to u}\, \mathrm{t}(u, \eta) \mathcal{F}_{u' \to \eta}.$$

Now recall from section 3.2.3 that the fibers of the vector bundle J^- consist mainly of isomorphic images of finite dimensional kernels of Fredholm operators acting on the extension of cone-Sobolev spaces defined by (3.12). The operator-valued symbols $\mathrm{t}(u, \eta)$

correspond to the projection of the cone-Sobolev space $\mathcal{K}^{s,\gamma}(\mathcal{X}^\wedge)$ onto the finite dimensional kernel of (3.12). By considering a local trivialization of J^- over an open subset $\Omega \subset \mathcal{E}$ and using the fact that $\mathrm{t}(u,\eta)$ are projections, it is possible to prove the following proposition.

Proposition 5.1. Locally on $\Omega \subset \mathcal{E}$, the trace operator $\mathcal{T}$ in (5.1) acts on each of the components of the stretched cotangent bundle $T^*_\wedge M$ as an integral operator with kernel in $C^\infty(\Omega \times \mathcal{X}^\wedge \times \Omega) \otimes \mathbb{C}^{N^-}$.

This proposition and its proof is contained in the more general result presented in [Sch98] proposition 3.4.6. The reader is referred to that book for details.

Given a operator-valued trace symbol $\mathrm{t}(u,\eta)$, the trace operator $\mathcal{T}$ is unique module negligible operators from the point of view of ellipticity and smoothness. See [NSSS06] section 6.1 for details.

Theorem 5.2. *Locally near M the moduli space $\mathfrak{M}(M,\Phi,\mathcal{T},\gamma)$ is homeomorphic to the zero set of a smooth map $\mathfrak{G}$ between smooth manifolds $\mathcal{M}_1$, $\mathcal{M}_2$ given as neighborhoods of zero in finite dimensional Banach spaces. The map $\mathfrak{G} : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ satisfies $\mathfrak{G}(0) = 0$ and $\mathfrak{M}(M,\Phi,\mathcal{T},\gamma)$ near M is a smooth manifold of finite dimension when $\mathfrak{G}$ is the zero map.*

Proof. As γ is an admissible weight we have that

$$\mathcal{A}_{\mathrm{DP}[0]} = \begin{bmatrix} \mathrm{DP}[0] & C \\ \mathcal{T} & B \end{bmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix} \tag{5.2}$$

is a Fredholm operator. Thus the cokernel is a finite dimensional space and it can be identified with a finite dimensional subspace in the codomain of $\mathcal{A}_{\mathrm{DP}[0]}$ (denoted as $\operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]}$) such that it splits the codomain in the following way

$$\begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix} = \operatorname{Im} \mathcal{A}_{\mathrm{DP}[0]} \oplus \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]}. \tag{5.3}$$

Now consider the Banach space

$$\begin{pmatrix} \mathcal{W}^{s,\gamma}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+}) \end{pmatrix} \oplus \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]}.$$

Consider the following extension $\hat{\mathrm{P}}$ of the deformation operator to this space

$$\hat{\mathrm{P}} : \begin{pmatrix} \mathcal{W}^{s,\gamma}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^s(\mathcal{E}, \mathbb{C}^{N^+}) \end{pmatrix} \oplus \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]} \longrightarrow \begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix}$$

given by

$$\hat{\mathrm{P}}\left(\begin{bmatrix}\Xi\\ g\end{bmatrix},\begin{bmatrix}v\\ w\end{bmatrix}\right)=\mathcal{A}_{\mathrm{P}}\begin{bmatrix}\Xi\\ g\end{bmatrix}+\begin{bmatrix}v\\ w\end{bmatrix},$$

where we are using the notation $\mathcal{A}_{\mathrm{P}}$ for the operator $\begin{bmatrix}\mathrm{P} & \mathrm{C}\\ \mathcal{T} & \mathrm{B}\end{bmatrix}$.

Hence

$$\mathrm{D}\hat{\mathrm{P}}[0]\left(\begin{bmatrix}\Xi\\ g\end{bmatrix},\begin{bmatrix}v\\ w\end{bmatrix}\right)=\begin{bmatrix}\mathrm{DP}[0] & \mathrm{C}\\ \mathcal{T} & \mathrm{B}\end{bmatrix}\begin{bmatrix}\Xi\\ g\end{bmatrix}+\begin{bmatrix}v\\ w\end{bmatrix}$$

and

$$\mathrm{Ker}\,\mathrm{D}\hat{\mathrm{P}}[0]=\mathrm{Ker}\,\mathcal{A}_{\mathrm{DP}[0]}\times\{0\}.$$

Observe that $\mathrm{D}\hat{\mathrm{P}}[0]$ is surjective and $\mathrm{Ker}\,\mathrm{D}\hat{\mathrm{P}}[0]$ is finite dimensional. Then

$$\begin{pmatrix}\mathcal{W}^{s,\gamma}(M,\bigwedge^\bullet T^*_\wedge M)\\ \oplus\\ H^s(\mathcal{E},\mathbb{C}^{N^+})\end{pmatrix}\oplus\mathrm{Coker}\,\mathcal{A}_{\mathrm{DP}[0]}=\left(\mathrm{Ker}\,\mathrm{D}\hat{\mathrm{P}}[0]\oplus N\right)\oplus\mathrm{Coker}\,\mathcal{A}_{\mathrm{DP}[0]}$$

for some closed subspace N.

By the Implicit Function Theorem 1.6 there exists $\mathcal{U}_1\subset\mathrm{Ker}\,\mathcal{A}_{\mathrm{DP}[0]}$, $\mathcal{U}_2=\mathcal{U}_2'\times\mathcal{U}_2''\subset N\oplus\mathrm{Coker}\,\mathcal{A}_{\mathrm{DP}[0]}$ and a smooth map $\mathfrak{G}_1\times\mathfrak{G}_2:\mathcal{U}_1\longrightarrow\mathcal{U}_2'\times\mathcal{U}_2''$ such that

$$\hat{\mathrm{P}}^{-1}(0)\cap(\mathcal{U}_1\times\mathcal{U}_2)=$$

$$\left\{\left(\begin{bmatrix}a\\ b\end{bmatrix},\mathfrak{G}_1\left(\begin{bmatrix}a\\ b\end{bmatrix}\right),\mathfrak{G}_2\left(\begin{bmatrix}a\\ b\end{bmatrix}\right)\right):\begin{bmatrix}a\\ b\end{bmatrix}\in\mathcal{U}_1\right\}\subset\mathrm{Ker}\,\mathrm{D}\hat{\mathrm{P}}[0]\oplus N\oplus\mathrm{Coker}\,\mathcal{A}_{\mathrm{DP}[0]}.$$

This give us a description of the elements in the null set of the non-linear operator $\hat{\mathrm{P}}$ in a neighborhood of zero in terms of elements in $\mathrm{Ker}\,\mathrm{D}\hat{\mathrm{P}}[0]$. In order to pass to solutions of the deformation operator P in terms of $\mathrm{Ker}\,\mathrm{DP}[0]$ we have the following.

Observe that

$$\left(\begin{bmatrix}a\\ b\end{bmatrix},\mathfrak{G}_1\left(\begin{bmatrix}a\\ b\end{bmatrix}\right),\mathfrak{G}_2\left(\begin{bmatrix}a\\ b\end{bmatrix}\right)\right)\in\hat{\mathrm{P}}^{-1}(0)\cap(\mathcal{U}_1\times\mathcal{U}_2),$$

implies

$$\hat{P}\left(\begin{bmatrix}a\\ b\end{bmatrix},\mathfrak{G}_1\left(\begin{bmatrix}a\\ b\end{bmatrix}\right),\mathfrak{G}_2\left(\begin{bmatrix}a\\ b\end{bmatrix}\right)\right)=\mathcal{A}_P\left(\begin{bmatrix}a\\ b\end{bmatrix},\mathfrak{G}_1\left(\begin{bmatrix}a\\ b\end{bmatrix}\right)\right)+\mathfrak{G}_2\left(\begin{bmatrix}a\\ b\end{bmatrix}\right)=0.$$

Hence the term $\mathfrak{G}_2\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ (that belongs to $\operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]}$) represents an obstruction to lifting the infinitesimal solution $\begin{bmatrix} a \\ b \end{bmatrix}$ to an authentic solution

$$\left(\begin{bmatrix} a \\ b \end{bmatrix}, \mathfrak{G}_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right)$$

of the non-linear operator $\mathcal{A}_{\mathrm{P}}$. Therefore if all obstructions vanish i.e. if

$$\mathfrak{G}_2 : \mathcal{U}_1 \subset \operatorname{Ker} \mathcal{A}_{\mathrm{DP}[0]} \longrightarrow \mathcal{U}_2'' \subset \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]} \tag{5.4}$$

is the zero map we have

$$\mathcal{A}_{\mathrm{P}}^{-1}(0) \cap (\mathcal{U}_1 \times \mathcal{U}_2') = \left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, \mathfrak{G}_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) : \begin{bmatrix} a \\ b \end{bmatrix} \in \mathcal{U}_1 \right\}.$$

and the set $\mathcal{A}_{\mathrm{P}}^{-1}(0) \cap (\mathcal{U}_1 \times \mathcal{U}_2')$ is diffeomorphic to $\mathcal{U}_1 \subset \operatorname{Ker} \mathcal{A}_{\mathrm{DP}[0]}$.

Consequently, if the obstructions vanish, small solutions of the non-linear boundary value problem (4.8) are given by

$$\mathcal{A}_{\mathrm{P}}^{-1}(0) \cap (\mathcal{U}_1 \times \mathcal{U}_2') \cap \begin{pmatrix} \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \\ \oplus \\ \{0\} \end{pmatrix} = \left\{ \big(\Xi_1, \mathfrak{G}_1(\Xi_1)\big) : \begin{bmatrix} \mathrm{P}_{\omega_{\mathbb{C}^N}} \oplus \mathrm{P}_{\operatorname{Im}\Omega} \\ \mathcal{T} \end{bmatrix} (\Xi_1 + \mathfrak{G}_1(\Xi_1)) = 0 \right\}.$$

This is a non-empty open neighborhood of zero in $\mathcal{W}^{s,\gamma}(M, T^*_\wedge M)$ diffeomorphic to an open set of the finite dimensional space

$$\operatorname{Ker} \begin{bmatrix} \mathrm{DP}[0] \\ \mathcal{T} \end{bmatrix}. \tag{5.5}$$

Thus we can conclude that when $\mathfrak{G}_2$ is the zero map the moduli space is a smooth manifold of finite dimension less or equal to the dimension of the kernel of the linear boundary value problem

$$\begin{bmatrix} \mathrm{DP}[0] \\ \mathcal{T} \end{bmatrix} : \mathcal{W}^{s,\gamma}(M, T^*_\wedge M) \longrightarrow \begin{matrix} \mathcal{W}^{s-1,\gamma-1}(M, \bigwedge^\bullet T^*_\wedge M) \\ \oplus \\ H^{s-1}(\mathcal{E}, J^-) \end{matrix}.$$

∎

Chapter 6

Conclusions and final remarks

The theorem 5.2 says that when the map (5.4)

$$\mathfrak{G}_2 : \mathcal{U}_1 \subset \operatorname{Ker} \mathcal{A}_{\mathrm{DP}[0]} \longrightarrow \mathcal{U}_2'' \subset \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]}$$

is the zero map, the moduli space $\mathfrak{M}(M, \Phi, \mathcal{T}, \gamma)$ is a smooth manifold of finite dimension. For every small solution Ξ_1 of the linearised boundary value problem $\mathcal{A}_{\mathrm{DP}[0]}$, the map $\mathfrak{G}_2$ gives us an obstruction

$$\mathfrak{G}_2(\Xi_2) \in \mathcal{U}_2'' \subset \operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]},$$

to lift the linearised solution to a solution of the non-linear deformation operator with boundary condition. When the obstruction space $\mathcal{U}_2''$ vanishes it follows immediately that there are no obstructions, as the map $\mathfrak{G}_2$ is trivially the zero map, and the moduli space is smooth and finite dimensional.

A careful analysis of the obstruction space is needed to determine under which conditions it vanishes. In [Joy04b], Joyce analyzed the obstruction space of the moduli space of deformations of special Lagrangian submanifolds with conical singularities. He found that the obstruction space depends only on the cones that model the singularities. In the edge singular case we expect a similar result i.e. the obstruction space depends only on the geometric structures that model the singularity, namely, the cone $\mathcal{X}^\wedge$ and the edge $\mathcal{E}$.

If the obstruction space vanishes (therefore $\operatorname{Coker} \mathcal{A}_{\mathrm{DP}[0]} = \{0\}$) the moduli space is a smooth manifold of finite dimension. The next step is to determine its expected dimension. From theorem 5.2 we only know that $\dim \mathfrak{M}(M, \Phi, \mathcal{T}, \gamma) \leq \dim \operatorname{Ker} \mathcal{A}_{\mathrm{DP}[0]} = \operatorname{Ind} \mathcal{A}_{\mathrm{DP}[0]}$. In order to compute the dimension we need to consider the index of edge-degenerate operators and Hodge theory in the edge singular context. In this direction we consider the material related to index theory in [NSSS06] chapter 5 quite relevant for this purpose. Moreover, some elements of Hodge theory on manifolds with edge singularities have been studied in [HM05] and [ST99]. These references might be helpful to compute the expected dimension of our moduli space $\mathfrak{M}(M, \Phi, \mathcal{T}, \gamma)$.

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Appendix A

In chapter 3 and 4 we used results from this appendix. Most of the results here are based on vector-valued Sobolev embeddings. In the first section of this appendix we recall the basics of vector-valued Sobolev embeddings and derive some consequences related to cone and edge-Sobolev spaces. In the second part we prove the estimate (A.12) following a similar result of Witt and Dreher [DW02]. This estimate implies the Banach Algebra property of our edge-Sobolev spaces on M, (A.42), and the regularity of the product of elements in $\mathcal{W}^{s,\gamma}(M)$, (A.43). In order to simplify the notation we will denote $a \approx b$ and $a \lesssim b$ if $a = \kappa b$ or $a \leq \kappa b$ respectively with a positive constant κ depending only on s and γ.

A.1 Vector-valued Sobolev embeddings

Let's consider the classical Sobolev spaces $W^{m,p}(\mathbb{R}^q)$ (see [Bre11] for a detailed introduction). A classical tool in the analysis of partial differential equations on $\mathbb{R}^q$ is the set of Sobolev embeddings, see [Bre11] section 9.3.

Theorem A.1. Let $m \in \mathbb{Z}$, $m > 1$ and $p \in [1, +\infty)$.

$$\text{If} \quad \frac{q}{p} > m \quad \text{then} \quad W^{m,p}(\mathbb{R}^q) \hookrightarrow L^k(\mathbb{R}^q) \quad \text{where } \frac{1}{k} = \frac{1}{p} - \frac{m}{q}. \tag{A.1}$$

$$\text{If} \quad \frac{q}{p} = m \quad \text{then} \quad W^{m,p}(\mathbb{R}^q) \hookrightarrow L^k(\mathbb{R}^q) \quad \text{for all} \quad k \in [p, +\infty). \tag{A.2}$$

$$\text{If} \quad \frac{q}{p} < m \quad \text{then} \quad W^{m,p}(\mathbb{R}^q) \hookrightarrow L^\infty(\mathbb{R}^q) \text{ and } \quad W^{m,p}(\mathbb{R}^q) \hookrightarrow \mathcal{C}^r(\mathbb{R}^q) \tag{A.3}$$

where $r = [s - \frac{q}{2}]$ i.e. r is the integer part of $s - \frac{q}{2}$.

In this appendix we are interested in the vector-valued version of this theorem i.e. given a Banach space $\mathfrak{B}$ we want a version for the $\mathfrak{B}$-valued Sobolev spaces $W^{m,p}(\mathbb{R}^q, \mathfrak{B})$. There are many books and monographs dealing with vector-valued spaces of all kinds like $L^p(\mathbb{R}^q, \mathfrak{B}), \mathcal{C}^k(\mathbb{R}^q, \mathfrak{B})$ and $\mathcal{S}(\mathbb{R}^q, \mathfrak{B})$, see [Trè67], [Jar81], [Ama95]. In many cases they work in the more general context where $\mathfrak{B}$ is a Fréchet or locally convex Hausdorff space. For our specific purposes we follow closely [Kre]. Here Kreuter analyzes carefully the validity of theorem A.1 for the spaces $W^{m,p}(\mathbb{R}^q, \mathfrak{B})$ where $\mathfrak{B}$ is a Banach space.

Recall that the vector-valued space of distributions is defined as the space of continuous operators from $C_0^\infty(\mathbb{R}^q)$ to $\mathfrak{B}$ i.e. $\mathcal{D}'(\mathbb{R}^q, \mathfrak{B}) := \mathcal{L}(C_0^\infty(\mathbb{R}^q), \mathfrak{B})$. The vector-valued

L^p-spaces, $L^p(\mathbb{R}^q, \mathfrak{B})$, are defined by means of the Bochner integral. The Bochner integral is constructed by means of $\mathfrak{B}$-valued step functions in a similar way to the standard Lebesgue integral. See [Abe12] appendix A.4 for details. The vector-valued $\mathcal{C}^k$-spaces, $\mathcal{C}^k(\mathbb{R}^q, \mathfrak{B})$, are defined with respect to the Fréchet derivative. The vector-valued Sobolev space is defined by

$$W^{m,p}(\mathbb{R}^q, \mathfrak{B}) := \left\{ f \in L^p(\mathbb{R}^q, \mathfrak{B}) : \partial^\alpha f \in L^p(\mathbb{R}^q, \mathfrak{B}) \quad \forall \quad |\alpha| \leq m \right\} \tag{A.4}$$

where the derivatives of f are taken in the distribution sense i.e. weak derivatives.

Here we recall the definition of the Radon-Nikodym property and some results related to it. It turns out that the key property that $\mathfrak{B}$ must satisfy in order to have vector-valued Sobolev embeddings for $W^{m,p}(\mathbb{R}^q, \mathfrak{B})$ is the Radon-Nikodym property. For extended details the reader is referred to [Kre] chapter 2.

Definition 27. A Banach space $\mathfrak{B}$ has the Radon-Nikodym property if every Lipschitz continuous function $f : I \longrightarrow \mathfrak{B}$ is differentiable almost everywhere, where $I \subset \mathbb{R}$ is an arbitrary interval.

Proposition A.2. Every reflexive space has the Radon-Nikodym property. In particular the spaces $L^p(\mathbb{R}^q)$ with $1 < p < \infty$ and Hilbert spaces have the Radon-Nikodym property.

Corollary A.3. The Sobolev embeddings in theorem A.1 are valid for the vector-valued Sobolev spaces $W^{m,p}(\mathbb{R}^q, L^p(\mathbb{R}^q))$ with $1 < p < \infty$ and $W^{m,p}(\mathbb{R}^q, \mathfrak{H})$ where $\mathfrak{H}$ is a Hilbert space.

As a consequence of these vector-valued results we have the following applications to cone and edge-Sobolev spaces.

Proposition A.4. If $f \in \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ (see (2.11)) and $s > \frac{m+1}{2}$ then there exists $C > 0$ depending only on s and γ such that we have the following estimate on $(0,1) \times \mathcal{X}$

$$\left| \partial_r^{\alpha'} \partial_\sigma^{\alpha''} f(r,\sigma) \right| \leq C \|f\|_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} \, r^{\gamma - \frac{m+1}{2} - |\alpha'|} \tag{A.5}$$

for all $(r,\sigma) \in (0,1) \times \mathcal{X}$ and $|\alpha'| + |\alpha''| \leq [s - \frac{m+1}{2}]$.

Proof. We can work locally on $\mathbb{R}^+ \times \mathcal{U}_\lambda$ where $\{\mathcal{U}_\lambda\}$ is a finite open covering of $\mathcal{X}$, $\{\varphi_\lambda\}$ is a subbordinate partition of unity and we consider $\omega\varphi_\lambda f$. For simplicity we write just f instead of $\omega\varphi_\lambda f$. At the end we take the smallest constant among those obtained for each element in the finite covering.

Take $f \in \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ then by (2.10) we have $S_{\gamma - \frac{m}{2}} f \in H^s(\mathbb{R}^{1+m})$. Therefore if $s > \frac{m+1}{2}$ by (A.3) we have $S_{\gamma - \frac{m}{2}} f \in L^\infty(\mathbb{R}^{1+m})$ and

$$\sup_{(t,\sigma) \in \mathbb{R}^{1+m}} \left| \partial^\alpha_{(t,\sigma)} (S_{\gamma - \frac{m}{2}} f)(t,\sigma) \right| \lesssim \left\| S_{\gamma - \frac{m}{2}} f \right\|_{H^s(\mathbb{R}^{1+m})} \lesssim \|f\|_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)} \tag{A.6}$$

for all $|\alpha| \leq [s - \frac{m+1}{2}]$. Now by definition (see (2.10)) $(S_{\gamma - \frac{m}{2}} f)(t,x) = e^{-(\frac{1}{2} - (\gamma - \frac{m}{2}))t} f(e^{-t}, x)$ with $r = e^{-t}$. Thus (A.5) follows immediately. ■

Proposition A.5. If $g \in \mathcal{W}^{s,\gamma}(M)$ (see (2.13)) and $s > \frac{m+1+q}{2} + \mathfrak{c}_\gamma$ where $\mathfrak{c}_\gamma$ is the constant defined in (2.14), then there exists $C' > 0$ depending only on s and γ such that we have the following estimate on $(0,1) \times \mathcal{X} \times \mathcal{E}$

$$\left|\partial_r^{\alpha'} \partial_{(\sigma,u)}^{\alpha''} g(r,\sigma,u)\right| \le C' \|g\|_{\mathcal{W}^{s,\gamma}(M)}\, r^{\gamma - \frac{m+1}{2} - |\alpha'|} \tag{A.7}$$

for all $(r,\sigma,u) \in (0,1) \times \mathcal{X} \times \mathcal{E}$ and $|\alpha'| + |\alpha''| \le [s - \frac{m+1}{2}]$.

Proof. We work locally on $(0,1) \times \mathcal{U}_\lambda \times \Omega_j$ as in proposition 3.2. Take $g \in \mathcal{W}^{s,\gamma}(M)$. Again by (A.3) and $s > \frac{m+1}{2}$ we have that for each $u \in \mathbb{R}^q$

$$(S_{\gamma - \frac{m}{2}} g)(u) \in H^s(\mathbb{R}^{1+m})$$

and

$$\left\|(S_{\gamma - \frac{m}{2}} g)(u)\right\|_{L^\infty(\mathbb{R}^{m+1})} \lesssim \left\|g(u)\right\|_{\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)}. \tag{A.8}$$

Now (2.15) implies $g \in H^{s-\mathfrak{c}_\gamma}(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge))$ and $s > \frac{q}{2} + \mathfrak{c}_\gamma$ together with (A.3) and corollary A.3 implies that we have a continuous embedding

$$H^{s-\mathfrak{c}_\gamma}(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)) \hookrightarrow L^\infty(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge))$$

as $\mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge)$ is a Hilbert space, see definition 12.

Consequently

$$\|g\|_{L^\infty(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge))} = \sup_{u \in \mathbb{R}^q} \left\{\left\|g(u)\right\|_{\mathcal{H}^{s,\gamma}}\right\} \tag{A.9}$$

$$\lesssim \|g\|_{H^{s-\mathfrak{c}_\gamma}(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(\mathcal{X}^\wedge))} \tag{A.10}$$

$$\lesssim \|g\|_{\mathcal{W}^{s,\gamma}(\mathcal{X}^\wedge \times \mathbb{R}^q)}. \tag{A.11}$$

Hence (A.8), (A.11) and the change of variable $r = e^{-t}$ implies (A.7) as in proposition A.4. ∎

A.2 Banach Algebra property of edge-Sobolev spaces

In [DW02] Witt and Dreher used a variant of the edge-Sobolev spaces we use in this thesis. In that paper they are interested in applications to weakly hyperbolic equations. Their spaces are defined on $(0,T) \times \mathbb{R}^n$. In this context they proved (proposition 4.1 in [DW02]) that their edge-Sobolev spaces have the structure of a Banach algebra. With some modifications and by using vector-valued Sobolev embeddings it is possible to extend their result to our edge-Sobolev spaces on M. This extension follows closely the proof of Witt and Dreher. For completeness we include the details of this extension in our context as we used the estimates (A.43) and (A.42) in Chapter 3 and 4.

Proposition A.6. Let $f, g \in \mathcal{W}^{s,\gamma}(M)$ with $s \in \mathbb{N}$ and $s > \frac{q+m+3}{2}$. Then $fg \in \mathcal{W}^{s,2\gamma - \frac{m+1}{2}}(M)$ and we have the following estimate

$$\|fg\|_{\mathcal{W}^{s,2\gamma - \frac{m+1}{2}}(M)} \le C \|f\|_{\mathcal{W}^{s,\gamma}(M)} \|g\|_{\mathcal{W}^{s,\gamma}(M)} \tag{A.12}$$

with a constant $C > 0$ depending only on s and γ.

Proof. By means of finite open covers and partitions of unity on $\mathcal{X}$ and $\mathcal{E}$ we need to estimate in terms of $\omega\varphi_\lambda\phi_j f$ and $\omega\varphi_\lambda\phi_j g$ as in proposition 3.3. To avoid unnecessary long expressions we will denote them simply by f and g. To save space in long expressions we use the notation $\hat{f}$ to denote the Fourier transformation with respect to the conormal variable η i.e. $\hat{f} = \mathcal{F}_{u\to\eta}f$. We will estimate on an open set $(0,\varepsilon)\times\mathcal{U}_\lambda\times\Omega_j$. Then the global estimate is obtained by adding these terms. Take $f,g\in C_0^\infty(M)$. By definition of our edge-Sobolev (2.12) norm and by (2.10) we have

$$\|fg\|^2_{\mathcal{W}^{s,2\gamma-\frac{m+1}{2}}(M)} \approx \int\limits_{\mathbb{R}^q_\eta}[\eta]^{2s}\left\|\kappa^{-1}_{[\eta]}\mathcal{F}_{u\to\eta}(fg)(\eta)\right\|^2_{\mathcal{H}^{s,2\gamma-\frac{m+1}{2}}(\mathbb{R}^+\times\mathbb{R}^m)}d\eta \tag{A.13}$$

$$\approx \sum_{|\alpha|\le s}\int\limits_{\mathbb{R}^q_\eta}\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma}[\eta]^{2s-(m+1)}\left|\partial^\alpha\left(e^{-(\frac{m+1}{2}-\gamma)2t}\mathcal{F}_{u\to\eta}(fg)([\eta]^{-1}e^{-t},\sigma,\eta)\right)\right|^2 d\sigma dt d\eta \tag{A.14}$$

Here we will estimate the term with $\alpha=0$. The estimates on the other terms $\alpha\neq 0$ are similar. For each term in (A.14) we have

$$\int\limits_{\mathbb{R}^q_\eta}\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma}[\eta]^{2s-(m+1)}\left|e^{-(\frac{m+1}{2}-\gamma)2t}\mathcal{F}_{u\to\eta}(fg)\right|^2 d\eta = \tag{A.15}$$

$$\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma}e^{-(\frac{m+1}{2}-\gamma)4t}\left\|[\eta]^{s-\frac{m+1}{2}}\mathcal{F}_{u\to\eta}(fg)\right\|^2_{L^2(\mathbb{R}^q_\eta)}d\sigma dt. \tag{A.16}$$

Now, the hypothesis $s>\frac{q+m+3}{2}$ allows us to use lemma 4.6 in [DW02]. Basically, this lemma implies that for fixed (t,σ) we have the following estimate

$$\begin{aligned}\left\|\Lambda(\eta)\widehat{fg(t,\sigma)}(\eta)\right\|_{L^2(\mathbb{R}^q_\eta)} \le{}& \left\|f(t,\sigma)(u)\right\|_{L^\infty(\mathbb{R}^q_u)}\cdot\left\|\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\right\|_{L^2(\mathbb{R}^q_\eta)}\\ &+\left\|g(t,\sigma)(u)\right\|_{L^\infty(\mathbb{R}^q_u)}\cdot\left\|\Lambda(\eta)\hat{f}(r,\sigma)(\eta)\right\|_{L^2(\mathbb{R}^q_\eta)}\\ &+C_0\left\|\Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta]\right\|_{L^2(\mathbb{R}^q_\eta)}\cdot\left\|\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\right\|_{L^2(\mathbb{R}^q_\eta)}\end{aligned}$$

with $C_0>0$ and $\Lambda(\eta)=[\eta]^{s-\frac{m+1}{2}}$.

Applying this estimate to (A.16) we have

$$\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma}e^{-(\frac{m+1}{2}-\gamma)4t}\left\|[\eta]^{s-\frac{m+1}{2}}\mathcal{F}_{u\to\eta}(fg)\right\|^2_{L^2(\mathbb{R}^q_\eta)}d\sigma dt\right)^{\frac{1}{2}} \tag{A.17}$$

$$\le\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma}\left(e^{-(\frac{m+1}{2}-\gamma)2t}\left\|f(t,\sigma)(u)\right\|_{L^\infty(\mathbb{R}^q_u)}\cdot\left\|\Lambda(\eta)\hat{g}(t,x)(\eta)\right\|_{L^2(\mathbb{R}^q_\eta)}+\right.\right. \tag{A.18}$$

$$+\, e^{-(\frac{m+1}{2}-\gamma)2t}\big\|g(t,\sigma)(u)\big\|_{L^\infty(\mathbb{R}^q_u)}\cdot\Big\|\Lambda(\eta)\hat{f}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^q_\eta)} \tag{A.19}$$

$$+\, e^{-(\frac{m+1}{2}-\gamma)2t}C_0\Big\|\Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta]\Big\|_{L^2(\mathbb{R}^q_\eta)}\cdot\big\|\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\big\|_{L^2(\mathbb{R}^q_\eta)}\Bigg)^2 d\sigma dt\Bigg)^{\frac{1}{2}}. \tag{A.20}$$

By the Minkowski inequality we have that (A.17) is less or equal to the following terms

$$\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma} e^{-(\frac{m+1}{2}-\gamma)4t}\big\|f(t,\sigma)(u)\big\|^2_{L^\infty(\mathbb{R}^q_u)}\cdot\big\|\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\big\|^2_{L^2(\mathbb{R}^q_\eta)}\, d\sigma dt\right)^{\frac{1}{2}} \tag{A.21}$$

$$+\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma} e^{-(\frac{m+1}{2}-\gamma)4t}\big\|g(t,\sigma)(u)\big\|^2_{L^\infty(\mathbb{R}^q_u)}\cdot\Big\|\Lambda(\eta)\hat{f}(t,\sigma)(\eta)\Big\|^2_{L^2(\mathbb{R}^q_\eta)}\, d\sigma dt\right)^{\frac{1}{2}} \tag{A.22}$$

$$+\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma} e^{-(\frac{m+1}{2}-\gamma)4t}C_0\Big\|\Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta]\Big\|^2_{L^2(\mathbb{R}^q_\eta)}\cdot\big\|\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\big\|^2_{L^2(\mathbb{R}^q_\eta)}\, d\sigma dt\right)^{\frac{1}{2}}, \tag{A.23}$$

hence, by the inequality in (A.17), we have

$$\left(\int\limits_{\mathbb{R}_t}\int\limits_{\mathbb{R}^m_\sigma} e^{-(\frac{m+1}{2}-\gamma)4t}\Big\|[\eta]^{s-\frac{m+1}{2}}\mathcal{F}_{u\to\eta}(fg)\Big\|^2_{L^2(\mathbb{R}^q_\eta)}\, d\sigma dt\right)^{\frac{1}{2}} \tag{A.24}$$

$$\leq \Big\|e^{-(\frac{m+1}{2}-\gamma)t}f(t,\sigma,u)\Big\|_{L^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^q_u)}\cdot\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.25}$$

$$+\Big\|e^{-(\frac{m+1}{2}-\gamma)t}g(t,\sigma,u)\Big\|_{L^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^q_u)}\cdot\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{f}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.26}$$

$$+\, C_0\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta]\Big\|_{L^\infty(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))}\cdot\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))}. \tag{A.27}$$

The edge-Sobolev norm of f and g written as in (A.16) implies that

$$\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{g}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))}\leq\|g\|_{\mathcal{W}^{s,\gamma}(M)} \tag{A.28}$$

and

$$\Big\|e^{-(\frac{m+1}{2}-\gamma)t}\Lambda(\eta)\hat{f}(t,\sigma)(\eta)\Big\|_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))}\leq\|f\|_{\mathcal{W}^{s,\gamma}(M)}, \tag{A.29}$$

hence we only need to deal with the L^∞ terms.

To analyze the L^∞ terms recall that by hypothesis $s>\frac{q}{2}$ so we have the standard continuous Sobolev embedding $H^s(\mathbb{R}^q)\hookrightarrow L^\infty(\mathbb{R}^q)$. Consequently for fixed (t,σ) we have

$$\Big\|e^{-(\frac{m+1}{2}-\gamma)t}g(t,\sigma)(\eta)\Big\|^2_{L^\infty(\mathbb{R}^q_u)}\lesssim\Big\|e^{-(\frac{m+1}{2}-\gamma)t}g(t,\sigma)(\eta)\Big\|^2_{H^s(\mathbb{R}^q_u)} \tag{A.30}$$

$$=\Big\|\langle\eta\rangle^s e^{-(\frac{m+1}{2}-\gamma)t}\hat{g}(t,\sigma)(\eta)\Big\|^2_{L^2(\mathbb{R}^q_\eta)}, \tag{A.31}$$

therefore

$$\left\| e^{-(\frac{m+1}{2}-\gamma)t} g(t,\sigma)(\eta) \right\|^2_{L^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^q_u)} \lesssim \left\| \langle\eta\rangle^s e^{-(\frac{m+1}{2}-\gamma)t} \hat{g}(t,\sigma)(\eta) \right\|^2_{L^\infty(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.32}$$

$$\lesssim \left\| \langle\eta\rangle^s e^{-(\frac{m+1}{2}-\gamma)t} \hat{g}(t,\sigma)(\eta) \right\|^2_{W^{s,2}(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.33}$$

$$= \sum_{|\beta|\le s} \left\| \langle\eta\rangle^s \partial^\beta_{(t,\sigma)} \left(e^{-(\frac{m+1}{2}-\gamma)t} \hat{g}(t,\sigma)(\eta) \right) \right\|^2_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.34}$$

$$\le \|g\|^2_{\mathcal{W}^{s,\gamma}(M)} . \tag{A.35}$$

In (A.33) we have used the vector-valued version of the standard Sobolev embedding (see section A.1). In the same way we obtain

$$\left\| e^{-(\frac{m+1}{2}-\gamma)t} f(t,\sigma)(\eta) \right\|^2_{L^\infty(\mathbb{R}^{m+1}\times\mathbb{R}^q_u)} \tag{A.36}$$

$$\lesssim \|f\|^2_{\mathcal{W}^{s,\gamma}(M)} . \tag{A.37}$$

Then (A.35) and (A.37) implies that (A.25) and (A.26) are bounded by

$$C(s,\gamma)\|f\|^2_{\mathcal{W}^{s,\gamma}(M)}\|g\|^2_{\mathcal{W}^{s,\gamma}(M)} . \tag{A.38}$$

Thus the only term remaining is (A.27). Again using the vector-valued Sobolev embedding we have

$$\left\| e^{-(\frac{m+1}{2}-\gamma)t} \Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta] \right\|^2_{L^\infty(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.39}$$

$$\lesssim \left\| e^{-(\frac{m+1}{2}-\gamma)t} \Lambda(\eta)\hat{f}(t,\sigma)(u)/[\eta] \right\|^2_{W^{s,2}(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \tag{A.40}$$

$$= \sum_{|\beta|\le s} \left\| \frac{\Lambda(\eta)}{[\eta]} \partial^\beta_{(t,x)} \left(e^{-(\frac{m+1}{2}-\gamma)t} \hat{f}(t,\sigma)(u) \right) \right\|^2_{L^2(\mathbb{R}^{m+1},L^2(\mathbb{R}^q_\eta))} \lesssim \|f\|^2_{\mathcal{W}^{s,\gamma}(M)} \tag{A.41}$$

as $\left|\frac{\Lambda(\eta)}{[\eta]}\right|^2 = [\eta]^{2s-(m+1)} \cdot [\eta]^{-2} \lesssim [\eta]^{2s-(m+1)}$. Thus (A.41) and (A.28) implies that (A.27) is bounded by (A.38). ■

Corollary A.7. If $s \in \mathbb{N}$ with $s > \frac{q+m+3}{2}$ and $\gamma \ge \frac{m+1}{2}$ then the edge Sobolev space $\mathcal{W}^{s,\gamma}(M)$ is a Banach algebra under point-wise multiplication i.e. given $f, g \in \mathcal{W}^{s,\gamma}(M)$ we have

$$\|fg\|_{\mathcal{W}^{s,\gamma}(M)} \le C'\|f\|_{\mathcal{W}^{s,\gamma}(M)}\|g\|_{\mathcal{W}^{s,\gamma}(M)} \tag{A.42}$$

with a constant C' depending only on s and γ.

Proof. By (A.12) we have $fg \in \mathcal{W}^{s,2\gamma-\frac{m+1}{2}}$. Note that $\gamma \ge \frac{m+1}{2}$ if and only if $2\gamma - \frac{m+1}{2} \ge \gamma$ from which the corollary follows immediately. ■

Corollary A.8. Let $f, g \in \mathcal{W}^{s,\gamma}(M)$ such that $s \in \mathbb{N}$ with $s > \frac{q+m+3}{2}$ and $\gamma > \frac{m+1}{2}$. Then

$$fg \in \mathcal{W}^{s,\gamma+\beta}(M) \tag{A.43}$$

for $\beta > 0$ given by $\beta = \gamma - \frac{m+1}{2}$.

Proof. By (A.12) we have $fg \in \mathcal{W}^{s,2\gamma-\frac{m+1}{2}}$. Moreover $\gamma > \frac{m+1}{2}$ implies $2\gamma - \frac{m+1}{2} = \gamma + \beta$ with $\beta = \gamma - \frac{m+1}{2} > 0$. ■

Curriculum Vitae

EDUCATION

- **2012-16 Ph.D. Mathematics.**

  The University of Western Ontario, London, Ontario, Canada.

  Advisors: Tatyana Barron (UWO) and Spiro Karigiannis (Waterloo).

  Thesis: MODULI SPACE AND DEFORMATIONS OF SPECIAL LAGRANGIAN SUBMANIFOLDS WITH EDGE SINGULARITIES.

- **2011-12 M.Sc. Mathematics.**

  The University of Western Ontario, London, Ontario, Canada.

- **2006-10 B.Sc. Mathematics (summa cum laude).**

  Universidad Autonoma Metropolitana, Mexico City, Mexico

RESEARCH INTERESTS

Partial Differential Equations, Microlocal Analysis and Differential Geometry, in particular: elliptic boundary value problems, Riemannian metrics with special holonomy, analysis on singular spaces, calibrated submanifolds, moduli spaces of geometric structures, microlocal analysis and parametrix methods on manifolds with boundary, singularities and/or non-compact ends.

EMPLOYMENT

- **Graduate Teaching Assistant**, University of Western Ontario, Sep. 2011-Aug. 2016.

- **Lecturer**, University of Western Ontario, Sep. 2015-Dec. 2015.

- **Lecturer**, National Polytechnic Institute, Mexico, Jan. 2011-July 2011.

- **Teaching Assistant**, Universidad Autonoma Metropolitana, Mexico, Nov. 2008-Aug. 2011.

RESEARCH SEMINAR PRESENTATIONS

Title: Moduli Space and Deformations of Special Lagrangian Submanifolds with edge singularities.

Presented at:

- Analysis Seminar, University of Western Ontario. Canada. March 2016.
- Geometry and Topology Seminar, McMaster University, Canada. April 2016.
- Geometry and Topology Seminar, University of Waterloo, Canada. April 2016.
- Séminaire de géométrie et topologie, Centre interuniversitaire de recherche en géométrie et topologie (CIRGET), Montreal, Canada. April 2016.

CONFERENCES ATTENDED

- **Fields Geometric Analysis Colloquium**, Fields Institute, Toronto, Canada. Spring 15.
- **Fields Geometric Analysis Colloquium**, Fields Institute, Toronto, Canada. Winter/Fall 14.
- **Minischool on Variational Problems in Geometry**, Fields Institute, Toronto. Nov. 2014.

SCHOLARSHIPS AND AWARDS

September 2011-August 2016:

- Western Graduate Research Scholarship (WGRS), the University of Western Ontario.
- Graduate Teaching Assistanship (GTA), the University of Western Ontario.
- Graduate Research Assistanship (GRA), the University of Western Ontario.

Medal for Academic Merit, Universidad Autonoma Metropolitana, Mexico, 2010.
Federal Government Scholarship for Undergraduate Studies, Mexico, 2007-2010.