Deformations of G_2 and $Spin(7)$ Structures on Manifolds

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Abstract

We analyze the 16 classes of G_2-structures on compact oriented 7-manifolds, and study infinitesmal and global deformations between these classes. We recover some known results from [**12**] on conformally equivalent G_2-structures using different methods and also present necessary and sufficient conditions for being able to conformally scale a G_2-structure in one class to a G_2-structure in a strictly smaller subclass.

We show that globally deforming a G_2-structure by a vector field w in a well defined way always yields a new G_2-structure which geometrically compresses the manifold in the w direction and expands it in the remaining 6 directions. This produces manifolds with new G_2-structures which have a long, thin "tubular" metric.

We also study the infinitesmal analogue of this vector field deformation. In this case corresponding to every vector field, we produce a closed path in the space of G_2-structures all corresponding to the same Riemannian metric. This is mentioned in [**5**] but here we present an explicit formula and demonstrate with two examples why this is a generalization of the phase freedom for the complex volume form in Calabi-Yau 3-folds and the hyper-Kähler rotation for $K3$ manifolds.

Similar results are also obtained in the $Spin(7)$ case. Here there are 4 classes of $Spin(7)$-structures. The relationship between the metric and the $Spin(7)$-structure is much more complicated and the analogous deformations depend on two vector fields v and w.

This work is dedicated to my mother
and to the memory of my father.

If everybody had parents like them,
the world would be a much better place.

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To quote the incomparable Jar Jar Binks: Meesa comin' home!

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Chapter 1

Introduction

1.1. Cross Product Structures

Special structures on manifolds in differential geometry are usually described by globally defined smooth sections of some tensor bundle, satisfying some pointwise algebraic conditions, and with potentially stronger global requirements. One such example of an additional structure that can be imposed on a smooth Riemannian manifold M of dimension n is that of an r-fold *cross product*. This is an *alternating* r-linear smooth map

$$B : \underbrace{TM \times \ldots \times TM}_{r \text{ copies}} \to TM$$

that is compatible with the metric in the sense that

$$|B(e_1, \ldots, e_r)|^2 = |e_1 \wedge \ldots \wedge e_r|^2$$

Such a cross product also gives rise to an $(r+1)$-form α given by

$$\alpha(e_1, \ldots, e_r, e_{r+1}) = g(B(e_1, \ldots, e_r), e_{r+1})$$

where $g(\cdot, \cdot)$ is the Riemannian metric.

Cross products on real vector spaces were classified by Brown and Gray in [**2**]. Global vector cross products on manifolds were first studied by Gray in [**19**]. They fall into four categories:

(1) When $r = n - 1$ and α is the volume form of the manifold. Under the metric identification of vector fields and one forms, this cross product corresponds to the *Hodge star* operator on $(n-1)$-forms. This is not an extra structure beyond that given by the metric.
(2) When $n = 2m$ and $r = 1$, we can have a one-fold cross product $J : TM \to TM$. Such a map satisfies $J^2 = -I$ and is an *almost complex structure*. The associated 2-form is the *Kähler form* ω.
(3) The first of two *exceptional cases* is a 2-fold cross product on a 7-manifold. Such a structure is called a G_2-structure, and the associated 3-form φ is called a G_2-form.
(4) The second exceptional case is a 3-fold cross product on an 8-manifold. This is called a $\mathit{Spin}(7)$-structure, and the associated 4-form Φ is called a $\mathit{Spin}(7)$-form.

In cases 2–4 the existence of these structures is a topological condition on M given in terms of the Stiefel-Whitney classes (see [**19, 31, 33**]). One can also study the restricted sub-class of such manifolds where the associated differential form α is *parallel* with respect to the Levi-Civita connection ∇. In case 1, the volume form is always parallel. For the almost complex structures J of case 2, $\nabla J = 0$ if and only if the manifold is Kähler, which is equivalent to $d\omega = 0$ *and* the almost complex structure is *integrable*. In this case, the Riemannian holonomy of the manifold is a subgroup of $U(m)$. For cases 3 and 4, the condition that the differential form be parallel is a non-linear differential equation. Manifolds with parallel G_2-structures have holonomy a subgroup of G_2 and manifolds with parallel $Spin(7)$-structures have holonomy a subgroup of $Spin(7)$, hence their names. One can also show (see [**1**]) that such manifolds are all *Ricci-flat*.

There is a sub-class of the Kähler manifolds which are Ricci-flat. Such manifolds possess a global non-vanishing holomorphic *volume form* Ω in addition to the Kähler form ω, and these two forms satisfy some relation. These manifolds are called *Calabi-Yau* manifolds as their existence was demonstrated by Yau's proof of the Calabi conjecture [**36**]:

Theorem 1.1.1 (Calabi-Yau, 1978)**.** *Let M be a compact complex manifold with vanishing first Chern class $c_1 = 0$. Then if ω is a Kähler form on M, there exists a unique Ricci-flat Riemannian metric g on M whose associated Kähler form is in the same cohomology class as ω.*

This theorem characterises those manifolds admitting Calabi-Yau metrics in terms of certain topological information. The equivalence is demonstrated by writing the Ricci-flat condition as a partial differential equation and proving existence and uniqueness of solutions. Calabi-Yau manifolds have holonomy a subgroup of $SU(m)$ and are characterized by *two* parallel forms, ω and Ω. In fact, they posses two parallel cross products: a 1-fold cross product J, and a complex analogue of case 1 above, where Ω plays the role of the volume form and the $(m-1)$-fold cross product is a *complex* Hodge star.

Calabi-Yau manifolds (at least in complex dimension 3) have long been of interest in string theory. More recently, manifolds with holonomy G_2 and $Spin(7)$ have also been studied. (See, for example, [**4, 5, 28, 29, 30**]). It would be useful to have an analogue of the Calabi-Yau theorem, or something similar, in the G_2 and $Spin(7)$ cases. There is a significant difference between the cases, however, which makes G_2 and $Spin(7)$ manifolds much more difficult to study.

An almost complex structure J does not by itself determine a metric. If we also have a Riemannian metric, then together the compatibility requirement yields the Kähler form $\omega(u, v) = g(Ju, v)$. In contrast, a 2-fold or 3-fold cross product structure *does* determine the metric uniquely, and thus also determines the associated 3-form φ or 4-form Φ. Because the metric and complex structure are "uncoupled" in the Calabi-Yau case, we can start with a *fixed* integrable complex structure J, and then look for different metrics (which correspond to different Kähler forms for the same J) which are Ricci-flat and make J parallel. As J is already integrable, it is parallel precisely when ω is closed, so we can simply look at different metrics which all correspond to closed Kähler forms, and from that set look for a Ricci-flat metric. Hence we can restrict ourselves to starting with a Kähler manifold, and looking at other Kahler metrics which could be Ricci-flat. The Calabi-Yau theorem then says that there exists precisely one such metric in each cohomology class which contains at least one Kähler metric.

In the G_2 and $Spin(7)$ cases, however, we cannot fix a cross product structure and then vary the metric to make it parallel. Once we choose a cross product, the metric is determined. In the Calabi-Yau case, we can start with $U(m)$ holonomy and describe the conditions for being able to obtain $SU(m)$ holonomy. For G_2 and $Spin(7)$, there is no intermediate starting class. A crucial ingredient in the proof of the Calabi-Yau theorem is the $\partial\bar{\partial}$ lemma, which allows us to write the difference of any two Kähler forms in terms of an unknown *function* f. Therefore as a first step towards an analogous result in the G_2 and $Spin(7)$ cases, we would like to determine the simplest data required to describe the relations between any two G_2 or $Spin(7)$ forms.

1.2. Overview of New Results

If we start with only a G_2-structure, not necessarily parallel, this gives us a 3-form which satisfies some "positive-definiteness" property, since it determines a Riemannian metric. In [**12**], Fernández and Gray classified such manifolds by looking at the decomposition of $\nabla\varphi$ into G_2-irreducible components. There are 16 such classes, with various inclusion relations between them. There is a similar decomposition in [**23**] of almost complex manifolds into subclasses. Some of these classes are: integrable (complex), symplectic, almost Kähler, and nearly Kähler. Thus these 16 subclasses of manifolds with a G_2-structure are analogues of these "weaker than Kähler" conditions. Similar studies by Fernánadez in [**10**] of the $Spin(7)$ case yield 4 subclasses of manifolds with a $Spin(7)$-structure.

As a first step in trying to determine an analogue for the Calabi conjecture in the G_2 case, we can study these various weaker subclasses and their deformations. If we start in one class, and change the 3-form φ in some way (which changes the metric too) we would like to know under what conditions this subclass is preserved, or more generally what subclass the new G_2-structure now belongs to. The space of 3-forms on a manifold with a G_2-structure decomposes into a direct sum of irreducible G_2-representations:

$$\wedge^3 = \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27}$$

where $\wedge^3_k$ is a k-dimensional vector space at each point on M. This decomposition depends on our initial 3-form φ_{o}, however. This again is in contrast to the decomposition on a complex manifold into forms of type (p, q), which depends only on the complex structure and does not change as we vary the Kähler (or metric) structure. We can consider a deformation $\varphi_0 \mapsto \varphi_0 + \eta$ in the G_2-structure, for $\eta \in \wedge^3_k$ and determine conditions on φ_{o} and η which preserve the subclass or change it in an interesting way.

If $\eta \in \wedge^3_1$, this corresponds to a conformal scaling of the metric, and one can explicitly describe which of the 16 classes are conformally invariant. (These results were already known to Fernández and Gray but here they are reproduced in a different way.) A new result in this case is the following:

Theorem 1.2.1. *Let $\theta_o = *_o(*_o d\varphi_o \wedge \varphi_o)$ be the canonical 1-form arising from a G_2-structure φ_o. Then if $\tilde{\varphi} = f^3\varphi_o$ for some non-vanishing function f, the new canonical 1-form $\tilde{\theta}$ differs from the old θ_o by an exact form:*

$$\tilde{\theta} = -12d(\log(f)) + \theta_o$$

Thus in the classes where θ is closed, (there are some and they are conformally invariant classes), we get a well-defined cohomology class in $H^1(M)$, invariant under conformal changes of metric. A similar result also holds in the $Spin(7)$ case.

If, however, we deform φ by an element $\eta \in \wedge^3_7$, then $\eta = w \lrcorner *_{\mathrm{o}} \varphi_{\mathrm{o}}$ for some vector field w, and in Section 3.2 we prove the following:

Theorem 1.2.2. *Under this deformation, the new metric on vector fields v_1 and v_2 is given by*

$$\langle v_1, v_2 \rangle_{\sim} = \frac{1}{(1 + |w|^2_o)^{\frac{2}{3}}} \left(\langle v_1, v_2 \rangle_o + \langle w \times v_1, w \times v_2 \rangle_o \right)$$

where $\times$ is the cross product associated to the G_2-structure.

From this one can write down non-trivial differential equations on the vector field w for certain subclasses to be preserved. It would be interesting to solve some of these equations for the unknown vector field w. This would mean that there were certain distinguished vector fields on some classes of manifolds with G_2-structures. The important result here, however, is that the new 3-form $\tilde{\varphi}$ is *always* positive-definite. That is, it always corresponds to a G_2-structure. This gives information about the structure of the open set $\wedge^3_+(M)$ of positive definite 3-forms on M.

If instead we deform φ in the $\wedge^3_7$ direction infinitesmally by the flow equation

$$\frac{\partial}{\partial t}\varphi_t = w \lrcorner *_t \varphi_t$$

then we show in Section 3.3 that the metric g *does not change* and also:

Theorem 1.2.3. *The solution is given by*

$$\varphi(t) = \varphi_0 + \frac{1-\cos(|w|t)}{|w|^2}(w \lrcorner * (w \lrcorner * \varphi_0)) + \frac{\sin(|w|t)}{|w|}(w \lrcorner * \varphi_0)$$

Hence the solution exists for all time and is a closed path *in* $\wedge^3(M)$*. Also, the path only depends on the unit vector field* $\pm\frac{w}{|w|}$*, and the norm* $|w|$ *only affects the speed of travel along this path.*

In [**5**] the fact that the space of G_2-structures which correspond to the *same* metric as a fixed G_2-structure yields an $\mathbb{RP}^7$ bundle over M is mentioned. This is the content of the above theorem, and we provide an explicit description of these G_2-structures in terms of vector fields on M. In addition, in the special cases of $M = N \times S^1$ or $M = L \times T^3$, where N is a Calabi-Yau 3-fold and L is a $K3$ surface, we show that this closed path of G_2-structures corresponds to the freedom of changing the *phase* of the holomorphic volume form $\Omega \mapsto e^{it}\Omega$ on N or performing a *hyperKähler rotation* on L. Thus this theorem can be seen as a generalization of these two situations.

The same kind of analysis can be done in the $Spin(7)$ case. Similar but more complicated results hold in this case and are presented in Section 5. Here there are only 4 subclasses but the decomposition of $\wedge^4$ into irreducible $Spin(7)$-representations is more complicated:

$$\wedge^4 = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \oplus \wedge^4_{35}$$

In this case it is the space $\wedge^4_7$ which infinitesmally gives a closed path of $Spin(7)$-structures all corresponding to the same metric. Global deformations in the $\wedge^4_7$ direction give a new $Spin(7)$-structure related to the old one by a complicated expression involving the triple cross product.

1.3. Notation and Conventions

Many of the calculations that follow use various relations between the interior product $\lrcorner$, the exterior product $\wedge$, and the Hodge star operator $*$. Readers unfamiliar with this can refer to Appendix 6. The appendix also contains some preliminary results about determinants that are used repeatedly in many of the proofs that follow.

In much of the computations there are two metrics present: an old metric g_{o} and a new metric $\tilde{g}$. Their associated volume forms, induced metrics on differential forms, and Hodge star operators are also identified by a subscript $_{\mathrm{o}}$ for old or a $\tilde{}$ for new. We also often use the metric isomorphism between vector fields and one-forms, and denote this isomorphism by the superscript $^{\#}$ for both the one-form $w^{\#}$ associated to the vector field w and the vector field $\alpha^{\#}$ associated to the one-form α. In the presence of two metrics, this isomorphism is *always* only used for the old metric g_{o}.

Chapter 2

Manifolds with a G_2-structure

2.1. G_2-structures

Let M be an oriented 7-manifold with a global 2-fold cross product structure. Such a structure will henceforth be called a G_2-structure. Its existence is a topological condition, given by the vanishing of the second Stiefel-Whitney class $w_2 = 0$. (See [**19, 31, 33**] for details.) This cross product $\times$ gives rise to an associated *Riemannian metric* g and an alternating 3-form φ which are related by:

$$\varphi(u, v, w) = g(u \times v, w). \tag{2.1}$$

This should be compared to the relation between a Kähler metric ω and a compatible almost complex structure J:

$$\omega(u, v) = g(Ju, v).$$

Note that in the Kähler case, the metric and the almost complex structure can be prescribed independently. This is *not* true in the case of manifolds with a G_2-structure, and this leads to many more complications. For a G_2-structure φ, near a point $p \in M$ we can choose local coordinates $x^1, \dots, x^7$ so that *at the point* p, we have:

$$\varphi_p = dx^{123} - dx^{167} - dx^{527} - dx^{563} - dx^{415} - dx^{426} - dx^{437} \tag{2.2}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. In these coordinates the metric at p is the standard Euclidean metric

$$g_p = \sum_{k=1}^{7} dx^k \otimes dx^k$$

and the Hodge star dual $*\varphi$ of φ is

$$(*\varphi)_p = dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} - dx^{2637} - dx^{1537} - dx^{1526} \tag{2.3}$$

The 3-forms on M that arise from a G_2-structure are called *positive* 3-forms or *non-degenerate*. We will denote this set by $\wedge^3_{\mathrm{pos}}$. The subgroup of $SO(7)$ that preserves φ_p is G_2. This is proved in [**4, 24**]. Hence at each point p, the set of G_2-structures at p is isomorphic to $GL(7, \mathbb{R})/G_2$, which is $49 - 14 = 35$ dimensional. Since $\wedge^3(\mathbb{R}^7)$ is also 35 dimensional, the set

$\wedge^3_{\text{pos}}(p)$ of positive 3-forms at p is an *open subset* of $\wedge^3_p$. We will determine some new information about the structure of $\wedge^3_{\text{pos}}$ in Section 3.2.

Remark 2.1.1. Note that in the $Spin(7)$ case the situation is very different. The set of 4-forms on an 8-manifold M that determine a $Spin(7)$-structure is *not* an open subset of $\wedge^4(M)$. This is discussed in Chapter 4.1.

2.2. G_2-Decomposition of $\bigwedge^*(M)$

All of the facts collected in this section are all well known and more details can be found in [**12, 31, 33**].

The group G_2 acts on $\mathbb{R}^7$, and hence acts on the spaces $\wedge^*$ of differential forms on M. One can decompose each space $\wedge^k$ into irreducible G_2-representations. The results of this decomposition are presented below (see [**12, 33**]). The notation $\wedge^k_l$ refers to an l-dimensional irreducible G_2-representation which is a subspace of $\wedge^k$. Also, "vol" will denote the volume form of M (determined by the metric g), and w is a vector field on M.

$$
\begin{aligned}
\wedge^0_1 &= \{f \in C^\infty(M)\} \\
\wedge^1_7 &= \{\alpha \in \Gamma(\wedge^1(M)\} \\
\wedge^2 &= \wedge^2_7 \oplus \wedge^2_{14} \\
\wedge^3 &= \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27} \\
\wedge^4 &= \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \\
\wedge^5 &= \wedge^5_7 \oplus \wedge^5_{14} \\
\wedge^6_7 &= \{w \lrcorner \,\mathrm{vol}\} \\
\wedge^7_1 &= \{f\,\mathrm{vol}; f \in C^\infty(M)\}
\end{aligned}
$$

Since $G_2 \in SO(7)$, the decomposition respects the *Hodge star* $*$ operator, and $*\wedge^k_l = \wedge^{7-k}_l$. Nevertheless, we will still describe the remaining cases explicitly, in several ways, as all the descriptions will be useful for us.

Before we describe $\wedge^k_l$ for $k = 2, 3, 4, 5$, let us describe some isomorphisms between these subspaces:

Proposition 2.2.1. *The map $\alpha \mapsto \varphi \wedge \alpha$ is an isomorphism between the following spaces:*

$$\begin{array}{rcl} \wedge^0_1 & \cong & \wedge^3_1 \\ \wedge^1_7 & \cong & \wedge^4_7 \\ \wedge^2_7 & \cong & \wedge^5_7 \\ \wedge^2_{14} & \cong & \wedge^5_{14} \\ \wedge^3_7 & \cong & \wedge^6_7 \\ \wedge^4_1 & \cong & \wedge^7_1 \end{array}$$

The map $\alpha \mapsto *\varphi \wedge \alpha$ is an isomorphism between the following spaces:

$$\begin{array}{rcl} \wedge^0_1 & \cong & \wedge^4_1 \\ \wedge^1_7 & \cong & \wedge^5_7 \\ \wedge^2_7 & \cong & \wedge^6_7 \\ \wedge^3_1 & \cong & \wedge^7_1 \end{array}$$

In addition, if α is a 1-form, we have the following identities:

$$\begin{aligned} * \left(\varphi \wedge *(\varphi \wedge \alpha)\right) &= -4\alpha && (2.4) \\ *\varphi \wedge * \left(\varphi \wedge \alpha\right) &= 0 \\ * \left(*\varphi \wedge *(*\varphi \wedge \alpha)\right) &= 3\alpha && (2.5) \\ \varphi \wedge * \left(*\varphi \wedge \alpha\right) &= -2 \left(*\varphi \wedge \alpha\right) && (2.6) \end{aligned}$$

Proof. Since the statements are pointwise, it is enough to check them in local coordinates using (2.2) and (2.3). This is tedious but straightforward. □

From these identities, we can prove the following lemma:

Lemma 2.2.2. *If α is a 1-form on M, then we have:*

$$\begin{aligned} |\varphi \wedge \alpha|^2 &= 4|\alpha|^2 && (2.7) \\ | * \varphi \wedge \alpha|^2 &= 3|\alpha|^2 && (2.8) \end{aligned}$$

Further, for any G_2-structure given by φ, we have:

$$|\varphi|^2 = 7. \qquad (2.9)$$

Proof. From equation (2.4), we have:

$$\begin{aligned} \varphi \wedge *(\varphi \wedge \alpha) &= -4 * \alpha \\ \alpha \wedge \varphi \wedge *(\varphi \wedge \alpha) &= -4\alpha \wedge *\alpha \\ -|\varphi \wedge \alpha|^2 \,\mathrm{vol} &= -4|\alpha|^2 \,\mathrm{vol} \end{aligned}$$

which proves (2.7). An analogous calculation using (2.5) yields (2.8). Finally, from (6.4), (6.5), and (6.9), we compute:

$$|\varphi|^2|\alpha|^2 = |*\varphi\wedge\alpha|^2 + |\varphi\wedge\alpha|^2 = 3|\alpha|^2 + 4|\alpha|^2 = 7|\alpha|^2$$

which gives $|\varphi|^2 = 7$. (This is also immediate from (2.2) and the fact that the coordinates were chosen so that the dx^k's are orthonormal at p.) □

We have the following relations between φ, $*\varphi$, and an arbitrary vector field w:

Lemma 2.2.3. *The following relations hold for any vector field w, where $w^\#$ is the associated 1-form (obtained from the metric isomorphism):*

$$\begin{aligned}
*\left(\varphi\wedge w^\#\right) &= w\lrcorner *\varphi &(2.10)\\
\left(\varphi\wedge w^\#\right) &= w\lrcorner\varphi &(2.11)\\
\varphi\wedge(w\lrcorner *\varphi) &= -4 * w^\# &(2.12)\\
*\varphi\wedge(w\lrcorner *\varphi) &= 0 &\\
*\varphi\wedge(w\lrcorner\varphi) &= 3 * w^\# &(2.13)\\
\varphi\wedge(w\lrcorner\varphi) &= -2 * (w\lrcorner\varphi) &(2.14)
\end{aligned}$$

Proof. Since $*^2 = 1$, these results follow from Lemma 6.0.9 and Proposition 2.2.1. □

We now explicitly describe the decomposition beginning with $k = 2, 5$.

$$\begin{aligned}
\wedge^2_7 &= \{w\lrcorner\varphi;\, w\in\Gamma(T(M))\} &(2.15)\\
&= \{\beta\in\wedge^2;\, *(\varphi\wedge\beta) = -2\beta\} &\\
&= \{\beta\in\wedge^2;\, *(*\varphi\wedge(*(*\varphi\wedge\beta))) = 3\beta\} &\\
\wedge^2_{14} &= \{\beta\in\wedge^2;\, *\varphi\wedge\beta = 0\} &(2.16)\\
&= \{\beta\in\wedge^2;\, *(\varphi\wedge\beta) = \beta\} &\\
&= \{\textstyle\sum a_{ij}e^i\wedge e^j;\, (a_{ij})\in\mathfrak{g}_2\} &
\end{aligned}$$

$$\begin{aligned}
\wedge^5_7 &= \{\alpha\wedge *\varphi;\, \alpha\in\wedge^1_7\} &(2.17)\\
&= \{\gamma\in\wedge^5;\, \varphi\wedge *\gamma = -2\gamma\} &\\
&= \{\gamma\in\wedge^5;\, *\varphi\wedge(*(*\varphi\wedge *\gamma)) = 3\gamma\} &\\
\wedge^5_{14} &= \{\gamma\in\wedge^5;\, \varphi\wedge *\gamma = \gamma\} &(2.18)\\
&= \{\gamma\in\wedge^5;\, *\varphi\wedge *\gamma = 0\} &
\end{aligned}$$

Notice that these subspaces are -2 and $+1$ eigenspaces of the operators $L(\beta) = *(\phi\wedge\beta)$ on $\wedge^2$ and $M(\gamma) = \phi\wedge *\gamma$ on $\wedge^5$. From this fact we get the following useful formulas for the projections π_k onto the k-dimensional representations, for $\beta \in \wedge^2$ and $\gamma \in \wedge^5$:

$$\begin{aligned} *(\varphi \wedge \beta) &= -2\pi_7(\beta) + \pi_{14}(\beta) \\ \pi_7(\beta) &= \frac{\beta - *(\varphi \wedge \beta)}{3} \\ \pi_{14}(\beta) &= \frac{2\beta + *(\varphi \wedge \beta)}{3} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \varphi \wedge *\gamma &= -2\pi_7(\gamma) + \pi_{14}(\gamma) \\ \pi_7(\gamma) &= \frac{\gamma - \varphi \wedge *\gamma}{3} \\ \pi_{14}(\gamma) &= \frac{2\gamma + \varphi \wedge *\gamma}{3} \end{aligned}$$

We now move on to the decompositions for $k = 3, 4$.

$$\begin{aligned} \wedge^3_1 &= \{f\varphi; f \in C^\infty(M)\} && (2.20) \\ &= \{\eta \in \wedge^3; \varphi \wedge (*(*\varphi \wedge \eta)) = 7\eta\} \\ \wedge^3_7 &= \{*(\varphi \wedge \alpha); \alpha \in \wedge^1_7\} && (2.21) \\ &= \{w \lrcorner * \varphi; w \in \Gamma(T(M))\} \\ &= \{\eta \in \wedge^3; *(\varphi \wedge *(\varphi \wedge \eta)) = -4\eta\} \\ \wedge^3_{27} &= \{\eta \in \wedge^3; \varphi \wedge \eta = 0 \text{ and } *\varphi \wedge \eta = 0\} && (2.22) \end{aligned}$$

$$\begin{aligned} \wedge^4_1 &= \{f *\varphi; f \in C^\infty(M)\} && (2.23) \\ &= \{\sigma \in \wedge^4; *\varphi \wedge (*(\varphi \wedge \sigma)) = 7\sigma\} \\ \wedge^4_7 &= \{\varphi \wedge \alpha; \alpha \in \wedge^1_7\} && (2.24) \\ &= \{\sigma \in \wedge^4; (\varphi \wedge *(\varphi \wedge *\sigma)) = -4\sigma\} \\ \wedge^4_{27} &= \{\sigma \in \wedge^4; \varphi \wedge \sigma = 0 \text{ and } \varphi \wedge *\sigma = 0\} && (2.25) \end{aligned}$$

2.3. The metric of a G_2-structure

From Lemma 2.2.2, we can obtain a formula for determining the metric g from the 3-form φ:

Proposition 2.3.1. *If v is a vector field on M, then*

$$(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = -6|v|^2 \operatorname{vol} \tag{2.26}$$

Proof. From Lemma 6.0.9 and (2.6) we have

$$v \lrcorner \varphi = *(v^{\#} \wedge *\varphi)$$

and

$$(v \lrcorner \varphi) \wedge \varphi = -2(v^{\#} \wedge *\varphi)$$

Thus we obtain

$$(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = -2|v^{\#} \wedge *\varphi|^2 \operatorname{vol} = -6|v|^2 \operatorname{vol}$$

where we have used (2.8). $\square$

Remark 2.3.2. Equation (2.26) can be used to give an alternative definition to the "positivity" condition. Let ζ be a nowhere vanishing 7-form on M, which exists since M is orientable. A 3-form φ is in $\wedge^3_{\mathrm{pos}}$ if and only if at every point p in M, the function $f(v)$ defined on $T_p(M)$ by the map $v \mapsto -(v \lrcorner \varphi_p) \wedge (v \lrcorner \varphi_p) \wedge \varphi_p = f(v)\zeta_p$ satisfies $f(v) \geq 0$ with equality if and only if $v = 0$.

By polarizing (2.26) in v, we obtain the relation:

$$(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi = -6\langle v, w \rangle \operatorname{vol}$$

From this equation we can obtain the metric.

Lemma 2.3.3. *Fix a vector field $v = v^k e_k$, where $e_1, e_2, \ldots, e_7$ is an oriented local frame of vector fields. The expression obtained from v by*

$$\frac{((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \ldots, e_7)}{\left(\det\left(((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \ldots, e_7)\right)\right)^{\frac{1}{9}}} \tag{2.27}$$

is homogeneous of order 2 in v, and independent of the choice of $e_1, \ldots, e_7$. As shown in the next theorem, up to a constant this is $|v|^2$.

Proof. The homogeneity of (2.27) of order 2 in v is clear. Now suppose we choose a different oriented basis $e'_1, e'_2, \ldots, e'_7$. Then we have

$$e'_i = P_{ij} e_j$$

and hence

$$(e'_i \lrcorner \varphi) \wedge (e'_j \lrcorner \varphi) \wedge \varphi = P_{ik} P_{jl} (e_k \lrcorner \varphi) \wedge (e_l \lrcorner \varphi) \wedge \varphi$$

Hence in the new basis the denominator of (2.27) changes by a factor of

$$\left(\det(P)^2 \det(P)^7\right)^{\frac{1}{9}} = \det(P)$$

and the numerator also changes by a factor of $\det(P)$, leaving the quotient invariant. $\square$

We have now give the expression for the metric in terms of the 3-form φ.

Theorem 2.3.4. *Let v be a tangent vector at a point p and let $e_1, e_2, \dots, e_7$ be any basis for T_pM. Then the length $|v|$ of v is given by*

$$|v|^2 = 6^{\frac{2}{9}} \frac{((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \dots, e_7)}{(\det(((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \dots, e_7)))^{\frac{1}{9}}} \tag{2.28}$$

Proof. We work in local coordinates at the point p. In this notation $g_{ij} = \langle e_i, e_j \rangle$ with $1 \leq i, j \leq 7$. Let $\det(g)$ denote the determinant of (g_{ij}). We have from (2.26) that

$$\begin{aligned}
((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi) &= -6 g_{ij} \,\mathrm{vol} \\
&= -6 g_{ij} \sqrt{\det(g)} e^1 \wedge \dots \wedge e^7 \\
\det(((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \dots, e_7)) &= (-6)^7 \det(g) \det(g)^{\frac{7}{2}} \\
&= -6^7 \det(g)^{\frac{9}{2}}
\end{aligned}$$

and since

$$\begin{aligned}
(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi &= -6|v|^2 \,\mathrm{vol} \\
&= -6|v|^2 \sqrt{\det(g)} e^1 \wedge e^2 \wedge \dots \wedge e^7 \\
((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi)(e_1, e_2, \dots, e_7) &= -6|v|^2 \det(g)^{\frac{1}{2}}
\end{aligned}$$

these two expressions can be combined to yield (2.28). □

2.4. The cross product operation

In this section we will describe the cross product operation on a manifold with a G_2-structure in terms of the 3-form φ, and present some useful relations.

Definition 2.4.1. Let u and v be vector fields on M. The *cross product*, denoted $u \times v$, is a vector field on M whose associated 1-form under the metric isomorphism satisfies:

$$(u \times v)^{\#} = v \lrcorner u \lrcorner \varphi \tag{2.29}$$

Notice that this immediately yields the relation between $\times$, φ, and the metric g:

$$g(u \times v, w) = (u \times v)^{\#}(w) = w \lrcorner v \lrcorner u \lrcorner \varphi = \varphi(u, v, w). \tag{2.30}$$

Another characterization of the cross product is obtained from this one using Lemma 6.0.9:

$$\begin{aligned}
(u \times v)^{\#} &= v \lrcorner u \lrcorner \varphi \qquad (2.31)\\
&= - * (v^{\#} \wedge *(u \lrcorner \varphi))\\
&= - * (v^{\#} \wedge u^{\#} \wedge * \varphi)\\
&= *(u^{\#} \wedge v^{\#} \wedge * \varphi)
\end{aligned}$$

Now since $u^{\#} \wedge v^{\#}$ is a 2-form, we can write it as $\beta_7 + \beta_{14}$, with $\beta_j \in \wedge^2_j$. Then we have, using (2.15) and (2.16):

$$\begin{aligned}
(u \times v)^{\#} \wedge * \varphi &= *(\beta_7 \wedge * \varphi) \wedge * \varphi \qquad (2.32)\\
&= 3 * \beta_7
\end{aligned}$$

Taking the norm of both sides, and using (2.8):

$$\left|(u \times v)^{\#} \wedge * \varphi\right|^2 = 3\left|(u \times v)^{\#}\right|^2 = 3|u \times v|^2 = 9|\beta_7|^2$$

from which we obtain

$$|\beta_7|^2 = \frac{1}{3}|u \times v|^2 \qquad (2.33)$$

Lemma 2.4.2. *Let u and v be vector fields. Then*

$$|u \times v|^2 = |u \wedge v|^2 \qquad (2.34)$$

Proof. With $\beta = u^{\#} \wedge v^{\#}$, we have from (2.15) and (2.16):

$$\begin{aligned}
\beta \wedge \varphi &= -2 * \beta_7 + * \beta_{14}\\
\beta \wedge \beta \wedge \varphi &= -2|\beta_7|^2 \operatorname{vol} + |\beta_{14}|^2 \operatorname{vol}\\
&= 0
\end{aligned}$$

since $\beta = u^{\#} \wedge v^{\#}$ is decomposable. So $|\beta_{14}|^2 = 2|\beta_7|^2$ and finally we obtain from (2.33):

$$|u \times v|^2 = 3|\beta_7|^2 = |\beta_7|^2 + |\beta_{14}|^2 = |\beta|^2 = |u \wedge v|^2$$

□

The following lemma will be used in Section 3.2 to determine how the metric changes under a deformation in the $\wedge^3_7$ direction.

Lemma 2.4.3. *The following identity holds for v and w vector fields:*

$$(v \lrcorner w \lrcorner * \varphi) \wedge (v \lrcorner w \lrcorner \varphi) \wedge * \varphi = -2|v \wedge w|^2 \operatorname{vol} \qquad (2.35)$$

Proof. We start with Lemma 6.0.9 to rewrite

$$\begin{aligned} v \lrcorner w \lrcorner * \varphi &= *(v^\# \wedge *(w \lrcorner * \varphi)) \\ &= - * (v^\# \wedge w^\# \wedge \varphi) \\ &= 2\beta_7 - \beta_{14} \end{aligned}$$

using the notation as above. From equations (2.29) and (2.32) we have

$$(v \lrcorner w \lrcorner \varphi) \wedge *\varphi = -3 * \beta_7$$

Combining these two equations and (2.33),

$$\begin{aligned} (v \lrcorner w \lrcorner * \varphi) \wedge (v \lrcorner w \lrcorner \varphi) \wedge *\varphi &= (2\beta_7 - \beta_{14}) \wedge (-3 * \beta_7) \\ &= -6|\beta_7|^2 \operatorname{vol} \\ &= -2|u \wedge v|^2 \operatorname{vol} \end{aligned}$$

which completes the proof. □

Finally, we prove a theorem which will be useful in Section 3.2 where we will use it to show that to first order, deforming a G_2-structure by an element of $\wedge^3_7$ does not change the metric.

Theorem 2.4.4. *Let u, v, w be vector fields. Then*

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) = 0.$$

Note that in terms of the decompositions in (2.15) *and* (2.21)*, this theorem says that the wedge product map*

$$\wedge^2_7 \times \wedge^2_7 \times \wedge^3_7 \to \wedge^7_1$$

is the zero map.

Proof. Since it is an 8-form,

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge *\varphi = 0.$$

Taking the interior product with w and rearranging,

$$\begin{aligned} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) &= -(w \lrcorner u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge *\varphi \\ &\quad - (u \lrcorner \varphi) \wedge (w \lrcorner v \lrcorner \varphi) \wedge *\varphi \end{aligned}$$

Now using (2.13), we get

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) = -3\,(w \lrcorner u \lrcorner \varphi) \wedge *v^\# - 3\,(w \lrcorner v \lrcorner \varphi) \wedge *u^\#$$

Finally, from (6.6), we have

$$\begin{aligned} &(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) \\ =\ & -3\,(u \lrcorner \varphi) \wedge * \left(w^\# \wedge v^\#\right) - 3\,(v \lrcorner \varphi) \wedge * \left(w^\# \wedge u^\#\right) \\ =\ & -3\varphi \wedge * \left(u^\# \wedge w^\# \wedge v^\#\right) - 3\varphi \wedge * \left(v^\# \wedge w^\# \wedge u^\#\right) \\ =\ & 0. \end{aligned}$$

□

2.5. The 16 classes of G_2-structures

According to the classification of Fernández and Gray in [**12**], a manifold with a G_2-structure has holonomy a subgroup of G_2 if and only if $\nabla\varphi = 0$, which they showed to be equivalent to

$$d\varphi = 0 \qquad \text{and} \qquad d * \varphi = 0.$$

They established this equivalence by decomposing the space W that $\nabla\varphi$ belongs to into irreducible G_2-representations, and identifying the invariant subspaces of W with isomorphic subspaces of $\wedge^*(M)$. This space W decomposes as

$$W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$$

where the subscript k denotes the dimension of the irreducible representation W_k. Now $d\varphi \in \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27}$ and $d * \varphi \in \wedge^5_7 \oplus \wedge^5_{14}$. Up to isomorphism, the projections $\pi_k(d\varphi)$ and $\pi_k(d{*}\varphi)$ are non-zero constant multiples of $\pi_k(\nabla\varphi)$. Therefore in the following we will consider $d\varphi$ and $d * \varphi$ instead of $\nabla\varphi$. Since both of these have a component in a 7-dimensional representation, they are multiples:

Lemma 2.5.1. *The following identity holds:*

$$\mu = *d\varphi \wedge \varphi = - * d * \varphi \wedge *\varphi \tag{2.36}$$

where we have defined the 6-form μ by the above two equal expressions. They are the components $\pi_7(d\varphi)$ and $\pi_7(d{*}\varphi)$ transferred to the isomorphic space $\wedge^6_7$.

Proof. See [**4**] for a proof of this fact. □

We prefer to work with the associated 1-form, $\theta = *\mu$. We will see later that in some subclasses this 1-form is closed or at least "partially closed."

Now we say a G_2-structure is in the class $W_i \oplus W_j \oplus W_k$ with i, j, k distinct where $\{i, j, k\} \subset \{1, 7, 14, 27\}$ if only the component of $d\varphi$ or $d * \varphi$ in the l-dimensional representation vanishes. Here $\{l\} = \{1, 7, 14, 27\} \setminus \{i, j, k\}$. Similarly the G_2-structure is in the class $W_i \oplus W_j$ if the k and l-dimensional components vanish, and in the class W_i if the other three components are zero. In this way we arrive at 16 classes of G_2-structures on a manifold. In Table 2.1 we describe the classes in terms of differential equations on the form φ. This classification first appeared in [**12**] and then in essentially this form in [**6**].

In Table 2.1, the function $h = \frac{1}{7} * (\varphi \wedge d\varphi)$ is the image of $\pi_1(d\varphi)$ in $\wedge^0$ under the isomorphism $\wedge^4_1 \cong \wedge^0_1$. The abbreviation "LC" stands for *locally conformal to* and means that for those classes, we can (at least locally)

TABLE 2.1. The 16 classes of G_2-structures

Class	Defining Equations	$d\theta$
$W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$	no relation on $d\varphi$, $d * \varphi$.	
$W_7 \oplus W_{14} \oplus W_{27}$	$d\varphi \wedge \varphi = 0$	$d\theta =?$
$W_1 \oplus W_{14} \oplus W_{27}$	$\theta = 0$	$\theta = 0$
$W_1 \oplus W_7 \oplus W_{27}$	$d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ or $\varphi \wedge (*d * \varphi) = -2d * \varphi$	$\pi_7(d\theta) = 0$
$W_1 \oplus W_7 \oplus W_{14}$	$d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$	$d\theta =?$
$W_{14} \oplus W_{27}$	$d\varphi \wedge \varphi = 0$ and $\theta = 0$	$\theta = 0$
$W_7 \oplus W_{27}$	$d\varphi \wedge \varphi = 0$ and $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$	$\pi_7(d\theta) = 0$
$W_7 \oplus W_{14}$	$d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$	$d\theta = 0$
$W_1 \oplus W_{27}$	$d * \varphi = 0$	$\theta = 0$
$W_1 \oplus W_{14}$	$d\varphi - h * \varphi = 0$	$\theta = 0$
$W_1 \oplus W_7$	$d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$ and $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$	$d\theta = 0$
W_{27}	$d\varphi \wedge \varphi = 0$ and $d * \varphi = 0$	$\theta = 0$
W_{14}	$d\varphi = 0$	$\theta = 0$
W_7	$d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ and $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$	$d\theta = 0$
W_1	$d\varphi - h * \varphi = 0$ and $d * \varphi = 0$	$\theta = 0$
$\{0\}$	$d\varphi = 0$ and $d * \varphi = 0$	$\theta = 0$

conformally change the metric to enter a strictly smaller subclass. This will be explained in Section 3.1.

We now prove the closedness or partial closedness of θ in the various classes as given in the final column of Table 2.1. The closedness of θ in the classes $W_1 \oplus W_7$ and $W_7 \oplus W_{14}$ was originally shown using a slightly different approch by Cabrera in [**6**].

Lemma 2.5.2. *If φ is in the classes W_7, $W_7 \oplus W_{14}$, or $W_1 \oplus W_7$, then $d\theta = 0$. Furthermore, if φ is in the classes $W_7 \oplus W_{27}$ or $W_1 \oplus W_7 \oplus W_{27}$ then $\pi_7(d\theta) = 0$.*

Proof. We begin by showing that if φ satisfies $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$, then $d\theta = 0$, and if φ satisfies $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$, then $\pi_7(d\theta) = 0$.

Suppose $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$. We differentiate this equation to obtain:

$$d\theta \wedge \varphi = \theta \wedge d\varphi = \theta \wedge \left(-\frac{1}{4}\theta \wedge \varphi\right) = 0$$

But wedge product with φ is an isomorphism from $\wedge^2$ to $\wedge^5$, so $d\theta = 0$. Now suppose $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$. Differentiating this equation yields

$$d\theta \wedge *\varphi = \theta \wedge d * \varphi = \theta \wedge \left(-\frac{1}{3}\theta \wedge *\varphi\right) = 0$$

But wedge product with $*\varphi$ is an isomorphism from $\wedge^2_7$ to $\wedge^6_7$, so $\pi_7(d\theta) = 0$.

Thus by comparing with Table 2.1, we have shown that in the classes $W_7 \oplus W_{14}$ and W_7, we have $d\theta = 0$. Also, in the classes $W_1 \oplus W_7 \oplus W_{27}$ and $W_7 \oplus W_{27}$ we have $\pi_7(d\theta) = 0$. We still have to show that θ is closed in the class $W_1 \oplus W_7$. We already have that $\pi_7(d\theta) = 0$, so we need only show that $\pi_{14}(d\theta) = 0$ in this case. We differentiate $d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$ to obtain

$$\begin{aligned} 0 &= \frac{1}{4}d\theta \wedge \varphi - \frac{1}{4}\theta \wedge d\varphi - dh \wedge *\varphi - hd * \varphi \\ &= \frac{1}{4}d\theta \wedge \varphi - \frac{1}{4}\theta \wedge \left(-\frac{1}{4}\theta \wedge \varphi + h * \varphi\right) - dh \wedge *\varphi - h\left(-\frac{1}{3}\theta \wedge *\varphi\right) \\ &= \frac{1}{4}d\theta \wedge \varphi + \alpha \wedge *\varphi \end{aligned}$$

for some 1-form α, where we have used the fact that $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ in this class. But $\alpha \wedge *\varphi$ is in $\wedge^5_7$, and since wedge product with φ is an isomorphism from $\wedge^2_7$ to $\wedge^5_7$, this shows that $\pi_{14}(d\theta) = 0$. $\square$

The inclusion relations among these various subclasses are analyzed in [**12, 6, 7, 11, 4, 5, 28, 29, 33**]. For all but one case, examples can be found of manifolds which are in a particular class but *not* in a strictly smaller subclass. For example, a manifold in the class W_{14} which does *not* have holonomy G_2 appears in [**11**]. There is one case of an inclusion in Table 2.1 which is *not* strict. This is given by the following result, which first appeared in [**6**].

Proposition 2.5.3. *The class $W_1 \oplus W_{14}$ equals $W_1 \cup W_{14}$ exactly.*

Proof. In the class $W_1 \oplus W_{14}$, we have $d\varphi - h*\varphi = 0$ (and by consequence $\theta = 0$). Differentiating this equation,

$$dh \wedge *\varphi = -hd * \varphi$$

If $h \neq 0$, then by dividing by h and using Proposition 2.2.1, we see that $d * \varphi \in \wedge^5_7$, so $\pi_{14}(d * \varphi) = 0$. But since we already have that $\theta = 0$, this means $d * \varphi = 0$ and hence φ is actually of class W_1 (nearly G_2). If $h = 0$ then $d\varphi = 0$ and φ is of class W_{14} (almost G_2). $\square$

Remark 2.5.4. Note that in the proof of the above proposition, we see that if φ is of class W_1 (nearly G_2), then $dh \wedge *\varphi = 0$, and so $dh = 0$ by

Proposition 2.2.1. Therefore in the nearly G_2 case, the function h is *locally constant*, or constant if the manifold M is connected. In [**20**] Gray showed that all nearly G_2 manifolds are actually *Einstein*.

In [**13, 14**], Fernández and Ugarte show that for manifolds with a G_2-structure in the classes $W_1 \oplus W_7 \oplus W_{27}$ ("integrable") or $W_7 \oplus W_{14}$, there exists a subcomplex of the deRham complex. They then show how to define analogues of Dolbeault cohomology of complex manifolds in these two cases, including analogues of $\bar{\partial}$-harmonic forms. They derive some properties of these cohomology theories and topological restrictions on the existence of G_2-structures in some strictly smaller subclasses.

Chapter 3

Deformations of a fixed G_2-structure

Let us begin with a fixed G_2-structure on a manifold M in a certain class. We are interested in how deforming the form φ affects the class. In other words, we are interested in what kinds of deformations preserve which classes of G_2-structures. Now since $\varphi \in \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27}$, there are three canonical ways to deform φ. For example, since $\wedge^3_1 = \{f\varphi\}$, adding to φ an element of $\wedge^3_1$ amounts to conformally scaling φ. This preserves the decomposition into irreducible representations in this case. However, since the decomposition does depend on φ (unlike the decomposition of forms into (p, q) types on a Kähler manifold) in general if we add an element of $\wedge^3_7$ or $\wedge^3_{27}$ the decomposition does change. So deforming in those two directions really only makes sense infinitesmally. However, we shall attempt to get as far as we can with an actual deformation before we restrict to infinitesmal deformations.

3.1. Conformal Deformations of G_2-structures

Let f be a smooth, nowhere vanishing function on M. For notational convenience, which will become evident, we will conformally scale φ by f^3. Let the new form $\tilde{\varphi} = f^3\varphi_{\mathrm{o}}$. We first compute the new metric $\tilde{g}$ and the new volume form $\mathrm{vol}_{\tilde{}}$ in the following lemma.

Lemma 3.1.1. *The metric g_o on vector fields, the metric g_o^{-1} on one forms, and the volume form* vol_o *transform as follows:*

$$\begin{aligned} \mathrm{vol}_{\tilde{}} &= f^7 \,\mathrm{vol}_o \\ \tilde{g} &= f^2 g_o \\ \tilde{g}^{-1} &= f^{-2} g_o^{-1} \end{aligned}$$

Proof. Using Proposition 2.3.1, we have in a local coordinate chart:

$$\begin{aligned} \tilde{g}(u, v)\mathrm{vol}_{\tilde{}} &= -\frac{1}{6}(u \lrcorner \tilde{\varphi}) \wedge (v \lrcorner \tilde{\varphi}) \wedge \tilde{\varphi} \\ &= f^9 g_{\mathrm{o}}(u, v)\,\mathrm{vol}_{\mathrm{o}} \\ \tilde{g}(u, v)\sqrt{\det(\tilde{g})}dx^1 \ldots dx^7 &= f^9 g_{\mathrm{o}}(u, v)\sqrt{\det(g_{\mathrm{o}})}dx^1 \ldots dx^7. \end{aligned}$$

Thus, taking determinants of the coefficients of both sides,

$$\begin{aligned} \det(\tilde{g})^{\frac{7}{2}}\det(\tilde{g}) &= f^{63}\det(g_{\mathrm{o}})^{\frac{7}{2}}\det(g_{\mathrm{o}}) \\ \sqrt{\det(\tilde{g})} &= f^{7}\sqrt{\det(g_{\mathrm{o}})} \end{aligned}$$

This gives $\mathrm{vol}_{\tilde{}} = f^7\,\mathrm{vol}_{\mathrm{o}}$, from which we can immediately see that $\tilde{g} = f^2 g_{\mathrm{o}}$ and $\tilde{g}^{-1} = f^{-2} g_{\mathrm{o}}$. □

Notice that the new metric $\tilde{g}$ is always a positive definite metric as long as f is non-vanishing, even if f is negative. However, the *orientation* of M changes for negative f since the volume form changes sign. Now we can determine the new Hodge star $\tilde{*}$ in terms of the old $*_{\mathrm{o}}$.

Lemma 3.1.2. *If α is a k-form, then $\tilde{*}\alpha = f^{7-2k} *_o \alpha$.*

Proof. Let α, β be k-forms. Then from Lemma 3.1.1 the new metric on k-forms is $\langle\ ,\ \rangle_{\tilde{}} = f^{-2k}\langle\ ,\ \rangle_{\mathrm{o}}$. From this we compute:

$$\begin{aligned} \beta \wedge \tilde{*}\alpha &= \langle \beta, \alpha\rangle_{\tilde{}}\mathrm{vol}_{\tilde{}} \\ &= f^{-2k}\langle \beta, \alpha\rangle_{\mathrm{o}} f^7\,\mathrm{vol}_{\mathrm{o}} \\ &= f^{7-2k}\beta \wedge *_{\mathrm{o}}\alpha. \end{aligned}$$

□

Corollary 3.1.3. *The new form $\tilde{\varphi}$ satisfies $\tilde{*}\tilde{\varphi} = f^4 *_o \varphi_o$.*

Proof. This follows immediately from Lemma 3.1.2. □

We can also see directly that under a conformal scaling, the norm of φ is unchanged, even though the metric changes when φ changes (in fact $|\varphi|^2 = 7$ from Lemma 2.2.2). To see this we compute:

$$\langle\tilde{\varphi}, \tilde{\varphi}\rangle_{\tilde{}} = (f^{-2})^3\left\langle f^3\varphi_{\mathrm{o}}, f^3\varphi_{\mathrm{o}}\right\rangle_{\mathrm{o}} = \langle\varphi_{\mathrm{o}}, \varphi_{\mathrm{o}}\rangle_{\mathrm{o}}.$$

Combining all our results so far yields:

Lemma 3.1.4. *We have the following relations:*

$$\begin{aligned} d\tilde{\varphi} &= 3f^2 df \wedge \varphi_o + f^3 d\varphi_o \\ d\tilde{*}\tilde{\varphi} &= 4f^3 df \wedge *_o\varphi_o + f^4 d *_o \varphi_o \\ \tilde{*}d\tilde{\varphi} &= 3f *_o (df \wedge \varphi_o) + f^2 *_o d\varphi_o \\ \tilde{*}d\tilde{*}\tilde{\varphi} &= 4 *_o (df \wedge *_o\varphi_o) + f *_o (d *_o \varphi_o) \end{aligned}$$

Proof. This follows from Lemma 3.1.2 and Corollary 3.1.3. □

From these results, we can determine which classes of G_2-structures are conformally invariant. We can also determine what happens to the 6-form μ from equation (2.36) as well as the associated 1-form $\theta = *\mu$. This is all given in the following theorem:

Theorem 3.1.5. *Under the conformal deformation $\tilde{\varphi} = f^3\varphi_o$, we have:*

$$\begin{aligned}
d\tilde{*}\tilde{\varphi} + \frac{1}{3}\tilde{\theta}\wedge\tilde{*}\tilde{\varphi} &= f^4\left(d *_o \varphi_o + \frac{1}{3}\theta_o \wedge *_o\varphi_o\right) &(3.1)\\
d\tilde{\varphi} + \frac{1}{4}\tilde{\theta}\wedge\tilde{\varphi} &= f^3\left(d\varphi_o + \frac{1}{4}\theta_o\wedge\varphi_o\right) &(3.2)\\
d\tilde{\varphi}\wedge\tilde{\varphi} &= f^6\left(d\varphi_o\wedge\varphi_o\right) &(3.3)\\
d\tilde{\varphi} + \frac{1}{4}\tilde{\theta}\wedge\tilde{\varphi} - \tilde{h}\tilde{*}\tilde{\varphi} &= f^3\left(d\varphi_o + \frac{1}{4}\theta_o\wedge\varphi_o - h_o *_o \varphi_o\right) &(3.4)\\
\tilde{\mu} &= -12 f^4 *_o df + f^5\mu_o &(3.5)\\
\tilde{\theta} &= -12 d(\log(f)) + \theta_o &(3.6)
\end{aligned}$$

Hence, we see (from Table 2.1) that the classes which are conformally invariant are exactly $W_7 \oplus W_{14} \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{14}$, $W_7 \oplus W_{27}$, $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and W_7. These are precisely the classes which have a W_7 component. (This conclusion was originally observed in [**12**] *using a different method.)*

Additionally, (3.6) *shows that since θ changes by an exact form, in the classes where $d\theta = 0$, we have a well defined cohomology class $[\theta]$ which is unchanged under a conformal scaling. These are the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and W_7.*

Proof. We begin by using Lemma 3.1.4 and (2.36) to compute $\tilde{\mu}$ and $\tilde{\theta}$:

$$\begin{aligned}
\tilde{\mu} &= \tilde{*}d\tilde{\varphi}\wedge\tilde{\varphi}\\
&= \left(3f *_o (df\wedge\varphi_o) + f^2 *_o d\varphi_o\right)\wedge f^3\varphi_o\\
&= 3f^4\varphi_o\wedge *_o(\varphi_o\wedge df) + f^5\mu_o\\
&= -12 f^4 *_o df + f^5\mu_o
\end{aligned}$$

where we have used (2.4) in the last step. Now from Lemma 3.1.2, we get:

$$\tilde{\theta} = \tilde{*}\tilde{\mu} = -12 f^{-1} df + \theta_o = -12 d(\log(f)) + \theta_o.$$

Now using the above expression for $\tilde{\theta}$, we have:

$$\begin{aligned} d\tilde{*}\tilde{\varphi} + \frac{1}{3}\tilde{\theta} \wedge \tilde{*}\tilde{\varphi} &= 4f^3 df \wedge *_{\mathrm{o}}\varphi_{\mathrm{o}} + f^4 d *_{\mathrm{o}} \varphi_{\mathrm{o}} \\ &\quad + \frac{1}{3}\left(-12f^{-1}df + \theta_{\mathrm{o}}\right) \wedge f^4 *_{\mathrm{o}} \varphi_{\mathrm{o}} \\ &= f^4 \left(d *_{\mathrm{o}} \varphi_{\mathrm{o}} + \frac{1}{3}\theta_{\mathrm{o}} \wedge *_{\mathrm{o}}\varphi_{\mathrm{o}} \right) \\ d\tilde{\varphi} + \frac{1}{4}\tilde{\theta} \wedge \tilde{\varphi} &= 3f^2 df \wedge \varphi_{\mathrm{o}} + f^3 d\varphi_{\mathrm{o}} + \frac{1}{4}\left(-12f^{-1}df + \theta_{\mathrm{o}}\right) \wedge f^3\varphi_{\mathrm{o}} \\ &= f^3 \left(d\varphi_{\mathrm{o}} + \frac{1}{4}\theta_{\mathrm{o}} \wedge \varphi_{\mathrm{o}} \right) \end{aligned}$$

and finally, since $\varphi_{\mathrm{o}} \wedge \varphi_{\mathrm{o}} = 0$,

$$d\tilde{\varphi} \wedge \tilde{\varphi} = \left(3f^2 df \wedge \varphi_{\mathrm{o}} + f^3 d\varphi_{\mathrm{o}}\right) \wedge f^3\varphi_{\mathrm{o}} = f^6 \left(d\varphi_{\mathrm{o}} \wedge \varphi_{\mathrm{o}}\right).$$

Finally, since $h = \frac{1}{7} * (\varphi \wedge d\varphi)$, we have

$$\begin{aligned} \tilde{h}\tilde{*}\tilde{\varphi} &= \frac{1}{7}\tilde{*}(\tilde{\varphi} \wedge d\tilde{\varphi}) f^4 *_{\mathrm{o}} \varphi_{\mathrm{o}} \\ &= \frac{1}{7} f^{-7} *_{\mathrm{o}} (f^6 \varphi_{\mathrm{o}} \wedge d\varphi_{\mathrm{o}}) f^4 *_{\mathrm{o}} \varphi_{\mathrm{o}} \\ &= f^3 h_{\mathrm{o}} *_{\mathrm{o}} \varphi_{\mathrm{o}} \end{aligned}$$

which yields (3.4) when combined with (3.2). This completes the proof. □

These results now enable us to give necessary and sufficient conditions for obtaining a closed or co-closed $\tilde{\varphi}$ by conformally scaling the original φ_{o}.

Theorem 3.1.6. *Let φ_o be a positive 3-form (associated to a G_2-structure). Under the conformal deformation $\tilde{\varphi} = f^3\varphi_o$, the new 3-form $\tilde{\varphi}$ satisfies*

- $d\tilde{\varphi} = 0 \iff \varphi_o$ *is in class $W_7 \oplus W_{14}$ and $12d\log(f) = \theta_o$.*
- $d *_o \varphi_o = 0 \iff \varphi_o$ *is in class $W_1 \oplus W_7 \oplus W_{27}$ and $12d\log(f) = \theta_o$.*

Note that in both cases, in order to have $\tilde{\varphi}$ be closed or co-closed after conformal scaling, the original 1-form θ_o has to be exact. *In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if φ_o is in the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, or W_7, where $d\theta_o = 0$.*

Proof. From Lemma 3.1.4, for $d\tilde{\varphi} = 0$, we need

$$\begin{aligned} d\tilde{\varphi} &= 3f^2 df \wedge \varphi_{\mathrm{o}} + f^3 d\varphi_{\mathrm{o}} = 0 \\ d\varphi_{\mathrm{o}} &= -3d\log(f) \wedge \varphi_{\mathrm{o}} \end{aligned}$$

which says that $d\varphi_{\mathrm{o}} \in \wedge^4_7$ by Proposition 2.2.1. Hence $\pi_1(d\varphi_{\mathrm{o}})$ and $\pi_{27}(d\varphi_{\mathrm{o}})$ both vanish and φ_{o} must be already at least of class $W_7 \oplus W_{14}$. Then to make $d\tilde{\varphi} = 0$, we need to eliminate the W_7 component, which requires $12d\log(f) = \theta_o$ by Theorem 3.1.5. Similarly, to make $d\tilde{*}\tilde{\varphi} = 0$, Lemma 3.1.4 gives

$$\begin{aligned} d\tilde{*}\tilde{\varphi} &= 4f^3 df \wedge *_o\varphi_o + f^4 d *_o \varphi_o = 0 \\ d *_o \varphi_o &= -4d\log(f) \wedge *_o\varphi_o \end{aligned}$$

which says $d *_o \varphi_o \in \wedge^5_7$ and $\pi_{14}(d *_o \varphi_o) = 0$ by Proposition 2.2.1. Thus φ_o must already be at least class $W_1 \oplus W_7 \oplus W_{27}$ and we need to choose f by $12d\log(f) = \theta_o$ to scale away the W_7 component. □

Remark 3.1.7. We have shown that the transformation $\tilde{\varphi} = f^3\varphi_o$ stays in a particular subclass as long as there is a W_7 component to that class. If there is, and the original θ_o is exact (which it will be in the simply-connected case), then we can choose f to scale away the W_7 component and enter a stricter subclass. Conversely, Theorem 3.1.5 shows that a conformal scaling by a non-constant f will always generate a non-zero W_7 component if we started with none. Hence, if we are trying to constuct metrics of holonomy G_2 on a simply-connected manifold, it is enough to construct a metric in the class W_7, since we can then conformally scale (uniquely) to obtain a metric of holonomy G_2. This is why the class W_7 is called *locally conformal* G_2.

3.2. Deforming φ by an element of $\wedge^3_7$

The type of deformation of φ that is next in line in terms of increasing complexity is to add an element of $\wedge^3_7$. This space is isomorphic to $\wedge^1_7 \cong \Gamma(T(M))$, so we can think of this process as deforming φ by a vector field. In fact, an element $\eta \in \wedge^3_7$ is of the form $w\lrcorner * \varphi$ for some vector field w, by (2.21). Let $\tilde{\varphi} = \varphi_o + tw\lrcorner *_o \varphi_o$, for $t \in \mathbb{R}$. We will develop formulas for the new metric $\tilde{g}$, the new Hodge star $\tilde{*}$, and other expressions entirely in terms of the old φ_o, the old $*_o$, and the vector field w. Note in this case the background decomposition into irreducible G_2-representations changes, and we will eventually linearize by taking $\frac{d}{dt}\Big|_{t=0}$ of our results.

Lemma 3.2.1. *In the expression*

$$-6|v|^2_{\tilde{}}\mathrm{vol}_{\tilde{}} = (v\lrcorner\tilde{\varphi}) \wedge (v\lrcorner\tilde{\varphi}) \wedge \tilde{\varphi}$$

which is a cubic polynomial in t*, the linear and cubic terms both vanish, and the coefficient of the quadratic term is*

$$-6|v \wedge w|^2_o \,\mathrm{vol}_o$$

Proof. The coefficient of t^3 is:

$$(v\lrcorner w\lrcorner *_o \varphi_o) \wedge (v\lrcorner w\lrcorner *_o \varphi_o) \wedge (w\lrcorner *_o \varphi_o)$$

This expression is zero because it arises by taking the interior product with w of the 8-form

$$(v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge (v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge *_{\rm o}\varphi_{\rm o} = 0.$$

The coefficient of t is:

$$(v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner \varphi_{\rm o}) \wedge (w \lrcorner *_{\rm o} \varphi_{\rm o}) + 2\,(v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o}$$

The first term vanishes by Theorem 2.4.4. For the second term, let us start with the 8-form

$$(v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} = 0$$

and take the interior product with w. This gives:

$$\begin{aligned}(v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} &= (w \lrcorner v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} \\ &\quad - (v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge (w \lrcorner \varphi_{\rm o})\end{aligned}$$

Again, the last term is zero by Theorem 2.4.4, and the coefficient of t becomes:

$$\begin{aligned}2\,(w \lrcorner v \lrcorner \varphi_{\rm o}) \wedge (v \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} &= 8\,(w \lrcorner v \lrcorner \varphi_{\rm o}) \wedge *_{\rm o} v^{\#} \\ &= -8\,(v \lrcorner w \lrcorner \varphi_{\rm o}) \wedge *_{\rm o} v^{\#} \\ &= -8\,(w \lrcorner \varphi_{\rm o}) \wedge *_{\rm o} \left(v^{\#} \wedge v^{\#}\right) \\ &= 0\end{aligned}$$

where we have used (2.12) and (6.6).

The coefficient of t^2 is:

$$(v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge ((v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} + 2(v \lrcorner \varphi_{\rm o}) \wedge (w \lrcorner *_{\rm o} \varphi_{\rm o})) \tag{3.7}$$

We will rewrite both of these terms using (2.12) and (2.13). First, we have

$$\begin{aligned}\varphi_{\rm o} \wedge (w \lrcorner *_{\rm o} \varphi_{\rm o}) &= -4 *_{\rm o} w^{\#} \\ (v \lrcorner \varphi_{\rm o}) \wedge (w \lrcorner *_{\rm o} \varphi_{\rm o}) - \varphi_{\rm o} \wedge (v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) &= -4 v \lrcorner *_{\rm o} w^{\#} \\ &= 4 *_{\rm o} (v^{\#} \wedge w^{\#}) \qquad (3.8)\end{aligned}$$

and similarly,

$$\begin{aligned}*_{\rm o}\varphi_{\rm o} \wedge (v \lrcorner \varphi_{\rm o}) &= 3 *_{\rm o} v^{\#} \\ (w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge (v \lrcorner \varphi_{\rm o}) + *_{\rm o}\varphi_{\rm o} \wedge (w \lrcorner v \lrcorner \varphi_{\rm o}) &= 3 w \lrcorner *_{\rm o} v^{\#} \\ &= 3 *_{\rm o} (v^{\#} \wedge w^{\#}) \qquad (3.9)\end{aligned}$$

Multiplying equation (3.9) by 3 and subtracting (3.8), we get:

$$(v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \varphi_{\rm o} + 2(v \lrcorner \varphi_{\rm o}) \wedge (w \lrcorner *_{\rm o} \varphi_{\rm o}) = 3(v \lrcorner w \lrcorner \varphi_{\rm o}) \wedge *_{\rm o}\varphi_{\rm o} + 5 *_{\rm o} (v^{\#} \wedge w^{\#})$$

Thus the quadratic term (3.7) can be rewritten as:

$$(v \lrcorner w \lrcorner *_{\rm o} \varphi_{\rm o}) \wedge \left(3(v \lrcorner w \lrcorner \varphi_{\rm o}) \wedge *_{\rm o}\varphi_{\rm o} + 5 *_{\rm o} (v^{\#} \wedge w^{\#})\right)$$

The second term is zero by (6.6). From Lemma 2.4.3 the statement follows. This completes the proof. $\square$

Before we can use Lemma 3.2.1 to obtain the new metric, we have to extract the new volume form.

Proposition 3.2.2. *With $\tilde{\varphi} = \varphi_o + w \lrcorner *_o \varphi_o$, the new volume form is*

$$\mathrm{vol}_{\sim} = (1 + |w|^2_o)^{\frac{2}{3}} \,\mathrm{vol}_o \tag{3.10}$$

Proof. We work in local coordinates. Let $e_1, e_2, \ldots, e_7$ be a basis for the tangent space, with $w = w^j e_j$, $g_{ij} = \langle e_i, e_j \rangle_{\mathrm{o}}$ and $\tilde{g}_{ij} = \langle e_i, e_j \rangle_{\sim}$. Then Lemma 3.2.1 says that

$$|v|^2_{\sim} \sqrt{\det(\tilde{g}_{ij})} = \left(|v|^2_{\mathrm{o}} + |v \wedge w|^2_{\mathrm{o}}\right) \sqrt{\det(g_{ij})}$$

Polarizing this equation, we have:

$$\begin{aligned} \langle v_1, v_2 \rangle_{\sim} \sqrt{\det(\tilde{g}_{ij})} &= \left(\langle v_1, v_2 \rangle_{\mathrm{o}} + \langle v_1, v_2 \rangle_{\mathrm{o}} |w|^2_{\mathrm{o}} - \langle v_1, w \rangle_{\mathrm{o}} \langle v_2, w \rangle_{\mathrm{o}}\right) \sqrt{\det(g_{ij})} \\ \tilde{g}_{ij} \sqrt{\det(\tilde{g}_{ij})} &= \left(g_{ij} + \langle e_i \wedge w, e_j \wedge w \rangle_{\mathrm{o}}\right) \sqrt{\det(g_{ij})} \end{aligned}$$

with $v_1 = e_i$ and $v_2 = e_j$. Now substituting $w = w^k e_k$ in the second term,

$$\begin{aligned} \langle e_i \wedge w, e_j \wedge w \rangle_{\mathrm{o}} &= w^k w^l \langle e_i \wedge e_k, e_j \wedge e_l \rangle_{\mathrm{o}} \\ &= w^k w^l \det \begin{pmatrix} \langle e_i, e_j \rangle_{\mathrm{o}} & \langle e_i, e_l \rangle_{\mathrm{o}} \\ \langle e_k, e_j \rangle_{\mathrm{o}} & \langle e_k, e_l \rangle_{\mathrm{o}} \end{pmatrix} \\ &= w^k w^l \left(g_{ij} g_{kl} - g_{il} g_{kj}\right) \\ &= w_l w^l g_{ij} - w_i w_j \\ &= |w|^2_{\mathrm{o}} g_{ij} - w_i w_j \end{aligned}$$

Thus we have

$$\tilde{g}_{ij} \sqrt{\det(\tilde{g}_{ij})} = \left(g_{ij}(1 + |w|^2_{\mathrm{o}}) - w_i w_j\right) \sqrt{\det(g_{ij})}$$

We take determinants of both sides of this equation, and use the fact that they are 7×7 matrices, to obtain

$$(\det(\tilde{g}_{ij}))^{\frac{9}{2}} = (\det(g_{ij}))^{\frac{7}{2}} \det\left(g_{ij}(1 + |w|^2_{\mathrm{o}}) - w_i w_j\right). \tag{3.11}$$

Using Lemma 6.0.11, the determinant on the right is

$$(1 + |w|^2_{\mathrm{o}})^7 \det(g_{ij}) + \sum_{k,l=1}^{7} (-1)^{k+l} (-w_k w_l) (1 + |w|^2_{\mathrm{o}})^6 (G_{kl}) \tag{3.12}$$

where G_{kl} is the $(k,l)^{\text{th}}$ *minor* of the matrix g_{ij}. Now with $w_l = w^m g_{ml}$, the expression (3.12) becomes

$$\begin{aligned}
&= (1+|w|^2_{\mathrm{o}})^7 \det(g_{ij}) - \sum_{k,l,m} (-1)^{k+l} w_k w^m g_{ml} (1+|w|^2_{\mathrm{o}})^6 G_{kl} \\
&= (1+|w|^2_{\mathrm{o}})^7 \det(g_{ij}) - \sum_{k,m} w_k w^m (1+|w|^2_{\mathrm{o}})^6 \left(\sum_l (-1)^{k+l} g_{ml} G_{kl} \right) \\
&= (1+|w|^2_{\mathrm{o}})^7 \det(g_{ij}) - \sum_{k,m} w_k w^m (1+|w|^2_{\mathrm{o}})^6 \delta_{km} \det(g_{ij}) \\
&= (1+|w|^2_{\mathrm{o}})^7 \det(g_{ij}) - |w|^2_{\mathrm{o}} (1+|w|^2_{\mathrm{o}})^6 \det(g_{ij}) \\
&= (1+|w|^2_{\mathrm{o}})^6 \det(g_{ij})
\end{aligned}$$

since the sum over l is the determinant of the matrix (g_{ij}) with the k^{th} row replaced by the m^{th} row, and so vanishes unless $k = m$, where it is $\det(g_{ij})$. Substituting this result into equation (3.11), we obtain

$$\begin{aligned}
(\det(\tilde{g}_{ij}))^{\frac{9}{2}} &= (\det(g_{ij}))^{\frac{7}{2}} (1+|w|^2_{\mathrm{o}})^6 \det(g_{ij}) \\
\sqrt{\det(\tilde{g}_{ij})} &= (1+|w|^2_{\mathrm{o}})^{\frac{2}{3}} \sqrt{\det(g_{ij})}
\end{aligned}$$

which completes the proof. $\square$

Now letting $t = 1$, with $\tilde{\varphi} = \varphi_{\mathrm{o}} + w \lrcorner *_{\mathrm{o}} \varphi_{\mathrm{o}}$, Lemma 3.2.1 and Proposition 3.2.2 yield

$$\begin{aligned}
|v|^2_{\sim} \mathrm{vol}_{\sim} &= \left(|v|^2_{\mathrm{o}} + |v \wedge w|^2_{\mathrm{o}}\right) \mathrm{vol}_{\mathrm{o}} \\
\langle v, v \rangle_{\sim} &= \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}}} (\langle v, v \rangle_{\mathrm{o}} + |v|^2_{\mathrm{o}} |w|^2_{\mathrm{o}} - \langle v, w \rangle^2_{\mathrm{o}})
\end{aligned}$$

Polarizing this equation, we obtain:

$$\langle v_1, v_2 \rangle_{\sim} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}}} (\langle v_1, v_2 \rangle_{\mathrm{o}} + \langle v_1, v_2 \rangle_{\mathrm{o}} |w|^2_{\mathrm{o}} - \langle v_1, w \rangle_{\mathrm{o}} \langle v_2, w \rangle_{\mathrm{o}}) \tag{3.13}$$

which by Lemma 2.4.2 can also be written as

$$\langle v_1, v_2 \rangle_{\sim} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}}} \left(\langle v_1, v_2 \rangle_{\mathrm{o}} + \langle w \times v_1, w \times v_2 \rangle_{\mathrm{o}}\right) \tag{3.14}$$

In local coordinates with $w = w^i e_i$, $g_{ij} = \langle e_i, e_j \rangle_{\mathrm{o}}$, and $w^{\#} = w_i e^i$, we see that

$$\tilde{g}_{ij} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}}} (g_{ij}(1+|w|^2_{\mathrm{o}}) - w_i w_j) \tag{3.15}$$

Proposition 3.2.3. *In local coordinates, the metric $\tilde{g}^{ij}$ on 1-forms is given by:*

$$\tilde{g}^{ij} = \frac{1}{(1+|w|^2_o)^{\frac{1}{3}}}(g^{ij} + w^i w^j)$$

Proof. We compute:

$$\begin{aligned}
\tilde{g}_{ij}\tilde{g}^{jk} &= \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}}}(g_{ij}(1+|w|^2_{\mathrm{o}}) - w_i w_j)\frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{1}{3}}}(g^{jk} + w^j w^k) \\
&= \frac{\big((g_{ij}g^{jk} + g_{ij}w^j w^k)(1+|w|^2_{\mathrm{o}}) - g^{jk}w_i w_j - w_i w_j w^j w^k\big)}{(1+|w|^2_{\mathrm{o}})} \\
&= \frac{1}{(1+|w|^2_{\mathrm{o}})}\big((\delta^k_i + w_i w^k)(1+|w|^2_{\mathrm{o}}) - w_i w^k - |w|^2_{\mathrm{o}} w_i w^k\big) \\
&= \delta^k_i
\end{aligned}$$

which completes the proof. □

Now with $\alpha = \alpha_i e^i$ and $\beta = \beta_j e^j$ two 1-forms, their new inner product is

$$\begin{aligned}
\langle \alpha, \beta \rangle_{\sim} = \alpha_i \beta_j \tilde{g}^{ij} &= \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{1}{3}}}(\alpha_i\beta_j g^{ij} + \alpha_i w^i \beta_j w^j) \\
&= \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{1}{3}}}(\langle \alpha, \beta \rangle_{\mathrm{o}} + (w \lrcorner \alpha)(w \lrcorner \beta)) \qquad (3.16)
\end{aligned}$$

From this expression we can derive a formula for the new metric $\langle\ ,\ \rangle_{\sim}$ on k-forms:

Theorem 3.2.4. *Let α, β be k-forms. Then*

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1+|w|^2_o)^{\frac{k}{3}}}(\langle \alpha, \beta \rangle_o + \langle w \lrcorner \alpha, w \lrcorner \beta \rangle_o) \qquad (3.17)$$

Proof. We have already established it for the case $k = 1$ in (3.16), and the case $k = 0$ is trivial. For the general case, we will prove the statement on *decomposable* forms and it follows in general by linearity. Let $\alpha = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}$ and $\beta = e^{j_1} \wedge e^{j_2} \wedge \ldots \wedge e^{j_k}$. Then by the definition of the metric on k-forms,

$$\langle \alpha, \beta \rangle_{\sim} = \det \begin{pmatrix} \langle e^{i_1}, e^{j_1} \rangle_{\sim} & \langle e^{i_1}, e^{j_2} \rangle_{\sim} & \ldots & \langle e^{i_1}, e^{j_k} \rangle_{\sim} \\ \langle e^{i_2}, e^{j_1} \rangle_{\sim} & \langle e^{i_2}, e^{j_2} \rangle_{\sim} & \ldots & \langle e^{i_2}, e^{j_k} \rangle_{\sim} \\ \vdots & & \ddots & \vdots \\ \langle e^{i_k}, e^{j_1} \rangle_{\sim} & \langle e^{i_k}, e^{j_2} \rangle_{\sim} & \ldots & \langle e^{i_k}, e^{j_k} \rangle_{\sim} \end{pmatrix}$$

Now from equation (3.16) each entry in the above matrix is of the form

$$\left\langle e^{i_a}, e^{j_b} \right\rangle_{\sim} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{1}{3}}} (g^{i_a j_b} + w^{i_a} w^{j_b})$$

and we have

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{k}{3}}} \det \begin{pmatrix} g^{i_1 j_1} + w^{i_1} w^{j_1} & \dots & g^{i_1 j_k} + w^{i_1} w^{j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} + w^{i_k} w^{j_1} & \dots & g^{i_k j_k} + w^{i_k} w^{j_k} \end{pmatrix}$$

Now we apply Lemma 6.0.11 to obtain

$$\langle \alpha, \beta \rangle_{\mathrm{o}} + \sum_{l,m=1}^{k} (-1)^{l+m} w^{i_l} w^{j_m} \left\langle e^{i_1} \wedge \dots \widehat{e^{i_l}} \dots \wedge e^{i_k}, e^{j_1} \wedge \dots \widehat{e^{j_m}} \dots \wedge e^{j_k} \right\rangle_{\mathrm{o}}$$

for the determinant above. Now with $w = w^i e_i$, we can take the interior product with both α and β:

$$\begin{aligned} w \lrcorner \alpha &= \sum_{l=1}^{k} (-1)^{l-1} w^{i_l} e^{i_1} \wedge \dots \widehat{e^{i_l}} \dots \wedge e^{i_k} \\ w \lrcorner \beta &= \sum_{m=1}^{k} (-1)^{m-1} w^{j_m} e^{j_1} \wedge \dots \widehat{e^{j_m}} \dots \wedge e^{j_k} \end{aligned}$$

and hence the sum over l and m above is just $\langle w \lrcorner \alpha, w \lrcorner \beta \rangle_{\mathrm{o}}$. Putting everything together, we arrive at (3.17):

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{k}{3}}} (\langle \alpha, \beta \rangle_{\mathrm{o}} + \langle w \lrcorner \alpha, w \lrcorner \beta \rangle_{\mathrm{o}})$$

□

To continue our analysis of the new G_2-structure $\tilde{\varphi}$, we now need to compute the new Hodge star $\tilde{*}$.

Theorem 3.2.5. *The Hodge star for the new metric on a k-form α is given by:*

$$\begin{aligned} \tilde{*}\alpha &= (1+|w|^2_o)^{\frac{2-k}{3}} \left(*_o \alpha + (-1)^{k-1} w \lrcorner (*_o (w \lrcorner \alpha)) \right) \qquad (3.18) \\ &= (1+|w|^2_o)^{\frac{2-k}{3}} \left(*_o \alpha + w \lrcorner (w^\# \wedge *_o \alpha) \right) \end{aligned}$$

Proof. The second form follows from the first from (6.2). Although it looks a little more cluttered, we prefer to use the first form for $\tilde{*}$. Notice that up to a scaling factor, the new star is given by 'twisting by w', taking the old star, then 'untwisting by w', and adding this to the old star. To establish this formula, let β be an arbitrary k-form and compute:

$$\begin{aligned}
\beta \wedge \tilde{*}\alpha &= \langle \beta, \alpha \rangle_{\sim} \mathrm{vol}_{\sim} \\
&= \frac{1}{(1+|w|^2_{\mathrm{o}})^{\frac{k}{3}}} (\langle \alpha, \beta \rangle_{\mathrm{o}} + \langle w \lrcorner \alpha, w \lrcorner \beta \rangle_{\mathrm{o}})(1+|w|^2_{\mathrm{o}})^{\frac{2}{3}} \mathrm{vol}_{\mathrm{o}} \\
&= (1+|w|^2_{\mathrm{o}})^{\frac{2-k}{3}} (\beta \wedge *_{\mathrm{o}} \alpha + (w \lrcorner \beta) \wedge *_{\mathrm{o}} (w \lrcorner \alpha))
\end{aligned}$$

Now if we take the interior product with w of the 8-form

$$\beta \wedge *_{\mathrm{o}}(w \lrcorner \alpha) = 0$$

we obtain

$$(w \lrcorner \beta) \wedge *_{\mathrm{o}}(w \lrcorner \alpha) = (-1)^{k-1} \beta \wedge (w \lrcorner (*_{\mathrm{o}}(w \lrcorner \alpha)))$$

and this completes the proof, since β is arbitrary. $\square$

At this point before continuing it is instructive to observe directly that $\tilde{*}^2 = 1$, as of course it should on a 7-manifold. It clarifies the necessity of all the factors of $(1+|w|^2_{\mathrm{o}})$. Let α be a k-form:

$$\begin{aligned}
\tilde{*}\alpha &= (1+|w|^2_{\mathrm{o}})^{\frac{2-k}{3}} \left(*_{\mathrm{o}}\alpha + (-1)^{k-1} w \lrcorner (*_{\mathrm{o}}(w \lrcorner \alpha)) \right) \\
\tilde{*}(\tilde{*}\alpha) &= (1+|w|^2_{\mathrm{o}})^{\frac{2-(7-k)}{3}} \left(*_{\mathrm{o}}(\tilde{*}\alpha) + (-1)^{(7-k)-1} w \lrcorner (*_{\mathrm{o}}(w \lrcorner (\tilde{*}\alpha))) \right) \\
&= (1+|w|^2_{\mathrm{o}})^{\frac{(k-5)+(2-k)}{3}} \big[*_{\mathrm{o}}(*_{\mathrm{o}}\alpha) + (-1)^{k-1} *_{\mathrm{o}} (w \lrcorner (*_{\mathrm{o}}(w \lrcorner (\alpha)))) \\
&\quad + (-1)^k w \lrcorner (*_{\mathrm{o}}(w \lrcorner (*_{\mathrm{o}}\alpha))) + (-1) w \lrcorner (*_{\mathrm{o}}(w \lrcorner w \lrcorner \ldots)) \big]
\end{aligned}$$

Now the last term is zero because of the two successive interior products with w. Using $*_{\mathrm{o}}^2 = 1$ we now have:

$$\tilde{*}(\tilde{*}\alpha) = \frac{\left[\alpha + (-1)^{k-1} *_{\mathrm{o}} (w \lrcorner (*_{\mathrm{o}}(w \lrcorner (\alpha)))) + (-1)^k w \lrcorner (*_{\mathrm{o}}(w \lrcorner (*_{\mathrm{o}}\alpha))) \right]}{(1+|w|^2_{\mathrm{o}})}$$

Now using equation (6.8), (with $n = 7$) we get

$$\tilde{*}(\tilde{*}\alpha) = (1+|w|^2_{\mathrm{o}})^{-1} \left(\alpha + |w|^2_{\mathrm{o}} \alpha \right) = \alpha$$

Again, just as in the conformal scaling case, we can show directly that the norm of φ is unchanged, even though the metric changes when φ changes:

$$|\tilde{\varphi}|^2_{\sim} = |\varphi_{\mathrm{o}}|^2_{\mathrm{o}}$$

We compute:

$$\begin{aligned}
|\tilde{\varphi}|^2_{\sim} &= \frac{1}{(1+|w|^2_{\circ})^{\frac{3}{3}}}\left(\langle\tilde{\varphi},\tilde{\varphi}\rangle_{\circ} + \langle w\lrcorner\tilde{\varphi}, w\lrcorner\tilde{\varphi}\rangle_{\circ}\right) \\
&= (1+|w|^2_{\circ})^{-1}\left(|\varphi_{\circ}|^2_{\circ} + 2\langle\varphi_{\circ}, w\lrcorner *_{\circ}\varphi_{\circ}\rangle_{\circ} + |w\lrcorner *_{\circ}\varphi_{\circ}|^2_{\circ} + |w\lrcorner\varphi_{\circ}|^2_{\circ}\right)
\end{aligned}$$

where we have used the fact that $w\lrcorner\tilde{\varphi} = w\lrcorner\varphi_{\circ}$. Now the second term vanishes because of the orthogonality of $\wedge^3_1$ and $\wedge^3_7$, and finally we use (6.9) to obtain:

$$|\tilde{\varphi}|^2_{\sim} = (1+|w|^2_{\circ})^{-1}\left(|\varphi_{\circ}|^2_{\circ} + |w|^2_{\circ}|\varphi_{\circ}|^2_{\circ}\right) = |\varphi_{\circ}|^2_{\circ}$$

as expected.

We now give a geometric interpretation of the transformation $\varphi_{\circ} \mapsto \varphi_{\circ} + w\lrcorner *_{\circ}\varphi_{\circ}$. From equation 3.13 for the new metric $\tilde{g}$, with $v_1 = v$ and $v_2 = w$, we have

$$\begin{aligned}
\langle v, w\rangle_{\sim} &= \frac{1}{(1+|w|^2_{\circ})^{\frac{2}{3}}}\left(\langle v,w\rangle_{\circ} + \langle v,w\rangle_{\circ}|w|^2_{\circ} - \langle v,w\rangle_{\circ}\langle w,w\rangle_{\circ}\right) \\
&= \frac{1}{(1+|w|^2_{\circ})^{\frac{2}{3}}}\langle v,w\rangle_{\circ}
\end{aligned}$$

Hence we see that all the distances are *shrunk* by a factor of $(1+|w|^2_{\circ})^{-\frac{2}{3}}$ in the direction of the vector field w. On the other hand, if either v_1 or v_2 is orthogonal to w in the old metric, then equation 3.13 gives

$$\begin{aligned}
\langle v_1, v_2\rangle_{\sim} &= \frac{1}{(1+|w|^2_{\circ})^{\frac{2}{3}}}\left(\langle v_1,v_2\rangle_{\circ} + \langle v_1,v_2\rangle_{\circ}|w|^2_{\circ} - 0\right) \\
&= (1+|w|^2_{\circ})^{\frac{1}{3}}\langle v_1,v_2\rangle_{\circ}
\end{aligned}$$

Thus in the directions perpendicular to the vector field w, the distances are *stretched* by a factor of $(1+|w|^2_{\circ})^{\frac{1}{3}}$. Therefore this new metric is expanded in the 6 directions perpendicular to w and is compressed in the direction parallel to w. This produces a *tubular* manifold. For example in the case of $M = N \times S^1$, where N is a Calabi-Yau 3-fold and the metric on M is the product metric, if we take $w = \frac{\partial}{\partial\theta}$ where θ is a coordinate on S^1, then the Calabi-Yau manifold N is expanded and the circle factor S^1 is compressed under $\varphi_{\circ} \mapsto \varphi_{\circ} + w\lrcorner *_{\circ}\varphi_{\circ}$. By replacing w by tw and letting $t \to \infty$, we can make this "tube" as long and thin as we want. The total volume, however, always increases as $(1+|w|^2_{\circ})^{\frac{2}{3}}$ by Proposition 3.2.2.

In general, determining the class of G_2-structure that $\tilde{\varphi}$ belongs to for $\tilde{\varphi} = \varphi_{\circ} + w\lrcorner\varphi_{\circ}$ involves some very complicated differential equations on the vector field w. However, since $\tilde{\varphi}$ is *always* a positive 3-form for any w, it may be interesting to study some of these differential equations in the simplest cases to determine if one can choose w to produce a $\tilde{\varphi}$ in a strictly smaller subclass. From Theorem 3.2.5, we have

$$\begin{aligned} \tilde{*}\tilde{\varphi} &= (1+|w|_o^2)^{-\frac{1}{3}} \left(*_o\tilde{\varphi} + w\lrcorner(*_o(w\lrcorner\tilde{\varphi}))\right) \\ &= (1+|w|_o^2)^{-\frac{1}{3}} \left(*_o\varphi_o + *_o(w\lrcorner *_o \varphi_o) + w\lrcorner *_o (w\lrcorner\varphi_o)\right) \end{aligned} \tag{3.19}$$

The presence of the $(1+|w|_o^2)^{-\frac{1}{3}}$ term in $\tilde{*}\tilde{\varphi}$ causes $d\tilde{*}\tilde{\varphi}$ to be very complicated unless $|w|_o^2$ is constant.

3.3. Infinitesmal deformations in the $\wedge^3_7$ direction

Now since the decomposition of the space of differential forms corresponding to the G_2-structure φ *changes* when we add something in $\wedge^3_7$, it makes more sense to consider a one-parameter family φ_t of G_2-structures, satisfying

$$\frac{\partial}{\partial t}\varphi_t = w\lrcorner *_t \varphi_t \tag{3.20}$$

for a fixed vector field w. That is, at each time t, we move in the direction $w\lrcorner *_t \varphi_t$ which is a 3-form in $\wedge^3_{7_t}$, the decomposition depending on t. Since the Hodge star $*_t$ is also changing in time, this is *a priori* a nonlinear equation. However, our first observation is that this is in fact not the case:

Proposition 3.3.1. *Under the flow described by equation* (3.20)*, the metric* g does not change*. Hence the volume form and Hodge star are also constant.*

Proof. From (2.26) which gives the metric from the 3-form, we have:

$$g_t(u,v)\,\mathrm{vol}_t = -\frac{1}{6}(u\lrcorner\varphi_t)\wedge(v\lrcorner\varphi_t)\wedge\varphi_t$$

Differentiating with respect to t, and using the differential equation (3.20),

$$\begin{aligned} &-6\frac{\partial}{\partial t}\left(g_t(u,v)\,\mathrm{vol}_t\right) \\ = \;& (u\lrcorner w\lrcorner *_t \varphi_t)\wedge(v\lrcorner\varphi_t)\wedge\varphi_t + (u\lrcorner\varphi_t)\wedge(v\lrcorner w\lrcorner *_t \varphi_t)\wedge\varphi_t \\ & + (u\lrcorner\varphi_t)\wedge(v\lrcorner\varphi_t)\wedge(w\lrcorner *_t \varphi_t) \end{aligned}$$

Now from the proof of Lemma 3.2.1 (the linear term) we see that this expression is zero, by polarizing. From this it follows easily by taking determinants that vol_t is constant and thus so is g_t and $*_t$. $\square$

Therefore we can replace $*_t$ by $*_0 = *$ and equation (3.20) is actually *linear*. Moreover, the flow determined by this linear equation gives a one-parameter family of G_2-structures each yielding the *same* metric g. Our equation is now

$$\frac{\partial}{\partial t}\varphi_t = w\lrcorner * \varphi_t = A\varphi_t$$

where A is the linear operator $\alpha \mapsto A\alpha = w\lrcorner * \alpha$ on $\wedge^3$.

Proposition 3.3.2. *The operator A is* skew-symmetric. *Furthermore, the eigenvalues λ of A and their multiplicities N_λ are:*

$$\begin{array}{rl} \lambda = 0 & N_\lambda = 21 \\ \lambda = i|w| & N_\lambda = 7 \\ \lambda = -i|w| & N_\lambda = 7 \end{array}$$

Proof. Let $e^1, e^2, \ldots e^{35}$ be a basis of $\wedge^3$. Then

$$\begin{aligned} A_{ij}\,\mathrm{vol} &= \left\langle e^i, Ae^j \right\rangle \mathrm{vol} \\ &= e^i \wedge *(w\lrcorner * e^j) \\ &= -e^i \wedge w^\# \wedge e^j \\ &= w^\# \wedge e^i \wedge e^j \\ &= -A_{ji}\,\mathrm{vol} \end{aligned}$$

since 3-forms anti-commute. Therefore A is diagonalizable over $\mathbb{C}$. Suppose now that $\alpha \in \wedge^3$ is an eigenvector with eigvenalue $\lambda = 0$. Then

$$\begin{aligned} A\alpha &= w\lrcorner * \alpha \\ &= - * (w^\# \wedge \alpha) \\ &= 0 \end{aligned}$$

so $w^\# \wedge \alpha = 0$ and hence $\alpha = w^\# \wedge \beta$ for some $\beta \in \wedge^2$. Therefore the multiplicity of $\lambda = 0$ is $\dim(\wedge^2) = 21$. If $A\alpha = \lambda\alpha$ for $\lambda \neq 0$, then $\alpha = \frac{1}{\lambda}(w\lrcorner * \alpha)$ and $w\lrcorner\alpha = 0$. Then we can write (6.8) as

$$|w|^2\alpha = -w\lrcorner * (w\lrcorner * \alpha) = -A^2\alpha = -\lambda^2\alpha$$

and hence $\lambda = \pm i|w|$. Since the eigenvalues come in complex conjugate pairs and there are $35 - 21 = 14$ remaining, there must be 7 of each. This completes the proof. ∎

Now if α is an eigenvector for $\frac{\partial}{\partial t}\alpha_t = A\alpha_t = \lambda\alpha_t$, then $\alpha(t) = e^{\lambda t}\alpha(0)$. Let $u_1, u_2, \ldots, u_{21}$ be a basis for the $\lambda = 0$ eigenspace, and $v_1, \ldots, v_7$ and

$\bar{v}_1, \ldots, \bar{v}_7$ be bases of *complex* eigenvectors corresponding to the $\lambda = +i|w|$ and $\lambda = -i|w|$ eigenspaces, respectively. We can write

$$\begin{aligned} \phi_0 &= \sum_{k=1}^{7} c_k v_k + \sum_{k=1}^{7} \bar{c}_k \bar{v}_k + \sum_{k=1}^{21} h_k u_k \\ &= \sum_{k=1}^{7} c_k v_k + \sum_{k=1}^{7} \bar{c}_k \bar{v}_k + \eta_0 \end{aligned}$$

where η_0 as defined by the above equation is the part of φ_0 in the kernel of A. Then the solution is given by

$$\begin{aligned} \varphi_t &= \sum_{k=1}^{7} c_k e^{i|w|t} v_k + \sum_{k=1}^{7} \bar{c}_k e^{-i|w|t} \bar{v}_k + \eta_0 \\ &= \cos(|w|t) \sum_{k=1}^{7} (c_k v_k + \bar{c}_k \bar{v}_k) + \sin(|w|t) \sum_{k=1}^{7} i(c_k v_k - \bar{c}_k \bar{v}_k) + \eta_0 \\ &= \cos(|w|t)\beta_0 + \sin(|w|t)\gamma_0 + \eta_0 \qquad (3.21) \end{aligned}$$

All that remains is to determine β_0, γ_0, and η_0 in terms of the initial condition φ_0. Substituting $t = 0$ into (3.21), we have

$$\varphi_0 = \beta_0 + \eta_0$$

Differentiating, we have

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_t &= -|w|\sin(|w|t)\beta_0 + |w|\cos(|w|t)\gamma_0 \\ A\varphi_t &= \cos(|w|t)A\beta_0 + \sin(|w|t)A\gamma_0 + A\eta_0 \end{aligned}$$

Comparing coefficients, we have

$$\begin{aligned} A\beta_0 &= |w|\gamma_0 \\ A\gamma_0 &= -|w|\beta_0 \\ A\eta_0 &= 0 \end{aligned}$$

From $\beta_0 = \varphi_0 - \eta_0$ and the equations above, we get

$$\gamma_0 = \frac{1}{|w|}(A\varphi_0)$$

and substituting this into the second equation, we obtain

$$\beta_0 = -\frac{1}{|w|^2}(A^2\varphi_0)$$

Finally, we can state the general solution:

Theorem 3.3.3. *The solution to the differential equation*

$$\frac{\partial}{\partial t}\varphi_t = w \lrcorner *_t \varphi_t$$

is given by

$$\varphi(t) = \varphi_0 + \frac{1-\cos(|w|t)}{|w|^2}(w \lrcorner * (w \lrcorner * \varphi_0)) + \frac{\sin(|w|t)}{|w|}(w \lrcorner * \varphi_0) \quad (3.22)$$

The solution exists for all time and is closed curve (an ellipse) in $\wedge^3$. Also, the path only depends on the unit vector field $\frac{w}{|w|}$, and the norm $|w|$ only affects the speed of travel along this curve.

Proof. This is all immediate from the above discussion. $\square$

Remark 3.3.4. In [**5**], it is shown that the set of G_2-structures on M which correspond to the same metric as that of a fixed G_2-structure φ_o is an $\mathbb{RP}^7$-bundle over the manifold M. The above theorem gives an explicit formula (3.22) for a path of G_2-structures all corresponding to the same metric g starting from an arbitrary vector field w on M.

Remark 3.3.5. This can also be compared to the Kähler case. Since the metric and the almost complex structure J are independent in this case, for a fixed metric g, the family of 2-forms $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ for varying J's are all Kähler forms corresponding to the same metric.

Remark 3.3.6. Even though the metric is unchanged under an infinitesmal deformation in the $\wedge^3_7$ direction, the class of G_2-structure *can* change. Therefore simply knowing that a metric on a 7-manifold arises from a G_2-structure and knowing the metric explicitly does not determine the class.

We now apply this theorem to two specific examples, where we will reproduce known results.

Example 3.3.7. Let N be a Calabi-Yau threefold, with Kähler form ω and holomorphic $(3,0)$ form Ω. The complex coordinates will be denoted by $z^j = x^j + iy^j$. Then there is a natural G_2-structure φ on the product $N \times S^1$ given by

$$\varphi = \mathrm{Re}(\Omega) - d\theta \wedge \omega \quad (3.23)$$

where θ is the coordinate on the circle S^1. This induces the product metric on $N \times S^1$, with the flat metric on S^1. With the orientation on $N \times S^1$ given by $(x^1, x^2, x^3, \theta, y^1, y^2, y^3)$, it is easy to check that

$$*\varphi = -d\theta \wedge \mathrm{Im}(\Omega) - \frac{\omega^2}{2}$$

Now let $w = \frac{\partial}{\partial\theta}$ be a globally defined non-vanishing vector field on S^1 with $|w| = 1$. Then we have

$$\begin{aligned} w \lrcorner * \varphi &= -\operatorname{Im}(\Omega) \\ *(w \lrcorner * \varphi) &= -d\theta \wedge \operatorname{Re}(\Omega) \\ w \lrcorner *(w \lrcorner * \varphi) &= -\operatorname{Re}(\Omega) \end{aligned}$$

Thus for this choice of vector field w, the flow in (3.22) is given by

$$\begin{aligned} \varphi_t &= \operatorname{Re}(\Omega) - d\theta \wedge \omega - (1 - \cos(t)) \operatorname{Re}(\Omega) - \sin(t) \operatorname{Im}(\Omega) \\ &= \cos(t) \operatorname{Re}(\Omega) - \sin(t) \operatorname{Im}(\Omega) - d\theta \wedge \omega \\ &= \operatorname{Re}(e^{it}\Omega) - d\theta \wedge \omega \end{aligned}$$

which is the canonical G_2 form on $N \times S^1$ where now the Calabi-Yau structure on N is given by $e^{it}\Omega$ and ω. It is well-known that we have this freedom of changing the holomorphic volume form Ω by a phase and preserving the Ricci-flat metric. Here it arises naturally using the flow described by (3.22) and the canonical vector field $w = \frac{\partial}{\partial\theta}$.

Example 3.3.8. As a second example, let W be a $K3$ surface, which is hyperKähler with hyperKähler triple ω_1, ω_2, and ω_3. With local coordinates $z^1 = -y^0 + iy^1$ and $z^2 = y^2 + iy^3$ these forms can be written as:

$$\begin{aligned} \omega_1 &= -dy^0 \wedge dy^1 + dy^2 \wedge dy^3 \\ \omega_2 &= -dy^0 \wedge dy^2 + dy^3 \wedge dy^1 \\ \omega_3 &= -dy^0 \wedge dy^3 + dy^1 \wedge dy^2 \end{aligned}$$

The volume form vol_W on W is given by $\frac{\omega_j^2}{2}$ for any $j = 1, 2, 3$. There is a natural G_2-structure φ on the product $W \times T^3$ given by

$$\varphi = d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 - d\theta^2 \wedge \omega_2 - d\theta^3 \wedge \omega_3 \tag{3.24}$$

where $\theta^1, \theta^2, \theta^3$ are coordinates on the torus T^3. This induces the product metric on $W \times T^3$, with the flat metric on T^3. With the orientation on $W \times T3$ given by $(\theta^1, \theta^2, \theta^3, y^0, y^1, y^2, y^3)$, it is easy to check that

$$*\varphi = \mathrm{vol}_W - d\theta^2 \wedge d\theta^3 \wedge \omega_1 - d\theta^3 \wedge d\theta^1 \wedge \omega_2 - d\theta^1 \wedge d\theta^2 \wedge \omega_3$$

Now let $w = \frac{\partial}{\partial\theta^1}$ be one of the globally defined non-vanishing vector fields on T^3 with $|w| = 1$. Then we have

$$\begin{aligned} w \lrcorner * \varphi &= d\theta^3 \wedge \omega_2 - d\theta^2 \wedge \omega_3 \\ *(w \lrcorner * \varphi) &= d\theta^1 \wedge d\theta^2 \wedge \omega_2 - d\theta^3 \wedge d\theta^1 \wedge \omega_3 \\ w \lrcorner *(w \lrcorner * \varphi) &= d\theta^2 \wedge \omega_2 + d\theta^3 \wedge \omega_3 \end{aligned}$$

Thus for this choice of vector field w, the flow in (3.22) is given by

$$\begin{aligned}\varphi_t &= d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 - d\theta^2 \wedge \omega_2 - d\theta^3 \wedge \omega_3 + \\ &\quad (1-\cos(t))\left(d\theta^2 \wedge \omega_2 + d\theta^3 \wedge \omega_3\right) + \sin(t)\left(d\theta^3 \wedge \omega_2 - d\theta^2 \wedge \omega_3\right) \\ &= d\theta^1 \wedge d\theta^2 \wedge d\theta^3 - d\theta^1 \wedge \omega_1 \\ &\quad - d\theta^2 \wedge (\cos(t)\omega_2 + \sin(t)\omega_3) - d\theta^3 \wedge (-\sin(t)\omega_2 + \cos(t)\omega_3)\end{aligned}$$

which is the canonical G_2 form on $W \times T^3$ where now the hyperKähler triple on W is given by $\tilde{\omega}_1 = \omega_1$, $\tilde{\omega}_2 = \cos(t)\omega_2 + \sin(t)\omega_3$, and $\tilde{\omega}_3 = -\sin(t)\omega_2 + \cos(t)\omega_3$. This is just a restatement of the fact that on a hyperKähler manifold, we have an S^2 worth of complex structures, and we can choose any triple I, J, K such that $IJ = K$ to obtain the three Kähler forms. The above construction corresponds to a *hyperKähler rotation* where $J \mapsto J\cos(t) + K\sin(t)$ and $K \mapsto -J\sin(t) + K\cos(t)$. This is rotation by an angle t around the axis in S^2 that represents the complex structure I. It is clear that we can hyperKähler rotate around any axis by taking $w = a^1\frac{\partial}{\partial\theta^1} + a^2\frac{\partial}{\partial\theta^2} + a^3\frac{\partial}{\partial\theta^3}$ where $(a^1)^2 + (a^2)^2 + (a^3)^2 = 1$ in the flow described by equation (3.22). All these hyperKähler structures on W yield the same metric, and hence determine the same metric on $W \times T^3$, as expected.

Chapter 4

Manifolds with a $Spin(7)$*-structure*

4.1. $Spin(7)$-structures

Let M be an oriented 8-manifold with a global 3-fold cross product structure. Such a structure will henceforth be called a $Spin(7)$-structure. Its existence is also given by the same topological condition, the vanishing of the second Stiefel-Whitney class $w_2 = 0$. (Again see [**19, 31, 33**] for details.) Similarly to the G_2 case, this cross product $X(\cdot,\cdot,\cdot)$ gives rise to an associated *Riemannian metric* g and an alternating 4-form Φ which are related by:

$$\Phi(a,b,c,d) = g(X(a,b,c),d). \tag{4.1}$$

As in the G_2 case, the metric and the cross product structure cannot be prescribed independently. We will see in Section 4.3 how the 4-form Φ determines the metric $g(\cdot,\cdot)$. For a $Spin(7)$-structure Φ, near a point $p \in M$ we can choose local coordinates $x^0, x^1 \ldots, x^7$ so that *at the point* p, we have:

$$\begin{aligned} \Phi_p \;=\; & dx^{0123} - dx^{0167} - dx^{0527} - dx^{0563} - dx^{0415} - dx^{0426} - dx^{0437} \\ & + dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} - dx^{2637} - dx^{1537} - dx^{1526} \end{aligned}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. In these coordinates the metric at p is the standard Euclidean metric

$$g_p = \sum_{k=1}^{8} dx^k \otimes dx^k$$

and $*\Phi = \Phi$, so Φ is *self-dual.* The 4-forms that arise from a $Spin(7)$-structure are called *positive* or *non-degenerate*, and this set is denoted $\wedge^4_{\text{pos}}$. The subgroup of $SO(8)$ that preserves Φ_p is $Spin(7)$. (see [**4**].) Hence at each point p, the set of $Spin(7)$-structures at p is isomorphic to $GL(8,\mathbb{R})/Spin(7)$, which is $64-21 = 43$ dimensional. This time, however, in contrast to the G_2 case, since $\wedge^4(\mathbb{R}^8)$ is 70 dimensional, the set $\wedge^4_{\text{pos}}(p)$ of positive 4-forms at p is *not* an open subset of $\wedge^4_p$. We will determine some new information about the structure of $\wedge^4_{\text{pos}}$ in Section 5.2.

4.2. $Spin(7)$-Decomposition of $\bigwedge^*(M)$

The facts collected in this section about the decomposition of the space of forms in the $Spin(7)$ case can also be found in [**10, 31, 33**].

There is an action of the group $Spin(7)$ on $\mathbb{R}^8$, and hence on the spaces $\wedge^*$ of differential forms on M. We can decompose each space $\wedge^k$ into irreducible $Spin(7)$-representations. The results of this decomposition are presented below. As before, the notation $\wedge^k_l$ refers to an l-dimensional irreducible $Spin(7)$-representation which is a subspace of $\wedge^k$, w is a vector field on M and vol is the volume form.

$$
\begin{aligned}
\wedge^0_1 &= \{f \in C^\infty(M)\} \\
\wedge^1_8 &= \{\alpha \in \Gamma(\wedge^1(M)\} \\
\wedge^2 &= \wedge^2_7 \oplus \wedge^2_{21} \\
\wedge^3 &= \wedge^3_8 \oplus \wedge^3_{48} \\
\wedge^4 &= \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \oplus \wedge^4_{35} \\
\wedge^5 &= \wedge^5_8 \oplus \wedge^5_{48} \\
\wedge^6 &= \wedge^6_7 \oplus \wedge^6_{21} \\
\wedge^7_8 &= \{w \lrcorner \mathrm{vol}\} \\
\wedge^8_1 &= \{f \,\mathrm{vol}; f \in C^\infty(M)\}
\end{aligned}
$$

This decomposition respects the *Hodge star* $*$ operator since $Spin(7) \in SO(8)$, so $*\wedge^k_l = \wedge^{8-k}_l$. Again, we will give explicit descriptions of the remaining cases.

Before we describe $\wedge^k_l$ for $k = 2, 3, 4, 5, 6$, there are some isomorphisms between these subspaces:

Proposition 4.2.1. *The map $\alpha \mapsto \Phi \wedge \alpha$ is an isomorphism between the following spaces:*

$$
\begin{aligned}
\wedge^0_1 &\cong \wedge^4_1 \\
\wedge^1_8 &\cong \wedge^5_8 \\
\wedge^2_7 &\cong \wedge^6_7 \\
\wedge^2_{21} &\cong \wedge^6_{21} \\
\wedge^3_8 &\cong \wedge^7_8 \\
\wedge^4_1 &\cong \wedge^8_1
\end{aligned}
$$

In addition, if α is a 1-form, we have the following identity:

$$* \, (\Phi \wedge *(\Phi \wedge \alpha)) = -7\alpha \tag{4.2}$$

Proof. These statements can be easily checked pointwise using local coordinates. □

From this identity, we can prove the following lemma:

Lemma 4.2.2. *If α is a 1-form on M, then we have:*

$$|\Phi \wedge \alpha|^2 = 7|\alpha|^2 \tag{4.3}$$

Further, for any $Spin(7)$-structure given by Φ, we have:

$$|\Phi|^2 = 14. \tag{4.4}$$

Proof. From (4.2), we have:

$$\begin{aligned}
\Phi \wedge *(\Phi \wedge \alpha) &= 7 * \alpha \\
\alpha \wedge \Phi \wedge *(\Phi \wedge \alpha) &= 7\alpha \wedge *\alpha \\
|\Phi \wedge \alpha|^2 \,\mathrm{vol} &= 7|\alpha|^2 \,\mathrm{vol}
\end{aligned}$$

which proves (4.3). From (6.9), we compute:

$$|\Phi|^2|\alpha|^2 = |(*\Phi) \wedge \alpha|^2 + |\Phi \wedge \alpha|^2 = 7|\alpha|^2 + 7|\alpha|^2 = 14|\alpha|^2$$

which gives $|\Phi|^2 = 14$, which can also be seen using the local coordinate expression for Φ and the fact that the coordinates were chosen so that the dx^k's were orthonormal at p. □

We have some relations between Φ and an arbitrary vector field w:

Lemma 4.2.3. *The following relations hold for any vector field w, where $w^\#$ is the associated 1-form:*

$$\begin{aligned}
*\left(\Phi \wedge w^\#\right) &= w \lrcorner \Phi \qquad (4.5)\\
\Phi \wedge (w \lrcorner \Phi) &= 7 * w^\# \qquad (4.6)
\end{aligned}$$

Proof. Since on an 8-manifold $*^2 = (-1)^k$ on k-forms, these results follow from Lemma 6.0.9 and Proposition 4.2.1. □

We now explicitly describe the decomposition of the space of forms, beginning with $k = 2, 6$. These should be compared to the G_2 case which were given in (2.15) – (2.18).

$$\begin{aligned}
\wedge^2_7 &= \{\beta \in \wedge^2; *(\Phi \wedge \beta) = -3\beta\} \qquad (4.7)\\
&= \{v^\# \wedge w^\# - *(v^\# \wedge w^\# \wedge \Phi); v, w \in \Gamma(T(M))\}\\
\wedge^2_{21} &= \{\beta \in \wedge^2; *(\Phi \wedge \beta) = \beta\} \qquad (4.8)\\
&= \{\sum a_{ij} e^i \wedge e^j; (a_{ij}) \in \mathfrak{spin}(7)\}
\end{aligned}$$

$$\begin{aligned}
\wedge^6_7 &= \{\mu \in \wedge^6; \Phi \wedge *\mu = -3\mu\} && (4.9)\\
\wedge^6_{21} &= \{\mu \in \wedge^6; \Phi \wedge *\mu = \mu\} && (4.10)
\end{aligned}$$

Note that these subspaces are -3 and $+1$ eigenspaces of the operators $G(\beta) = *(\Phi \wedge \beta)$ on $\wedge^2$ and $M(\mu) = \Phi \wedge *\mu$ on $\wedge^6$. From this fact we get the following useful formulas for the projections π_k onto the k-dimensional representations, for$\beta \in \wedge^2$ and $\mu \in \wedge^6$:

$$\begin{aligned}
*(\Phi \wedge \beta) &= -3\pi_7(\beta) + \pi_{21}(\beta) && (4.11)\\
\pi_7(\beta) &= \frac{\beta - *(\Phi \wedge \beta)}{4}\\
\pi_{21}(\beta) &= \frac{3\beta + *(\Phi \wedge \beta)}{4}
\end{aligned}$$

and

$$\begin{aligned}
\Phi \wedge *\mu &= -3\pi_7(\mu) + \pi_{21}(\mu)\\
\pi_7(\mu) &= \frac{\mu - \Phi \wedge *\mu}{4}\\
\pi_{21}(\mu) &= \frac{3\mu + \Phi \wedge *\mu}{4}
\end{aligned}$$

We now move on to the decompositions for $k = 3, 4, 5$. For $k = 3$, we have:

$$\begin{aligned}
\wedge^3_8 &= \{*(\Phi \wedge \alpha); \alpha \in \wedge^1_8\} && (4.12)\\
&= \{w \lrcorner \Phi; w \in \Gamma(T(M))\}\\
&= \{\eta \in \wedge^3; *(\Phi \wedge *(\Phi \wedge \eta)) = -7\eta\}\\
\wedge^3_{48} &= \{\eta \in \wedge^3; \Phi \wedge \eta = 0\} && (4.13)
\end{aligned}$$

For $k = 5$, the decomposition is:

$$\begin{aligned}
\wedge^5_8 &= \{\Phi \wedge \alpha; \alpha \in \wedge^1_8\} && (4.14)\\
\wedge^5_{48} &= \{\mu \in \wedge^5; \Phi \wedge *\mu = 0\} && (4.15)
\end{aligned}$$

And finally, the middle dimension $k = 4$ decomposes as:

$$\begin{aligned}
\wedge^4_1 &= \{f\Phi; f \in C^\infty(M)\} && (4.16)\\
&= \{\eta \in \wedge^4; \Phi \wedge (*(\Phi \wedge \eta)) = 14\eta\} \\
\wedge^4_7 &= \{v^\# \wedge (w\lrcorner\Phi) - w^\# \wedge (v\lrcorner\Phi); v, w \in \Gamma(T(M))\} && (4.17)\\
&= \{v\lrcorner(w^\# \wedge \Phi) - w\lrcorner(v^\# \wedge \Phi); v, w \in \Gamma(T(M))\} \\
&= \{w\lrcorner * (v\lrcorner\Phi) - v\lrcorner * (w\lrcorner\Phi); v, w \in \Gamma(T(M))\} \\
\wedge^4_{27} &= \{\sigma \in \wedge^4; *\sigma = \sigma, \sigma \wedge \Phi = 0, \sigma \wedge \tau = 0 \quad \forall \tau \in \wedge^4_7\} && (4.18)\\
\wedge^4_{35} &= \{\sigma \in \wedge^4; *\sigma = -\sigma\} && (4.19)
\end{aligned}$$

4.3. The metric of a *Spin*(7)-structure

Here the situation differs significantly from the G_2 case. Because Φ is self-dual equation (4.2) gives us only one useful identity rather than the four identities in equations (2.4) – (2.6). In particular it was equation (2.6) which enabled us to prove Proposition 2.3.1 to obtain a formula for the metric from the 3-form φ in the G_2 case.

The prescription for obtaining the metric from the 4-form Φ in the *Spin*(7) case is much more complicated. We begin with the following Lemma, which is analogous to Proposition 2.3.1 in the G_2 case, and whose proof is similar to the proof of Lemma 2.4.3.

Lemma 4.3.1. *The following identity holds for v and w vector fields:*

$$(v\lrcorner w\lrcorner\Phi) \wedge (v\lrcorner w\lrcorner\Phi) \wedge \Phi = -6|v \wedge w|^2 \,\mathrm{vol} \tag{4.20}$$

Proof. Let $\beta = v^\# \wedge w^\# = \beta_7 + \beta_{21}$ using the decompositions in (4.7) and (4.8). From Lemma 6.0.9 we can write

$$\begin{aligned}
v\lrcorner w\lrcorner\Phi &= *(v^\# \wedge *(w\lrcorner\Phi)) \\
&= - * (v^\# \wedge w^\# \wedge \Phi) \\
&= 3\beta_7 - \beta_{21}
\end{aligned}$$

where we have used the self-duality $*\Phi = \Phi$ and the characterizations of $\wedge^2_7$ and $\wedge^2_{21}$. Similarly we obtain

$$\begin{aligned}
(v\lrcorner w\lrcorner\Phi) \wedge \Phi &= (3\beta_7 - \beta_{21}) \wedge \Phi \\
&= -9 * \beta_7 - *\beta_{21}
\end{aligned}$$

Combining the two and using the orthogonality of the decompositions, we have

$$\begin{aligned}
(v\lrcorner w\lrcorner\Phi) \wedge (v\lrcorner w\lrcorner\Phi) \wedge \Phi &= (3\beta_7 - \beta_{21}) \wedge (-9 * \beta_7 - *\beta_{21}) && (4.21)\\
&= (-27|\beta_7|^2 + |\beta_{21}|^2) \,\mathrm{vol}
\end{aligned}$$

Now since $\beta = v^{\#} \wedge w^{\#}$ is decomposable, from (4.7) and (4.8) we obtain:

$$\begin{aligned} \beta \wedge \Phi &= -3 * \beta_7 + *\beta_{21} \\ \beta \wedge \beta \wedge \Phi &= -3|\beta_7|^2 \operatorname{vol} + |\beta_{21}|^2 \operatorname{vol} \\ &= 0 \end{aligned}$$

Hence $|\beta_{21}|^2 = 3|\beta_7|^2$ and

$$\begin{aligned} |v \wedge w|^2 = |\beta|^2 &= |\beta_7|^2 + |\beta_{21}|^2 \\ &= |\beta_7|^2 + 3|\beta_7|^2 = 4|\beta_7|^2 \end{aligned}$$

Finally, substituting this into (4.21), we have

$$\begin{aligned} (v \lrcorner w \lrcorner \Phi) \wedge (v \lrcorner w \lrcorner \Phi) \wedge \Phi &= (-27|\beta_7|^2 + |\beta_{21}|^2) \operatorname{vol} \\ &= -24|\beta_7|^2 \operatorname{vol} \\ &= -6|v \wedge w|^2 \operatorname{vol} \end{aligned}$$

which completes the proof. □

Remark 4.3.2. Equation (4.20) can be used to give an alternative definition to the "positivity" condition. Let ξ be a nowhere vanishing 8-form on M, which exists since M is orientable. A 4-form Φ is in $\wedge^4_{\text{pos}}$ if and only if at every point p in M, the function $f(v)$ which is defined on the product $T_p(M) \times T_p(M)$ by the map $(v, w) \mapsto -(v \lrcorner w \lrcorner \Phi_p) \wedge (v \lrcorner w \lrcorner \Phi_p) \wedge \Phi_p = f(v)\xi_p$ satisfies $f(v) \geq 0$ with equality if and only if $v \wedge w = 0$.

If we polarize (4.20) in w, we obtain the useful equation:

$$\begin{aligned} (v \lrcorner w_1 \lrcorner \Phi) \wedge (v \lrcorner w_2 \lrcorner \Phi) \wedge \Phi &= -6\langle v \wedge w_1, v \wedge w_2 \rangle \operatorname{vol} \\ &= -6\big(|v|^2 \langle w_1, w_2\rangle - \langle v, w_1\rangle \langle v, w_2 \rangle\big) \operatorname{vol} \end{aligned} \tag{4.22}$$

From this equation we can obtain the metric.

Lemma 4.3.3. *Fix a* non-zero *vector field* $v = v^k e_k$*, where* $e_0, e_1, e_2, \ldots, e_7$ *is an oriented local frame of vector fields. Without loss of generality assume* $v^0 \neq 0$*. The expression obtained from* v *by*

$$\frac{\left(\det\left(\left(\left(e_i \lrcorner v \lrcorner \Phi\right) \wedge \left(e_j \lrcorner v \lrcorner \Phi\right) \wedge \left(v \lrcorner \Phi\right)\right)\left(e_1, e_2, \ldots, e_7\right)\right)\right)^{\frac{1}{3}}}{\left(\left(\left(v \lrcorner \Phi\right) \wedge \Phi\right)\left(e_1, e_2, \ldots, e_7\right)\right)^3} \tag{4.23}$$

is an expression which is homogeneous of order 4 *in* v*, and is independent of the choice of* $e_0, \ldots, e_7$*. We will see in the next theorem that up to a constant, this is* $|v|^4$*.*

Proof. The expression $(e_i \lrcorner v \lrcorner \Phi) \wedge (e_j \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi)$ is cubic in v, so after taking the 7×7 determinant and the cube root, the numerator of (4.23) is of order 7 in v. Since the denominator is cubic in v, the whole expression is homogeneous of degree 4 in v. Now suppose we extend v to an oriented basis $v, e'_1, e'_2, \dots, e'_7$ in a different way. Then

$$e'_i = P_{ij}e_j + Q_i v$$

and we have

$$(e'_i \lrcorner v \lrcorner \Phi) \wedge (e'_j \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi) = P_{ik}P_{jl}(e_k \lrcorner v \lrcorner \Phi) \wedge (e_l \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi)$$

Hence in the new basis the numerator of (4.23) changes by a factor of

$$\left(\det(P)^2 \det(P)^7\right)^{\frac{1}{3}} = \det(P)^3$$

and the denominator also changes by a factor of $\det(P)^3$, so the quotient is invariant. □

Remark 4.3.4. It is interesting to note how different this expression is from the G_2 case in Lemma 2.3.3.

We now derive the expression for the metric in terms of the 4-form Φ in the *Spin*(7) case.

Theorem 4.3.5. *Let v be a tangent vector at a point p and let $e_0, e_1, \dots, e_7$ be any oriented basis for T_pM, so that $\mathrm{vol}(e_0, e_1, \dots, e_7) > 0$. Then the length $|v|$ of v is given by*

$$|v|^4 = -\frac{(7)^3}{(6)^{\frac{7}{3}}} \frac{\left(\det\left(\left((e_i \lrcorner v \lrcorner \Phi) \wedge (e_j \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi)\right)(e_1, e_2, \dots, e_7)\right)\right)^{\frac{1}{3}}}{\left(\left((v \lrcorner \Phi) \wedge \Phi\right)(e_1, e_2, \dots, e_7)\right)^3}$$

Proof. We work in local coordinates at the point p. In this notation $g_{ij} = \langle e_i, e_j \rangle$ with $0 \le i, j \le 7$. Let $\det_8(g)$ denote the 8×8 determinant of (g_{ij}) and let $\det_7(g)$ denote the 7×7 determinant of the submatrix where $1 \le i, j \le 7$. Using the fact that $\Phi^2 = 14\,\mathrm{vol} = 14\sqrt{\det_8(g)}e^0 \wedge e^1 \dots e^7$, and writing $v = v^k e_k$, we compute

$$\begin{aligned} A(v) &= \left((v \lrcorner \Phi) \wedge \Phi\right)(e_1, e_2, \dots, e_7) \\ &= 7v^0\sqrt{\det_8(g)} \end{aligned} \tag{4.24}$$

Now $\langle v, e_i \rangle = v^k g_{kj} = v_j$. We also have the 7×7 matrix

$$\begin{aligned} B_{ij}(v) &= \left((e_i \lrcorner v \lrcorner \Phi) \wedge (e_j \lrcorner v \lrcorner \Phi) \wedge (v \lrcorner \Phi)\right)(e_1, e_2, \dots, e_7) \\ &= -6\left(|v|^2 g_{ij} - v_i v_j\right) v^0 \sqrt{\det_8(g)} \end{aligned} \tag{4.25}$$

where we have used Lemma 4.3.1. Now consider the 7×7 matrix $(|v|^2 g_{ij} - v_i v_j)$ and its determinant:

$$\det \begin{pmatrix} |v|^2 g_{11} - v_1 v_1 & \dots & |v|^2 g_{17} - v_1 v_7 \\ \vdots & \ddots & \vdots \\ |v|^2 g_{71} - v_7 v_1 & \dots & |v|^2 g_{77} - v_7 v_7 \end{pmatrix}$$

By Lemma 6.0.11, this is

$$\det\left(|v|^2 g_{ij}\right) + \sum_{r,s=1}^{7} (-1)^{r+s} \left(-v_r v_s\right) |v|^{12} (G_{rs}) \tag{4.26}$$

where G_{rs} is the $(r,s)^{\text{th}}$ *minor* of the *submatrix* of (g_{ij}) for $1 \leq i,j \leq 7$. Now with $v_s = v^t g_{ts}$, the expression (4.26) becomes

$$\begin{aligned}
&= |v|^{14}\det{}_7(g) - \sum_{r,s=1}^{7}\sum_{t=0}^{7}(-1)^{r+s} v_r v^t g_{ts} |v|^{12} G_{rs} \\
&= |v|^{12}\left(|v|^2\det{}_7(g) - \sum_{r,t=1}^{7} v_r v^t \left(\sum_{s=1}^{7}(-1)^{r+s} g_{ts} G_{rs}\right)\right. \\
&\qquad\qquad \left. - \sum_{r,s=1}^{7}(-1)^{r+s} v_r v^0 g_{0s} G_{rs}\right) \\
&= |v|^{12}\left(|v|^2\det{}_7(g) - \sum_{r=1}^{7}\sum_{t=0}^{7} v_r v^t \delta_{tr}\det{}_7(g) - \sum_{r,s=1}^{7}(-1)^{r+s} v_r v^0 g_{0s} G_{rs}\right) \\
&= |v|^{12}\left(|v|^2\det{}_7(g) - \sum_{r=1}^{7} v_r v^r \det{}_7(g) - \sum_{r,s=1}^{7}(-1)^{r+s} v_r v^0 g_{0s} G_{rs}\right) \\
&= |v|^{12}\left(v^0 v_0 \det{}_7(g) + \sum_{r,s=1}^{7}(-1)^{r+s+1} v_r v^0 g_{0s} G_{rs}\right)
\end{aligned}$$

since the sum over s is the determinant of the 7×7 submatrix (g_{ij}) for $1 \leq i,j \leq 7$ with the r^{th} row replaced by the t^{th} row, and so vanishes unless $r = t$, where it is $\det_7(g)$. We have also used the fact that $|v|^2 = v_k v^k$ in the above calculation. We can further manipulate the above expression by writing $v_0 = v^k g_{0k}$ and $v_r = v^k g_{kr}$. We obtain

$$\begin{aligned}
&= |v|^{12} v^0 \left(\sum_{k=0}^{7} g_{0k} v^k \det_7(g) + \sum_{r,s=1}^{7} \sum_{k=0}^{7} (-1)^{r+s+1} g_{kr} v^k g_{0s} G_{rs} \right) \\
&= |v|^{12} v^0 \left[v^0 \left(g_{00} \det_7(g) + \sum_{r,s=1}^{7} (-1)^{r+s+1} g_{0r} g_{0s} G_{rs} \right) \right. \\
&\quad \left. + \sum_{k=1}^{7} v^k \left(g_{0k} \det_7(g) - \sum_{s=1}^{7} g_{0s} \left(\sum_{r=1}^{7} (-1)^{r+s} g_{rk} G_{rs} \right) \right) \right] \\
&= |v|^{12} v^0 \left(v^0 \det_8(g) + \sum_{k=1}^{7} v^k \left(g_{0k} \det_7(g) - \sum_{s=1}^{7} g_{0s} \delta_{ks} \det_7(g) \right) \right) \\
&= |v|^{12} v^0 v^0 \det_8(g)
\end{aligned}$$

where we have used Lemma 6.0.12 in the first step above. Returning to (4.25), we have now shown that

$$\begin{aligned}
\det B_{ij}(v) &= (-6)^7 |v|^{12} (v^0)^2 \det_8(g) (v^0)^7 (\det_8(g))^{\frac{7}{2}} \\
&= (-6)^7 |v|^{12} (v^0)^9 (\det_8(g))^{\frac{9}{2}}
\end{aligned}$$

and hence

$$(\det B_{ij}(v))^{\frac{1}{3}} = (-6)^{\frac{7}{3}} |v|^4 (v^0)^3 (\det_8(g))^{\frac{3}{2}}.$$

Finally, since from (4.24) we have

$$(A(v))^3 = (7)^3 (v^0)^3 (\det_8(g))^{\frac{3}{2}}$$

these two expressions can be combined to yield

$$|v|^4 = -\frac{(7)^3}{(6)^{\frac{7}{3}}} \frac{(\det B_{ij}(v))^{\frac{1}{3}}}{(A(v))^3} \tag{4.27}$$

which completes the proof. □

We now collect some facts about various 2-forms which can be constructed from a pair of vector fields v and w.

Proposition 4.3.6. *Let v and w be two vector fields. Define the 2-form $\beta = v^\# \wedge w^\# = \beta_7 + \beta_{21}$. Then we can construct two other 2-forms $v \lrcorner w \lrcorner \Phi$ and $*((v \lrcorner \Phi) \wedge (w \lrcorner \Phi))$ from v and w and these are related to β by*

$$\begin{aligned}
v \lrcorner w \lrcorner \Phi &= 3\beta_7 - \beta_{21} &(4.28)\\
*((v \lrcorner \Phi) \wedge (w \lrcorner \Phi)) &= 2\beta_7 - 6\beta_{21} &(4.29)
\end{aligned}$$

Furthermore, we have the following relations between these 2-forms:

$$\begin{aligned}
(u\lrcorner w\lrcorner\Phi)\wedge(v^\#\wedge w^\#)\wedge\Phi &= -3\langle u\wedge w, v\wedge w\rangle\,\mathrm{vol} && (4.30)\\
(u^\#\wedge w^\#)\wedge(v\lrcorner\Phi)\wedge(w\lrcorner\Phi) &= -4\langle u\wedge w, v\wedge w\rangle\,\mathrm{vol} && (4.31)\\
(u\lrcorner w\lrcorner\Phi)\wedge(v\lrcorner w\lrcorner\Phi)\wedge\Phi &= -6\langle u\wedge w, v\wedge w\rangle\,\mathrm{vol} && (4.32)
\end{aligned}$$

Note that this also implies that the expressions on the left above are both symmetric in u and v.

Proof. In the proof of Lemma 4.3.1 we established that

$$\begin{aligned}
v\lrcorner w\lrcorner\Phi &= -*(v^\#\wedge w^\#\wedge\Phi)\\
&= 3\beta_7-\beta_{21}
\end{aligned}$$

Also we had $|\beta_{21}|^2 = 3|\beta_7|^2$ and $|v\wedge w|^2 = |\beta|^2 = 4|\beta_7|^2$. Hence we have that $(v\lrcorner w\lrcorner\Phi)\wedge\Phi = -9*\beta_7 - *\beta_{21}$. Then

$$\begin{aligned}
(v\lrcorner w\lrcorner\Phi)\wedge(v\lrcorner w\lrcorner\Phi)\wedge\Phi &= (3\beta_7-\beta_{21})\wedge(-9*\beta_7-*\beta_{21})\\
&= -27|\beta_7|^2\,\mathrm{vol}+|\beta_{21}|^2\,\mathrm{vol}\\
&= -24|\beta_7|^2\,\mathrm{vol}\\
&= -6|v\wedge w|^2\,\mathrm{vol}
\end{aligned}$$

from which (4.32) now follows by polarization. Now since $\Phi\wedge\Phi = 14\,\mathrm{vol}$, we have

$$(w\lrcorner\Phi)\wedge\Phi = 7w\lrcorner\,\mathrm{vol} = 7*w^\#$$

Taking the interior product on both sides with v,

$$\begin{aligned}
(v\lrcorner w\lrcorner\Phi)\wedge\Phi-(w\lrcorner\Phi)\wedge(v\lrcorner\Phi) &= 7v\lrcorner*w^\#\\
&= -7*(v^\#\wedge w^\#)\\
(3\beta_7-\beta_{21})\wedge\Phi+(v\lrcorner\Phi)\wedge(w\lrcorner\Phi) &= -7*\beta_7-7*\beta_{21}\\
-9*\beta_7-*\beta_{21}+(v\lrcorner\Phi)\wedge(w\lrcorner\Phi) &= -7*\beta_7-7*\beta_{21}
\end{aligned}$$

which can be rearranged to give (4.29). Next we compute

$$\begin{aligned}
(v\lrcorner w\lrcorner\Phi)\wedge(v^\#\wedge w^\#)\wedge\Phi &= -*\left(v^\#\wedge w^\#\wedge\Phi\right)\wedge(v^\#\wedge w^\#\wedge\Phi)\\
&= (3\beta_7-\beta_{21})\wedge(-3*\beta_7+*\beta_{21})\\
&= -9|\beta_7|^2\,\mathrm{vol}-|\beta_{21}|^2\,\mathrm{vol}\\
&= -12|\beta_7|^2\,\mathrm{vol}\\
&= -3|v\wedge w|^2\,\mathrm{vol}
\end{aligned}$$

which after polarizing $v\mapsto u+v$, yields (4.30). Note the fact that the expression is symmetric in u and v is clear from the first line above. In the next section we will use this result in the forms

$$(u^\#\wedge w^\#\wedge\Phi)\wedge*(v^\#\wedge w^\#\wedge\Phi) = 3\langle u\wedge w, v\wedge w\rangle\,\mathrm{vol} \qquad (4.33)$$

Since it is not immediately apparent that expression (4.31) is symmetric in u and v we will not use a polarization argument here. Instead we establish it directly. We start with an appropriate 9-form (which is automatically zero) and take the interior product with v:

$$\begin{aligned} 0 &= u^\# \wedge w^\# \wedge \Phi \wedge (w \lrcorner \Phi) \\ 0 &= \langle u, v\rangle w^\# \wedge \Phi \wedge (w \lrcorner \Phi) - \langle v, w\rangle u^\# \wedge \Phi \wedge (w \lrcorner \Phi) \\ &\quad + u^\# \wedge w^\# \wedge (v \lrcorner \Phi) \wedge (w \lrcorner \Phi) + u^\# \wedge w^\# \wedge \Phi \wedge (v \lrcorner w \lrcorner \Phi) \\ &= 7\left(|w|^2\langle u, v\rangle - \langle u, w\rangle\langle v, w\rangle\right) \mathrm{vol} + \left(u^\# \wedge w^\# \wedge (v \lrcorner \Phi) \wedge (w \lrcorner \Phi)\right) \\ &\quad - 3\langle u \wedge w, v \wedge w\rangle \,\mathrm{vol} \end{aligned}$$

where we have used (4.30) in the last step. This now can be rearranged to give (4.31). We will have occasion to use this relation in the form

$$(\Phi \wedge u^\#) \wedge * \left(v^\# \wedge w^\# \wedge (w \lrcorner \Phi)\right) = 4\langle u \wedge w, v \wedge w\rangle \,\mathrm{vol} \tag{4.34}$$

in the next section. □

4.4. The triple cross product operation

In this section we will describe the triple cross product operation on a manifold with a $Spin(7)$-structure in terms of the 4-form Φ, and present some useful relations.

Definition 4.4.1. Let u, v, and w be vector fields on M. The *triple cross product*, denoted $X(u, v, w)$, is a vector field on M whose associated 1-form under the metric isomorphism satisfies:

$$(X(u, v, w))^\# = w \lrcorner v \lrcorner u \lrcorner \Phi \tag{4.35}$$

This immediately yields the relation between X, Φ, and the metric g:

$$g(X(u, v, w), y) = (X(u, v, w))^\#(y) = y \lrcorner w \lrcorner v \lrcorner u \lrcorner \Phi = \Phi(u, v, w, y). \tag{4.36}$$

We can obtain another useful characterization of the triple cross product from this one using Lemma 6.0.9:

$$\begin{aligned} (X(u, v, w))^\# &= w \lrcorner v \lrcorner u \lrcorner \Phi \qquad (4.37) \\ &= *(w^\# \wedge *(v \lrcorner u \lrcorner \Phi)) \\ &= *(w^\# \wedge v^\# \wedge *(u \lrcorner \Phi)) \\ &= - * (w^\# \wedge v^\# \wedge u^\# \wedge *\Phi) \\ &= *(u^\# \wedge v^\# \wedge w^\# \wedge \Phi) \end{aligned}$$

Note the similarity to the G_2 case given by (2.31).

Since $u^\# \wedge v^\# \wedge w^\#$ is a 3-form, we can write it as $\gamma_8 + \gamma_{48}$, with $\gamma_j \in \wedge^3_j$. Using (4.12) and (4.13) we see that:

$$\begin{aligned} (X(u,v,w))^\# \wedge \Phi &= *(\gamma_7 \wedge \Phi) \wedge \Phi \\ &= 7 * \gamma_7 \end{aligned} \tag{4.38}$$

Taking the norm of both sides, and using (4.3):

$$|(X(u,v,w))^\# \wedge \Phi|^2 = 7|(X(u,v,w))^\#|^2 = 7|X(u,v,w)|^2 = 49|\gamma_7|^2$$

from which we obtain

$$|\gamma_7|^2 = \frac{1}{7}|X(u,v,w)|^2 \tag{4.39}$$

We can now establish the following Lemma.

Lemma 4.4.2. *Let u, v, and w be vector fields. Then*

$$|X(u,v,w)|^2 = |u \wedge v \wedge w|^2 \tag{4.40}$$

Proof. First we note that

$$\begin{aligned} |u \wedge v \wedge w|^2 &= \det \begin{pmatrix} |u|^2 & \langle u,v\rangle & \langle u,w\rangle \\ \langle u,v\rangle & |v|^2 & \langle v,w\rangle \\ \langle u,w\rangle & \langle v,w\rangle & |w|^2 \end{pmatrix} \\ &= |u|^2|v|^2|w|^2 + 2\langle u,v\rangle\langle v,w\rangle\langle u,w\rangle \\ &\quad - |u|^2\langle v,w\rangle^2 - |v|^2\langle u,w\rangle^2 - |w|^2\langle u,v\rangle^2 \end{aligned}$$

as we will have to identify an expression of this form several times in what follows. Starting from (4.38), and using (4.13),

$$\begin{aligned} 7|\gamma_7|^2 \operatorname{vol} &= \gamma_7 \wedge \Phi \wedge *(\gamma_7 \wedge \Phi) \\ &= \Phi \wedge u^\# \wedge v^\# \wedge w^\# \wedge * \left(u^\# \wedge v^\# \wedge w^\# \wedge \Phi\right) \end{aligned}$$

We will use Lemma 6.0.9 three times, each time observing that all but one term will be a multiple of $|u \wedge v \wedge w|^2 \operatorname{vol}$. The first application yields

$$\begin{aligned} & w^\# \wedge * \left(u^\# \wedge v^\# \wedge w^\# \wedge \Phi\right) = \\ & \langle u,w\rangle * (v^\# \wedge w^\# \wedge \Phi) - \langle v,w\rangle * (u^\# \wedge w^\# \wedge \Phi) \\ & + |w|^2 * (u^\# \wedge v^\# \wedge \Phi) - * \left(u^\# \wedge v^\# \wedge w^\# \wedge (w \lrcorner \Phi)\right) \end{aligned}$$

Taking the wedge product on the left of the above expression with $\Phi \wedge u^\# \wedge v^\#$ and using (4.33) we have

$$\begin{aligned} & 7|\gamma_7|^2 \operatorname{vol} \\ = \; & -3\langle u,w\rangle\langle u \wedge v, w \wedge v\rangle \operatorname{vol} - 3\langle v,w\rangle\langle u \wedge v, u \wedge w\rangle \operatorname{vol} \\ & + 3|w|^2|u \wedge v|^2 \operatorname{vol} - (\Phi \wedge u^\# \wedge v^\#) \wedge * \left(u^\# \wedge v^\# \wedge w^\# \wedge (w \lrcorner \Phi)\right) \\ = \; & 3|u \wedge v \wedge w|^2 \operatorname{vol} - (\Phi \wedge u^\# \wedge v^\#) \wedge * \left(u^\# \wedge v^\# \wedge w^\# \wedge (w \lrcorner \Phi)\right) \end{aligned}$$

Now we apply Lemma 6.0.9 again on the last term:

$$\begin{aligned}
&v^{\#} \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (w \lrcorner \Phi)\big) = \\
&-\langle u, v\rangle * (v^{\#} \wedge w^{\#} \wedge (w \lrcorner \Phi)) + |v|^2 * (u^{\#} \wedge w^{\#} \wedge (w \lrcorner \Phi)) \\
&- \langle v, w\rangle * (u^{\#} \wedge v^{\#} \wedge (w \lrcorner \Phi)) + *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big)
\end{aligned}$$

Now taking wedge product on the left with $-(\Phi \wedge u^{\#})$ and using (4.34),

$$\begin{aligned}
&-(\Phi \wedge u^{\#} \wedge v^{\#}) \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (w \lrcorner \Phi)\big) = \\
&4\langle u, v\rangle\langle u \wedge w, v \wedge w\rangle \operatorname{vol} - 4|v|^2 |u \wedge w|^2 \operatorname{vol} \\
&+ 4\langle v, w\rangle\langle u \wedge v, u \wedge w\rangle \operatorname{vol} \\
&- (\Phi \wedge u^{\#}) \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big) \\
= \quad & -4|u \wedge v \wedge w|^2 \operatorname{vol} - (\Phi \wedge u^{\#}) \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big)
\end{aligned}$$

Combining our computations so far we have

$$7|\gamma_7|^2 \operatorname{vol} = -|u \wedge v \wedge w|^2 \operatorname{vol} - (\Phi \wedge u^{\#}) \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big)$$

Now using Lemma 6.0.9 one last time on the final term,

$$\begin{aligned}
&u^{\#} \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big) = \\
&|u|^2 * (v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)) - \langle u, v\rangle * (u^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)) \\
&+ \langle u, w\rangle * (u^{\#} \wedge v^{\#} \wedge (v \lrcorner w \lrcorner \Phi)) - *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (u \lrcorner v \lrcorner w \lrcorner \Phi)\big)
\end{aligned}$$

Taking the wedge product on the left with Φ and this time using (4.30),

$$\begin{aligned}
&-(\Phi \wedge u^{\#}) \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (v \lrcorner w \lrcorner \Phi)\big) = \\
&3|u|^2|v \wedge w|^2 \operatorname{vol} - 3\langle u, v\rangle\langle u \wedge w, v \wedge w\rangle \operatorname{vol} \\
&- 3\langle u, w\rangle\langle u \wedge v, w \wedge v\rangle \operatorname{vol} + \Phi \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (u \lrcorner v \lrcorner w \lrcorner \Phi)\big) \\
= \quad & 3|u \wedge v \wedge w|^2 \operatorname{vol} + \Phi \wedge *\big(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (u \lrcorner v \lrcorner w \lrcorner \Phi)\big)
\end{aligned}$$

which now give us

$$\begin{aligned}
7|\gamma_7|^2 \operatorname{vol} &= 2|u \wedge v \wedge w|^2 \operatorname{vol} + u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge (u \lrcorner v \lrcorner w \lrcorner \Phi) \wedge \Phi \\
&= 2|u \wedge v \wedge w|^2 \operatorname{vol} - \gamma_7 \wedge \Phi \wedge *(u^{\#} \wedge v^{\#} \wedge w^{\#} \wedge \Phi) \\
&= 2|u \wedge v \wedge w|^2 \operatorname{vol} - \gamma_7 \wedge \Phi \wedge *(\gamma_7 \wedge \Phi) \\
&= 2|u \wedge v \wedge w|^2 \operatorname{vol} - 7|\gamma_7|^2 \operatorname{vol}
\end{aligned}$$

from which we have $|u \wedge v \wedge w|^2 = 7|\gamma_7|^2 = |X(u, v, w)|^2$ from (4.39). $\square$

4.5. The 4 classes of $Spin(7)$-structures

Similar to the classification of G_2-structures by Fernández and Gray in [**12**], Fernández studied $Spin(7)$-structures in [**10**]. In this case, the results are slightly different because a 4-form Φ which determines a $Spin(7)$-structure is self-dual. Such a manifold has holonomy a subgroup of $Spin(7)$ if and only if $\nabla\Phi = 0$, which Fernández showed to be equivalent to

$$d\Phi = 0.$$

Again this equivalence was established by decomposing the space W that $\nabla\Phi$ belongs to into irreducible $Spin(7)$-representations, and comparing the invariant subspaces of W to the isomorphic spaces in $\wedge^*(M)$. In the $Spin(7)$ case this space W decomposes as

$$W = W_8 \oplus W_{48}$$

where again the subscript k denotes the dimension of the irreducible representation W_k. Again in analogy with the G_2 case, we have a canonically defined 7-form ζ and 1-form θ, given by

$$\zeta = *d\Phi \wedge \Phi \tag{4.41}$$

$$\theta = *\zeta = *\left(*d\Phi \wedge \Phi\right) \tag{4.42}$$

Note that $\theta = 0$ when the manifold has holonomy contained in $Spin(7)$, and more generally θ vanishes if $\pi_8(d\Phi) = 0$. We will see below that in this case the form θ is closed.

This time we have only 4 classes of $Spin(7)$-structures: the classes $\{0\}$, W_8, W_{48}, and $W = W_8 \oplus W_{48}$. Table 4.1 describes the classes in terms of differential equations on the form Φ. Unlike the G_2 case, the inclusions between these classes are all strict, and this is discussed in [**10**].

TABLE 4.1. The 4 classes of $Spin(7)$-structures

Class	Defining Equations	$d\theta$
$W_8 \oplus W_{48}$	no relation on $d\Phi$.	
W_8	$d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$	$d\theta = 0$
W_{48}	$\theta = 0$	$\theta = 0$
$\{0\}$	$d\Phi = 0$	$\theta = 0$

Remark 4.5.1. Note that in the $Spin(7)$ case, there is no analogue of an "integrable" structure, nor are there analogues of *almost* or *nearly* $Spin(7)$-structure as there are in the G_2 case. An almost $Spin(7)$ manifold ($d\Phi = 0$) automatically has holonomy $Spin(7)$. And $d\Phi$ does not have a one-dimensional component which would give us the analogue of a nearly G_2-structure.

We now prove the closedness of θ in the class W_8 as given in the final column of Table 4.1.

Lemma 4.5.2. *If* Φ *satisfies* $d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$, *then* $d\theta = 0$.

Proof. Suppose $d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$. We differentiate this equation to obtain:

$$d\theta \wedge \Phi = \theta \wedge d\Phi = \theta \wedge \left(-\frac{1}{7}\theta \wedge \Phi\right) = 0$$

But wedge product with Φ is an isomorphism from $\wedge^2$ to $\wedge^6$, so $d\theta = 0$. $\square$

Chapter 5

Deformations of a fixed $Spin(7)$*-structure*

We begin with a fixed $Spin(7)$-structure on a manifold M in a certain class. We will deform the form Φ and see how this affects the class. This time there are only 4 classes, and only two intermediate classes. However, the ways we can deform Φ in the $Spin(7)$ case are more complicated. Since $\Phi \in \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \oplus \wedge^4_{35}$, there are now four canonical ways to deform the 4-form Φ. Again, since $\wedge^4_1 = \{f\Phi\}$, adding to Φ an element of $\wedge^4_1$ amounts to conformally scaling Φ. This preserves the decomposition into irreducible representations. In all other case, however, since the decomposition does depend on Φ the decomposition will change for the other kinds of deformations. We will see that analogously to the G_2 case, adding something in $\wedge^4_7$ infinitesmally will give us a path in the space of positive 4-forms, all corresponding to the same metric.

5.1. Conformal Deformations of $Spin(7)$-structures

Let f be a smooth, nowhere vanishing function on M. We conformally scale Φ by f^4, for notational convenience. Denote the new form by $\tilde{\Phi} = f^4\Phi_{\mathrm{o}}$. We first compute the new metric $\tilde{g}$ and the new volume form $\mathrm{vol}_{\tilde{}}$ in the following lemma.

Lemma 5.1.1. *The metric g_o on vector fields, the metric g_o^{-1} on one forms, and the volume form vol_o transform as follows:*

$$\begin{aligned} \tilde{g} &= f^2 g_o \\ \tilde{g}^{-1} &= f^{-2} g_o^{-1} \\ \mathrm{vol}_{\tilde{}} &= f^8\,\mathrm{vol}_o \end{aligned}$$

Proof. We substitute $\tilde{\Phi} = f^4\Phi_{\mathrm{o}}$ into equations (4.24) and (4.25) to obtain

$$\begin{aligned} \tilde{A}(v) &= \big((v \lrcorner f^4\Phi_{\mathrm{o}}) \wedge f^4\Phi_{\mathrm{o}}\big)\,(e_1, e_2, \ldots, e_7) \\ &= f^8 A_{\mathrm{o}}(v) \\ \tilde{B}_{ij}(v) &= \big((e_i \lrcorner v \lrcorner f^4\Phi_{\mathrm{o}}) \wedge (e_j \lrcorner v \lrcorner f^4\Phi_{\mathrm{o}}) \wedge (v \lrcorner f^4\Phi_{\mathrm{o}})\big)\,(e_1, e_2, \ldots, e_7) \\ &= f^{12}(B_{\mathrm{o}})_{ij}(v) \end{aligned}$$

Substituting these expressions into (4.27) we compute

$$\begin{aligned} |v|_{\sim}^4 &= -\frac{(7)^3}{(6)^{\frac{7}{3}}} \frac{(f^{12})^{\frac{7}{3}} (\det{(B_{\mathrm{o}})_{ij}(v)})^{\frac{1}{3}}}{(f^8)^3 (A_{\mathrm{o}}(v))^3} \\ &= f^4 |v|_{\mathrm{o}}^4 \end{aligned}$$

from which we have $|v|_{\sim}^2 = f^2 |v|_{\mathrm{o}}^2$ and the remaining conclusions now follow. □

We now determine the new Hodge star $\tilde{*}$ in terms of the old $*_{\mathrm{o}}$.

Lemma 5.1.2. *If α is a k-form, then $\tilde{*}\alpha = f^{8-2k} *_o \alpha$.*

Proof. Let α, β be k-forms. Then from Lemma 5.1.1 the new metric on k-forms is $\langle\ ,\ \rangle_{\sim} = f^{-2k} \langle\ ,\ \rangle_{\mathrm{o}}$. From this we compute:

$$\begin{aligned} \beta \wedge \tilde{*}\alpha &= \langle \beta, \alpha \rangle_{\sim} \mathrm{vol}_{\sim} \\ &= f^{-2k} \langle \beta, \alpha \rangle_{\mathrm{o}} f^8 \,\mathrm{vol}_{\mathrm{o}} \\ &= f^{8-2k} \beta \wedge *_{\mathrm{o}}\alpha. \end{aligned}$$

□

From this we obtain the following:

Lemma 5.1.3. *The exterior derivative of the new 4-form $d\tilde{\Phi}$ and $\tilde{*}d\tilde{\Phi}$ are*

$$\begin{aligned} d\tilde{\Phi} &= 4f^3 df \wedge \Phi_o + f^4 d\Phi_o \\ \tilde{*}d\tilde{\Phi} &= 4f *_o (df \wedge \Phi_o) + f^2 *_o d\Phi_o \end{aligned}$$

Proof. This is immediate from $\tilde{\Phi} = f^4 \Phi_{\mathrm{o}}$ and Lemma 5.1.2. □

We can also see directly that under a conformal scaling, the norm of Φ is unchanged, even though the metric changes when Φ changes (in fact we know that $|\Phi|^2 = 14$ for any $Spin(7)$-structure from Lemma 4.2.2). To see this we compute:

$$\langle \tilde{\Phi}, \tilde{\Phi} \rangle_{\sim} = (f^{-2})^4 \left\langle f^4 \Phi_{\mathrm{o}}, f^4 \Phi_{\mathrm{o}} \right\rangle_{\mathrm{o}} = \langle \Phi_{\mathrm{o}}, \Phi_{\mathrm{o}} \rangle_{\mathrm{o}}.$$

From these results, we determine which classes of $Spin(7)$-structures are conformally invariant. We can also determine what happens to the 7-form ζ and the associated 1-form $\theta = *\zeta$. This is all given in the following theorem:

Theorem 5.1.4. *Under the conformal deformation $\tilde{\Phi} = f^4 \Phi_o$, we have:*

$$\begin{aligned} d\tilde{\Phi} + \frac{1}{7}\tilde{\theta} \wedge \tilde{\Phi} &= f^4 \left(d\Phi_o + \frac{1}{7}\theta_o \wedge \Phi_o \right) && (5.1) \\ \tilde{\zeta} &= -28 f^5 *_o df + f^6 \zeta_o && (5.2) \\ \tilde{\theta} &= -28 d(\log(f)) + \theta_o && (5.3) \end{aligned}$$

Hence, we see from Table 4.1 and equation (5.1) *that only the class W_8 is preserved under a conformal deformation of Φ. (This part was originally proved in* [**10**] *using a different method.) Also,* (5.3) *shows that θ changes by an exact form, so in the class W_8, where θ is closed, we have a well defined cohomology class $[\theta]$ which is unchanged under a conformal scaling.*

Proof. We begin by using Lemma 5.1.3 and (4.41) to compute $\tilde{\zeta}$ and $\tilde{\theta}$:

$$\begin{aligned} \tilde{\zeta} &= \tilde{*} d\tilde{\Phi} \wedge \tilde{\Phi} \\ &= \left(4f *_o (df \wedge \Phi_o) + f^2 *_o d\Phi_o \right) \wedge f^4 \Phi_o \\ &= 4 f^5 \Phi_o \wedge *_o (\Phi_o \wedge df) + f^6 \zeta_o \\ &= -28 f^5 *_o df + f^6 \zeta_o \end{aligned}$$

where we have used (4.2) in the last step. Now from Lemma 5.1.2, we get:

$$\tilde{\theta} = \tilde{*}\tilde{\zeta} = -28 f^{-1} df + \theta_o = -28 d(\log(f)) + \theta_o.$$

Now using the above expression for $\tilde{\theta}$, we have:

$$\begin{aligned} d\tilde{\Phi} + \frac{1}{7}\tilde{\theta} \wedge \tilde{\Phi} &= 4 f^3 df \wedge \Phi_o + f^4 d\Phi_o + \frac{1}{7}\left(-28 f^{-1} df + \theta_o \right) \wedge f^4 \Phi_o \\ &= f^4 \left(d\Phi_o + \frac{1}{7}\theta_o \wedge \Phi_o \right) \end{aligned}$$

which completes the proof. □

The next result gives necessary and sufficient conditions for being able to achieve holonomy $Spin(7)$ by conformally scaling.

Theorem 5.1.5. *Let Φ_o be a positive 4-form (associated to a $Spin(7)$-structure). Under the conformal deformation $\tilde{\Phi} = f^4 \Phi_o$, the new 4-form $\tilde{\Phi}$ satisfies $d\tilde{\Phi} = 0$ if and only if Φ_o is already at least class W_8 and $28 d \log(f) = \theta_o$. Hence in order to have $\tilde{\Phi}$ be closed (and hence correspond to holonomy $Spin(7)$), the original 1-form θ_o has to be* exact. *In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if Φ_o is in the class W_8, since $d\theta_o = 0$.*

Proof. From Lemma 5.1.3, for $d\tilde{\varphi} = 0$, we need

$$\begin{aligned} d\tilde{\Phi} &= 4f^3 df \wedge \Phi_{\mathrm{o}} + f^4 d\Phi_{\mathrm{o}} = 0 \\ d\Phi_{\mathrm{o}} &= -4d\log(f) \wedge \Phi_{\mathrm{o}} \end{aligned}$$

which says that $d\Phi_{\mathrm{o}} \in \wedge^5_8$ by Proposition 4.2.1. Hence $\pi_{48}(d\Phi_{\mathrm{o}}) = 0$ so Φ_{o} must be already of class W_8. Then to make $d\tilde{\Phi} = 0$, we need to eliminate the W_8 component, which requires $28d\log(f) = \theta_{\mathrm{o}}$ by Theorem 5.1.4. □

Remark 5.1.6. Note that if we start with a $Spin(7)$-structure Φ_{o} that is already holonomy $Spin(7)$, then Theorem 5.1.4 shows that a conformal scaling by a non-constant f will always generate a non-zero W_8 component.

5.2. Deforming Φ by an element of $\wedge^4_7$

We continue our analogy with the G_2 case and now deform the $Spin(7)$ 4-form Φ by an element of $\wedge^4_7$. According to (4.17), an element $\sigma_7 \in \wedge^4_7$ is determined by *two* vector fields v and w. In fact, we have $\sigma_7 = v^{\#} \wedge (w \lrcorner \Phi_{\mathrm{o}}) - w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}})$. Now let $\tilde{\Phi} = \Phi_{\mathrm{o}} + t\left(v^{\#} \wedge (w \lrcorner \Phi_{\mathrm{o}}) - w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}})\right)$, for $t \in \mathbb{R}$. We will develop formulas for the new metric $\tilde{g}$, the new Hodge star $\tilde{*}$, and other expressions entirely in terms of the old Φ_{o}, the old $*_{\mathrm{o}}$, and the vector fields v and w. Note that once again the background decomposition into irreducible $Spin(7)$-representations changes, and we will eventually linearize by taking $\frac{d}{dt}\big|_{t=0}$ of our results.

Using the notation of Theorem 4.3.5, we first prove the following.

Proposition 5.2.1. *Let $\sigma_7 = \left(v^{\#} \wedge (w \lrcorner \Phi_o) - w^{\#} \wedge (v \lrcorner \Phi_o)\right)$. Under the transformation $\tilde{\Phi} = \Phi_o + \sigma_7$, we have*

$$\tilde{\Phi}^2 = \left(1 + \frac{4}{7}|v \wedge w|^2_o\right)\Phi^2_o \tag{5.4}$$

Proof. We compute

$$\begin{aligned} \tilde{\Phi}^2 &= \left(\Phi_{\mathrm{o}} + v^{\#} \wedge (w \lrcorner \Phi_{\mathrm{o}}) - w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}})\right)^2 \\ &= \Phi^2_{\mathrm{o}} + 2v^{\#} \wedge (w \lrcorner \Phi_{\mathrm{o}}) \wedge \Phi_{\mathrm{o}} - 2w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}}) \wedge \Phi_{\mathrm{o}} \\ &\quad - 2v^{\#} \wedge (w \lrcorner \Phi_{\mathrm{o}}) \wedge w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}}) \\ &= \Phi^2_{\mathrm{o}} + 14v^{\#} \wedge *_{\mathrm{o}} w^{\#} - 14w^{\#} \wedge *_{\mathrm{o}} v^{\#} - 2v^{\#} \wedge w^{\#} \wedge (v \lrcorner \Phi_{\mathrm{o}}) \wedge (w \lrcorner \Phi_{\mathrm{o}}) \\ &= \Phi^2_{\mathrm{o}} + 8|v \wedge w|^2_{\mathrm{o}} \operatorname{vol}_{\mathrm{o}} \end{aligned}$$

where we have used both (4.6) and (4.31). Now since $\varphi^2_{\mathrm{o}} = 14 \operatorname{vol}_{\mathrm{o}}$ we have

$$\tilde{\Phi}^2 = 14 \operatorname{vol}_{\mathrm{o}} + 14\left(\frac{4}{7}\right)|v \wedge w|^2 \operatorname{vol}_{\mathrm{o}} = \left(1 + \frac{4}{7}|v \wedge w|^2_{\mathrm{o}}\right)\Phi^2_{\mathrm{o}}$$

which completes the proof. □

Corollary 5.2.2. *Under the transformation* $\tilde{\Phi} = \Phi_o + \sigma_7$*, the expression* $A_o(u) = ((u\lrcorner\Phi_o) \wedge \Phi_o)(e_1, e_2, \ldots, e_7)$ *changes by*

$$\begin{aligned} \tilde{A}(u) &= \left((u\lrcorner\tilde{\Phi}) \wedge \tilde{\Phi}\right)(e_1, e_2, \ldots, e_7) \\ &= \left(1 + \frac{4}{7}|v \wedge w|_o^2\right) A_o(u) \end{aligned} \tag{5.5}$$

Proof. This follows from Proposition 5.2.1 by taking the interior product of both sides with u. □

We continue the computation of the expressions needed to describe the new metric with the following lemma.

Lemma 5.2.3. *With* $\tilde{\Phi} = \Phi_o + t\sigma$*, in the expression*

$$\left(e_i\lrcorner u\lrcorner\tilde{\Phi}\right) \wedge \left(e_i\lrcorner u\lrcorner\tilde{\Phi}\right) \wedge \tilde{\Phi}$$

which is a cubic polynomial in t*, the linear term vanishes.*

Proof. The coefficient of t is

$$2(e_i\lrcorner u\lrcorner\sigma) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o} + (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \sigma \tag{5.6}$$

We start with (4.32) which says that

$$-6|e_i \wedge u|^2 \operatorname{vol}_\mathrm{o} = (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o}$$

Taking the interior product with w and then wedging with $v^\#$, we obtain:

$$\begin{aligned} -6|e_i \wedge u|^2 *_\mathrm{o} w^\# &= 2(w\lrcorner e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o} \\ &\quad + (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (w\lrcorner\Phi_\mathrm{o}) \\ -6|e_i \wedge u|^2 \langle v, w\rangle \operatorname{vol}_\mathrm{o} &= 2v^\# \wedge (w\lrcorner e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o} \\ &\quad + (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge v^\# \wedge (w\lrcorner\Phi_\mathrm{o}) \end{aligned} \tag{5.7}$$

Now we can also compute that

$$\begin{aligned} e_i\lrcorner u\lrcorner(v^\# \wedge (w\lrcorner\Phi_\mathrm{o})) &= e_i\lrcorner\left(\langle u, v\rangle(w\lrcorner\Phi_\mathrm{o}) - v^\# \wedge (u\lrcorner w\lrcorner\Phi_\mathrm{o})\right) \\ &\quad = \langle u, v\rangle(e_i\lrcorner w\lrcorner\Phi_\mathrm{o}) - \langle e_i, v\rangle(u\lrcorner w\lrcorner\Phi_\mathrm{o}) \\ &\quad - v^\# \wedge (e_i\lrcorner u\lrcorner w\lrcorner\Phi_\mathrm{o}) \end{aligned}$$

Taking the wedge product of this expression with $2(e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o}$ and rearranging,

$$\begin{aligned} &2v^\# \wedge (w\lrcorner e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o} \\ = \;&2e_i\lrcorner u\lrcorner\left(v^\# \wedge (w\lrcorner\Phi_\mathrm{o})\right) \wedge (e_i\lrcorner u\lrcorner\Phi_\mathrm{o}) \wedge \Phi_\mathrm{o} \\ &+ 12\langle u, v\rangle\langle e_i \wedge w, e_i \wedge u\rangle \operatorname{vol}_\mathrm{o} \\ &+ 12\langle e_i, v\rangle\langle e_i \wedge u, w \wedge u\rangle \operatorname{vol}_\mathrm{o} \end{aligned}$$

where we have used (4.32) again. Now substituting the above expression into (5.7),

$$\begin{aligned}
& 2e_i \lrcorner u \lrcorner \left(v^{\#} \wedge (w \lrcorner \Phi_o)\right) \wedge (e_i \lrcorner u \lrcorner \Phi_o) \wedge \Phi_o \\
& + (e_i \lrcorner u \lrcorner \Phi_o) \wedge (e_i \lrcorner u \lrcorner \Phi_o) \wedge v^{\#} \wedge (w \lrcorner \Phi_o) \\
= \quad & -6 \left(|e_i \wedge u|^2 \langle v, w\rangle + 2\langle u, v\rangle \langle e_i \wedge w, e_i \wedge u\rangle \right. \\
& \left. + 2\langle e_i, v\rangle \langle e_i \wedge u, w \wedge u\rangle\right) \mathrm{vol}_o \\
= \quad & -6 \left(|e_i \wedge u|^2 \langle v, w\rangle + 2|e_i|^2 \langle u, v\rangle \langle u, w\rangle + 2|u|^2 \langle e_i, v\rangle \langle e_i, w\rangle\right) \mathrm{vol}_o \\
& + 12 \left(\langle u, v\rangle \langle e_i, u\rangle \langle e_i, w\rangle + \langle e_i, v\rangle \langle e_i, u\rangle \langle u, w\rangle\right) \mathrm{vol}_o
\end{aligned}$$

The right hand side of the above expression is visible symmetric in v and w, so if we interchange the v's and w's on the left hand side and take the difference, we get zero. But since $\sigma = v^{\#} \wedge (w \lrcorner \Phi_o) - w^{\#} \wedge (v \lrcorner \Phi_o)$, this is exactly the linear term (5.6). Hence we have shown the vanishing of the linear term. □

In analogy with the G_2 case, we expect the cubic term to vanish as well, and the quadratic term should be related to the triple cross product. The complexity of these terms is much greater this time because we have two vector fields instead of one. However, the above results are enough to proceed with this deformation infinitesmally.

5.3. Infinitesmal deformations in the $\wedge^4_7$ direction

Again continuing the analogy with the G_2 case, we now consider *infinitesmal deformations* in the $\wedge^4_7$ direction. Consider a one-parameter family Φ_t of $Spin(7)$-structures, satisfying

$$\frac{\partial}{\partial t}\Phi_t = w \lrcorner *_t (v \lrcorner \Phi_t) - v \lrcorner *_t (w \lrcorner \Phi_t) \tag{5.8}$$

for a pair of vector fields v and w. That is, at each time t, we move in the direction of a 4-form in $\wedge^4_{7_t}$, since the decomposition of $\wedge^4$ depends on Φ_t and hence is changing in time. Since the Hodge star $*_t$ is also changing in time, this is again *a priori* a nonlinear equation. However, just like in the G_2 case, it is not:

Proposition 5.3.1. *Under the flow described by equation* (5.8)*, the metric* g does not change*. Hence the volume form and Hodge star are also constant.*

Proof. Since we have shown in Lemma 5.2.3 and Corollary 5.2.2 that the first order term in the new metric vanishes, the proposition follows. □

Therefore we can replace $*_t$ by $*_0 = *$ and equation (5.8) is actually *linear*. Moreover, the flow determined by this linear equation gives a one-parameter family of G_2-structures each yielding the *same* metric g. Our equation is now

$$\frac{\partial}{\partial t}\Phi_t = w\lrcorner * (v\lrcorner\Phi_t) - v\lrcorner * (w\lrcorner\Phi_t) = B\Phi_t$$

where B is the linear operator $\alpha \mapsto B\alpha = w\lrcorner * (v\lrcorner\alpha) - v * (w\lrcorner\alpha)$ on $\wedge^4$.

Proposition 5.3.2. *The operator B is* skew-symmetric*. Furthermore, the eigenvalues λ of B are $\lambda = 0, \pm i|v \wedge w|$.*

Proof. Let $e^1, e^2, \dots e^{70}$ be a basis of $\wedge^4$. Then

$$\begin{aligned} B_{ij}\,\mathrm{vol} &= \left\langle e^i, Be^j \right\rangle \mathrm{vol} \\ &= e^i \wedge * \left(v\lrcorner(w^\# \wedge e^j) - w\lrcorner(v^\# \wedge e^j)\right) \\ &= v^\# \wedge e^i \wedge *(w^\# \wedge e^j) - w^\# \wedge e^i * (v^\# \wedge e^j) \\ &= w^\# \wedge e^j * (v^\# \wedge e^i) - w^\# \wedge e^i * (v^\# \wedge e^j) \\ &= -B_{ji}\,\mathrm{vol} \end{aligned}$$

and hence B is diagonalizable over $\mathbb{C}$. In order to find the eigenvalues of B, we first compute some powers of B, using the fact that $B\alpha$ can also be written as $B\alpha = v^\# \wedge (w\lrcorner\alpha) - w^\# \wedge (v\lrcorner\alpha)$:

$$\begin{aligned} B^2\alpha &= v\lrcorner(w^\# \wedge B\alpha) - w\lrcorner(v^\# \wedge B\alpha) \\ &= v\lrcorner\left(w^\# \wedge v^\# \wedge (w\lrcorner\alpha)\right) + w\lrcorner\left(v^\# \wedge w^\# \wedge (v\lrcorner\alpha)\right) \\ &= \langle v, w\rangle \left(v^\# \wedge (w\lrcorner\alpha) + w^\# \wedge (v\lrcorner\alpha)\right) - |v|^2 w^\# \wedge (w\lrcorner\alpha) \\ &\quad - |w|^2 v^\# \wedge (v\lrcorner\alpha) - 2v^\# \wedge w^\# \wedge (v\lrcorner w\lrcorner\alpha) \end{aligned}$$

and thus

$$\begin{aligned} B^3\alpha &= v\lrcorner(w^\# \wedge B^2\alpha) - w\lrcorner(v^\# \wedge B^2\alpha) \\ &= v\lrcorner\left(\langle v, w\rangle w^\# \wedge v^\# \wedge (w\lrcorner\alpha) - |w|^2 w^\# \wedge v^\# \wedge (v\lrcorner\alpha)\right) \\ &\quad - w\lrcorner\left(\langle v, w\rangle v^\# \wedge w^\# \wedge (v\lrcorner\alpha) - |v|^2 v^\# \wedge w^\# \wedge (w\lrcorner\alpha)\right) \\ &= \langle v, w\rangle^2 B\alpha - |v|^2|w|^2 B\alpha + (|v|^2 - |w|^2) v^\# \wedge w^\# \wedge (v\lrcorner w\lrcorner\alpha) \end{aligned}$$

after some simplification. Performing one more iteration, we obtain:

$$B^4\alpha = \left(\langle v, w\rangle^2 - |v|^2|w|^2\right) B^2\alpha$$

which gives $\lambda^4 = \left(\langle v, w\rangle^2 - |v|^2|w|^2\right)\lambda^2$. Therefore the non-zero eigvenvalues are $\lambda = \pm i|v \wedge w|$. This completes the proof. $\square$

Now we proceed exactly as in the G_2 case. If we replace A by B and the non-zero eigenvalues by $\pm i|v \wedge w|$, then all the remaining calculations of Section 3.3 carry through. Therefore we have

$$\begin{aligned}\Phi_t &= -\frac{1}{|v\wedge w|^2}\cos(|v\wedge w|t)B^2\Phi_0 + \frac{1}{|v\wedge w|}\sin(|v\wedge w|t)B\Phi_0 \\ &\quad + \Phi_0 + \frac{1}{|v\wedge w|^2}B^2\Phi_0\end{aligned}$$

which we summarize as the following theorem.

Theorem 5.3.3. *The solution to the differential equation*

$$\frac{\partial}{\partial t}\Phi_t = w\lrcorner * (v\lrcorner\Phi_t) - v\lrcorner * (w\lrcorner\Phi_t)$$

is given by

$$\Phi(t) = \Phi_0 + \frac{1-\cos(|v\wedge w|t)}{|v\wedge w|^2}B^2\Phi_0 + \frac{\sin(|v\wedge w|t)}{|v\wedge w|}B\Phi_0 \tag{5.9}$$

where $B\alpha = v\lrcorner(w^\# \wedge \alpha) - w\lrcorner(v^\# \wedge \alpha)$. The solution exists for all time and is closed curve (an ellipse) in $\wedge^4$.

Proof. This follows from the above discussion. □

Remark 5.3.4. In [**5**], it is shown that the set of $Spin(7)$-structures on M which correspond to the same metric as that of a fixed $Spin(7)$-structure Φ_o is a rank 7 bundle over the manifold M. The above theorem gives an explicit formula (5.9) for a path of $Spin(7)$-structures all corresponding to the same metric g starting from a pair of vector fields v and w on M.

Remark 5.3.5. Again, even though the metric is unchanged under an infinitesmal deformation in the $\wedge^4_7$ direction, the class of $Spin(7)$-structure *can* change.

We now apply this theorem to three specific examples, where we will again reproduce known results.

Example 5.3.6. Let N be a Calabi-Yau fourfold, with Kähler form ω and holomorphic $(4,0)$ form Ω. The complex coordinates will be denoted by $z^j = x^j + iy^j$. Then N has a natural $Spin(7)$-structure Φ on it given by

$$\Phi = \mathrm{Re}(\Omega) - \frac{\omega^2}{2} \tag{5.10}$$

It is easy to check in local coordinates that $\omega \in \wedge^2_7$ in the $Spin(7)$ decomposition. If we take two local orthonormal vector fields v and w for which

$\pi_7(v^\# \wedge w^\#) = \omega$, then one can compute that

$$\begin{aligned} w\lrcorner *(v\lrcorner\Phi) - v\lrcorner *(w\lrcorner\Phi) = \quad B\Phi &= -\operatorname{Im}(\Omega) \\ B^2\Phi &= -\operatorname{Re}(\Omega) \end{aligned}$$

Thus for vector fields v and w which correspond to ω, the flow in (5.9) is given by

$$\begin{aligned} \Phi_t &= \operatorname{Re}(\Omega) - \frac{\omega^2}{2} - (1-\cos(t))\operatorname{Re}(\Omega) - \sin(t)\operatorname{Im}(\Omega) \\ &= \cos(t)\operatorname{Re}(\Omega) - \sin(t)\operatorname{Im}(\Omega) - \frac{\omega^2}{2} \\ &= \operatorname{Re}(e^{it}\Omega) - \frac{\omega^2}{2} \end{aligned}$$

which is the canonical $Spin(7)$ form on N where now the Calabi-Yau structure is given by $e^{it}\Omega$ and ω. Thus we arrive at the phase freedom for Calabi-Yau fourfolds.

Example 5.3.7. A subset of the Calabi-Yau fourfolds are the hyperKähler manifolds of complex dimension four. Suppose N is hyperKähler with hyperKähler triple ω_1, ω_2, and ω_3. Then $\Omega = \frac{(\omega_2+i\omega_3)^2}{2}$ is a holomorphic $(4,0)$-form in the complex structure corresponding to ω_1. Therefore we have

$$\begin{aligned} \operatorname{Re}(\Omega) &= \frac{\omega_2^2-\omega_3^2}{2} \\ \operatorname{Im}(\Omega) &= \omega_2\omega_3 \end{aligned}$$

where we are ommitting the $\wedge$ symbols since everyting here commutes. By comparing with the calculation of Example 5.3.6, taking the element of $\wedge^4_7$ corresponding to ω_1, we have

$$\begin{aligned} \Phi_t &= \frac{\omega_2^2-\omega_3^2}{2}\cos(t) - \omega_2\omega_3\sin(t) - \frac{\omega_1^2}{2} \\ &= \frac{\tilde{\omega}_2^2-\tilde{\omega}_3^2}{2} - \frac{\omega_1^2}{2} \end{aligned}$$

where $\tilde{\omega}_2 = \cos(\frac{t}{2})\omega_2 - \sin(\frac{t}{2})\omega_3$, and $\tilde{\omega}_3 = \sin(\frac{t}{2})\omega_2 + \cos(\frac{t}{2})\omega_3$. This is again a hyperKähler rotation. Of course, the $Spin(7)$-structure is not actually used in this example, only the relation between the Calabi-Yau and the hyperKähler structures.

Example 5.3.8. Finally, consider a 7-manifold M with a G_2-structure φ. We can put a $Spin(7)$-structure Φ on the product $M \times S^1$ given by

$$\Phi = d\theta \wedge \varphi + *_7\varphi$$

where $*_7\varphi$ is the 4-form dual to φ on M. This induces the product metric on $M \times S^1$, with the flat metric on S^1. Now let $v = \frac{\partial}{\partial\theta}$ be a globally defined non-vanishing vector field on S^1 with $|v| = 1$. Choose another vector field w on M. Then one computes

$$\begin{aligned} w \lrcorner * (v \lrcorner \Phi) - v \lrcorner * (w \lrcorner \Phi) = \quad B\Phi &= d\theta \wedge (w \lrcorner *_7 \varphi) + *_7(w \lrcorner *_7 \varphi) \\ B^2\Phi &= d\theta \wedge (w \lrcorner *_7 (w \lrcorner *_7 \varphi)) \\ &\quad + *_7 (w \lrcorner *_7 (w \lrcorner *_7 \varphi)) \end{aligned}$$

Then the flow in (5.9) gives

$$\Phi_t = d\theta \wedge \varphi_t + *_7\varphi_t$$

where φ_t is the flow given by (3.22) for the vector field w. Thus in the product case $M \times S^1$ we recover the results of Section 3.3.

Chapter 6

Appendix: Linear Algebra Identities

Here we collect together various identities involving the exterior and interior products and the Hodge star operator. Also some useful identities involving determinants are also proved. We establish the results in the general case of a Riemannian manifold M of dimension n, although only the cases $n = 7, 8$ are used in the text. Let $\langle\ ,\ \rangle$ denote the metric on M, as well as the induced metric on forms. In all that follows, α and γ are k-forms, β is a $(k-1)$-form, w is a vector field, and $w^\#$ is the 1-form dual to w in the given metric. That is,

$$|w|^2 = \langle w, w\rangle = w^\#(w) = \left\langle w^\#, w^\# \right\rangle$$

Now $*$ takes k-forms to $(n-k)$-forms, and is defined by

$$\langle \alpha, \gamma\rangle \operatorname{vol} = \alpha \wedge *\gamma = \gamma \wedge *\alpha$$

We also have

$$*^2 = (-1)^{k(n-k)} \tag{6.1}$$

on k-forms.

Lemma 6.0.9. *We have the following four identities:*

$$\begin{aligned} *(w \lrcorner \alpha) &= (-1)^{k+1}(w^\# \wedge *\alpha) && (6.2)\\ (w \lrcorner \alpha) &= (-1)^{nk+n} * (w^\# \wedge *\alpha) && (6.3)\\ *(w \lrcorner * \alpha) &= (-1)^{nk+n+1}(w^\# \wedge \alpha) && (6.4)\\ (w \lrcorner * \alpha) &= (-1)^{k} * (w^\# \wedge \alpha) && (6.5) \end{aligned}$$

Proof. We compute:

$$\begin{aligned} \langle \beta, w \lrcorner \alpha\rangle \operatorname{vol} &= \beta \wedge *(w \lrcorner \alpha)\\ &= (w \lrcorner \alpha)(\beta^\#) \operatorname{vol}\\ &= \alpha(w \wedge \beta^\#) \operatorname{vol}\\ &= \left\langle \alpha, w^\# \wedge \beta \right\rangle \operatorname{vol}\\ &= (w^\# \wedge \beta) \wedge *\alpha\\ &= (-1)^{k-1}\beta \wedge (w^\# \wedge \alpha) \end{aligned}$$

Since β is arbitrary, (6.2) follows. Substituting $*\alpha$ for α, and using (6.1), we obtain (6.4). The other two are obtained by taking $*$ of both sides of the first two identities. $\square$

Note that from the above proof we have also the useful relation

$$(X \lrcorner \alpha) \wedge *\beta = \alpha \wedge * \left(X^{\#} \wedge \beta\right) \tag{6.6}$$

for any k-form α, $(k-1)$-form β, and vector field X.

The next lemma gives further relations between a k-form α and a vector field w.

Lemma 6.0.10. *With the same notation as above, we have the following three identities:*

$$\begin{aligned} |w|^2\alpha &= w^{\#} \wedge (w \lrcorner \alpha) + w \lrcorner (w^{\#} \wedge \alpha) && (6.7) \\ |w|^2\alpha &= (-1)^{nk+1} * (w \lrcorner (*(w \lrcorner \alpha))) && (6.8) \\ &\quad + (-1)^{nk+n+1} (w \lrcorner * (w \lrcorner * \alpha)) \\ |w|^2|\alpha|^2 &= |w \lrcorner \alpha|^2 + |w \lrcorner * \alpha|^2 && (6.9) \end{aligned}$$

Proof. Using the fact that the interior product is an anti-derivation, we have

$$w \lrcorner (w^{\#} \wedge \alpha) = (w \lrcorner w^{\#}) \wedge \alpha - w^{\#} \wedge (w \lrcorner \alpha)$$

which is equation (6.7) since $w \lrcorner w^{\#} = |w|^2$. Now (6.8) follows from this one using the identities of Lemma 6.0.9. The last identity can also be obtained this way, but it is faster to proceed as follows:

$$\begin{aligned} 0 &= w^{\#} \wedge \alpha \wedge *\alpha \\ 0 &= w \lrcorner (w^{\#} \wedge \alpha \wedge *\alpha) \\ &= |w|^2 \alpha \wedge *\alpha - w^{\#} \wedge (w \lrcorner \alpha) \wedge *\alpha + (-1)^{k+1} w^{\#} \wedge \alpha \wedge (w \lrcorner * \alpha) \\ &= |w|^2 |\alpha|^2 \operatorname{vol} - (w \lrcorner \alpha) \wedge *(w \lrcorner \alpha) - (w \lrcorner * \alpha) \wedge *(w \lrcorner * \alpha) \end{aligned}$$

using Lemma 6.0.9, which can be rearranged to give (6.9). $\square$

The next lemma about determinants is used many times in the computation of the metrics and volume forms arising from G_2 and $Spin(7)$-structures.

Lemma 6.0.11. *Let (a_{ij}) be an $n \times n$ matrix, (w_i) a $n \times 1$ vector, and C a scalar. Consider the matrix*

$$b_{ij} = CA_{ij} \pm w_i w_j$$

Its determinant is given by

$$\det(b_{ij}) = C^n \det(a_{ij}) \pm \sum_{k,l=1}^{n} (-1)^{k+l} w_k w_l C^{n-1} A_{kl} \tag{6.10}$$

where A_{kl} is the $(k,l)^{th}$ minor *of the matrix a_{ij}. That is, it is the determinant of (a_{ij}) with the k^{th} row and l^{th} column removed*

Proof. The determinant of (b_{ij}) is

$$\det \begin{pmatrix} Ca_{11} \pm w_1 w_1 & \dots & Ca_{1n} \pm w_1 w_n \\ \vdots & \ddots & \vdots \\ Ca_{n1} \pm w_n w_1 & \dots & Ca_{nn} \pm w_n w_n \end{pmatrix}$$

Since the determinant is a linear function of the *columns* of a matrix, we can write the above determinant as a sum of determinants where each column is of one of these two forms:

$$\begin{pmatrix} Ca_{1k} \\ Ca_{2k} \\ \vdots \\ Ca_{nk} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \pm w_1 w_k \\ \pm w_2 w_k \\ \vdots \\ \pm w_7 w_k \end{pmatrix}$$

Each of the determinants which has at least two columns of $\pm w_l w_k$'s will have at least two proportional columns and hence will vanish. So the only non-zero contributions come from the determinant with all Ca_{kl}'s and the n determinants with only *one* columns of $\pm w_l w_k$'s. Therefore we are left with:

$$\det (Ca_{ij}) + \sum_{k,l=1}^{n} (-1)^{k+l} (\pm w_k w_l) C^{n-1}(A_{kl}) \tag{6.11}$$

In equation (6.11) the first term is the determinant with all columns of Ca_{kl}'s, the sum over l is the sum over the n different determinants with a single column of $\pm w_k w_l$'s, in the l^{th} column, and in the sum over k we expand each of those determinants along the l^{th} column. This completes the proof. □

The following lemma is used in the derivation of the metric from the 4-form Φ in the $Spin(7)$ case in Theorem 4.3.5.

Lemma 6.0.12. *Let (g_{ij}) be an $n \times n$ symmetric matrix, with $0 \le i, j \le n-1$ and let its determinant be denoted* $\det_n(g)$. *Let $det_{n-1}(g)$ be the determinant of the $(n-1) \times (n-1)$ submatrix of (g_{ij}) where $1 \le i, j \le n-1$, and G_{ij} be the $(n-2) \times (n-2)$ minors of this submatrix. Then we have*

$$\det_n(g) = g_{00}\det_{n-1}(g) + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}(-1)^{r+s+1} g_{0r} g_{0s} G_{rs} \tag{6.12}$$

Proof. We begin by expanding $\det_n(g)$ along the first row:

$$\det_n(g) = g_{00}\det_{n-1}(g) + \sum_{r=1}^{n-1}(-1)^{1+(r+1)}g_{0r}\tilde{G}_{0r}$$

where $\tilde{G}_{ij}$ is the determinant of the $(n-1) \times (n-1)$ submatrix of (g_{ij}) obtained by deleting the 0^{th} row and the r^{th} column. We expand each $\tilde{G}_{0r}$ along the first column:

$$\tilde{G}_{0r} = \sum_{s=1}^{n-1}(-1)^{1+s}g_{s0}G_{sr}$$

since when we take the determinant of $\tilde{G}_{0r}$ with the 0^{th} column and s^{th} row removed, this is precisely G_{sr}. Combining these two expressions and using the symmetry of (g_{ij}) yields equation (6.12). $\square$

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