QUASI-EXACTLY SOLVABLE SYSTEMS IN QUANTUM MECHANICS

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1. Introduction

In non-relativistic quantum mechanics, the state of a system is characterized by a wave function Ψ which is the solution of the second order linear partial differential *Schrödinger equation*:

$$H_{\text{op}}\Psi = E\Psi$$

where H_{op} is a Hermitian (self-adjoint) second order linear differential operator known as a Schrödinger operator and E is the eigenvalue of the partial differential equation, identified with the total energy of the system. In general, the form of the operator H_{op} will include a term that depends on the interaction potential being considered, and then the solutions to the equation that correspond to bound states of the system are associated with a discrete set of energy eigenvalues. For this reason we say that the system is *quantized*. The wave function Ψ can be interpreted as the probability amplitude of the particle's position. That is, $|\Psi|^2$ is the probability density. For this to be possible, Ψ must be square-integrable, and all such square integrable wave functions are called *normalizable*. Therefore we usually restrict attention to normalizable solutions of the Schrödinger equation. In general the wave function Ψ can depend on a number of variables, both those describing position in physical space, and others describing internal degrees of freedom such as particle *spin* and even more exotic properties such as quark flavour or colour. In general this forces the wave function Ψ to be a column vector, or *spinor*. These internal degrees of freedom will be ignored in the subsequent discussion, and we will instead simply consider potentials that depend only on position coordinates of a collection of particles. In this case, Ψ is simply a complex-valued scalar function of its arguments. For a system consisting of only one particle in three dimensions interacting with a potential $V(x, y, z)$, the form of H_{op} is:

$$H_{\text{op}} = -\nabla^2 + V(x, y, z) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z)$$

(in normalized units where $\frac{\hbar^2}{2m} = 1$, where $\hbar$ is Planck's constant and m is the mass of the particle.) In general the Schrödinger equation cannot be solved exactly to obtain the entire spectrum of eigenvalues except for a small number of potentials, notably the harmonic oscillator and the hydrogen atom (Coulomb) potentials, which are nevertheless of extreme theoretical importance. For most potentials, the equation must be solved numerically to obtain the spectrum. However, recent research [15, 10] has discovered an intermediate class of potentials, in which it is possible to obtain a finite part of the eigenvalue spectrum algebraically, that is as

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solutions of a polynomial equation, while the rest must still be obtained numerically. For this reason, they are referred to as *quasi-exactly solvable* potentials. These potentials are characterized by the fact that for them, the operator H_{op} can be expressed as a second (or higher) degree polynomial in the generators of a finite-dimensional Lie algebra that possesses a finite-dimensional representation space. This will be made more precise later. In this paper we will give a brief overview of the idea of quasi-exact solvabilty and summarize the main results that are known. In addition, we will apply the results to some specific examples.

2. Finite-dimensional Lie Algebras

We will need some elementary definitions.

Definition 2.1. *Lie algebra* $\mathfrak{g}$ is a vector space over a field, here taken as the real numbers $\mathbb{R}$, together with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *bracket*, denoted $[v, w]$ which satisfies the following properties:

i) Bilinearity: $\forall\ u, v, w \in \mathfrak{g}\ \ \forall\ t, s \in \mathbb{R}$,
$[u, tv + sw] = t[u, v] + s[u, w]$
$[tu + sv, w] = t[u, w] + s[v, w]$

ii) $[v, v] = 0\ \ \forall\ v \in \mathfrak{g}$

iii) The bracket satisfies the *Jacobi identity* $\forall\ u, v, w \in \mathfrak{g}$,
$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

Note that by properties i) and ii),

$$0 = [v + w, v + w] = [v, v] + [v, w] + [w, v] + [w, w] = 0 + [v, w] + [w, v] + 0$$

which shows that the bracket is *skew-symmetric*, $[v, w] = -[w, v]$.

Definition 2.2. *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ onto the space of linear operators on a vector space W, which is called the representation space. That is, if we denote the vector space of linear maps on W by $L(W)$, then a representation is a map ρ such that:

i) $\rho(v) \in L(W)\ \ \forall\ v \in \mathfrak{g}$

ii) $\rho\left([v, w]\right) = [\rho(v), \rho(w)] \equiv \rho(v) \circ \rho(w) - \rho(w) \circ \rho(v)$

iii) $\rho(sv + tw) = s\rho(v) + t\rho(w)\ \ \forall\ v, w \in \mathfrak{g}\ \ \forall\ s, t \in \mathbb{R}$

where we have defined the bracket in $L(W)$ by $[A, B] = A \circ B - B \circ A$, the *commutator* of A and B. A simple calculation shows that the commutator satisfies the Jacobi identity, and thus a representation preserves Lie algebraic structure.

For our purposes, the vector space W will be taken to be a subspace of the space of smooth (infinitely differentiable) functions. Then, since differentiation is a linear operation, a possible representation of a Lie algebra consists of first order linear differential operators. This is the representation that we will use because we want to express the Schrödinger operator H_{op}, which is a linear differential operator, as a polynomial in first order differential operators.

We will now illustrate these ideas with a simple example, which is in fact the most important case that we will consider. The Lie algebra $\mathfrak{sl}(2)$ can be defined, in a representation of 2×2 real matrices, as those matrices with trace zero. Thus we impose one contraint on a 4-dimensional vector space, and hence $\mathfrak{sl}(2)$ will be a 3-dimensional Lie algebra. A basis is given by:

$$J^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad J^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad J^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Simple computations yield the following commutation relations:

$$[J^+, J^-] = -2J^3 \qquad [J^+, J^3] = -J^+ \qquad [J^-, J^3] = J^-$$

Thus we do indeed have a Lie algebra since the basis elements are closed under the commutator. We have $\mathbb{R}^2$ as a representation space. Now we can get a representation of this Lie algebra in terms of differential operators on a subspace of the space of smooth functions [15] if we make the identification $x \iff \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $y \iff \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where x and y are real independent variables. With this identification,

$$\begin{aligned} J^+x &= -y & J^+y &= 0 \\ J^-x &= 0 & J^-y &= x \\ J^3x &= -\frac{1}{2}x & J^3y &= \frac{1}{2}y \end{aligned}$$

From the actions of the basis elements on x and y we can see by inspection that a representation in terms of differential operators is given by:

$$J^+ = -y\frac{d}{dx} \qquad J^- = x\frac{d}{dy} \qquad J^3 = \frac{1}{2}\left(y\frac{d}{dy} - x\frac{d}{dx}\right)$$

However, having this representation in terms of differential operators, we can extend it to be defined on an n-dimensional subspace of the space of smooth functions of x and y given by the linearly independent basis elements $x^k y^{n-k}$ for $k = 0, 1, 2, \ldots, n$, since

$$\begin{aligned} [J^+, J-]f(x,y) &= J^+(J^-f(x,y)) - J^-(J^+f(x,y)) \\ &= -y\frac{d}{dx}\left(x\frac{\partial f}{\partial y}\right) + x\frac{d}{dy}\left(y\frac{\partial f}{\partial x}\right) \\ &= -y\frac{\partial f}{\partial y} - xy\frac{\partial^2 f}{\partial x \partial y} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial y \partial x} \\ &= -2J^3 f(x,y) \end{aligned}$$

and analogous calculations show that the other two commutation relations are also still satisfied in this basis. Now we want to have a representation on a subspace of smooth functions of a single real variable, so we define $\xi = \frac{y}{x}$, then the set $\{x^n y^0, x^{n-1}y^1, \ldots, x^0 y^n\}$ becomes $x^n\{1, \xi, \xi^2, \ldots, \xi^n\}$. Now we see that

$$\begin{aligned} J^+x^k y^{n-k} &= -y\frac{d}{dx}\left(x^n \xi^{n-k}\right) \\ &= -x\xi\Big(nx^{n-1}\xi^{n-k} + x^n(n-k)\xi^{n-k-1}\left(-\frac{y}{x^2}\right)\Big) \\ &= x^n\left(\xi^2\frac{d}{d\xi} - n\xi\right)\xi^{n-k} \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} J^- x^k y^{n-k} &= x^n \left(\frac{d}{d\xi}\right) \xi^{n-k} \\ J^3 x^k y^{n-k} &= x^n \left(\xi \frac{d}{d\xi} - \frac{n}{2}\right) \xi^{n-k} \end{aligned}$$

Thus, if we now consider the factor x^n as a constant which we will subsequently ignore, and ξ as the only independent variable, we finally obtain a representation of $\mathfrak{sl}(2)$ on the space of polynomials of degree at most n given by:

$$\begin{aligned} J^+ &= \xi^2 \frac{d}{d\xi} - n\xi \\ J^- &= \frac{d}{d\xi} \\ J^3 &= \xi \frac{d}{d\xi} - \frac{n}{2} \end{aligned}$$

These three first order differential operators form the basis of a 3-dimensional Lie algebra $\mathfrak{h}$ of differential operators that leave the subspace W spanned by $\{1, \xi, \xi^2, \ldots, \xi^n\}$ invariant. That is, if $f(\xi) \in W$, and $J \in \mathfrak{h}$, then $J(f(\xi)) \in W$. Similar constructions can also be done for other Lie algebras. One other example is $\mathfrak{so}(3)$, which can be represented by 3×3 skew-symmetric matrices. These are important for higher dimensional quasi-exactly solvable problems, because as we shall see, for the case of one (real or complex) variable, all the quasi-exactly solvable systems arise from the above $\mathfrak{sl}(2)$ representation.

3. Quasi-Exact Solvability

The idea behind the phenomenon of quasi-exact solvability [17] is that there exists a finite-dimensional subspace W of the infinite-dimensional space of solutions to the Schrödinger equation that is *invariant* under the Schrödinger operator H_{op}. That is, if $\Psi \in W$, then $H_{\text{op}}\Psi \in W$. We can describe H_{op} as an infinite-dimensional square matrix in a basis $\{\phi_i\}$ of the solution space, where the $(i,j)^{\text{th}}$ entry of the matrix H is $H_{ij} = \langle \phi_i, H_{\text{op}} \phi_j \rangle$, where $\langle\ ,\ \rangle$ denotes the standard complex inner product on the vector space of smooth functions. If the first N elements of the basis are taken to be a basis $\{\phi_1, \phi_2, \ldots, \phi_N\}$ for W, then the matrix representing H_{op} assumes a block-diagonal form, with an $N \times N$ block in the upper left which is the restriction of H_{op} to the invariant subspace W, while the lower right block remains infinite-dimensional:

$$H_{\text{op}} = \begin{pmatrix} H_{1,1} & H_{1,2} & \ldots & H_{1,N} & 0 & 0 & \ldots \\ H_{2,1} & H_{2,2} & \ldots & 0 & 0 & \ldots & \\ \vdots & \vdots & \ddots & \vdots & 0 & 0 & \ldots \\ H_{N,1} & H_{N,2} & \ldots & H_{N,N} & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & H_{N+1,N+1} & H_{N+1,N+2} & \ldots \\ 0 & 0 & \ldots & 0 & H_{N+2,N+1} & H_{N+2,N+2} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In this form each block can be diagonalized separately, so we can obtain N of the energy eigenvalues algebraically using standard methods of linear algebra. The basic problem of determining which potentials are quasi-exactly solvable, then, is to classify the Schrödinger operators which admit such a finite dimensional invariant subspace. This problem is then further reduced by considering the Schrödinger operators that are in the *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ of first order differential operators. This means that the Schrödinger operator H_{op} can be written as a polynomial in the generators (basis elements) of the Lie algebra $\mathfrak{g}$, which are first order differential operators. These are called *Lie algebraic* differential operators. If $\mathfrak{g}$ itself is quasi-exactly solvable, the elements of $\mathfrak{g}$ themselves leave a finite dimensional subspace invariant, and thus any element of its universal enveloping algebra is also quasi-exactly solvable.

Hence the remaining question is when is a Lie algebra of first order differential operators quasi-exactly solvable. In order to classify such algebras, we need to define an appropriate equivalence relation between differential operators.

4. Equivalence of Differential Operators

The important properties to preserve in transforming a Lie algebra of differential operators into another to which we want it to be equivalent [13] in some sense is of course the Lie algebraic structure, or commutation relations, as well as the solutions to the Schrödinger equation. That is, if Ψ is the solution to the equation before the transformation, we need an explicit description of the solution Ψ' to the equation after transformation that leads to the same eigenvalue. We illustrate this in the one-dimensional case:

Definition 4.1. Let $T = f(x)\frac{d}{dx} + g(x)$ be a first order differential operator. Then the operator $\tilde{T} = \tilde{f}(\tilde{x})\frac{d}{d\tilde{x}} + \tilde{g}(\tilde{x})$ is *equivalent* to T if it is related to it by the following transformation:

$$\begin{aligned} \tilde{x} &= \varphi(x) \\ \tilde{T}(x) &= e^{\sigma(x)} \circ T(x) \circ e^{-\sigma(x)} \end{aligned}$$

where $\varphi(x)$ is a smooth invertible function and $\sigma(x)$ is a smooth function of x. Now if $\Psi(x)$ is an eigenfunction for the equation $\tilde{T}\Psi = E\Psi$, then if we define $\tilde{\Psi}(\tilde{x}) = \tilde{\Psi}(\varphi(x)) = e^{\sigma(x)}\Psi(x)$, we preserve the eigenvalue equation, $\tilde{T}\tilde{\Psi} = E\tilde{\Psi}$. It is also easy to see that this definition of equivalence preserves the commutator between operators: if

$$U = [T, S] = T \circ S - S \circ T$$

then

$$\begin{aligned} [\tilde{T}, \tilde{S}] = \tilde{T} \circ \tilde{S} - \tilde{S} \circ \tilde{T} &= e^{\sigma(x)}(T \circ S - S \circ T)e^{-\sigma(x)} \\ &= \widetilde{[T, S]} = \tilde{U} \end{aligned}$$

With this definition of equivalence we can solve for the coefficient functions $\tilde{f}(\tilde{x})$ and $\tilde{g}(\tilde{x})$ in terms of $f(x)$ and $g(x)$:

$$\begin{aligned}
\tilde{T} &= e^{\sigma(x)}\left(f(x)\frac{d}{dx} + g(x)\right)e^{-\sigma(x)} \\
&= e^{\sigma(x)}e^{-\sigma(x)}\left(f(x)\left(\frac{d}{dx} - \sigma'(x)\right) + g(x)\right) \\
&= f(x)\left(\frac{d\tilde{x}}{dx}\right)\frac{d}{d\tilde{x}} + g(x) - \sigma'(x) \\
&= \varphi'(x)f(x)\frac{d}{d\tilde{x}} + g(x) - \sigma'(x)
\end{aligned}$$

Hence we can immediately identify:

$$\begin{aligned}
\tilde{f}(\tilde{x}) &= \tilde{f}(\varphi(x)) = \varphi'(x)f(x) \\
\tilde{g}(\tilde{x}) &= \tilde{g}(\varphi(x)) = g(x) - \sigma'(x)
\end{aligned}$$

The same notion of equivalence of differential operators carries through unchanged for higher dimensions when $x = (x_1, x_2, \ldots, x_n)$. Two Lie algebras of differential operators are said to be equivalent if the two bases are related by this kind of a transformation.

In general the operator T to which we apply this type of transformation will be the Schrödinger operator H_{op} itself, thus ensuring we preserve the solutions in the sense defined above. Since H_{op} is a second order differential operator, we must do a similar but more involved calculation to obtain the transformed coefficient functions. We are interested in finding which Schrödinger operators are equivalent in the above sense to a second order differential operator that is a polynomial in the generators of a quasi-exactly solvable Lie algebra, so we would like to know the most general form of a second order differential operator that is equivalent to a Schrödinger operator. The result, in one dimension, is that in fact *any* second order operator whose coefficient function for the second derivative term is always negative can be so mapped [14] to a Schrödinger operator:

Theorem 4.2. *Let*

$$H_{op} = -f(x)\frac{d^2}{dx^2} - g(x)\frac{d}{dx} - h(x)$$

be a second order differential operator with $f(x) > 0$ always. Then H_{op} is equivalent to the Schrödinger operator

$$\tilde{H}_{op} = -\frac{d^2}{d\tilde{x}^2} + V(\tilde{x})$$

under the transformation given by

$$\begin{aligned}
\tilde{x} = \varphi(x) &= \int^x \frac{ds}{\sqrt{f(s)}} \\
\sigma(x) &= -\frac{1}{4}\log|f(x)| + \int^x \frac{g(s)ds}{2f(s)}
\end{aligned}$$

The potential energy $V(\tilde{x})$ is given by:

$$V(\tilde{x}) = V(\varphi^{-1}(x)) = \frac{3(f')^2 - 8gf' + 4g^2}{16f} - h + \frac{1}{2}g' - \frac{1}{4}f''$$

Proof. We proceed as before, suppressing the explicit x-dependence to simplify notation,

$$\begin{aligned}
\tilde{H}_{\mathrm{op}} &= e^{\sigma} H_{\mathrm{op}} e^{-\sigma} = e^{\sigma}\left(-f\frac{d^2}{dx^2} - g\frac{d}{dx} - h\right)e^{-\sigma} \\
&= e^{\sigma}\left(-f\frac{d}{dx}\left(e^{-\sigma}\frac{d}{dx} - \sigma' e^{-\sigma}\right) - g\left(e^{-\sigma}\frac{d}{dx} - \sigma' e^{-\sigma}\right) - he^{-\sigma}\right) \\
&= e^{\sigma}\left(-f\left(e^{-\sigma}\frac{d^2}{dx^2} - 2\sigma' e^{-\sigma}\frac{d}{dx} + ((\sigma')^2 - \sigma'')e^{-\sigma}\right) - g\left(e^{-\sigma}\frac{d}{dx} - \sigma' e^{-\sigma}\right) - he^{-\sigma}\right) \\
&= -f\frac{d^2}{dx^2} + (2f\sigma' - g)\frac{d}{dx} + \left(f\sigma'' - f(\sigma')^2 + g\sigma' - h\right)
\end{aligned}$$

Now we transform the derivative operators according to the chain rule:

$$\begin{aligned}
\frac{d}{dx} &= \left(\frac{d\tilde{x}}{dx}\right)\frac{d}{d\tilde{x}} = \varphi'\frac{d}{d\tilde{x}} \\
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\varphi'\frac{d}{d\tilde{x}}\right) = \varphi''\frac{d}{d\tilde{x}} + (\varphi')^2\frac{d^2}{d\tilde{x}^2}
\end{aligned}$$

Substituting these results we obtain

$$\tilde{H}_{\mathrm{op}} = -f(\varphi')^2\frac{d^2}{d\tilde{x}^2} + (2f\sigma'\varphi' - g\varphi' - f\varphi'')\frac{d}{d\tilde{x}} + \left(f\sigma'' - f(\sigma')^2 + g\sigma' - h\right)$$

To obtain an operator in Schrödinger form we see that we require

$$f(\varphi')^2 = 1 \qquad \Longleftrightarrow \qquad \varphi(x) = \int^x \frac{ds}{\sqrt{f(s)}}$$

and to enforce the vanishing of the first order derivative,

$$\begin{aligned}
2f\sigma'\varphi' - g\varphi' - f\varphi'' = 0 \qquad &\Longleftrightarrow \qquad 2\sigma' f^{\frac{1}{2}} - gf^{-\frac{1}{2}} + \frac{1}{2}f^{-\frac{1}{2}}f' = 0 \\
&\Longleftrightarrow \sigma'(x) = \frac{g(x)}{2f(x)} - \frac{1}{4}\frac{f'(x)}{f(x)} \\
&\Longleftrightarrow \sigma(x) = -\frac{1}{4}\log|f(x)| + \int^x \frac{g(s)ds}{2f(s)}
\end{aligned}$$

Finally, substituting the expressions for $\sigma(x)$ and $\varphi(x)$ into the multiplication operator yields the above expression for $V(\tilde{x})$. □

In general it may be difficult to explicitly determine the required change of variables because the procedure involves evaluating elliptic integrals.

At this point there is an important distinction between one-dimensional problems (where the potential V depends on a single variable, and multi-dimensional problems, for the case of the Schrödinger equation. This is because in higher dimensions, under this notion of equivalence, it is no longer true that every second order differential operator is equivalent to a Schrödinger operator. One can find second order differential operators which are quasi-exactly solvable but are not equivalent to Schrödinger operators and hence do not admit a direct physical interpretation. Necessary and sufficient conditions for an operator to be equivalent to a Schr̈odinger operator in higher dimensions involve interpreting the coefficient functions of the $\frac{\partial^2}{\partial x_i \partial x_j}$ terms as defining a positive definite Riemannian metric [9, 10], and so multi-dimensional problems naturally lead to consideration of the curvature of space, which in the language of general relativity means that we are considering the mechanical system in the presence of a gravitational field. For this reason we shall examine the one-dimensional case in more detail and only briefly mention the known results in higher dimensions.

An important fact that must be taken into account when considering equivalence of Lie algebras of differential operators is that while equivalence does in some sense preserve the solutions to the Schrödinger equation, it does *not* necessarily preserve their normalizability. That is, if Ψ is a square-integrable wave function solution to $H_{\text{op}}\Psi = E\Psi$, there is no guarantee that under equivalence, the transformed wave function $e^{\sigma(x)}\Psi$ will also be square-integrable. This would be assured if $\sigma(x)$ were purely imaginary, so that the transformation was unitary and hence preserved norms in the vector space of smooth functions. Since we allow $\sigma(x)$ to be real, the eigenfunctions of a Lie algebraic quasi-exactly solvable operator with normalizable wave functions may lose their normalizability upon transforming to Schrödinger form. In the one-dimensional case, explicit necessary and sufficient conditions for ensuring the normalizability of the wave functions of a quasi-exactly solvable Schrödinger operator are known [14]. These involve algebraic constraints on the coefficients of the polynomial of first order differential operators that makes up the Schrödinger operator.

5. Lie Algebras of First Order Differential Operators

Now we can describe Lie algebras of first order differential operators. Each element of the Lie algebra will be the sum of a differentiation operator and a multiplication operator. That is, if $\mathfrak{g}$ is the Lie algebra of first order differential operators, on an n-dimensional space with local coordinates $x = (x_1, x_2, \ldots, x_n)$, then an element T^j of $\mathfrak{g}$ can be written as: $T^j = v^j + \eta^j(x)$ where

$$v^j = \sum_{i=1}^{n} \xi^{ij}(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_i}$$

is a derivative operator, with the ξ^{ij} being smooth functions. We can choose a basis for our finite dimensional Lie algebra of differential operators to be of the form:

$$\begin{aligned} T^1 &= v^1 + \eta^1(x), T^2 = v^2 + \eta^2(x), \ldots, T^r = v^r + \eta^r(x) \\ T^{r+1} &= \zeta^1(x), T^{r+2} = \zeta^2(x), \ldots, T^{r+s} = \zeta^s(x) \end{aligned}$$

Since $\mathfrak{g}$ is to be a Lie algebra, it must be closed under the Lie bracket, which is the commutator, $[T^i, T^j] = T^iT^j - T^jT^i$, where the equality means that the two sides give the same result when acting on a smooth function. Thus we have:

$$\begin{aligned} [T^i, T^{r+j}]f &= (v^i + \eta^i)(\zeta^j)f - (\zeta^j)(v^i + \eta^i)f \\ &= v^i(\zeta^j f) + \eta^i\zeta^j f - \zeta^j v^i(f) - \zeta^j\eta^i f \\ &= \zeta^j v^i(f) - \zeta^j v^i(f) + v^i(\zeta^j)f \end{aligned}$$

Hence we must have $v^i(\zeta^j) \in \mathfrak{g} \quad \forall i = 1, \ldots, r \quad \forall j = 1, \ldots s$. Thus the s smooth functions $\zeta^j(x)$ must be a basis for an s-dimensional $\mathfrak{h}$-module M. Here the binary product between elements of $\mathfrak{h}$ and M is given by the action of the derivative operator on the element of M. That is, if $v = \sum_{i=1}^{n} \xi^i(x)\frac{\partial}{\partial x_i}$ where $v \in \mathfrak{h}$, and

$\zeta(x) \in M$, then

$$v(\zeta) = \sum_{i=1}^{n} \xi^i(x) \frac{\partial \zeta}{\partial x_i} = \sum_{i=1}^{s} c_i \zeta^i(x) \in M$$

for some constants c_i. Also, it is easy to see that closure under the bracket forces the r elements v^j to be a basis for an r-dimensional Lie algebra $\mathfrak{h}$ of derivative operators with Lie bracket given by the commutator. We need to impose additional conditions on the smooth functions $\eta^j(x)$ in order that the T^i's in fact span a Lie algebra. We require that $(T^iT^j - T^jT^i) \in \mathfrak{g} \quad \forall i,j = 1,\dots,r+s$. Note however that since the ζ^i's are part of the basis for $\mathfrak{g}$, if we change the η^i's by adding any linear combination of the ζ^i's, then we do not change the linear span of the operators and hence still have the same Lie algebra $\mathfrak{g}$. Thus any conditions we impose on the functions $\eta^i(x)$ must be considered only on equivalences classes of smooth functions modulo M. That is, we consider η to be *equivalent* to $\eta' \mod M$ if and only if $\eta - \eta' \in M$. With this in mind we define the *quotient module* C^∞/M as the set of equivalence classes of smooth functions modulo M. Now we can determine explicitly the condition on the η^i's. To simplify the notation, we define the linear mapping $F : \mathfrak{h} \to C^\infty/M$ by the rule $F(v^i) \equiv \langle F; v^i \rangle = \eta^i$. Thus if T is an arbitrary differential operator in $\mathfrak{g}$, we have

$$T = \sum_{i=1}^{r+s} c_i T^i = \sum_{i=1}^{r} c_i v^i + \left(\sum_{i=1}^{r} c_i \eta^i(x) + \sum_{j=1}^{s} c_{r+j} \zeta^j(x) \right) = v^T + \eta^T(x)$$

Applying the mapping F,

$$\langle F; v^T \rangle = \sum_{i=1}^{r} c_i \langle F; v^i \rangle = \sum_{i=1}^{r} c_i \eta^i(x) \equiv \eta^T(x) \mod M$$

Thus we see that the mapping F is well defined on $\mathfrak{h}$. We can now demand that the bracket is closed in $\mathfrak{g}$. Let $T = v + \eta(x), S = w + \xi(x) \in \mathfrak{g}$. Then

$$\begin{aligned}
[T,S]f \quad &= (TS - ST)f = (v+\eta(x))(w+\xi(x))(f) - (w+\xi(x))(v+\eta(x))(f) \\
&= (vw - wv)(f) + \eta w(f) + v(\xi f) + \eta\xi f - \xi v(f) - w(\eta f) - \xi\eta f \\
&= (vw - wv)(f) + \eta w(f) + \xi v(f) + v(\xi) f - \xi v(f) - \eta w(f) - w(\eta) f \\
&= (vw - wv)(f) + (v(\xi) - w(\eta)) f
\end{aligned}$$

Thus by its action on an arbitrary smooth function f, we can identify:

$$\begin{aligned}
U = [T,S] \quad &= v^U + \eta^U(x) \\
&= (vw - wv) + (v(\xi) - w(\eta)) \\
&= [v,w] + (v\langle F; w\rangle - w\langle F; v\rangle)
\end{aligned}$$

Now we can ensure that $\mathfrak{g}$ is a Lie algebra (closed under the bracket) by applying the mapping F to the derivative operator component of $[T,S]$, which should give a function that is equivalent to the multiplication operator part of $[T,S]$ modulo M. Hence, we finally obtain our needed condition:

$$v\langle F; w\rangle - w\langle F; v\rangle - \langle F; [v,w]\rangle \equiv 0 \qquad \mod M \qquad \forall v,w \in \mathfrak{h}$$

We can express the above condition in the context of the cohomology theory of Lie algebras [12], of which we will describe only the bare minimum that we will require.

6. Cohomology Theory of Lie Algebras

Definition 6.1. For $i \geq 1$, an *i-cochain* F is a skew-symmetric i-linear mapping from $\mathfrak{h}$ into M. That is, it is a mapping $F : (v_1, v_2, \ldots, v_i) \mapsto F(v_1, v_2, \ldots, v_i) \in M$ where $v_j \in h$, $\forall j = 1, 2, \ldots, i$. For fixed $v_1, v_2, \ldots, v_{q-1}, v_{q+1}, \ldots, v_i$, the map $v_q \mapsto F(v_1, v_2, \ldots, v_i)$ is a linear map of $\mathfrak{h}$ into M. The skew-symmetry means that $F \to -F$ if any two of the v_j's are interchanged. If $i = 0$, we define a 0-cochain as a constant function from $\mathfrak{h}$ to M. That is, it is a fixed element σ of M.

The space of all i-cochains on $\mathfrak{h}$ into M is denoted $C^i(\mathfrak{h}, M)$ and is a vector space under the usual definitions of addition and scalar multiplication of functions. We can define an operator δ, called the *coboundary operator*, that takes an i-cochain to an $(i+1)$-cochain. This cochain δF is called the *coboundary* of F. An i-cochain F is called a *cocyle* if $\delta F = 0$, and it is called a *coboundary* if $F = \delta G$ for some $(i-1)$-cochain G. The set $Z^i(\mathfrak{h}, M)$ of i-cocyles is the nullspace of δ and is thus a subspace of $C^i(\mathfrak{h}, M)$. Similarly, the set $B^i(\mathfrak{h}, M)$ of i-coboundaries is a subspace of $C^i(\mathfrak{h}, M)$ since it is the image under δ of $C^{i-1}(\mathfrak{h}, M)$. It is a general fact that $B^i \subseteq Z^i$. That is, all coboundaries are cocyles. This is a consequence of the fact that $\delta^2 = 0$, which is straightforward but tedious to verify directly from the general formula for δF. For our purposes, we will only require the expressions for the coboundary of zero and one-cochains. Let σ be a zero-cochain, $v \in \mathfrak{h}$. Then the coboundary of σ is the one-cochain defined by:

$$(\delta\sigma)(v) = v(\sigma)$$

Let F be a one-cochain, $v, w \in \mathfrak{h}$, then the coboundary of F is the two-cochain defined by:

$$(\delta F)(v, w) = v(F(w)) - w(F(v)) - F([v, w])$$

This is easily seen to be linear and skew-symmetric. Substituting $F = \delta\sigma$ above gives:

$$(\delta^2\sigma)(v, w) = vw(\sigma) - wv(\sigma) - (vw - wv)(\sigma) = 0 \qquad \forall v, w \in \mathfrak{h}$$

verifying the fact that $\delta^2 = 0$ in this case.

Thus by identifying our linear map F above as a one-cochain from $\mathfrak{h}$ to C^∞/M, we see that F must be a cocyle. Now we demonstrate that if two Lie algebras of differential operators are equivalent, then they differ by a coboundary term:

$$\begin{aligned}
\tilde{T}f \quad &= e^\sigma (v + \eta) e^{-\sigma} f \\
&= e^\sigma \left(e^{-\sigma} v(f) - v(\sigma) e^{-\sigma} f + \eta f e^{-\sigma}\right) \\
&= (v + \eta - v(\sigma))\, f \\
&= Tf - \delta\sigma f
\end{aligned}$$

Thus, two Lie algebras of differential operators will be equivalent if they correspond to the same 1-cocycle F modulo a 1-coboundary. We define the first *cohomology space* of $\mathfrak{h}$ relative to C^∞/M as the quotient space $H^1(\mathfrak{h}, C^\infty/M) \equiv Z^1(\mathfrak{h}, C^\infty/M)/B^1(\mathfrak{h}, C^\infty/M)$. The elements $[F]$ of the first cohomology space are called *cohomology classes*. We can summarize all of the above results in the following theorem:

Theorem 6.2. *Equivalence classes of finite dimensional Lie algebras of first order differential operators are in one-to-one correspondence with equivalence classes of triples* $[\mathfrak{h}, M, [F]]$*, where* $\mathfrak{h}$ *is a finite dimensional Lie algebra of derivative operators,*

M is a finite-dimensional $\mathfrak{h}$-module of smooth functions, and $[F]$ is a cohomology class in $H^1(\mathfrak{h}, C^\infty/M)$.

We have now determined the necessary steps to classify quasi-exactly solvable Lie algebras of first order differential operators. First, one must classify all possible finite-dimensional Lie algebras $\mathfrak{h}$ of derivative operators, and for each find all possible finite-dimensional $\mathfrak{h}$-modules M. Then for each pair $(\mathfrak{h}, M)$ we must determine the first cohomology space. This classifies the Lie algebras of first order differential operators. Now the condition that such a Lie algebra is in fact quasi-exactly solvable is that it admits a finite-dimensional $\mathfrak{g}$-module W of smooth functions. It is easy to see that a Lie algebra of first order differential operators cannot be quasi-exactly solvable unless the $\mathfrak{h}$-module M is either the trivial module consisting of the zero function, or the constant functions with basis $\{1\}$. This is because if M contained non-constant functions, then by applying those elements of $\mathfrak{g}$ that were purely multiplication operators, which we have seen must be elements of M, to elements of the $\mathfrak{g}$-module W, we can generate an arbitrarily large number of smooth functions that are linearly independent, and so if W is an invariant subspace, it cannot be finite-dimensional. Also, it is clear that if $\mathfrak{g}$ is a quasi-exactly solvable Lie algebra with $\mathfrak{h}$-module $M = \{0\}$, then we can extend it to a quasi-exactly solvable Lie algebra of first order differential operators with $\mathfrak{h}$-module $M = \{1\}$ since constant function multiplication operators obviously leave any subspace W invariant. Thus we may restrict our attention to $M = \{1\}$ for classifying the quasi-exactly solvable Lie algebras of first order differential operators.

7. One-dimensional Classification

The finite-dimensional Lie algebras of derivative operators were classified [1] in 1880 up to changes of variables $\tilde{x} = \varphi(x)$ by Sophus Lie, for the case of one real or complex variable and for two complex variables. This classification was restricted to so-called *non-singular* Lie algebras, which means that at no point in the space does every derivative operator in the Lie algebra vanish simultaneously. Using these results as a starting point, Kamran and Olver classified the finite-dimensional Lie algebras of differential operators in one real or complex variable [13, 14] in 1990. Out of the three possibilities, only one of these Lie algebras could be made quasi-exactly solvable, and only if the cohomology parameter was *quantized*. That is, only a discrete set of cohomology parameters is allowed if the Lie algebra is to be quasi-exactly solvable [10]. With this restriction, the only quasi-exactly solvable Lie algebra of first order differential operators in one real or complex variable is in fact the $\mathfrak{sl}(2)$ algebra introduced earlier with the space of polynomials of degree at most n being the finite-dimensional invariant subspace.

8. Current results in higher dimensions

The classification of quasi-exactly solvable finite-dimensional Lie algebras of first order differential operators in two complex variables was completed in 1991 by González-Lopez, Kamran, and Olver [4, 5, 6]. There are 24 classes of finite-dimensional Lie algebras of differential operators in two complex variables and demanding that they are quasi-exactly solvable imposes a similar quantization of cohomology restriction that was recently interpreted in terms of algebraic geometry [3]. Lie algebras of derivative operators in two real variables were classified in the same year [7]. Finally, in 1995, finite-dimensional Lie algebras of first order differential operators and their quasi-exact solvability in two real variables were classified [11]. No work has as yet been attempted in higher dimensions.

Historically, results on one-dimensional quasi-exactly solvable systems were also achieved independently by Turbiner and Ushveridze [18, 17, 15] in the late 1980's. Ushveridze later proposed a different procedure [21] for generating quasi-exactly solvable systems. This method involves starting with a set of analytic functions that depend on some numerical parameters that are fitted to make the functions compatible with the Schrödinger equation being considered. From these functions one can obtain the quasi-exactly solvable Schrödinger operator and associated eigenfunctions simultaneously. This procedure has been shown to be essentially equivalent to the methods described here although some aspects of the relationship between the two methods are still not completely understood.

9. Examples of Quasi-Exactly Solvable Potentials

We now apply the above results to some specific examples. In one dimension, we have seen that all quasi-exactly solvable Schrödinger operators are polynomials in the generators of the $\mathfrak{sl}(2)$ algebra given above. The finite-dimensional invariant subspace is the space of polynomials of degree at most n. Note that in constructing the Schrödinger operator, if the parameter n cancels out from everywhere except for perhaps a constant term, which just translates that spectrum of eigenvalues and can be ignored, then we can find a finite-dimensional invariant subspace of arbitrarily large dimension. Thus the problem becomes in fact *exactly solvable*, in that we can alegbraically determine as many of the eigenvalues as we want. Hence we expect this to occur for the well-known exactly solvable potentials, which we will now demonstrate for a few cases [21]:

Harmonic Oscillator: The harmonic oscillator potential is given by $V(x) = \frac{1}{2}\omega^2 x^2$, where ω is the oscillation frequency. In terms of the $\mathfrak{sl}(2)$ generators, the Schrödinger operator can be written:

$$\begin{aligned} H_{\text{op}} &= -J^- J^- - \omega J^3 \\ &= -\frac{d^2}{d\xi^2} - \omega\xi\frac{d}{d\xi} + \frac{n\omega}{2} \end{aligned}$$

Applying the theorem for transforming this to Schrödinger form, we have $f(\xi) = 1$, $g(\xi) = \omega\xi$, $h(\xi) = -\frac{n\omega}{2}$. Substituting into the expression for the transformed operator gives:

$$\tilde{H}_{\text{op}} = -\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2 + \frac{(n+1)\omega}{2}$$

Morse Potential: The exactly solvable *Morse* potential is given by $V(x) = e^{-2\alpha x} - 2e^{-\alpha x}$, for some parameter α. The polynomial in $\mathfrak{sl}(2)$ generators given by

$$H_{\text{op}} = -\alpha^2 J^3 J^3 + 2\alpha J^- - \big(2\alpha + \alpha^2(n+1)\big)J^3$$

can be similarly transformed to the Schrödinger operator

$$\tilde{H}_{\text{op}} = -\frac{d^2}{dx^2} + e^{-2\alpha x} - 2e^{-\alpha x} + C$$

where the constant term depends on α and n.

Pöschl-Teller Potential: Another exactly solvable system is given by the *Pöschl-Teller* potential, $V(x) = -\frac{1}{\cosh^2(\alpha x)}$ for some parameter α. This can be constructed from

$$H_{\text{op}} = -\alpha^2 J^3 J^3 - \alpha^2 J^- J^- + \alpha^2 \left(\left(1 + \frac{4}{\alpha^2} \right)^{\frac{1}{2}} - (n+1) \right) J^3$$

which transforms to

$$\tilde{H}_{\text{op}} = -\frac{d^2}{dx^2} - \frac{1}{\cosh^2(\alpha x)} + C$$

Anharmonic oscillator: An example of a quasi-exactly solvable potential that is *not* exactly solvable is the one-dimensional *anharmonic* oscillator. We will construct the general form of this potential in reverse, starting from a polynomial of generators that forms a quasi-exactly solvable second order differential operator:

$$\begin{aligned} H_{\text{op}} &= -4J^3 J^- - 2(n+1)J^- + \sqrt{2}\nu J^+ + \sqrt{2}\mu J^3 \\ &= -4\xi \frac{d^2}{d\xi^2} + \left(\sqrt{2}\nu\xi^2 + \sqrt{2}\mu\xi - 2 \right) \frac{d}{d\xi} - \sqrt{2}n\nu\xi - \frac{n\mu}{\sqrt{2}} \end{aligned}$$

where we can identify $f(\xi) = 4\xi$, $g(\xi) = 2 - \sqrt{2}\nu\xi^2 - \sqrt{2}\mu\xi$, and $h(\xi) = \sqrt{2}n\nu\xi + \frac{n\mu}{\sqrt{2}}$. The expression for the potential then becomes:

$$V(x) = \frac{\nu^2}{8}x^6 + \frac{\mu\nu}{4}x^4 + \left(\frac{\mu^2}{8} - \nu \left(n + \frac{3}{4} \right) \right) x^2 + C$$

Thus by suitably choosing the parameters n, μ and ν, we can express many different anharmonic oscillator potentials in Lie algebraic form, but the parameter n will always be present. We can only obtain $n+1$ of the eigenvalues algebraically.

Generalized Coulomb Potential: For a final one-dimensional example [2], consider a particle moving in a central potential. In spherical coordinates, if the potential V depends only on the radial coordinate r, then it can be easily shown that assuming a variables-separable solution of the form $\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$, the Schrödinger equation separates into two equations, one being the radial component:

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \right) R(r) = ER(r)$$

where l is a non-negative integer that arises from demanding that the solution to the angular equation be periodic in the angular variables. This corresponds to the angular momentum of the particle. Specifically, we will consider a *generalized Coulomb potential* given by:

$$V(r) = \frac{\alpha}{r^{\frac{1}{2}}} + \frac{\gamma}{r} + \frac{\beta}{r^{\frac{3}{2}}}$$

The ordinary Coulomb potential for the interaction between two point charges is given simply by the $\frac{1}{r}$ term. We want to map this Schrödinger operator to an equivalent operator that will be a polynomial in the $\mathfrak{sl}(2)$ generators. To do this, we will use the transformation

$$\begin{aligned} x &= \varphi(r) = \sqrt{r} \\ \tilde{\Psi}(x) &= \tilde{\Psi}(\sqrt{r}) = e^{\sigma(r)}\Psi(r) \end{aligned}$$

where

$$\sigma(r) = \sqrt{-E}r + \frac{\alpha}{\sqrt{-E}}\sqrt{r} - (l+1)\log(r)$$

This satisfies the smoothness requirements as long as we restrict attention to the domain $r > 0$ and we expect $E < 0$ for a bound state solution. With this transformation, the Schrödinger operator becomes:

$$\begin{aligned}\tilde{H}_{\text{op}} &= e^{\sigma(r)}\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\alpha}{r^{\frac{1}{2}}} + \frac{\gamma}{r} + \frac{\beta}{r^{\frac{3}{2}}}\right)e^{-\sigma(r)} \\ &= -e^{\sigma(r)}\frac{d}{dr}\left(e^{-\sigma(r)}\frac{d}{dr} - \sigma'(r)e^{-\sigma(r)}\right) + \frac{l(l+1)}{r^2} + \frac{\alpha}{r^{\frac{1}{2}}} + \frac{\gamma}{r} + \frac{\beta}{r^{\frac{3}{2}}} \\ &= -\frac{d^2}{dr^2} + 2\sigma'(r)\frac{d}{dr} + \sigma''(r) - (\sigma'(r))^2 + \frac{l(l+1)}{r^2} + \frac{\alpha}{r^{\frac{1}{2}}} + \frac{\gamma}{r} + \frac{\beta}{r^{\frac{3}{2}}}\end{aligned}$$

Substituting the expression for $\sigma(r)$, we obtain, after some simplification,

$$\begin{aligned}\tilde{H}_{\text{op}} = \quad & -\frac{d^2}{dr^2} + 2\left(\sqrt{-E} + \frac{\alpha r^{-\frac{1}{2}}}{2\sqrt{-E}} - \frac{(l+1)}{r}\right)\frac{d}{dr} + E \\ & + \left(\frac{\alpha(l+\frac{3}{4})}{\sqrt{-E}} + \beta\right)\frac{1}{r^{\frac{3}{2}}} + \left(\frac{\alpha^2}{4E} + 2(l+1)\sqrt{-E} + \gamma\right)\frac{1}{r}\end{aligned}$$

Finally, we transform the derivative operators and substitute $r = x^2$:

$$\begin{aligned}\frac{d}{dr} &= \frac{1}{2}r^{-\frac{1}{2}}\frac{d}{dx} \\ \frac{d^2}{dr^2} &= \frac{1}{4r}\frac{d^2}{dx^2} - \frac{1}{4}r^{-\frac{3}{2}}\frac{d}{dx}\end{aligned}$$

The Schrödinger operator $\tilde{H}_{\text{op}}$ becomes:

$$\begin{aligned}\tilde{H}_{\text{op}} = \quad & -\frac{1}{4x^2}\frac{d^2}{dx^2} + \left(\frac{\sqrt{-E}}{x} + \frac{\alpha}{2\sqrt{-E}x^2} - \frac{(l+\frac{3}{4})}{x^3}\right)\frac{d}{dx} + E \\ & + \left(\frac{\alpha(l+\frac{3}{4})}{\sqrt{-E}} + \beta\right)\frac{1}{x^3} + \left(\frac{\alpha^2}{4E} + 2(l+1)\sqrt{-E} + \gamma\right)\frac{1}{x^2}\end{aligned}$$

Now we simplify the equation $\tilde{H}_{\text{op}}\tilde{R} = E\tilde{R}$ by multiplying through by $-4x^3$ and letting $\hat{H}_{\text{op}} = -4x^3\tilde{H}_{\text{op}}$. The spectral problem then becomes $\hat{H}_{\text{op}}\tilde{R}(r) = 4\beta\tilde{R}(r)$, with $\hat{H}_{\text{op}}$ given by:

$$\begin{aligned}\hat{H}_{\text{op}} = \quad & x\frac{d^2}{dx^2} + \left(4l + 3 - \frac{2\alpha x}{\sqrt{-E}} - 4\sqrt{-E}x^2\right)\frac{d}{dx} \\ & - \left(4\gamma + 8(l+1)\sqrt{-E} + \frac{\alpha^2}{E}\right)x - \frac{(4l+3)\alpha}{\sqrt{-E}}\end{aligned}$$

Since we want to write the second order differential operator that appears on the left hand side of this equation in terms of the $\mathfrak{sl}(2)$ generators, we need to ensure that the operator in fact does leave the space of polynomials of degree at most n invariant. The only terms that cause potential difficulty are the terms proportional to x and to $x^2\frac{d}{dx}$. We must demand that when acting on x^n these terms give zero. This will ensure that the operators leave the space of polynomials of degree at most n invariant, and then we should be able to express it as a polynomial in the $\mathfrak{sl}(2)$ generators. So we demand that

$$-4\sqrt{-E}x^2\frac{d}{dx}x^n - \left(4\gamma + 8(l+1)\sqrt{-E} + \frac{\alpha^2}{E}\right)x^{n+1} = 0$$

Thus, we have the restriction

$$\frac{\alpha^2}{E} + 4\gamma + 4\sqrt{-E}(2l+2+n) = 0$$

This equation must be satisfied by γ, α, and l to find the first $n+1$ eigenvalues algebraically. Substituting this equation into the above expression, the operator finally becomes:

$$\begin{aligned}\hat{H}_{\rm op} = \quad & \left(x\frac{d}{dx} - \frac{n}{2}\right)\frac{d}{dx} - 4\sqrt{-E}\left(x^2\frac{d^2}{dx^2} - nx\right) - \frac{2\alpha}{\sqrt{-E}}\left(x\frac{d}{dx} - \frac{n}{2}\right) \\ & + \left(4l+3+\frac{n}{2}\right)\frac{d}{dx} - \frac{(4l+3+n)\alpha}{\sqrt{-E}} \\ & = J^3J^- - 4\sqrt{-E}J^+ - \frac{2\alpha}{\sqrt{-E}}J^3 + \left(4l+3+\frac{n}{2}\right)J^- - \frac{(4l+3+n)\alpha}{\sqrt{-E}}\end{aligned}$$

in terms of the $\mathfrak{sl}(2)$ generators. Note the dependence on n shows that this system is *not* exactly solvable. This example illustrates the obvious fact that quasi-exact solvability is by no means restricted to the Schrödinger equation or even to quantum mechanics. It applies to any spectral problem. The differential operators do not even have to be second order, although the application of these methods to higher order quasi-exactly solvable operators has not been attempted and would be much more involved.

10. Future Directions

Research in the area of quasi-exactly solvable quantum mechanical systems still holds many interesting possibilities for the future. In higher dimensions, there is an interesting connection between the algebraic structure of the equation and the geometric structure of the underlying space that must be considered when analyzing equivalence of differential operators [17, 9, 10].

Another area that holds much promise is for Schrödinger equations that have multi-component wavefunction solutions, for example in the case of an electron in a magnetic field, where the solution is a two component spinor. Already quasi-exactly solvable systems have been discovered in this context [17, 15], using an extension of Lie algebras known as *graded* or *supersymmetric* algebras. There are also connections being discovered between quasi-exact solvability and the theory of orthogonal polynomials [19, 20], as well as two-dimensional conformal field theory [16].

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