GENERALIZED SYMMETRIES, CONSERVATION LAWS, AND NÖETHER'S THEOREM IN CLASSICAL MECHANICS

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Abstract. The ideas of conservation laws of systems of differential equations and of generalized variational symmetries of such systems are studied from the point of view of classical mechanics. Nöether's theorem is proved in this case, demonstrating a bijective correspondence between generalized symmetries and conservation laws. Some specific examples from mechanics are also considered.

Contents

1. Introduction

In classical mechanics, conservation laws, or constants of the motion, are expressions involving the time and generalized position and velocity coordinates (and possibly higher order derivatives) which are constant on the curves determined by the solution of the equations of motion. Simple examples include the conservation of linear momentum in the absence of external forces, conservation of angular momentum in the absence of external torques, or conservation of total mechanical energy if the potential energy does not explicitly depend on time. We are interested in studying systematic methods for computing conservation laws of a given mechanical system.

An unconstrained mechanical system of N particles involves $3N$ independent degrees of freedom and the motion of the system is described by a curve in $\mathbb{R}^{3N}$. We will consider systems constrained by $k < 3N$ functional relations on the position variables. These are called *holonomic* constraints and define an $n = 3N - k$ dimensional hypersurface of $\mathbb{R}^{3N}$ which will be denoted M^n. This hypersurface is parametrized by n coordinates $\{q^\alpha, \alpha = 1, 2, \dots, n\}$ and its tangent bundle TM^n by $2n$ coordinates $\{(q^\alpha, q'^\alpha), \alpha = 1, 2, \dots, n\}$. The motion of the system is described by

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the solution of the Euler-Lagrange equations for the system, given by the vanishing of the *Euler operator* $E(L)$:

$$E(L) = (E_1(L), \dots, E_n(L)) = 0$$

where

$$E_\alpha(L) = \frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial q'^\alpha}\right) \qquad \alpha = 1, 2, \dots, n \tag{1.1}$$

and L is the *Lagrangian* of the system, being the kinetic energy minus the potential energy expressed in terms of the generalized coordinates and velocities q^α, q'^α.

An important result that will be used repeatedly in the subsequent development is the following:

Lemma 1.1. *The Euler-Lagrange equations $E(L)$ vanish identically for a Lagrangian L if and only if L is a total time derivative of a smooth function P of time and the generalized coordinates and velocities.*

$$E(L) \equiv 0 \qquad \Longleftrightarrow \qquad L = \frac{d}{dt} P(t, q^\alpha, q'^\alpha) \tag{1.2}$$

The proof that if L is a total time derivative, then its Euler-Lagrange equations vanish identically is a trivial computation. The converse can be easily proved by assuming that $E(L) \equiv 0$ and then using the explicit equation 1.1 for $E(L)$ to obtain conditions on L. We need to use Poincaré's Lemma that if $\frac{\partial h^\alpha}{q^\beta} - \frac{\partial h^\beta}{q^\alpha} = 0$, for a smooth vector field $(h^1, \dots, h^n)$, then $h^\alpha = \frac{\partial f}{\partial q^\alpha}$ for some smooth function f. This can be used to show that $L = \frac{df}{dt}$ for some smooth f.

2. Vector Fields on $\mathbb{R} \times M^n$

We denote a smooth vector field X on $\mathbb{R} \times M^n$ by:

$$X = \xi(t, q^\alpha)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{n} \varphi^\alpha(t, q^\beta)\frac{\partial}{\partial q^\alpha}$$

where ξ and φ^α are smooth functions of t and the generalized positions q^α for all $\alpha = 1, \dots, n$. The integral curves of this vector field are a one parameter group of transformations of the space $\mathbb{R} \times M^n$.

We can extend the vector field X to a vector field $\mathrm{pr}^{(1)} X$ on the larger space $\mathbb{R} \times TM^n$ by demanding that the transformation of $\mathbb{R} \times TM^n$ given by the integral curves of $\mathrm{pr}^{(1)} X$ actually transform the velocities to velocities. That is, the new q'^α's should be the time derivatives of the new q^α's on the solution curves of the equations of motion. This gives the *first prolongation* of X, denoted $\mathrm{pr}^{(1)} X$.

Theorem 2.1. *The first prolongation* $\mathrm{pr}^{(1)} X$ *is given by:*

$$\mathrm{pr}^{(1)} X = X + \sum_{\alpha=1}^{n} \eta^\alpha \frac{\partial}{\partial q'^\alpha}$$

where the coefficient functions η^α are given by the polongation formula*:*

$$\eta^\alpha = \frac{d}{dt}(\varphi^\alpha - \xi q'^\alpha) + \xi q''^\alpha \qquad \alpha = 1, \dots, n \tag{2.1}$$

Proof. The proof of this theorem requires some knowledge of differential forms, *Lie derivatives*, and pull-backs. Understanding the proof is not essential to the understanding of the remainder of this exposition. □

On TM^n, the condition that the q'^α's are the time derivatives of the q^α's on the solution curves can be satisfied by demanding that the *contact forms* $dq^\alpha - q'^\alpha dt$ are pulled back to zero by the solution curve $\gamma(t)$, for $\alpha = 1, \ldots, n$ That is, $\gamma^*(dq^\alpha - q'^\alpha dt) = 0$. If we denote the integral curves of $\mathrm{pr}^{(1)} X$ through a point x_0 by $\exp(\varepsilon \, \mathrm{pr}^{(1)} X)x_0$, where ε is the parameter on the curve, then the transformed solution curve $\tilde{\gamma}(t)$ is given by $\tilde{\gamma}(t) = \exp(\varepsilon \, \mathrm{pr}^{(1)} X) \circ \gamma(t)$, and by demanding that the contact forms are pulled back to zero by the transformed curve $\tilde{\gamma}$, we have

$$(\exp(\varepsilon \, \mathrm{pr}^{(1)} X) \circ \gamma)^*(dq^\alpha - q'^\alpha dt) = 0$$

$$\gamma^*(\exp(\varepsilon \, \mathrm{pr}^{(1)} X)^*(dq^\alpha - q'^\alpha dt)) = 0$$

Now by subtracting this equation from $\gamma^*(dq^\alpha - q'^\alpha dt) = 0$, we obtain

$$\gamma^*\Big(\exp(\varepsilon \, \mathrm{pr}^{(1)} X)^*(dq^\alpha - q'^\alpha dt) - (dq^\alpha - q'^\alpha dt)\Big) = 0$$

If we divide both side of the equation by ε and let $\varepsilon \to 0$, the left side becomes the pull back of the Lie derivative with respect to $\mathrm{pr}^{(1)} X$ of the contact form. Hence this derivative is pulled back by γ to zero, so it must be a linear combination of contact forms. Thus, we must have

$$\mathcal{L}_{\mathrm{pr}^{(1)} X}(dq^\alpha - q'^\alpha dt) = \sum_{\beta=1}^{n} \lambda^\alpha_\beta (dq^\beta - q'^\beta dt)$$

for some constants λ^α_β, where $\mathcal{L}$ denotes Lie differentiation. Now we verify that equation (2.1) for $\mathrm{pr}^{(1)} X$ satisfies this condition using the homotopy formula for the Lie derivative of a differential form ω with respect to a vector field X,

$$\mathcal{L}_X(\omega) = X \lrcorner (d\omega) + d(X \lrcorner \omega)$$

where d is exterior differentiation and $X \lrcorner \omega$ is the interior product of X with ω. A routine calculation now yields the desired result, with $\lambda^\alpha_\beta = \frac{\partial \varphi^\alpha}{\partial q^\beta} - q'^\alpha \frac{\partial \xi}{\partial q^\beta}$.

There is another way of writing the first prolongation $\mathrm{pr}^{(1)} X$ that will be useful for what is to come. If we define $Q^\alpha = \varphi^\alpha - \xi q'^\alpha$ for all α, where $Q = (Q^1, \ldots, Q^n)$ is called the *characteristic* of the vector field X, then

$$\begin{aligned}
\mathrm{pr}^{(1)} X &= \xi \frac{\partial}{\partial t} + \sum_\alpha \varphi^\alpha \frac{\partial}{\partial q^\alpha} - \sum_\alpha \xi q'^\alpha \frac{\partial}{\partial q^\alpha} + \sum_\alpha \xi q'^\alpha \frac{\partial}{\partial q^\alpha} \\
&\quad + \sum_\alpha \left(\frac{dQ^\alpha}{dt}\right) \frac{\partial}{\partial q'^\alpha} + \sum_\alpha \xi q''^\alpha \frac{\partial}{\partial q'^\alpha} \\
&= \sum_\alpha Q^\alpha \frac{\partial}{\partial q^\alpha} + \sum_\alpha \left(\frac{dQ^\alpha}{dt}\right) \frac{\partial}{\partial q'^\alpha} + \xi \left(\frac{\partial}{\partial t} + \sum_\alpha q'^\alpha \frac{\partial}{\partial q^\alpha} + \sum_\alpha q''^\alpha \frac{\partial}{\partial q'^\alpha}\right) \\
&= \sum_\alpha Q^\alpha \frac{\partial}{\partial q^\alpha} + \sum_\alpha \left(\frac{dQ^\alpha}{dt}\right) \frac{\partial}{\partial q'^\alpha} + \xi \frac{d}{dt}
\end{aligned}$$

since the expression in brackets is the total time derivative on $\mathbb{R} \times TM^n$. The vector field $X_Q = \sum_\alpha Q^\alpha \frac{\partial}{\partial q^\alpha}$ is known as the *evolutionary form* of X and its prolongation is given immediately by equation (2.1) above as:

$$\mathrm{pr}^{(1)} X_Q = \sum_{\alpha=1}^{n} Q^\alpha \frac{\partial}{\partial q^\alpha} + \sum_{\alpha=1}^{n} \left(\frac{dQ^\alpha}{dt}\right) \frac{\partial}{\partial q'^\alpha} \tag{2.2}$$

Comparison with the above expression for the prolongation of X gives

$$\operatorname{pr}^{(1)} X = \operatorname{pr}^{(1)} X_Q + \xi \frac{d}{dt} \tag{2.3}$$

which will be useful later.

3. Conservation Laws

For simplicity, we will consider conservation laws only up to first order derivatives. A *conservation law* is an expression:

$$\frac{d}{dt} P(t, q^\alpha, q'^\alpha)$$

which vanishes on the solutions of the Euler-Lagrange equations, where P is a smooth function of its variables. That is, P is a constant on the solution curves. If $\frac{d}{dt}P$ has the form

$$\frac{d}{dt} P = \sum_{\alpha=1}^{n} Q^\alpha(t, q^\beta, q'^\beta) E_\alpha(L) \tag{3.1}$$

then clearly P is constant on the solution curves. Here the Q^α's are smooth functions, and the n-tuple $Q = (Q^1, \ldots, Q^n)$ is called the *characteristic* of the conservation law. We will see that it is no coincidence that this is the same symbol and name used in the definition of the evolutionary form of a vector field. In fact it can be shown [1] that all conservation laws are equivalent to conservation laws in the above characteristic form given in equation (3.1), in the sense that their difference is a *trivial* conservation law, meaning that either P itself vanishes on the solutions, or $\frac{dP}{dt}$ vanishes identically. We will assume this result in the subsequent development.

4. Nöether's Theorem

Definition 4.1. The vector field $X = \xi \frac{\partial}{\partial t} + \sum_\alpha \varphi^\alpha \frac{\partial}{\partial q^\alpha}$ is a *variational symmetry* of the Lagrangian L if and only if there exists a smooth function $B = B(t, q^\alpha, q'^\alpha)$ such that:

$$\operatorname{pr}^{(1)} X(L) + L \frac{d\xi}{dt} = \frac{dB}{dt} \tag{4.1}$$

The motivation for the above definition stems from the fact that if equation (4.1) is satisfied with $B = 0$, then it can be shown that the transformation of $\mathbb{R} \times M^n$ induced by the integral curves of X actually leaves the integral $\int L(t, q^\alpha(t), q'^\alpha(t)) dt$ of the Lagrangian function invariant. This fact is not important for the construction of conservation laws, since there exist many conservation laws arising from vector fields that do not satisfy this invariance condition. We are now in a position to state and prove Nöether's theorem.

Theorem 4.2 (E. Nöether)**.** *Let the vector field X be a variational symmetry of the Lagrangian L. Then the characteristic $Q = (Q^1, \ldots, Q^n)$ of X is also the characteristic of a conservation law for the Euler-Lagrange equations. That is, there exists a smooth function $P = P(t, q^\alpha, q'^\alpha)$ such that*

$$\frac{dP}{dt} = \sum_{\alpha=1}^{n} Q^\alpha E_\alpha(L) \tag{4.2}$$

(This justifies the use of the same name for two seemingly different objects.)

Proof. Since X is a variational symmetry, it satisfies equation (4.1). We rewrite $\mathrm{pr}^{(1)} X$ in terms of the prolongation of its evolutionary form X_Q in equation (2.3) and substitute the expression for $\mathrm{pr}^{(1)} X_Q$ given by equation (2.2).

$$\begin{aligned} \frac{dB}{dt} &= \mathrm{pr}^{(1)} X(L) + L\frac{d\xi}{dt} \\ &= \mathrm{pr}^{(1)} X_Q(L) + \xi\frac{dL}{dt} + L\frac{d\xi}{dt} \\ &= \sum_\alpha Q^\alpha \frac{\partial L}{\partial q^\alpha} + \sum_\alpha \left(\frac{dQ^\alpha}{dt}\right)\frac{\partial L}{\partial q'^\alpha} + \frac{d}{dt}(\xi L) \end{aligned}$$

We now integrate the second term by parts, writing

$$\left(\frac{dQ^\alpha}{dt}\right)\frac{\partial L}{\partial q'^\alpha} = \frac{d}{dt}\left(Q^\alpha\frac{\partial L}{\partial q'^\alpha}\right) - Q^\alpha\frac{d}{dt}\left(\frac{\partial L}{\partial q'^\alpha}\right)$$

We thus obtain:

$$\frac{d}{dt}\left(\xi L + \sum_\alpha Q^\alpha\frac{\partial L}{\partial q'^\alpha}\right) + \sum_\alpha Q^\alpha\left(\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial q'^\alpha}\right)\right) = \frac{dB}{dt}$$

$$\frac{dP}{dt} = \sum_{\alpha=1}^{n} Q^\alpha E_\alpha(L)$$

with P given by

$$P = B - \xi L - \sum_\alpha Q^\alpha\frac{\partial L}{\partial q'^\alpha} \tag{4.3}$$

which proves the theorem. □

5. Examples

We can now demonstrate the well known conservation laws of classical mechanics using Nöether's theorem. For example, we consider the vector field $X = \frac{\partial}{\partial t}$. By the variational symmetry criterion in equation (4.1), since one readily computes that $\mathrm{pr}^{(1)} X = X$, we see that X will be a variational symmetry if and only if $\frac{\partial L}{\partial t} = \frac{dB}{dt}$ for some smooth function B. In particular, if the Lagrangian L does not explicitly depend on time, then we have a constant of the motion (with $B = 0$). Specifically, let $L = \frac{1}{2}\sum_\alpha m_\alpha(q'^\alpha)^2 - V(q^\alpha)$, where m_α is the mass of the particle described by the coordinate q^α. This means that the kinetic energy is a diagonal quadratic form in the velocities and the potential is independent of the velocities. A direct application of equation (4.3) yields $P = \frac{1}{2}\sum_\alpha m_\alpha(q'^\alpha)^2 + V(q^\alpha)$ is a constant of the motion, which can be immediately identified as the total mechanical energy of the system.

In an exactly analogous manner, we can show that if the vector field $X = \frac{\partial}{\partial q^\alpha}$ is a variational symmetry of the same Lagrangian L above, then the conjugate momentum $p_\alpha = \frac{\partial L}{\partial q'^\alpha}$ is a conserved quantity. In both of these and many other examples, the smooth function B in equation (4.1) vanishes. However, there are also many examples when this is not the case.

Consider the vector field $X = \sum_\alpha t\frac{\partial}{\partial q^\alpha}$. The integral curves of this vector field are a one-parameter group of transformations called *Galilean boosts*. Actually, this is only a special case of a more general vector field. We consider the special case of a free system, so the potential energy V is zero. The Lagrangian in this case is $L = \frac{1}{2}\sum_\alpha m_\alpha (q'^\alpha)^2$. We have

$$\operatorname{pr}^{(1)} X(L) + L\frac{d\xi}{dt} = \sum_\alpha m_\alpha q'^\alpha = \frac{d}{dt}\left(\sum_\alpha m_\alpha q^\alpha\right)$$

So equation (4.1) is satisfied with $B = \sum_\alpha m_\alpha q^\alpha$. The associated conservation law is easily computed to be

$$P = \sum_\alpha m_\alpha q^\alpha - \sum_\alpha t m_\alpha q'^\alpha$$

If this expression is divided through by the total mass $M = \sum_\alpha m_\alpha$, then this constant of the motion simply expresses the fact that in a free mechanical system, the center of mass of the system moves in a straight line with a constant velocity.

6. Generalized Symmetries

Emmy Nöether recognized that these methods can be extended significantly by allowing the coefficient functions of the vector fields X to depend not only on the time and position coordinates, but also on velocities and even higher order derivatives. Of course, the integral curves of the vector fields are no longer realizable in $\mathbb{R} \times M^n$, but this has no effect on the statement and proof of Nöether's Theorem for X to give rise to a conservation law. We will see, in fact, that this generalization allows us to obtain a one to one correspondence between vector fields that are variational symmetries of a Lagrangian and conservation laws of the system.

Definition 6.1. A *generalized vector field* X is an expression of the form

$$X = \xi(t, q^\alpha, q'^\alpha, \ldots)\frac{\partial}{\partial t} + \sum_{\alpha=1}^{n} \varphi^\alpha(t, q^\beta, q'^\beta, \ldots)\frac{\partial}{\partial q^\alpha}$$

where the coefficient functions are smooth functions of the q^α's, t, and all the higher order derivatives of the q^α's. The first prolongation $\operatorname{pr}^{(1)} X$ of this vector field is obtained in exactly the same way as before, using equation (2.1).

That these generalized vector fields still satisfy Nöether's Theorem is immediate upon inspection of its proof. That we now have a one to one correspondence between variational symmetries and conservation laws will require some more work. To this end, we need to define the Fréchet derivative and its adjoint operator.

7. The Fréchet derivative and its adjoint

Let Ω^n denote the space of n-tuples of smooth functions of $(t, q^\alpha, q'^\alpha, \ldots)$. That is, if $P \in \Omega^n$, then $P = (P^1, \ldots, P^n)$, where P^i is a smooth function of $(t, q^\alpha, q'^\alpha, \ldots)$ for $i = 1, \ldots, n$.

Definition 7.1. Let $P \in \Omega^n$. Then the *Fréchet derivative* of P is the differential operator $D_P : \Omega^n \to \Omega^n$ such that if $Q \in \Omega^n$, then

$$D_P(Q) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} P(t, q^\alpha + \varepsilon Q^\alpha, q'^\alpha + \varepsilon\frac{dQ^\alpha}{dt}, \ldots)$$

That is, in P we replace each derivative $q^{(m)\alpha}$ by $q^{(m)\alpha} + \varepsilon\frac{d^m Q^\alpha}{dt^m}$, and then differentiate P with respect to ε and set $\varepsilon = 0$. For our purposes, we will not be taking Fréchet derivatives of any n-tuples involving more than first derivatives, and in this case we can actually compute an expression for D_P as a matrix operator. By definition, the i^{th} component of $D_P(Q)$ is:

$$(D_P(Q))_i = \sum_{\alpha=1}^{n} \left(\frac{\partial P^i}{\partial q^\alpha} Q^\alpha + \frac{\partial P^i}{\partial q'^\alpha} \left(\frac{dQ^\alpha}{dt} \right) \right)$$

Hence we can write

$$(D_P(Q))_i = \sum_{j=1}^{n} (D_P)_{ij} Q^j$$

where

$$(D_P)_{ij} = \frac{\partial P^i}{\partial q^j} + \left(\frac{\partial P^i}{\partial q'^j} \right) \frac{d}{dt} \tag{7.1}$$

Fréchet derivatives are computationally useful because they are related to the prolongation of vector fields that are in evolutionary form.

Lemma 7.2. *Let P, $Q \in \Omega^n$, and let Q depend only up to first derivatives in the q^α's. Then we have*

$$D_P(Q) = \mathrm{pr}^{(1)} X_Q(P) \quad \textit{where} \quad X_Q = \sum_\alpha Q^\alpha \frac{\partial}{\partial q^\alpha}$$

The proof is immediate from the definition of the prolongation of the evolutionary vector field X_Q and equation (7.1) for D_P above. Note that if we had allowed Q to depend on higher order derivatives, then we would require the prolongation of X_Q to the space of higher order derivatives as well. We are now in a position to define the adjoint of a Fréchet derivative.

Definition 7.3. Let $P \in \Omega^n$. Let D_P be the Fréchet derivative of P, which is an operator $D_P : \Omega^n \to \Omega^n$. Then the *adjoint* operator of D_P, denoted D_P^*, is the unique differential operator $D_P^* : \Omega^n \to \Omega^n$ such that

$$E\left(\sum_i Q^i (D_P(R))_i \right) = E\left(\sum_i (D_P^*(Q))_i R^i \right) \qquad \forall Q, R \in \Omega^n$$

If we restrict the dependence of the n-tuples Q and R to at most first order derivatives in the q^α's, then we can obtain a simple expression for D_P^* by using the fact that the Euler operator $E(L)$ is linear and hence the difference between the two expressions in brackets above must be a total time derivative by equation (1.2). We can know integrate by parts to obtain:

$$\sum_i Q^i (D_P(R))_i = \sum_{ij} Q^i (D_P)_{ij} R^j = \sum_{ij} (D_P^*)_{ij} Q^i R^j = \sum_j (D_P^*(Q))_j R^j$$

where

$$(D_P^*)_{ij} = \left(\frac{\partial P^j}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial P^j}{\partial q'^i} \right) \right) - \left(\frac{\partial P^j}{\partial q'^i} \right) \frac{d}{dt}$$

The uniqueness of D_P^* is a direct consequence of the method with which it is explicitly constructed.

The derivation of the above expression is a straightforward excercise. The importance of considering this adjoint operator lies in the following relation between D_P^* and the Euler operator $E(L)$.

Lemma 7.4. *Let P, $Q \in \Omega^n$, and assume that P and Q depend only up to first order in the derivatives of the q^α's. Then*

$$E\left(\sum_i P^i Q^i\right) = D^*_P(Q) + D^*_Q(P) \tag{7.2}$$

Proof. We verify the result for each component:

$$\begin{aligned}
E_\alpha\left(\sum_i P^i Q^i\right) &= \frac{\partial}{\partial q^\alpha}\left(\sum_i P^i Q^i\right) - \frac{d}{dt}\left(\frac{\partial}{\partial q'^\alpha}\left(\sum_i P^i Q^i\right)\right) \\
&= \sum_i \left(\left(\frac{\partial P^i}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial P^i}{\partial q'^\alpha}\right) - \left(\frac{\partial P^i}{\partial q'^\alpha}\right)\frac{d}{dt}\right)Q^i\right) \\
&\quad + \sum_i \left(\left(\frac{\partial Q^i}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial Q^i}{\partial q'^\alpha}\right) - \left(\frac{\partial Q^i}{\partial q'^\alpha}\right)\frac{d}{dt}\right)P^i\right) \\
&= \sum_i (D^*_P)_{\alpha i} Q^i + \sum_i (D^*_Q)_{\alpha i} P^i = (D^*_P(Q))_\alpha + (D^*_Q(P))_\alpha
\end{aligned}$$

which proves the claim. Note that if P and Q were not restricted to first order derivatives, we can still prove a similar result but we would first have to extend the Euler operator to second order or higher. Since all mechanical Lagrangians of interest are only first order, we will not do this. □

We can now use the adjoint of the Fréchet derivative to obtain a necessary and sufficient condition for an n-tuple Q to be the characteristic of a conservation law.

Theorem 7.5. *The n-tuple Q is the characteristic of a conservation law if and only if*

$$D^*_Q(E(L)) + D^*_{E(L)}(Q) = 0 \tag{7.3}$$

Proof. Recall that Q is the characteristic of a conservation law if and only if $\frac{d}{dt}P = \sum_\alpha Q^\alpha E_\alpha(L)$. Now using equation (1.2), the Euler operator acting on both sides of this expression must vanish. Now by equation (7.2) the result follows immediately. □

In order to demonstrate the one to one correspondence between variational symmetries and conservation laws, we must show that X is a variational symmetry if and only if it satisfies the same equation (7.3). First we need the following proposition.

Proposition 7.6. *The vector field X is a variational symmetry of L if and only if its evolutionary form X_Q is.*

Proof. **Proof.** Since X is a variational symmetry, it satisfies equation (4.1). We use equation (2.3) to write

$$\begin{aligned}
\mathrm{pr}^{(1)} X(L) + L\frac{d\xi}{dt} = \frac{dB}{dt} &= \mathrm{pr}^{(1)} X_Q(L) + \xi\frac{dL}{dt} + L\frac{d\xi}{dt} \\
&= \mathrm{pr}^{(1)} X_Q(L) + \frac{d}{dt}(\xi L)
\end{aligned}$$

Hence, we have

$$\mathrm{pr}^{(1)} X_Q(L) = \frac{d}{dt}B'$$

where $B' = B - \xi L$. Thus X_Q as a vector field also satisfies equation (4.1), and is a variational symmetry of L. The converse is proved similarly. We can now prove the final result. □

Theorem 7.7. *The evolutionary vector field X_Q is a variational symmetry of L if and only if*

$$D_Q^*(E(L)) + D_{E(L)}^*(Q) = 0$$

which is the same as equation (7.3)*.*

Proof. Since X_Q is a variational symmetry, by equation (4.2) we have

$$\sum_{\alpha=1}^{n} Q^\alpha E_\alpha(L) = \frac{dP}{dt}$$

Now if we take Euler operators of both sides, the right side vanishes by equation (1.2) and using equation (7.2) for the left side we obtain

$$D_Q^*(E(L)) + D_{E(L)}^*(Q) = 0$$

Conversely, suppose that the above equation holds. From the steps in the proof of Nöether's Theorem, we had that

$$\mathrm{pr}^{(1)} X_Q(L) = \sum_\alpha Q^\alpha E_\alpha(L) + \frac{d}{dt}\left(\sum_\alpha Q^\alpha \frac{\partial L}{\partial q'^\alpha}\right)$$

Now if we take Euler operators of both sides, the right hand side vanishes by hypothesis. Therefore, by the now much used equation (1.2), we have that $\mathrm{pr}^{(1)} X_Q$ must be a total time derivative, so it satisfies the variational symmetry condition of equation (4.1) and thus by the previous proposition, the vector field X itself is also a variational symmetry. This proves the theorem. □

We can now state the general form of Nöether's Theorem.

Theorem 7.8 (E. Nöether)**.** *A generalized vector field X is a variational symmetry of the Lagrangian L if and only if its characteristic Q is the characteristic of a conservation law* $\frac{dP}{dt} = \sum_\alpha Q^\alpha E_\alpha(L)$.

All the details in the proof of this theorem have now been established.

8. Example

Consider an unconstrained system of one particle in a gravitational field. The configuration space is specified by three position coordinates and three velocities, which we shall denote $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, the dot denoting differentiation with respect to time. The Lagrangian for this system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\mu}{(x^2 + y^2 + z^2)^{1/2}}$$

where m is the mass of the particle and μ is the strength of the gravitational field. One can readily calculate that the total mechanical energy and also the three components of the angular momentum vector are constants of the motion corresponding to variational symmetries given by the vector fields whose integral curves represent time translation and rotation about the three independent coordinate axes in $\mathbb{R}^3$. However, there are also three more constants of the motion, which correspond to vector fields that are not generators of geometric transformations, since their coefficient functions depend on the velocities as well. These are true generalized symmetries. One of these vector fields is given by

$$X_1 = (y\dot{y} + z\dot{z})\frac{\partial}{\partial x} + (\dot{x}y - 2x\dot{y})\frac{\partial}{\partial y} + (\dot{x}z - 2x\dot{z})\frac{\partial}{\partial z}$$

Since the Lagrangian L is completely symmetric with respect to the cyclic permutation of the variables $x \to y \to z$, the other variational symmetries X_2 and X_3 are obtained by permuting the variables in X_1. These vector fields are already in evolutionary form, so the variational symmetry condition in equation (4.1) reduces to checking that $\mathrm{pr}^{(1)} X(L)$ is a total time derivative.

The first prolongation of X_1 is easily computed from equation (2.2), and a routine calculation yields

$$\mathrm{pr}^{(1)} X_1(L) = m\ddot{x}(y\dot{y}+z\dot{z}) + m\ddot{y}(\dot{x}y-2x\dot{y}) + m\ddot{z}(\dot{x}z-2x\dot{z}) + \frac{\mu}{r^3}(\dot{x}(y^2+z^2) - xy\dot{y} - xz\dot{z})$$

Now a close inspection of this reveals that the right side can be written in the form

$$\frac{d}{dt}\Big(m\dot{x}(y\dot{y} + z\dot{z}) - mx(\dot{y}^2 + \dot{z}^2) + \frac{\mu x}{r}\Big) = \frac{dB}{dt}$$

Thus there exists a smooth function B such that X_1 satisfies the variational symmetry criterion, and so there is an associated conservation law. Note that the fact that the above expression is a total time derivative could have been determined by applying the Euler operator E to this expression, after extending E to allow for functions that can depend on second derivatives. This is straightforward, and a similar result to equation (1.2) can be proved, showing that a function of up to second derivatives in the q^α's is a total time derivative if and only if the extended Euler operator acting on the function is identically zero.

Now we can use equation (4.3) to compute the conservation law for this vector field, and the result is:

$$P = -m\dot{x}(y\dot{y} + z\dot{z}) + mx(\dot{y}^2 + \dot{z}^2) + \frac{\mu x}{r}$$

If we let $R_x = -P$ then obviously R_x is a conserved quantity on the solution curves of the Euler-Lagrange equations. We can obtain exactly analogous conserved quantities R_y and R_z from the two other generalized vector fields X_2 and X_3 that are also variational symmetries of L, by cyclically permuting $x \to y \to z$. Putting these three constants of the motion together, we obtain a *conserved vector* $\mathbf{R} = (R_x, R_y, R_z)$, which can be put into a simple form by defining the position vector $\mathbf{r} = (x, y, z)$ and the velocity vector $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$. With these definitions, it is easy to check that

$$\mathbf{R} = m\dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) - \frac{\mu\mathbf{r}}{||\mathbf{r}||}$$

where $\times$ denotes the standard vector cross product in $\mathbb{R}^3$. Now if we multiply top and bottom of the first term by the mass m, then we can write

$$\mathbf{R} = \frac{1}{m}\mathbf{p} \times \mathbf{L} - \frac{\mu\mathbf{r}}{||\mathbf{r}||}$$

where $\mathbf{p} = m\dot{\mathbf{r}}$ is the linear momentum of the particle and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum of the particle. The vector $\mathbf{R}$ is called the *Runge-Lenz vector*, and it can be shown that physically it points in the direction of the major axis of the conic section that is the orbit of the particle (the perihelion point of the orbit), and its magnitude is $\mu\varepsilon$, where ε is the eccentricity of the orbit. [2]

In this example, we explicitly demonstrated the certain generalized vector fields were variational symmetries of the Lagrangian. However, in practice, what is done is that a vector field X whose coefficients are arbitrary functions is assumed to be a variational symmetry, and one then obtains a system of partial differential equations for the coefficient functions which must be satisfied if X were indeed a variational symmetry. The highest order of the derivatives that the coefficient functions are allowed to depend upon must be decided upon in advance. In this way all generalized symmetries up to a certain order for any particular Lagrangian can always be found, in principle, by solving these partial differential equations in each case. Of course, in practice, this is not always easy.

The methods outlined above for determining conservation laws for systems of differential equations arising from a variational principle can also be extended to systems of partial differential equations (where there is more than one independent variable t), and also to cases of systems that are not derivable from a variational principle. The concept of a conservation law for the system is generalized from that of a constant on the solution curves (in the one-dimensional case) to expressions which are total divergences that vanish on solutions. One of the many uses of this symmetry analysis of systems of differential equations is in constructing explicit solutions. In the variational case, if X is a variational symmetry satisfying equation (4.1) with $B = 0$, then it can be shown that the integral curves of X transform solutions of the associated Euler-Lagrange equations to new solutions. Thus, knowledge of the variational symmetries of a Lagrangian and one solution of the system allows us to generate more solutions.

References

[1] P.J. Olver, *Applications of Lie Groups to Differential Equations*, 1993, 266-272.
[2] H. Goldstein, *Classical Mechanics*, 1980, 102-105.