THE INVERSE SCATTERING TRANSFORM AND INTEGRABILITY OF NONLINEAR EVOLUTION EQUATIONS

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Abstract. A method for integrating nonlinear partial differential equations is discussed which can be viewed as a nonlinear analogue of the Fourier transform. It involves associating the solution of the nonlinear equation to a linear eigenvalue equation whose eigenvalues are constants of the motion for the original equation. The solution of the original equation plays the role of a potential in the eigenvalue problem. Thus the solution is mapped to scattering data of the eigenvalue equation and the time evolution of these data is trivially computed. Inverse scattering techniques are then applied to obtain the solution to the original equation. It is also shown how these equations can be viewed as completely integrable infinite dimensional Hamiltonian systems in classical mechanics.

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1. Introduction

Since the study of differential equations began, almost all progress until recently had been made only for *linear* equations: those in which at most one power of the unknown function or its derivatives appeared in each term of the equation. The main reason for this is that linear equations obey the *superposition principle*. That is, since differentiation is a linear operation, any linear combination of solutions of a linear equation is again a solution of the equation. Hence the methods of Fourier series and Fourier transforms were developed to express general solutions in terms of sums or integrals of certain basic solutions.

However, in the last 30 years, there has been enormous progress in the study of nonlinear equations and in exact methods for their solution. In this expostion, we will give a semi-historical account of some of this work, specifically concerning the inverse scattering transform, which among other things can be viewed of as a nonlinear analogue of the Fourier transform.

There is much more to the inverse scattering transform than we discuss in this paper. Consideration of one-dimensional periodic problems solvable by this method reveals connections with algebraic geometry and Riemann surfaces. The inverse scattering transfrom is also connected to Bäcklund transformations, which relate solutions of partial differential equations to solutions of other equations. Much work has also more recently been done for higher dimensional problems. In addition, the main feature of these equations, the existence of particle-like solutions called solitons is only briefly mentioned in passing. More extensive treatments beyond this simple introduction can be found in [11], [12], [13], and [14].

For the sake of simplicity, we will restrict attention to partial differential equations in $1 + 1$ dimensions, meaning that there is one variable t that should be thought of as representing time ($t \geq 0$) and one variable $x \in \mathbb{R}$ representing a spatial dimension.

Definition 1.1. An *evolution equation* is a partial differential equation for an unknown function $u(x, t)$ of the form

$$u_t = K(u) \tag{1.1}$$

where $K(u)$ is an expression involving only u and its derivatives with respect to x. If this expression is nonlinear, equation (1.1) is called a nonlinear evolution equation.

Definition 1.2. A $\mathcal{C}^\infty$ function $u(x, t)$ on $\mathbb{R}$ (where t is regarded as a smoothly varying parameter) is said to *decay sufficiently rapidly* if u and all its x-derivatives go to zero as $|x| \to \infty$.

Remark. To avoid technical arguments that only serve to distract us from the ideas we are concerned with, we will always assume that the solutions $u(x, t)$ of the evolution equations in question decay sufficiently rapidly, so that boundary terms upon integration by parts will vanish. Note that most of the results we shall prove require only the first two or three x-derivatives of u to go to zero as $|x| \to \infty$, but we will not be concerned with trying to identify the weakest necessary hypotheses in each case.

The technique of inverse scattering as a method for integrating nonlinear evolution equations was first discovered in 1967 in the course of studying solutions to the Korteweg-deVries equation. We begin by reviewing this early work.

2. The KdV Equation

2.1. **Preliminaries.** The Korteweg-deVries equation, hereafter abbreviated the KdV equation, is perhaps the simplest nonlinear partial differential equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.1}$$

where $u = u(x,t)$ is a function of two variables. The KdV equation is extremely important as it arises in many physical contexts. It can be used to describe waves in shallow water (for which it was first discovered in 1895), anharmonic nonlinear lattices, gas dynamics, and hydromagnetic and ion-acoustic waves in cold plasma, for example. The interested reader is referred to [13] for the physical derivations. Note that the coefficients in front of the three terms are somewhat arbitrary and were chosen for future notational simplicity, since we can rescale our coordinates by letting $u = au'$, $x = bx'$, and $t = ct'$ and the equation becomes

$$acu'_{t'} + 6a^2bu'u'_{x'} + a^3b^3u'_{x'x'x'} = 0$$

Hence by suitably choosing a, b, and c, we can obtain any real coefficients.

Another observation that should be made about the KdV equation is that it is Galilean invariant, meaning that if $u(x,t)$ is a solution, then so is $u(x - 6ct, t) + c$ for any $c \in \mathbb{R}$, as can be easily verified. Thus we obtain a one parameter family of solutions.

Lemma 2.1. *Solutions of the KdV equation that decay sufficiently rapidly are uniquely determined by the initial data.*

Proof. Let u, v, be two solutions of equation (2.1), and let $w = u - v$. Substitution easily yields the equation

$$w_t + 6uw_x + 6wv_x + w_{xxx} = 0.$$

If we multiply the equation by w and integrate over all x, then after integrating by parts on the second term and using the fact that w and all its x-derivatives go to zero, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx + 6\int_{-\infty}^{\infty}\left(v_x - \frac{1}{2}u_x\right) w^2 dx = 0$$

Letting $E(t) = \frac{1}{2}\int w^2 dx$, and $M = \sup|6v_x - 3u_x| < \infty$, we have $E(t) \le E(0)e^{Mt}$. Since $E(0) = 0$, we have $E(t) \equiv 0$ and hence $w \equiv 0$, so $u = v$ for all times t. □

2.2. **Soliton Solutions.** We can ask if there are any permanent wave solutions of the KdV equation of the form $u(x,t) = f(x - ct)$. Substituting this into equation (2.1), we obtain

$$f''' + 6ff' - cf' = 0$$

which can be immediately integrated once to get

$$f'' + 3f^2 - cf = m$$

for some constant m. Now multiplying by f' and integrating once more,

$$\frac{1}{2}(f')^2 + f^3 - \frac{c}{2}f^2 - 2mf = n$$

for some other constant of integration n. This can actually be used to solve for f implicitly in terms of elliptic integrals, but we will consider only solutions where

$f(x)$ decays sufficiently rapidly, which forces m and n to be zero. The differential equation for f can now be integrated directly, and the result is:

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right) \tag{2.2}$$

Note that since the solution exists only for a wave speed $c > 0$, these *solitary wave* solutions always travel to the right, and the propogation speed is proportional to the wave amplitude, with larger waves moving faster.

Numerical experiments in 1965 by Kruskal and Zabusky (see [13]) showed that when two solitary wave solutions pass through each other, they emerge with their shape unchanged and a relative phase shift. Since their interactions were particle-like, these solutions were named *solitons.* Because nonlinear equations do not obey a superposition principle, special solutions like solitons were not expected to play a special role. However, the experiments by Kruskal and Zabusky also revealed that any solution of the KdV equation which vanishes asymptotically must in some sense be made up of a finite number of solitons.

2.3. **The Modified KdV Equation.** A similar equation that will play an important role in what follows is known as the *Modified KdV equation*, abbreviated the MKdV equation:

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{2.3}$$

The inspiration for the inverse scattering method came when Miura [2] discovered the following ingenious nonlinear transformation that related the solutions of the KdV and MKdV equations. If we let $u = v^2 - iv_x$, then one easily verifies that:

$$u_t + 6uu_x + u_{xxx} = \left(2v - i\frac{\partial}{\partial x}\right)\left(v_t + 6v^2 v_x + v_{xxx}\right)$$

Hence a solution $v(x,t)$ of the MKdV equation gives rise to a solution $u(x,t)$ of the KdV equation by the Miura transformation. Note that the transformation only works in one direction.

2.4. **Conservation Laws.** In the course of attempting to solve the KdV equation exactly, it was discovered that the equation has an infinite sequence of nontrivial conservation laws, which we shall presently define.

Definition 2.2. A *conservation law* associated to a differential equation in $1+1$ dimensions is an expression of the form

$$T_t + X_x = 0 \tag{2.4}$$

where T and X are functions of t, x, u and derivatives of u. T is called the *conserved density* and $-X$ is called the *flux* of T.

A *local* conservation law depends only on u and its derivatives, and not explicitly on x and t. This is the case, for example, if X and T are polynomials in u and its derivatives. For these conservation laws, we can integrate equation (2.4) to see that

$$I = \int_{-\infty}^{\infty} T\,dx$$

is a constant: $I_t = 0$. We say I is a *constant of the motion*, or an *integral* of the differential equation.

Remark. It is possible to have constants of the motion that do not arise from conservation laws. An example of this will be seen below when we see that the eigenvalues of a certain linear differential operator will be constants of the motion for the KdV equation.

The first of the conservation laws for the KdV equation can be easily seen by direct examination of equation (2.1):

$$u_t + (3u^2 + u_{xx})_x = 0$$

Another can be obtained by multiplying the equation by u:

$$\left(\frac{1}{2}u^2\right)_t + \left(2u^3 + \left(uu_{xx} - \frac{1}{2}u_x^2\right)\right)_x = 0$$

We will now describe a method presented in [3] by Miura, Gardner, and Kruskal for generating an infinite sequence of conservation laws for the KdV equation. If we define u by

$$u = w + i\varepsilon w_x + \varepsilon^2 w^2 \tag{2.5}$$

where ε is some parameter, then one can readily verify that

$$u_t + 6uu_x + u_{xxx} = \left(1 + 2\varepsilon^2 w + i\varepsilon\frac{d}{dx}\right)\left(w_t + 6(w + \varepsilon^2 w^2)w_x + w_{xxx}\right).$$

Thus we can map solutions w of the differential equation on the right hand side of the above equation to solutions of the KdV equation. This is nothing more than a generalization of the MKdV transformation defined in Section 2.3. Now let w be a solution of the partial differential equation given in the right hand side above. Then we can immediately read off the conservation law

$$w_t + \left(3w^2 + 2\varepsilon^2 w^3 + w_{xx}\right)_x = 0. \tag{2.6}$$

Thus $\int w dx$ is a constant of the motion, by integration of equation (2.6), and using the fact that w decays sufficiently rapidly. Now by formally writing $w = w_0 + w_1\varepsilon + w_2\varepsilon^2 + \ldots$, and substituting into equation (2.5), we can equate powers of ε to get $w = u - iu_x\varepsilon - (u^2 + u_{xx})\varepsilon^2 \ldots$, which we can substitute into $\int w dx$ and equation (2.6) to obtain a conservation law which is a formal power series in ε. Thus, since the KdV equation is independent of this parameter, the coefficients of each power of ε are conservation laws for the KdV equation. These calculuations give, for example, the following third nontrivial conservation law:

$$\left(u^3 - \frac{1}{2}u_x^2\right)_t + \left(\frac{9}{2}u^4 + 3u^2 u_{xx} - 6uu_x^2 - u_x u_{xxx} + \frac{1}{2}u_{xx}^2\right)_x = 0 \tag{2.7}$$

This particular conservation law will play a very important role in Section 7. Note that the presence of infinitely many conservation laws is not a feature of all evolution equations solvable by the inverse scattering method that will be described below, although it is true for the KdV and MKdV equations. What does happen, however, as we shall see in Section 7 is that there are infinitely many constants of the motion, not necessarily arising from conservation laws, which characterize the equations as completely integrable Hamiltonian systems.

2.5. **Exact Solution by Inverse Scattering.** Since this section is only meant to give a historical introduction to the inverse scattering transform, we will only sketch the original solution of the KdV equation as given in [1] and more explicitly in [8], and will leave the more rigorous description of the method to Sections 5 and 6. In 1967, the team of Gardner, Greene, Kruskal, and Miura, hereafter abbreviated GGKM, discovered a method of exactly solving the initial value problem for the KdV equation. Again we consider only initial data that decay sufficiently rapidly.

We begin by recalling the Miura transformation $u = v^2 - iv_x$ of Section 2.3 that transforms solutions of the MKdV equation into solutions of the KdV equation. Now equation (2.3) is a Ricatti equation, and there exists a well known procedure for linearizing such equations: If we let $v = \frac{-i\varphi_x}{\varphi}$, then the equation defining the Miura transformation becomes $u = \frac{-\varphi_{xx}}{\varphi}$. The Galilean invariance of the KdV equation lets us replace u by $u + \lambda$, and after some rearranging, we get:

$$\varphi_{xx} + (\lambda + u(x,t))\,\varphi = 0 \tag{2.8}$$

which is exactly the one dimensional time independent Schrödinger equation of quantum mechanics, and has been studied since the 1930's. There are two kinds of solutions to this equation that interest us. The *bound* states (so named because in quantum mechanics they correspond to particles whose total energy is negative), which correspond to negative eigenvalues and are square integrable, so in particular they decay as $|x| \to \infty$. There is always a countable, discrete family of these solutions. There is also a continuum of unbounded states corresponding to all possible positive eigenvalues. These solutions are asymptotically periodic waves as $|x| \to \infty$. Under the hypothesis that the potential $u(x,t)$ decays sufficiently rapidly, one can show that there is in fact only a finite number N of negative eigenvalues.

Since $u(x,t)$ evolves according the KdV equation, we can determine the time evolution of $\lambda(t)$ and $\varphi(x,t)$ in equation (2.8). Let us first consider the bound state solutions. These eigenfunctions φ_n satisfy $\varphi_n \to 0$ as $|x| \to \infty$, and φ_n is square integrable. Solving equation (2.8) for u and substituting into the KdV equation, one obtains:

$$\lambda_{nt}{\varphi_n}^2 + (\varphi_n Q_x - {\varphi_n}_x Q)_x = 0 \tag{2.9}$$

where $Q = {\varphi_n}_t + {\varphi_n}_{xxx} - 3(\lambda_n - u){\varphi_n}_x$. Now if we integrate equation (2.9) and use the fact that the bound state eigenfunctions are square integrable, we see that $\lambda_{nt} = 0$. Thus the discrete eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of equation (2.8) are constants of the motion in the KdV equation. Now using this fact, we can directly integrate equation (2.9) to obtain

$$Q = C(t)\varphi_n + D(t)\varphi_n \int_0^x \frac{ds}{\varphi_n(s)^2} \tag{2.10}$$

We will choose to normalize the eigenfunctions φ_n with respect to the standard norm on $L^2(\mathbb{R})$ by demanding that

$$\int_{-\infty}^{\infty} \varphi_n^2 dx = 1 \tag{2.11}$$

Let us write the discrete eigenvalues as $\lambda_n = (i\kappa_n)^2$, with $\kappa_n > 0$. The function $\psi_n(x) = \varphi_n \int_0^x \frac{ds}{\varphi_n(s)^2}$ is a solution of equation (2.9) that is linearly independent to φ_n. But we know that asymptotically, since the potential $u(x,t) \to 0$, the solutions look like linear combinations of $e^{\pm\kappa_n x}$. Now since $\varphi_n \to 0$ as $|x| \to \infty$, we see that ψ_n blows up, so we must have $D = 0$. If we now multiply equation (2.10) by φ_n and integrate, then using equation (2.11), we see that $C = 0$ for the discrete eigenvalues.

Now we define the components $c_n(t)$ of the *scattering data* that correspond to the discrete eigenvalues by

$$\varphi_n(x) \sim c_n(t)e^{-\kappa_n x} \qquad \text{as } |x| \to \infty \tag{2.12}$$

Substituting this into equation (2.10) and using the fact that $u \to 0$ as $|x| \to \infty$, and easy calculation yields:

$$\frac{dc_n}{dt} = 4\kappa_n^3 c_n(t) \tag{2.13}$$

which is trivial to integrate for the time evolution of $c_n(t)$.

We can follow a similar such procedure with the continuum of positive eigenvalues and their corresponding eigenfunctions. For large $|x|$, since $u \to 0$, the $\lambda = k^2 > 0$ solutions φ of equation (2.8) are asymptotically linear combinations of $e^{\pm ikx}$. This time since we have a continuum of positive eigenvalues, we can simply choose that $\lambda_t = 0$ in equation (2.9) and study the resulting eigenfunctions. Our goal is to eventually solve the KdV equation. We will do this by determining what is known as the *scattering data* for the eigenvalue problem in equation (2.8). One part of these data is the collection of functions $c_n(t)$ defined above. The rest of the data consists of a pair of functions $a(k,t)$ and $b(k,t)$ defined by imposing the following asymptotic boundary conditions:

$$\varphi \sim e^{-ikx} + \rho(k,t)e^{ikx} \qquad \text{as } x \to +\infty \tag{2.14}$$

$$\varphi \sim \tau(k,t)e^{-ikx} \qquad \text{as } x \to -\infty \tag{2.15}$$

where $\rho(k,t)$ and $\tau(k,t)$ are known as the *reflection* and *transmission coefficients*, respectively. The quantum mechanical interpretation of these conditions is that they represent steady-state radiation coming from $x = +\infty$ only. Actually, we will only need the reflection coefficient $\rho(k,t)$ to solve the KdV equation. Now the boundary condition at $x \to +\infty$ gives us that $C(k) = 4ik^3$ and $D(k) = 0$ in equation (2.10) and substitution of equations (2.14) and (2.15) into equation (2.10) gives us

$$\frac{d\rho}{dt} = 8ik^3\rho \qquad \frac{d\tau}{dt} = 0 \tag{2.16}$$

which are again trivial to integrate.

We can now use equations (2.13) and (2.16) to completely determine the time evolution of the *scattering data*:

$$S = \left\{ (\kappa_n, c_n)_1^N, \rho(k,t), \tau(k,t), k \in \mathbb{R} \right\}$$

The initial condition $u(x,0)$ gives us $S(0)$, and we can determine $S(t)$ for all time $t > 0$, so all that remains to solve the initial value problem for the KdV equation is to invert the scattering data $S(t)$ to get the potential $u(x,t)$ in equation (2.8). The important point is that in the Schrödinger equation, the variable t is only a parameter, and the scattering data evolves with t according to the KdV equation. Determination of the potential $u(x,t)$ from knowledge of the scattering data $S(t)$ is called the inverse problem of scattering theory for the Schrödinger equation. It involves a linear integral equation, known as the Gel'fand-Levitan-Marchenko equation. The details can be found in Section 5.3.

It turns out that if the reflection coefficient is initially $\rho(k, o) = 0$, we can obtain the solution explicitly entirely in terms of the c_n's and κ_n's. These types of solutions describe the interaction of a finite number of solitons. Each soliton has an amplitude and speed characterized by the eigenvalue κ_n and a position characterized by $c_n(t)$ (see [8]). In fact the potentials corresponding to soliton solutions are always reflectionless. It can be shown that the soliton solution of equation (2.2) corresponds to one discrete eigenvalue and a zero reflection coefficient.

Thus we have in principle the method of the inverse scattering transform: we map the solution u of the KdV equation to a potential in a Schrödinger equation, for which we can determine the initial scattering data and determine its time evolution. Then we invert the process to determine the potential $u(x, t)$ that gives these scattering data in the Schrödinger equation. The main reason why the method works is that the time evolution of the scattering data is easily computed. This happens because the scattering data are defined as $|x| \to \infty$, where u is known to approach zero. Thus the problem is reduced to solving *linear* ordinary differential equations for the time evolution of the scattering data and a *linear* integral equation where t is nothing more than a parameter.

The fact that the inverse scattering transform can be thought of as a nonlinear analogue of the Fourier transform can be seen by recalling the method of solution for the *linearized* KdV equation: $u_t + u_{xxx} = 0$. Taking Fourier transforms of both sides, we get:

$$\begin{aligned}\hat{u}(k,t) &= \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx \\ \hat{u}(k,0) &= \int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx \\ \hat{u}_t &= ik^3\hat{u}\end{aligned}$$

where we use the fact that $u \to 0$ as $|x| \to \infty$ and integrate by parts to get the time evolution of $\hat{u}$. Thus the partial differential equation is transformed to an infinite number of *ordinary* differential equations in t with k as a parameter, just as in the inverse scattering method used to solve the KdV equation. In Section 7 we will intrepret this in terms of action-angle variables for completely integrable systems in Hamiltonian mechanics. Now we can find $\hat{u}(k, t)$ for all times t and invert to recover $u(x, t)$:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{ikx}dx$$

.

We will now summarize our results in a way which will become useful in Section 3. Combining equation (2.8) with equation (2.10) and simplifying, we are led to the following system of equations for φ:

$$L\varphi = \lambda\varphi = \left(-\frac{\partial^2}{\partial x^2} - u(x,t)\right) \tag{2.17}$$

$$B\varphi = \varphi_t = -4\varphi_{xxx} - 3u_x\varphi - 6u\varphi_x + C\varphi \tag{2.18}$$

At this point it is instructive to see that we can obtain again the fact that $\lambda_t = 0$ for the discrete eigenvalues (where $C = 0$) by cross differentiating these equations and using the fact that $\varphi_{xxt} = \varphi_{txx}$.

3. The Lax Approach

In 1968, Lax discovered a formalism [4] for describing integrable nonlinear evolution equations that are amenable to exact solution by the method of inverse scattering. He presented a general principle for associating nonlinear evolution equations with linear operators so that the eigenvalues of the linear operator are constants of the motion for the nonlinear equation, as is the case with the KdV equation and the Schrödinger operator.

Consider the system of equations (2.17) and (2.18) involving the differential operators L and B, and view these operators as acting on $L^2(\mathbb{R})$, the space of square integrable funtions on the real line. Lax observed that the fact that the spectrum of L did not change with time could be shown to be equivalent to the statement that the operators $L(0)$ and $L(t)$, which are both self-adjoint in this case, are unitarily equivalent. That is, there exists an operator $U(t)$ such that $U^*U = I$, the identity operator, and

$$L(t)U(t) = U(t)L(0). \tag{3.1}$$

Now let $\varphi(x, 0, \lambda)$ be an eigenfunction of $L(0)$ with eigenvalue λ. Let $\varphi(x, t, \lambda) = U(t)\varphi(x, 0, \lambda)$. Then

$$\begin{aligned} L(t)\varphi(x, t, \lambda) &= L(t)U(t)\varphi(x, 0, \lambda) \\ &= U(t)L(0)\varphi(x, 0, \lambda) \\ &= U(t)\lambda\varphi(x, 0, \lambda) \\ &= \lambda\varphi(x, t, \lambda). \end{aligned}$$

Hence $\varphi(x, t, \lambda)$ is an eigenfunction of $L(t)$ with eigenvalue λ. If we differentiate equation (3.1) with respect to t, we obtain

$$L_t(t)U(t) + L(t)U_t(t) = U_t(t)L(0)$$

Now multiplying on the right by U^*, and using $UU^* = I$,

$$\begin{aligned} L_t(t) + L(t)U_t(t)U^*(t) &= U_t(t)L(0)U^*(t) \\ &= U_t(t)U^*(t)U(t)L(0)U^*(t) \\ &= U_t(t)U^*(t)L(t)U(t)U^*(t) \\ &= U_t(t)U^*(t)L(t) \end{aligned}$$

Hence, letting $B = U_tU^*$, we finally obtain:

$$L_t = BL - LB = [B, L] \tag{3.2}$$

which is known as the *Lax equation*, and L and B are called a *Lax pair*. If one similarly differentiates the equation $U^*U = I$, we get the identity $B^* = -B$, and thus B is *skew-adjoint*. The skew-adjoint property is needed to preserve unitarity, as we show in the following lemma.

Lemma 3.1. *Let $U(t)$ be a one parameter family of linear operators, such that $U_t = BU$, for some skew-adjoint one parameter family $B(t)$. Then if $U(0)$ is unitary, $U(t)$ is unitary for all t.*

Proof. Let $v_1, v_2 \in V$. Define $w_i = U(t)v_i$. Then $w_{it} = B(t)w_i$, and

$$\begin{aligned}\langle w_1, w_{2t}\rangle &= \langle w_1, Bw_2\rangle \\ &= \langle B^* w_1, w_2\rangle \\ &= \langle -Bw_1, w_2\rangle \\ &= -\langle w_{1t}, w_2\rangle .\end{aligned}$$

Thus we have

$$\begin{aligned}\langle w_1, w_2\rangle &= \langle U(t)v_1, U(t)v_2\rangle \\ &= \langle U(0)v_1, U(0)v_2\rangle \\ &= \langle v_1, v_2\rangle\end{aligned}$$

where we have used the fact that $U(0)$ is unitary in the last equality. Thus $U(t)$ is norm preserving, and $U(t)U^*(t) = U(0)U^*(0) = I$, so $U(t)$ is unitary. □

Lax chose equation (3.2) as the starting point for his formalism, and showed that this equation implied the spectrum of the eigenvalue problem $L\varphi = \lambda\varphi$ was invariant in time.

Theorem 3.2. *Let $L(t)$ be a one parameter family of self-adjoint linear operators defined on some Hilbert space V. Suppose that the discrete eigenvalues of $L\varphi = \lambda\varphi$ and their corresponding eigenfunctions are continuously differentiable with respect to t. Further suppose there is a one parameter family of operators $B(t)$ such that $L_t = [B, L]$. Then $\lambda_t = 0$, the spectrum of L is invariant.*

If λ is a simple eigenvalue (multiplicity one), then $\varphi_t = (B + C)\,\varphi$ for some arbitrary continuous function $C(t)$. If $B+C$ is skew-adjoint, then $\|\varphi\|$ is independent of t.

Proof. Differentiating $L\varphi = \lambda\varphi$, we have

$$L_t\varphi + L\varphi_t = \lambda_t\varphi + \lambda\varphi_t.$$

Now using $L_t = [B, L]$, and $BL\varphi = \lambda B\varphi$, we get

$$(L - \lambda)\,(\varphi_t - B\varphi) - \lambda_t\varphi = 0. \tag{3.3}$$

Taking the inner product on the left with φ and using the fact that L is self-adjoint,

$$\begin{aligned}\lambda_t\|\varphi\|^2 &= \langle \varphi, (L - \lambda)\,(\varphi_t - B\varphi)\rangle \\ &\langle (L - \lambda)\,\varphi, (\varphi_t - B\varphi)\rangle \\ &0.\end{aligned}$$

Thus the spectrum is invariant. Now looking back at equation (3.3), we see that if λ is a simple eigenvalue, since $\varphi_t - B\varphi$ is in the eigenspace, we have $\varphi_t = B\varphi + C\varphi$ for some $C(t)$. Finally,

$$\frac{\partial}{\partial t}\|\varphi\|^2 = \langle \varphi, (B + C)\varphi\rangle + \langle (B + C)\varphi, \varphi\rangle$$

and hence if $B + C$ is skew-adjoint, $\|\varphi\|^2$ is independent of t. □

In the case of the KdV equation as described in Section 2.5, we chose $\|\varphi\| = 1$, and calculation shows that B is already skew-adjoint, so $C = 0$. From equations (2.17) and (2.18), we see that the choices

$$L = -\frac{d^2}{dx^2} - u(x,t)$$
$$B = -4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right) + C$$

consitute a Lax pair for the KdV equation. Indeed, one can easily compute that $L_t = [B, L]$ in this case gives exactly the KdV equation (2.1).

Thus in general if $u_t = K(u)$ is an evolution equation, we try to associate to it a *self-adjoint* operator L and a *skew-adjoint* operator B, which evolve with time, that satisfy the Lax equation (3.2). By the above remarks this shows that the eigenvalues of L are a set of integrals (constants of the motion) for $u_t = K(u)$. The main drawback to this approach is that it requires one to successfully guess the right L and B for a given equation to show that the equation would be solvable by the inverse scattering approach.

One advantage, however, is that given a self-adjoint L, there is a somewhat systematic albeit complicated way of finding a sequence of evolution equations for which this given L is unitarily equivalent for all t and hence describes *isospectral flow* (constancy of the eigenvalues). We illustrate this method using the Schrödinger operator $L = -\frac{d^2}{dx^2} - u(x,t)$. One easily computes $L_t = -u_t$, and so we need to find a skew-adjoint B such that $[B, L] = -u_t$. If we try $B_0 = \frac{d}{dx}$ we get isospectral flow for the evolution equation $u_t + u_x = 0$, since in this case $[B_0, L] = u_x$. If we try the skew-adjoint operator $B_1 = a\frac{d^3}{dx^3} + b\frac{d}{dx} + \frac{d}{dx}b$, with a and b to be determined, then we have

$$[B_1, L] = 3au_x\frac{d^2}{dx^2} + 3au_{xx}\frac{d}{dx} + au_{xxx} + 2bu_x - 4b_x\frac{d^2}{dx^2} - 4b_{xx}\frac{d}{dx} - b_{xxx}.$$

Thus we see that if we choose $a = 4$ and $b = -3u(x,t)$, we recover the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ giving isospectral flow for L. It is easy to see how this can be generalized by trying more complicated (higher order) skew-adjoint B's and finding corresponding nonlinear evolution equations whose flow leaves the eigenvalues of L invariant. In fact there is an infinite sequence of B's that are connected with each odd order of $\frac{\partial}{\partial x}$, and hence an infinite family of evolution equations whose flows U_t all leave invariant the spectrum of the *same eigenvalue problem*. We will see this again in greater generality in Section 6.

4. Other Integrable Nonlinear Evolution Equations

4.1. **The Non-Linear Schrödinger Equation.**

In 1972, Zakharov and Shabat [6] studied the *Nonlinear Schrödinger equation*, hereafter abreviated the NLS equation:

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{4.1}$$

This equation describes the stationary two-dimensional self-focusing and the associated transverse instability of a plane monochromatic wave (see [13]). Unlike the linear Schrödinger equation, it contains a soliton solution, thereby embodying the concept of a wave packet. It represents a balance between linear dispersion, which tends to break up the wave packet, and a focusing effect of the cubic nonlinearity, produced by self interaction of the wave with itself.

Zakharov and Shabat found a Lax pair for this equation and showed that one can solve it using the inverse scattering technique. This was indeed an important discovery, not only because it was a second nonlinear evolution equation solvable by this technique, but also because the associated linear eigenvalue problem that one has to consider was *not* the linear Schrödinger equation in this case.

Remark. There is no connection between the linear and nonlinear Schrödinger equations as they appear in this discussion. The former is the linear eigenvalue problem associated to the KdV nonlinear evolution equation, while the latter is itself a nonlinear evolution equation that is solvable by the inverse scattering technique, to which is associated a different linear eigenvalue problem.

Zakharov and Shabat actually considered a slightly more general version of the NLS equation, with the factor of 2 appearing in the third term replaced by $\frac{2}{1-p^2}$ for some $p \neq \pm 1$. The Lax pair that they discovered for this equation consisted of the following pair of 2×2 matrix differential operators:

$$L = i\begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$$
$$B = ip\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} \frac{|u|^2}{1+p} & iu_x^* \\ -iu_x & -\frac{|u|^2}{1-p} \end{pmatrix}$$

With these choices, the Lax equation $L_t = [B, L]$ is satisfied. Note that if $p = 0$, we are reduced to the NLS equation as defined in equation (4.1). In this case it is easy to see that L and B are indeed self-adjoint and skew-adjoint, respectively, although this is *not* true in general for $p \neq 0$. However, it is still true in the general case that we have isospectral flow for the eigenvalue equation $L\mathbf{\Phi} = \lambda\mathbf{\Phi}$ for $\mathbf{\Phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ if u evolves according to the NLS equation, since the statement and proof of Theorem 3.2 only assumed the validity of the Lax equation (3.2). The asymptotic characteristics of the eigenfunctions (the scattering data) can be calculated at any instant of time from their initial values, and then $u(x,t)$ can be reconstructed at any time t by inverse scattering.

In their paper, Zakharov and Shabat presented the theory of inverse scattering for a general 2×2 eigenvalue problem, which they obtained from the above eigenvalue problem by the following change of variables.

$$\varphi_1 = \sqrt{1-p}e^{-i\left(\frac{\lambda}{1-p^2}\right)x}\psi_2 \qquad \varphi_2 = \sqrt{1+p}e^{-i\left(\frac{\lambda}{1-p^2}\right)x}\psi_1$$
$$q = \frac{iu}{\sqrt{1-p^2}} \qquad \zeta = \frac{\lambda p}{1-p^2}$$

It is an easy calculation to determine that this change of variables results in the following eigenvalue problem:

$$\psi_{1x} + i\zeta\psi_1 = q(x,t)\psi_2$$
$$\psi_{2x} - i\zeta\psi_2 = -q^*(x,t)\psi_1$$

They made a further generalization by replacing the $-q^*$ in the second equation by another arbitrary function. We can write the general Zakharov-Shabat equations as:

$$\varphi_{1x} + i\zeta\varphi_1 = u(x,t)\varphi_2 \tag{4.2}$$

$$\varphi_{2x} - i\zeta\varphi_2 = v(x,t)\varphi_1 \tag{4.3}$$

where the *potentials* $u(x,t)$ and $v(x,t)$ are taken as usual to be rapidly decaying smooth functions. We shall study the general theory of inverse scattering for these equations in Section 5.

Remark. If we take $v = -1$ in equations (4.2) and (4.3) above, and let $\varphi_2 = \psi$, then after some simplification, the system of equations reduces to

$$\psi_{xx} + u(x,t)\psi + \zeta^2\psi = 0$$

which is just the linear Schrödinger equation that was used to solve the KdV equation.

4.2. **The Sine-Gordon Equation.** In 1973, Ablowitz, Kaup, Newell, and Segur, (hereafter abbreviated AKNS) applied the Zakharov-Shabat inverse scattering formalism to the *Sine-Gordon* equation [7]:

$$u_{xt} = \sin(u) \tag{4.4}$$

This equation describes the propogation of ultra-short optical pulses in resonant media, and also arises in statistical mechanics and condensed matter physics. In fact it had also been studied long ago in connection with the theory of surfaces of constant negative curvature (see [13]).

AKNS considered the Zakharov-Shabat equations (4.2) and (4.3) with the substitutions $u \to -\frac{1}{2}u_x$, $v \to \frac{1}{2}u_x$, and chose the following time evolution for the eigenfunctions φ_1, φ_2:

$$(\varphi_1)_t = \frac{i}{4\zeta}\left(\varphi_1\cos(u) + \varphi_2\sin(u)\right) \tag{4.5}$$

$$(\varphi_2)_t = \frac{i}{4\zeta}\left(\varphi_1\sin(u) - \varphi_2\cos(u)\right) \tag{4.6}$$

With this choice for the time evolution of the eigenfunctions, if we assume isospectral flow, one can easily check under what conditions the two sets of equations (4.2)-(4.3) and (4.5)-(4.6) are consistent by cross differentiating and equating mixed partial derivatives. The result is that u must evolve according to the Sine-Gordon equation 4.4. Hence the methods of inverse scattering can be applied to this equation using the Zakharov-Shabat formalism which we shall describe in the next section.

5. Inverse Scattering

5.1. **The Scattering Data.** We will know outline the general methods and results of inverse scattering for the linear Schrödinger equation and the general Zakharov-Shabat eigenvalue problems, including the derivation of the linear integral equation that characterizes the inverse scattering problem. We begin with the Zakharov-Shabat equations (4.2) and (4.3). As usual, the potentials u and v are assumed to decay sufficiently rapidly. This is of vital importance in the results that are to follow, as it will be used many times. To define the scattering data, we study the following four solutions to these equations, given as 2×1 column vectors, which are defined by their asymptotic behaviour, where $\zeta = \xi + i\eta$ is the eigenvalue:

$$\mathbf{\Phi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x} \text{ as } |x| \to -\infty \qquad \mathbf{\Psi} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x} \text{ as } |x| \to +\infty \tag{5.1}$$

$$\tilde{\mathbf{\Phi}} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\zeta x} \text{ as } |x| \to -\infty \qquad \tilde{\mathbf{\Psi}} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x} \text{ as } |x| \to +\infty \tag{5.2}$$

Note that construction of the scattering data from the eigenvalue problem and inverse scattering is studied at a fixed time t, where t is only a parameter. Thus we will omit the explicit time dependence in all equations in this section. The study of the time evolution of the scattering data is a separate problem that requires the use of the nonlinear evolution equation, and this does not concern us in this section. Now one easily computes the *Wronskians* $W(\mathbf{\Phi}, \mathbf{\Psi}) = \varphi_1\psi_2 - \varphi_2\psi_1$ to be

$$W(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}) = -1 \tag{5.3}$$

$$W(\mathbf{\Psi}, \tilde{\mathbf{\Psi}}) = -1. \tag{5.4}$$

Thus both $\mathbf{\Phi}, \tilde{\mathbf{\Phi}}$ and $\mathbf{\Psi}, \tilde{\mathbf{\Psi}}$ are pairs of linearly independent solutions, so we may write

$$\mathbf{\Phi} = a(\zeta)\tilde{\mathbf{\Psi}} + b(\zeta)\mathbf{\Psi} \tag{5.5}$$

$$\tilde{\mathbf{\Phi}} = -\tilde{a}(\zeta)\mathbf{\Psi} + \tilde{b}(\zeta)\tilde{\mathbf{\Psi}} \tag{5.6}$$

for some functions a, b, $\tilde{a}$, and $\tilde{b}$, where the minus sign is chosen for future notational convenience.

Now using equations (5.3), (5.4), (5.5), and (5.6), one easily verifies the relation

$$a\tilde{a} + b\tilde{b} = 1$$

Lemma 5.1. *The Zakharov-Shabat eigenvalue problem of equations* (4.2) *and* (4.3) *with the boundary conditions for* $\mathbf{\Phi}$ *of equations* (5.1) *and* (5.2) *is equivalent to the integral equation*

$$\varphi_1(x,\zeta)e^{i\zeta x} = 1 + \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz u(y)v(z)e^{2i\zeta(y-z)}\varphi_1(z,\zeta)e^{i\zeta z}. \tag{5.7}$$

Proof. Differentiating the first of the Zakharov-Shabat equations and substituting the second, we have:

$$\varphi_{1xx} + \zeta^2\varphi_1 = u_x\varphi_2 + uv\varphi_1.$$

Now it is easy to check that equation (5.7) satisfies this equation as long as we define

$$\varphi_2(x) = \int_{-\infty}^{x} v(z)e^{i\zeta(x-z)}\varphi_1(z,\zeta)dz$$

and with this choice the other Zakharov-Shabat equation is also satisfied. The constant term arises from the boundary condition that $\varphi_1 e^{i\zeta x} \to 1$ as $|x| \to -\infty$. □

One can now use this integral equation to show (see [11]) that if u and v decay faster than any exponential, then the components of $\mathbf{\Phi}e^{i\zeta x}$, $\mathbf{\Psi}e^{-i\zeta x}$, $\tilde{\mathbf{\Phi}}e^{-i\zeta x}$, and $\tilde{\mathbf{\Psi}}e^{i\zeta x}$ are *entire* functions of ζ. This implies in particular that $a(\zeta)$, $\tilde{a}(\zeta)$, $b(\zeta)$, and $\tilde{b}(\zeta)$ are entire functions.

Remark. It is not necessary to make these rather strict assumptions on u and v to make progress on the inverse scattering problem, but since we are only outlining the main ideas, we shall make this simplifying assumption.

Proposition 5.2. *The asymptotic behaviour of $a(\zeta)$ and $\tilde{a}(\zeta)$ for large $|\zeta|$ is given by*

$$a(\zeta) = 1 - \frac{1}{2i\zeta}\int_{-\infty}^{\infty} u(y)v(y)dy + \mathcal{O}\left(\frac{1}{\zeta^2}\right)$$
$$\tilde{a}(\zeta) = 1 + \frac{1}{2i\zeta}\int_{-\infty}^{\infty} u(y)v(y)dy + \mathcal{O}\left(\frac{1}{\zeta^2}\right).$$

Proof. Using Lemma 5.1 and its proof, integration by parts gives large $|\zeta|$ expansions for $\mathbf{\Phi}e^{i\zeta x}$, and analogous integral equations for $\tilde{\mathbf{\Phi}}$, $\mathbf{\Psi}$, and $\tilde{\mathbf{\Psi}}$ give similar asymptotic expansions. Now use of equations (5.5) and (5.6) yields the desired result. The details are elementary and are left to the reader. □

The scattering data also contains information about the *discrete eigenvalues*. These are defined to be the points ζ_j and $\tilde{\zeta}_j$ where $a(\zeta)$ and $\tilde{a}(\zeta)$ vanish, respectively. They are discrete since a and $\tilde{a}$ have only isolated zeroes. We will soon see why these points are significant. At these points ζ_k, $\tilde{\zeta}_k$, examination of equations (5.5) and (5.6) shows that $\mathbf{\Phi}(x,\zeta_k) = D_k\mathbf{\Psi}(x,\zeta_k)$ and $\tilde{\mathbf{\Phi}}(x,\tilde{\zeta}_k) = \tilde{D}_k\tilde{\mathbf{\Psi}}(x,\tilde{\zeta}_k)$ for some constants $D_k = b(\zeta_k)$ and $\tilde{D}_k = \tilde{b}(\tilde{\zeta}_k)$.

Theorem 5.3. *Under the hypothesis that ensure analyticity of a and $\tilde{a}$ in the entire complex plane, there are only finitely many discrete eigenvalues for the Zakharov-Shabat problem.*

Proof. By Proposition 5.2, both a and $\tilde{a}$ approach 1 as $|\zeta| \to \infty$. Thus these functions are bounded away from zero outside some bounded region. Since the zeroes are isolated and lie in a compact set, there can only be a finite number of them. □

An important special case occurs when $v = u^*$. Then it is easy to check that the following symmetries hold:

(5.8) $$\tilde{\mathbf{\Psi}}(x,\zeta) = \begin{pmatrix}\psi_2^*(x,\zeta^*)\\ \psi_1^*(x,\zeta^*)\end{pmatrix} \qquad \tilde{\mathbf{\Phi}}(x,\zeta) = \begin{pmatrix}-\varphi_2^*(x,\zeta^*)\\ -\varphi_1^*(x,\zeta^*)\end{pmatrix}$$

(5.9) $$\tilde{a}(\zeta) = a^*(\zeta^*) \qquad \tilde{b}(\zeta) = -b^*(\zeta^*)$$

(5.10) $$\tilde{\zeta}_k = \zeta_k^* \qquad \tilde{D}_k = -D_k^* \quad k = 1,\ldots,N$$

5.2. **Inverse Scattering for the Zakharov-Shabat Equations.** We are now in a position to derive the inverse scattering equations for the Zakharov-Shabat problem. Using the boundary conditions in equations (5.1) and (5.2), we can write down the following integral representations for $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$:

(5.11) $$\mathbf{\Psi} = \begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x} + \int_x^{\infty}\boldsymbol{K}(x,s)e^{i\zeta s}ds$$

(5.12) $$\tilde{\mathbf{\Psi}} = \begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x} + \int_x^{\infty}\tilde{\boldsymbol{K}}(x,s)e^{-i\zeta s}ds$$

The integral terms involving $\boldsymbol{K}$ and $\tilde{\boldsymbol{K}}$ represent the difference between the asymptotic behaviour as $x \to \infty$ and the true eigenfunctions.

Theorem 5.4. *The kernels* $\boldsymbol{K}$ *and* $\tilde{\boldsymbol{K}}$ *are independent of the eigenvalue* ζ*.*

Proof. We will assume the result and show that we can indeed solve for $\boldsymbol{K}$ and $\tilde{\boldsymbol{K}}$ in this case. Substituting the expression for $\boldsymbol{\Psi}$ into the Zakharov-Shabat equations, we obtain

$$\begin{aligned}
0 &= -K_1(x,x)e^{i\zeta x} + \int_x^\infty K_{1x}(x,s)e^{i\zeta s}ds + i\zeta \int_x^\infty K_1(x,s)e^{i\zeta s}ds \\
&\quad - u(x)e^{i\zeta x} - u(x)\int_x^\infty K_2(x,s)e^{i\zeta s}ds \\
0 &= i\zeta e^{i\zeta x} + \int_x^\infty K_{2x}(x,s)e^{i\zeta s}ds - K_2(x,x)e^{i\zeta x} \\
&\quad - i\zeta e^{i\zeta x} - i\zeta \int_x^\infty K_2(x,s)e^{i\zeta s}ds - v(x)\int_x^\infty K_1(x,s)e^{i\zeta s}ds
\end{aligned}$$

Now in the first equation we integrate the third term by parts and similarly for the fourth term in the second equation. This gives the following two equations:

$$\begin{aligned}
0 &= \int_x^\infty e^{i\zeta s}\left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial s}\right)K_1(x,s) - u(x)K_2(x,s)\right]ds \\
&\quad - \left(u(x) + 2K_1(x,x)\right)e^{i\zeta x} + \lim_{s\to\infty} K_2(x,s)e^{i\zeta s} \\
0 &= \int_x^\infty e^{i\zeta s}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right)K_2(x,s) - v(x)K_1(x,s)\right]ds - \lim_{s\to\infty} K_2(x,s)e^{i\zeta s}.
\end{aligned}$$

Now if we insist on imposing the boundary conditions that $K_1(x,x) = -\frac{1}{2}u(x)$ and $\lim_{s\to\infty}\boldsymbol{K}(x,s) = 0$, then we see that necessary and sufficient conditions (by continuity) for the Zakharov-Shabat equations to be satisfied is that

$$\begin{aligned}
\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial s}\right)K_1(x,s) - u(x)K_2(x,s) &= 0 \\
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right)K_2(x,s) - v(x)K_1(x,s) &= 0
\end{aligned}$$

We can now see that a solution exists by introducing the new coordinates $\mu = \frac{1}{2}(x+s)$ and $\nu = \frac{1}{2}(x-s)$ in which the equations and their boundary conditions become

$$\begin{aligned}
&\frac{\partial}{\partial \nu}K_1(\mu,\nu) - u(\mu+\nu)K_2(\mu,\nu) = 0 \qquad && K_1(\mu,0) = -\frac{1}{2}u(\mu) \\
&\frac{\partial}{\partial \mu}K_2(\mu,\nu) - v(\mu+\nu)K_1(\mu,\nu) = 0 \qquad && \lim_{s\to\infty}\boldsymbol{K}(\mu,\nu) = 0
\end{aligned}$$

Now from the theory of integral equations the solution exists and is unique. The argument for $\tilde{\boldsymbol{K}}$ is similar. □

Let us rewrite equations (5.5) and (5.6) as

$$\frac{\boldsymbol{\Phi}(x,\zeta)}{a(\zeta)} = \tilde{\boldsymbol{\Psi}}(x,\zeta) + \frac{b(\zeta)}{a(\zeta)}\boldsymbol{\Psi}(x,\zeta)$$

and substitute this into equations (5.11) and (5.12). We have

(5.13)
$$\frac{\boldsymbol{\Phi}(x,\zeta)}{a(\zeta)} = \begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x} + \int_x^\infty \tilde{\boldsymbol{K}}(x,s)e^{-i\zeta s}ds + \frac{b(\zeta)}{a(\zeta)}\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x} + \int_x^\infty \boldsymbol{K}(x,s)e^{i\zeta s}ds.$$

All the zeroes of $a(\zeta)$ lie in a bounded region, so we can choose a contour $\mathcal{C}$ starting at $\zeta = -\infty + i0^+$ and ending at $\zeta = +\infty + i0^+$ (in case there are any zeroes on the real axis) that passes above all the zeroes of $a(\zeta)$. Now we use the fact that the delta function can be represented as $\delta(x) = \frac{1}{2\pi}\int_{\mathcal{C}} e^{i\zeta x} d\zeta$ and operate on equation (5.13) with the operator given by $\frac{1}{2\pi}\int_{\mathcal{C}} \left(e^{i\zeta y} \cdot\right) d\zeta$ for $y > x$ to obtain, after interchanging integrals:

$$I = \tilde{\boldsymbol{K}}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^\infty \boldsymbol{K}(x,s) F(s+y) ds \tag{5.14}$$

$$F = \frac{1}{2\pi}\int_{\mathcal{C}} \frac{b(\zeta)}{a(\zeta)} e^{i\zeta x} d\zeta \tag{5.15}$$

$$I \equiv \frac{1}{2\pi}\int_{\mathcal{C}} \frac{\boldsymbol{\Phi}(x,s)}{a(s)} e^{i\zeta y} d\zeta \tag{5.16}$$

Since $\boldsymbol{\Phi} e^{i\zeta x}$ is analytic and $y > x$, we can close the contour $\mathcal{C}$ by adding a semicircle of radius R to get a total integral of zero and the path we added contributes nothing in the limit, since the integral along that path is bounded by a term of the form $e^{-R(y-x)}$ which goes to zero as $R \to \infty$. Here we use the fact that $\boldsymbol{\Phi}$ and $a(\zeta)$ are both bounded. Thus we can conclude that $I \equiv 0$. A similar analysis of $\tilde{\boldsymbol{\Phi}} = -\tilde{a}\boldsymbol{\Psi} + \tilde{b}\tilde{\boldsymbol{\Psi}}$ gives the analogous equation

$$0 = \boldsymbol{K}(x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{F}(x+y) - \int_x^\infty \tilde{\boldsymbol{K}}(x,s) \tilde{F}(s+y) ds \tag{5.17}$$

$$\tilde{F}(x) = \frac{1}{2\pi}\int_{\tilde{\mathcal{C}}} \frac{\tilde{b}(\zeta)}{\tilde{a}(\zeta)} e^{-i\zeta x} d\zeta \tag{5.18}$$

where $\tilde{\mathcal{C}}$ is an analogous contour passing below all the zeroes of $\tilde{a}(\zeta)$. A special case occurs when both a and $\tilde{a}$ do not vanish on the real axis and have only isolated, simple zeroes. Then by closing the contours along the real axis and applying the residue theorem, one obtains:

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i\sum_{j=1}^{N} \frac{b(\zeta_j)}{a'(\zeta_j)} e^{i\zeta_j x}$$

$$\tilde{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{b}(\xi)}{\tilde{a}(\xi)} e^{-i\xi x} d\xi + i\sum_{j=1}^{\tilde{N}} \frac{\tilde{b}(\tilde{\zeta}_j)}{\tilde{a}'(\tilde{\zeta}_j)} e^{-i\tilde{\zeta}_j x}$$

We can put the equations (5.14)–(5.16) and (5.17)–(5.18) into a single matrix equation by defining

$$\mathcal{K} = \begin{pmatrix} \tilde{K}_1 & K_1 \\ \tilde{K}_2 & K_2 \end{pmatrix} \qquad \mathcal{F} = \begin{pmatrix} 0 & -\tilde{F} \\ F & 0 \end{pmatrix}.$$

The equations then become

$$\mathcal{K}(x,y) + \mathcal{F}(x+y) + \int_x^\infty \mathcal{K}(x,s)\mathcal{F}(s+y) ds = 0 \tag{5.19}$$

The physically significant case $v = u^*$ offers several simplifications. In addition to equations (5.8), (5.9), and (5.10) we have:

$$\tilde{F}(x) = -F^*(x) \qquad \tilde{\boldsymbol{K}}(x,y) = \begin{pmatrix} K_2^*(x,y) \\ K_1^*(x,y) \end{pmatrix}$$

and equation (5.19) becomes

$$K_1(x,y) + F^*(x+y) + \int_x^\infty \int_x^\infty K_1(x,z)F(z+s)F^*(s+y)dsdz = 0$$

which is the integral equation for inverse scattering in this case. The potential u is given by

$$u(x) = -2K_1(x,x)$$

Again, the theory of integral equations assures us of existence and uniqueness of solutions for this equation. See [11] for details. This completes the overview of inverse scattering for the Zakharov-Shabat eigenvalue problem.

5.3. **Inverse Scattering for the Linear Schrödinger Equation.** We will now discuss some of the details of inverse scattering for the linear Schrödinger equation. A very extensive treatment was done by Deift and Trubowitz in [10]. Now the eigenvalue problem is

$$\varphi_{xx} + (\lambda + u(x))\,\varphi = 0 \tag{5.20}$$

and to define the scattering data we consider solutions to this equation satisfying the following asymptotic behaviour:

$$\begin{aligned} &\varphi \sim e^{-ikx} && \text{as } x \to -\infty \\ &\psi \sim e^{ikx} && \text{as } x \to +\infty \\ &\tilde{\psi} \sim e^{-ikx} && \text{as } x \to +\infty \end{aligned}$$

where $\lambda = k^2$. Calculation of the Wronskian of ψ and $\tilde{\psi}$ shows that they are linearly independent for $k \neq 0$. Thus there exist $a(k)$, $b(k)$ such that

$$\varphi(x,k) = a(k)\tilde{\psi}(x,k) + b(k)\psi(x,k) \tag{5.21}$$

To compare with the analysis in Section 2.5, the reflection and transmission coefficients are given by $\rho(k) = \frac{b(k)}{a(k)}$ and $\tau(k) = \frac{1}{a(k)}$. The discrete eigenvalues $\lambda_n = -\kappa_n^2$, are defined where $\varphi_n(x) = \varphi(x, i\kappa_n)$ and $\psi_n(x) = \psi(x, i\kappa_n)$ both vanish as $|x| \to \infty$ and hence correspond to bound states. These occur at the zeroes of $a(k)$, where $\varphi_n = D_n\psi_n$ for some constant D_n. If we define $C_n = \frac{D_n}{a'(i\kappa_n)} = \frac{b(i\kappa_n)}{a'(i\kappa_n)}$, and follow a similar development as in Section 5.2, we obtain the following results for the inverse scattering problem. First we define

$$F(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(k)e^{ik\xi}dk - i\sum_{n=1}^{N} C_n e^{-\kappa_n\xi}$$

and then solve the Gel'fan-Levitan-Marchenko linear integral equation for $K(x,y)$ with $y \geq x$,

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(y+z)dz = 0$$

subject to the boundary condition that $K(x,z) \to 0$ as $z \to \infty$. Then the potential $u(x,t)$ that gives rise to these scattering data is given by

$$u(x,t) = 2\frac{d}{dx}K(x,x).$$

All that remains is to relate the C_n's to the c_n's of the scattering data as defined in Section 2.5. This is given by the following lemma.

Lemma 5.5. *With the definitions of C_n and c_n given above and in equation* (2.12), *the following relation holds:*

$$-iC_n = c_n^2$$

Proof. We use the fact that in the linear Schrödinger equation, the discrete eigenvalues are simple and occur only on the imaginary axis (see [10]). A simple calculation using equations (5.20) and (5.21) yields the following relation at $k = i\kappa_n$:

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial k}\frac{\partial \varphi}{\partial x} - \varphi\frac{\partial^2 \varphi}{\partial x \partial k}\right) = 2i\kappa_n \varphi^2 \tag{5.22}$$

As $x \to +\infty$, we have $\varphi_n \sim D_n e^{-\kappa_n x}$ and $\partial_k \varphi_n \sim a'(i\kappa_n)$. Also, as $x \to -\infty$, the eigenfunction φ_n and all its x and k derivatives go to zero exponentially. Putting these results into equation (5.22) gives the following equation:

$$a'(i\kappa_n) = -i\left(\frac{\int_{-\infty}^{\infty}\varphi_n^2 dx}{D_n}\right).$$

Using $a'(i\kappa_n) = \frac{D_n}{C_n}$, we have

$$-iC_n = \frac{D_n^2}{\int_{-\infty}^{\infty}\varphi_n^2 dx} = c_n^2$$

where we have used the fact that the definitions of c_n and D_n agree when the eigenfunctions have been normalized to unit norm. □

6. Ablowitz-Kaup-Newell-Segur Formalism

6.1. **Time Evolution of the Scattering Data.** We will present a method, originally devised by AKNS, of obtaining, given any suitable linear eigenvalue problem, nonlinear evolution equations solvable by the inverse scattering method which keep its spectrum invariant. In general, the associated evolution equations can then be solved if the inverse scattering procedure can be carried out for the particular eigenvalue problem.

The method is very general, but we will concentrate on the case when $\mathbf{\Phi}$ is a 2×1 column vector of functions with components φ_1 and φ_2. They began by assuming that the eigenvalue problem is that proposed by Zakharov and Shabat from equations (4.2) and (4.3), and considered the most general linear time evolution for the function $\mathbf{\Phi}$:

$$\varphi_{1x} = -i\zeta\varphi_1 + u(x,t)\varphi_2 \qquad \varphi_{1t} = A\varphi_1 + B\varphi_2 \tag{6.1}$$

$$\varphi_{2x} = i\zeta\varphi_2 + v(x,t)\varphi_1 \qquad \varphi_{2t} = C\varphi_1 + D\varphi_2 \tag{6.2}$$

where A, B, C, and D are scalar functions independent of $\mathbf{\Phi}$, but of course can and in general will be functions of u and v and their various derivatives. If we choose $v = -1$, we get the linear Schrödinger equation as remarked in Section 4.1 and if we let $v = \pm u^*$, we also get some physically significant evolution equations. We want to derive evolution equations for u and v that will leave the spectrum of equations (6.1) and (6.2) invariant under their flow. Thus we assume $\zeta_t = 0$ and proceed by cross-differentiating equations (6.1) and (6.2) and equating the mixed partial derivatives. This gives the following system of equations for A, B, C, and D:

$$A_x = uC - vB \tag{6.3}$$

$$C_x - 2i\zeta C = v_t + v(A - D) \tag{6.4}$$

$$B_x + 2i\zeta B = u_t - u(A - D) \tag{6.5}$$

$$-D_x = uC - vB \tag{6.6}$$

We can choose $A = -D$ to simplify to just three equations in three unknowns. We will restrict our attention to the following boundary conditions:

$$B, C \to 0 \qquad \text{as } |x| \to \infty \tag{6.7}$$

$$A \to \Omega(\zeta) \qquad \text{as } |x| \to \infty \tag{6.8}$$

for some function $\Omega(\zeta)$. Note this is not a very restrictive hypothesis since all the integrable nonlinear evolution equations described thus far satisfy these requirements.

Now solutions of these *compatibility* equations (6.3)–(6.6) give rise to evolution equations for u and v. If we try a polynomials in ζ for A, B, and C of up to second order, then a routine but tedious calculation yields the following equations:

$$-\frac{1}{2}\mu u_{xx} = u_t - \mu u^2 v$$
$$\frac{1}{2}\mu v_{xx} = v_t + \mu u v^2$$

where μ is a constant that was the coefficient of ζ^2 in the expansion of A. This is a coupled pair of nonlinear evolution equations similar to the nonlinear Schrödinger equation, and indeed we obtain exactly the NLS equation if we choose $v = -u^*$ and $\mu = 2i$. With these choices equations (6.3)–(6.6) are satisfied when $u(x,t)$ evolves according to the NLS equation.

In an exactly analogous method, if we try third order polynomials in ζ we can, with proper choice of the coefficients, reproduce the KdV and MKdV equations. Similarly expanding in negative powers of ζ will yield the Sine-Gordon equation and other physically interesting nonlinear evolution equations. See [11] for details.

We can now determine the time evolution of the scattering data for the AKNS equations. We will consider the same eigenfunctions and their asymptotic behaviour as described in equations (5.1) and (5.2).

Definition 6.1. The *time dependent* eigenfunctions that satisfy the AKNS equations (6.1) and (6.2) are given by:

$$\mathbf{\Phi}^{(t)} = \mathbf{\Phi} e^{\Omega(\zeta)t} \qquad \mathbf{\Psi}^{(t)} = \mathbf{\Psi} e^{-\Omega(\zeta)t} \tag{6.9}$$

$$\tilde{\mathbf{\Phi}}^{(t)} = \tilde{\mathbf{\Phi}} e^{-\Omega(\zeta)t} \qquad \tilde{\mathbf{\Psi}}^{(t)} = \tilde{\mathbf{\Psi}} e^{\Omega(\zeta)t} \tag{6.10}$$

If we differentiate equations (6.9) and (6.10) we have, for example,

$$\frac{\partial \mathbf{\Phi}^{(t)}}{\partial t} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \mathbf{\Phi}^{(t)} \implies \frac{\partial \mathbf{\Phi}}{\partial t} = \begin{pmatrix} A - \Omega(\zeta) & B \\ C & -A - \Omega(\zeta) \end{pmatrix} \mathbf{\Phi}.$$

If we use equations (6.7) and (6.8) and the fact that as $|x| \to \infty$, $\mathbf{\Phi} \sim \begin{pmatrix} ae^{-i\zeta x} \\ be^{i\zeta x} \end{pmatrix}$, we see that

$$a_t = 0 \qquad\qquad b_t = -2\Omega(\zeta)b$$

which can be trivially integrated to obtain:

$$a(\zeta,t) = a(\zeta,0) \qquad\qquad b(\zeta,t) = b(\zeta,0)e^{-2\Omega(\zeta)t}$$

Thus we have obtained the time evolution of a and b. Similarly we can determine the time evolution of $\tilde{a}$ and $\tilde{b}$ using $\tilde{\mathbf{\Phi}}$. The results are:

$$\tilde{a}_t = 0 \qquad\qquad \tilde{b}_t = 2\Omega(\zeta)\tilde{b}$$

All that remains is to determine the time evolution of the scattering data corresponding to the discrete eigenvalues.

$$\begin{aligned} C_j(t) &= \frac{b(\zeta_j,t)}{a'(\zeta_j,t)} \\ &= \frac{b(\zeta_j,0)}{a'(\zeta_j,0)}e^{-2\Omega(\zeta)t} \\ &= C_j(0)e^{-2\Omega(\zeta)t} \end{aligned}$$

and similarly $\left(C_j^*\right)_t = 2\Omega(\zeta)C_j^*$.

Just as in the solution of the KdV equation in Section 2.5, we see that the time evolution of the scattering data is an infinite set of uncoupled ordinary differential equations.

6.2. **The General AKNS Evolution Equations.** We can put this general class of nonlinear evolution equations associated to the Zakharov-Shabat problem into a form that will be used in Section 7. Examination of equations (6.1) and (6.2) shows that the eigenvalue equations are equivalent to:

$$\left(\varphi_1^2\right)_x + 2i\zeta\varphi_1^2 = 2u\varphi_1\varphi_2 \tag{6.11}$$

$$\left(\varphi_2^2\right)_x - 2i\zeta\varphi_2^2 = 2v\varphi_1\varphi_2 \tag{6.12}$$

$$\left(\varphi_1\varphi_2\right)_x = u\varphi_2^2 + v\varphi_1^2 \tag{6.13}$$

With the boundary conditions of equations (6.7) and (6.8), we can write

$$\begin{aligned} \varphi_1\varphi_2 &= I_-\left(u\varphi_2^2 + v\varphi_1^2\right) \\ \left(\varphi_1^2\right)_x &= -2i\zeta\varphi_1^2 + 2uI_-\left(u\varphi_2^2 + v\varphi_1^2\right) \\ \left(\varphi_2^2\right)_x &= 2i\zeta\varphi_2^2 + 2vI_-\left(u\varphi_2^2 + v\varphi_1^2\right) \end{aligned}$$

where we have defined the integral operator $I_-(\cdot) = \int_{-\infty}^x (\cdot)dy$. Now let us denote by $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ the solutions satisfying equations (5.1) and (5.2). Then we can write

$$\zeta\mathbf{\Phi}_i = \mathcal{L}\mathbf{\Phi}_i \qquad i = 1,2$$

where we have defined

$$\mathbf{\Phi}_1 = \begin{pmatrix} \varphi_1^2 \\ \varphi_2^2 \end{pmatrix} \qquad \mathbf{\Phi}_2 = \begin{pmatrix} \tilde{\varphi}_1^2 \\ \tilde{\varphi}_2^2 \end{pmatrix} \qquad \mathcal{L} = \frac{1}{2i}\begin{pmatrix} -\partial_x + 2uI_-v & 2uI_-u \\ -2vI_-v & \partial_x - 2vI_-u \end{pmatrix}. \tag{6.14}$$

Hence if $\Omega(\zeta)$ is analytic at ζ_0, then $\Omega(\zeta)\mathbf{\Phi}_i = \Omega(\mathcal{L})\Phi_i$ for $i = 1,2$ whenever $|\zeta - \zeta_0| < R$, where R is the radius of convergence at ζ_0.

Now let us define

$$\Delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \qquad N = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

with this notation, equations (6.1) and (6.2) (with $D = -A$) become:

$$\begin{aligned} \mathbf{\Phi}_x &= i\zeta\Delta\mathbf{\Phi} + N\mathbf{\Phi} \\ \mathbf{\Phi}_t &= Q\mathbf{\Phi} \end{aligned}$$

Now cross differentiating this equation and as always assuming isospectral flow ($\zeta_t = 0$), we obtain

$$N_t = Q_x + i[Q, \Delta] + [Q, N] \tag{6.15}$$

This equation can be greatly simplified by considering $S = P^{-1}QP$, where P is the 2×2 matrix $P = \begin{pmatrix} \mathbf{\Phi} & \tilde{\mathbf{\Phi}} \end{pmatrix}$. We know P is invertible because by equation (5.3), $\det(P) = W(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}) = -1$. Substituting $Q = PSP^{-1}$ into equation (6.15) gives, after much cancellation:

$$S_x = P^{-1}N_tP$$

which can be integrated directly to give

$$S = \Omega(\zeta)\Delta + \int_{-\infty}^{x} P^{-1}N_tPdy \tag{6.16}$$

where we have used the boundary condition $\lim_{x\to-\infty} S = \lim_{x\to-\infty} P^{-1}QP = \Omega(\zeta)\Delta$, from equations (5.1) and (5.2). Thus we have found expressions for A, B, and C in terms of u, v, $\mathbf{\Phi}$, and $\tilde{\mathbf{\Phi}}$.

Lemma 6.2. *Using the notation given in equations* (5.1), (5.2), (5.5), *and* (5.6) *for the scattering data,*

$$\lim_{x\to+\infty} S = \Omega(\zeta)\begin{pmatrix} a\tilde{a} - b\tilde{b} & 2\tilde{a}\tilde{b} \\ 2ab & -a\tilde{a} + b\tilde{b} \end{pmatrix}.$$

Proof. Asymptotically, as $x \to +\infty$, we have $\mathbf{\Phi} \sim \begin{pmatrix} ae^{-i\zeta x} \\ be^{i\zeta x} \end{pmatrix}$ and $\tilde{\mathbf{\Phi}} \sim \begin{pmatrix} \tilde{b}e^{-i\zeta x} \\ -\tilde{a}e^{i\zeta x} \end{pmatrix}$. Thus we have

$$\lim_{x\to\infty} S = \lim_{x\to\infty} P^{-1}QP = \lim_{x\to\infty} \begin{pmatrix} \tilde{a}e^{i\zeta x} & \tilde{b}e^{-i\zeta x} \\ be^{i\zeta x} & -ae^{-i\zeta x} \end{pmatrix}\begin{pmatrix} A & B \\ C & -A \end{pmatrix}\begin{pmatrix} ae^{-i\zeta x} & \tilde{b}e^{-i\zeta x} \\ be^{i\zeta x} & -\tilde{a}e^{i\zeta x} \end{pmatrix}$$

from which the result follows trivially. □

If we also compute $\lim_{x\to\infty} S$ using equation (6.16) and the explicit expressions for the entries of $P^{-1}N_tP$, and equates the two results, a lengthy calculation yields the following from the off-diagonal terms:

$$\begin{aligned} \Omega(\zeta)2\tilde{a}\tilde{b} &= \int_{-\infty}^{\infty} \left(\tilde{\varphi}_1^2 v_t - \tilde{\varphi}_2^2 u_t\right) dx \\ \Omega(\zeta)2ab &= \int_{-\infty}^{\infty} \left(-\varphi_1^2 v_t + \varphi_2^2 u_t\right) dx \end{aligned}$$

Now integrating equation (6.13), and using the asymptotic behaviour of the eigenfunctions, we see that

$$\begin{aligned}\int_{-\infty}^{\infty}\left(u\varphi_2^2+v\varphi_1^2\right)dx &= \int_{-\infty}^{\infty}(\varphi_1\varphi_2)_x dx\\ &= \varphi_1\varphi_2\Big|_{-\infty}^{\infty} = ab\end{aligned}$$

The analogous integral with φ_1 and φ_2 replaced by $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ gives $-\tilde{a}\tilde{b}$. We can conveniently combine these last two results in the following equation:

$$\int_{-\infty}^{\infty}\left[\begin{pmatrix} v_t \\ -u_t\end{pmatrix}+2\Omega(\zeta)\begin{pmatrix} v\\ u\end{pmatrix}\right]\cdot\mathbf{\Phi}_i dx = 0 \qquad i=1,2 \tag{6.17}$$

where the $\cdot$ denotes the usual dot product on 2-vectors. We can now use the fact that for an entire $\Omega(\zeta)$, we can write $\Omega(\zeta)\mathbf{\Phi}_i = \Omega(\mathcal{L})\mathbf{\Phi}_i$, but we would like to convert the integral expression above to something that is the integral of the dot product of some 2-vector with $\mathbf{\Phi}_i$. Hence, we need to get the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ so that we can write

$$\begin{aligned}\int_{-\infty}^{\infty}\begin{pmatrix} v\\ u\end{pmatrix}\cdot\Omega(\zeta)\mathbf{\Phi}_i dx &= \int_{-\infty}^{\infty}\begin{pmatrix} v\\ u\end{pmatrix}\cdot\Omega(\mathcal{L})\mathbf{\Phi}_i dx\\ &= \int_{-\infty}^{\infty}\Omega(\mathcal{L}^*)\begin{pmatrix} v\\ u\end{pmatrix}\cdot\mathbf{\Phi}_i dx.\end{aligned}$$

It is important to realize that there are two inner products here—the dot product of 2×1 column vectors, and the usual L^2 inner product given by integration of the product of two functions. Now the adjoint of a 2×2 matrix operator with respect to the first of these inner products is simply the conjugate transpose of the matrix, but we still have to get the adjoint with respect to the integration inner product of each of the entries in the transposed matrix. Simple integration by parts shows that $\partial_x^* = -\partial x$, a fact we used many times implicitly in Section 3.

Lemma 6.3. *If $\alpha(x)$, $\beta(x)$ are two functions, then $(\alpha I_-\beta)^* = \beta I_+\alpha$, where the operator I_+ is defined by $I_+f = \int_x^\infty f dx$.*

Proof. By direct computation,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\left(\alpha I_-\beta g\right)(x)dx &= \int_{-\infty}^{\infty} f(x)\alpha(x)\int_{-\infty}^{x}\beta(y)g(y)dydx\\ &= \int_{-\infty}^{\infty}\beta(y)g(y)\int_{y}^{\infty}\alpha(x)f(x)dxdy\\ &\int_{-\infty}^{\infty}\left(\beta I_+\alpha f\right)(x)g(x)dx.\end{aligned}$$

□

Now using Lemma 6.3 and the remarks preceding it, we see immediately from equation (6.14) that

$$\mathcal{L}^* = \frac{1}{2i}\begin{pmatrix}\partial_x + 2vI+-u & -2vI_+v\\ 2uI_+u & -\partial_x - 2uI_+v\end{pmatrix} \tag{6.18}$$

Thus, from equation (6.17), we see that a *sufficient* condition for the AKNS compatibility equations (6.3)–(6.6) with $D = -A$ to be satisfied is that

$$\begin{pmatrix} v_t \\ -u_t \end{pmatrix} + 2\Omega(\mathcal{L}^*)\begin{pmatrix} v \\ u \end{pmatrix} = 0 \tag{6.19}$$

It can be shown (see [11]) that this equation is also *necessary* if $u = v^*$ (the physically significant case), u decays sufficiently rapidly, A, B, and C satisfy the boundary conditions of equations (6.7) and (6.7), and $\Omega(\zeta)$ is entire. Equation (6.19) is called the general AKNS evolution equation.

7. The Hamiltonian Formulation

7.1. **Review of Hamiltonian Mechanics.** In classical Hamiltonian mechanics, motion of a mechanical system with N degrees of freedom is described by a *phase space* parametrized by $2N$ coordinates q_j, p_j for $j = 1 \ldots N$, and the evolution of these coordinates is given by the *Hamilton equations of motion*,

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \qquad\qquad \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \tag{7.1}$$

where $H = H(q_j, p_j)$ is called the Hamiltonian function for the system. We can define the *Poisson bracket* $\{F, G\}$ of two functions F, G, of the coordinates by

$$\{F, G\} = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right) \tag{7.2}$$

One can substitute the Hamilton equations (7.1) into the definition of the Poisson bracket to obtain the following relations:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \tag{7.3}$$

$$\{q_i, p_j\} = \delta_{ij} \tag{7.4}$$

If we let $F = H$ in equation (7.2) above and use the Hamilton equations (7.1), we can express the equations of motion as

$$\frac{dG}{dt} + \{H, G\} = 0. \tag{7.5}$$

We will now rewrite Hamilton's equations [9] in a form that will be more suitable to their infinite dimensional generalization. We define

$$u = \begin{pmatrix} p_1 \\ \vdots \\ p_N \\ q_1 \\ \vdots \\ q_N \end{pmatrix} \qquad\qquad H_u = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_N} \\ \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_N} \end{pmatrix}$$

With this notation, it is easy to verify that equations (7.1) become

$$\frac{du}{dt} = J_0 H_u$$

where the $2N \times 2N$ matrix J_0 is defined by

$$J_0 = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

and I_N is the $N \times N$ identity matrix. Now a simple computation shows that we can rewrite the Poisson bracket of two functions F and G in terms of the standard inner product $\langle , \rangle$ on $\mathbb{R}^{2N}$ as follows:

$$\{F, G\} = \langle F_u, J_0 G_u \rangle \tag{7.6}$$

It will be also useful to note for Section 7.2 that we can write

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} H(u + \varepsilon v) = \sum_{j=1}^{N} \left(\frac{\partial H}{\partial p_j} v_j + \frac{\partial H}{\partial q_j} v_{N+j} \right) = \langle H_u, v \rangle. \tag{7.7}$$

It can be shown (see [9]) that an arbitrary system $\frac{dv}{dt} = JG_v$ can be put into Hamiltonian form by a linear change of variables if and only if the there exists a nonsingular matrix T such that $TJT^* = J_0$. This is true if and only if J is non-singular and skew-adjoint. For the purposes of our generalization to infinite dimensions, we will drop the non-singular requirement for the following definition.

Definition 7.1. A system of differential equations is said to be *Hamiltonian* if it is of the form

$$\frac{du}{dt} = JH_u$$

for some skew-adjoint linear operator J and some Hamiltonian function H.

From equation (7.5), we see that a function F is constant along the flow of the Hamiltonian system if $\{F, H\} = 0$. In fact, the converse can also be shown to hold [9] as well. Thus, when this condition $\{F, H\} = 0$ holds, the Hamiltonian flows of F and H *commute*.

Definition 7.2. A finite dimensional ($2N$ variables) Hamiltonian system is said to be *completely integrable* if it admits N constants of the motion F_i, $i = 1 \ldots N$, with $F_1 = H$ such that $\{F_i, F_j\} = 0$ for all i, j and which are independent in the sense that the gradients F_{i_u} are linearly independent.

Remark. When two functionals F_i and F_j satisfy $\{F_i, F_j\} = 0$, the functionals are said to be in *involution.* Thus a Hamiltonian system is completely integrable if there exist N linearly independent gradients that are in involution.

The *Liouville Theorem* says any completely integrable system can be canonically transformed (preserving the Hamiltonian structure) to new coordinates known as *action-angle variables* in which the system is completely separable. The equations take the form

$$\frac{dJ_i}{dt} = \frac{\partial H}{\partial \theta_i} = 0 \qquad \qquad \frac{d\theta_i}{dt} = \frac{\partial H}{\partial J_i} = \omega_i$$

The action variables J_i are functions of the F_i and hence constant in time, and the angles θ_i evolve linearly in time: $\theta_i = \omega_i t + a_i$ for some *constants* ω_i, $i = 1 \ldots N$. Thus the equations of motion can be integrated by quadratures in the case of a completely integrable system.

7.2. **Infinite Dimensional Hamiltonian Systems.** We are now ready to generalize the preceding discussions to infinite dimensional systems and to describe the relation to integrable nonlinear evolution equations. Our phase space now will consist of all real valued $\mathcal{C}^\infty$ functions that decay sufficiently rapidly. The inner product on this space is the usual one given by

$$\langle u(x), v(x)\rangle = \int_{-\infty}^{\infty} u(x)v(x)dx.$$

In the infinite dimensional case the coordinates are given by a function $u(x)$, with x being the continuous analogue of the index j in the finite dimensional case. In this situation the gradient H_u is also called the *functional derivative* of H and is denoted $\frac{\delta H}{\delta u}$. We can determine an expression for it by analogy with equation (7.7):

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} H(u+\varepsilon v) = \langle H_u, v\rangle = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u} v(x)dx \tag{7.8}$$

In fact we can show in general that

$$\frac{\partial H}{\partial s} = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u}\frac{\partial u}{\partial s}dx. \tag{7.9}$$

We can use equation (7.8) to derive an explicit expression for the functional derivative. In the infinite dimensional case, the Hamiltonian H is given by an integral over all x of a Hamiltonian *density* $\mathcal{H}$,

$$H(u) = \int_{-\infty}^{\infty} \mathcal{H}\left(u(x)\right) dx$$

Now by repeated integration by parts, and the use of the fact that u decays sufficiently rapidly, equation (7.8) allows us to make the identification:

$$\frac{\delta H}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial u_{xx}} - \frac{\partial^3}{\partial x^3}\frac{\partial}{\partial u_{xxx}} + \cdots\right)\mathcal{H}$$

In analogy with Definition 7.1, an infinite dimensional Hamiltonian system is defined to be a a system of the form

$$u_t = J\frac{\delta H}{\delta u} \tag{7.10}$$

for some (possibly singular) skew-adjoint linear operator J. We will choose $J = \frac{d}{dx}$, which is skew-adjoint with respect to the inner product we have defined, as can easily be checked by integration by parts. It is definitely singular, however.

Remark. We can also formulate the definition of an infinite dimensional Hamiltonian system in a form closely resembling equations (7.1):

$$\frac{dv}{dt} = -\frac{\delta H}{\delta u} \qquad\qquad \frac{du}{dt} = \frac{\delta H}{\delta v} \tag{7.11}$$

for a pair $u(x)$, $v(x)$ of *conjugate variables.* In this form the Poisson bracket $\{F, G\}$ of two functions F, G, becomes

$$\{F, G\} = \int_{-\infty}^{\infty} \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u}\right). \tag{7.12}$$

This is the form of the equations that we will use in Section 7.3.

Returning to equation (7.10), we will choose $J = \frac{d}{dx}$, which is skew-adjoint with respect to the inner product we have defined. It is definitely singular, however. With this choice of J and the Hamiltonian density $\mathcal{H}\mathcal{F}_2 = -u^3 + \frac{1}{2}u_x^2$, equation (7.10) becomes the KdV equation $u_t + 6uu_x + u_{xxx} = 0$. This was first discovered in this form [5] by Zakharov and Faddeev. Note that this Hamiltonian is preciseley the constant of the motion for the KdV equation corresponding to the third non-trivial conservation law given in equation (2.7). The choice $\mathcal{H} = \mathcal{F}_1 = \frac{1}{2}u^2$, which corresponds to another constant of the motion for the KdV equation, gives the equation $u_x + u_t = 0$. Now a routine computation using equation (7.6) and $F_i = \int \mathcal{F}_i$ shows that $\{F_1, F_2\} = 0$, so the two Hamiltonian flows commute verifying again that F_1 is a constant of the motion for the KdV equation. In fact we already know that the KdV equation has an infinite set of conserved quantities, and if we consider the infinite sequence of evolution equations that arise from considering these constants of the motion as Hamiltonians, then we see that the flows of all these equations commute with each other. We are led to speculate that we have what can be defined as an infinite dimensional completely integrable system, and in fact we will see that this is the case in general in Section 7.3.

It is an interesting exercise to see how this formulation of the problem allows us to compute the constancy of the discrete eigenvalues of the Schrödinger operator under the flow determined by the KdV equation. We have

$$L\varphi = \lambda\varphi \tag{7.13}$$

with $L = -\frac{d^2}{dx^2} - u(x,t)$, and the eigenfunctions corresponding to the discrete eigenvalues are normalized so that $\langle \varphi, \varphi \rangle = 1$. Differentiating this identity gives $\langle \varphi, \varphi_t \rangle = 0$. If we replace u by $u + \varepsilon v$, and differentiate equation (7.13) with respect to ε, we obtain

$$L\varphi_t - v\varphi = \lambda\varphi_t + \lambda_t\varphi$$

Taking the inner product of this equation with φ and simplifying, we get

$$\lambda_t = \langle -\varphi^2, v \rangle$$

from which it follows using equation (7.8) that $\frac{\delta\lambda}{\delta u} = -\varphi^2$. Now by definition of the Poisson bracket,

$$\begin{aligned} \{H, \lambda\} &= \left\langle \frac{\delta H}{\delta u}, J\frac{\delta\lambda}{\delta u} \right\rangle \\ &= \int_{-\infty}^{\infty} (3u^2 + u_{xx})(2\varphi\varphi_x)dx \end{aligned}$$

where we have used the explicit forms of the Hamiltonian density for the KdV equation the operator J. Now repeated integration by parts and use of equation (7.13) eventually gives the result that $\{H, \lambda\} = 0$, and so the eigenvalue λ is a constant of the motion, as we determined in Section 2.5.

7.3. **The I.S.T. as a Canonical Transformation.** In this section we will show that the inverse scattering transform is a canonical transformation which converts a nonlinear evolution equation into an infinite sequence of separated ordinary differential equations for the action-angle variables, which can be integrated trivially.

Definition 7.3. A transformation from a set of coordinates $q(x)$, $p(x)$ to a new set $Q(x)$, $P(x)$ is called *canonical* if it preserves the Poisson brackets (Hamiltonian structure):

$$\{Q(x), Q(y)\} = \{P(x), P(y)\} = 0$$
$$\{Q(x), P(y)\} = \delta(x - y)$$

It is easy to see that these are the infinite dimensional generalizations of the finite dimensional commutation relations in equations (7.3) and (7.4). We now present the main result of this section.

Theorem 7.4. *If $\Omega(\zeta)$ is an entire function, then the general AKNS evolution equation* (6.19) *represents an infinite dimensional Hamiltonian system, in the sense of equations* (7.11)*, where u and v play the role of conjugate variables. The map $(u, v) \to S$, where S is the scattering data, is a canonical transformation. The Hamiltonian is given by*

$$H(u, v) = i \sum_{n=0}^{\infty} \omega_n i^n \alpha_n(u, v)$$

where ω_n and α_n are defined by

$$\Omega(\zeta) = \frac{1}{2i} \sum_{n=0}^{\infty} (-2\zeta)^n \omega_n \qquad\qquad \log a(\zeta) = \sum_{n=0}^{\infty} \frac{\alpha_n}{(2i\zeta)^{n+1}}.$$

Proof. This rather lengthy proof will be split into several steps, and we will merely sketch the ideas.

Step 1. We derive expressions for $\frac{\delta a(\zeta)}{\delta u(x)}$ and $\frac{\delta a(\zeta)}{\delta v(x)}$. From equation (4.2) and (4.3), and the asymptotic behaviour given in equation (5.1), we can easily see that

$$\varphi_1(x, \zeta) e^{i\zeta x} = 1 + \int_{-\infty}^{x} u(y) \varphi_2(y, \zeta) e^{i\zeta y} dy.$$

Using the characterization of the functional derivative given in equation (7.9), we can immediately read off that

$$\frac{\delta \varphi_1(x)}{\delta u(y)} = \theta(x - y) \varphi_2(y, \zeta) e^{i\zeta(y-x)}$$

where $\theta(x)$ is the Heaviside step function. Taking the limit as y approaches x from below, we get

$$\frac{\delta \varphi_1(x)}{\delta u(x)} = \varphi_2(x, \zeta) \tag{7.14}$$

and similar calculations determine the rest of the functional derivatives:

$$\frac{\delta \varphi_2(x)}{\delta v(x)} = \varphi_1(x, \zeta) \tag{7.15}$$

$$\frac{\delta \varphi_1(x)}{\delta v(x)} = 0 \qquad\qquad \frac{\delta \varphi_2(x)}{\delta u(x)} = 0 \tag{7.16}$$

$$\frac{\delta \psi_{1,2}(x)}{\delta u(x)} = 0 \qquad\qquad \frac{\delta \psi_{1,2}(x)}{\delta v(x)} = 0 \tag{7.17}$$

Direct differentiation with respect to x and use of the Zakharov-Shabat equations shows that $\varphi_1\psi_2 - \varphi_2\psi_1$ is a constant, and the asymptotic information as $x \to \infty$ determines this constant to be $a(\zeta)$, so by equations (7.14)–(7.17), we find

$$\frac{\delta a(\zeta)}{\delta u(x)} = \varphi_2(x,\zeta)\psi_2(x,\zeta) \qquad \frac{\delta a(\zeta)}{\delta v(x)} = -\varphi_1(x,\zeta)\psi_1(x,\zeta). \tag{7.18}$$

Step 2. We derive expressions for $\frac{\delta \log a(\zeta)}{\delta u(x)}$ and $\frac{\delta \log a(\zeta)}{\delta v(x)}$. Examination of the Zakharov-Shabat equations shows they can be rearranged in the following form.

$$(\varphi_1\psi_1)_x + 2i\zeta\varphi_1\psi_1 = u\left(\varphi_1\psi_2 + \varphi_2\psi_1\right) \tag{7.19}$$

$$(\varphi_2\psi_2)_x - 2i\zeta\varphi_2\psi_2 = v\left(\varphi_1\psi_2 + \varphi_2\psi_1\right) \tag{7.20}$$

$$\left(\varphi_1\psi_2 + \varphi_2\psi_1\right)_x = 2u\varphi_2\psi_2 + 2v\varphi_1\psi_1 \tag{7.21}$$

Now integrating equation (7.21) and using the asymptotic behaviour, we get the relation

$$\varphi_1\psi_2 + \varphi_2\psi_1 = a(\zeta) - 2\int_x^\infty u\varphi_2\psi_2 + v\varphi_1\psi_1. \tag{7.22}$$

Substituting equation (7.22) back into equations (7.19) and (7.20) and rearranging, we have

$$\zeta\begin{pmatrix}\varphi_2\psi_2\\ -\varphi_1\psi_1\end{pmatrix} = \mathcal{L}^*\begin{pmatrix}\varphi_2\psi_2\\ -\varphi_1\psi_1\end{pmatrix} - \frac{a(\zeta)}{2i}\begin{pmatrix}v\\ u\end{pmatrix}$$

where $\mathcal{L}^*$ was defined in equation (6.18). We can at least locally, within some radius of convergence, invert this equation to get

$$\begin{pmatrix}\varphi_2\psi_2\\ -\varphi_1\psi_1\end{pmatrix} = -\frac{a(\zeta)}{2i\zeta}\left(1 - \frac{\mathcal{L}^*}{\zeta}\right)^{-1}\begin{pmatrix}v\\ u\end{pmatrix} = -\frac{a(\zeta)}{2i\zeta}\sum_{n=0}^\infty\left(\frac{\mathcal{L}^*}{\zeta}\right)^n\begin{pmatrix}v\\ u\end{pmatrix}. \tag{7.23}$$

Now using equation (7.23) and equations (7.18) from Step 1, we obtain:

$$\frac{\delta \log a(\zeta)}{\delta u(x)} = -\frac{1}{2i\zeta}\sum_{n=0}^\infty\left(\frac{\mathcal{L}^*}{\zeta}\right)^n v \qquad \frac{\delta \log a(\zeta)}{\delta v(x)} = -\frac{1}{2i\zeta}\sum_{n=0}^\infty\left(\frac{\mathcal{L}^*}{\zeta}\right)^n u$$

Step 3. Using the results of Step 2, we can now directly verify that

$$\frac{\delta H}{\delta u} = -v_t \qquad \frac{\delta H}{\delta v} = u_t$$

with the definition of H given in the statement of the theorem. We show the first equation, for example.

$$\frac{\delta \log a(\zeta)}{\delta u(x)} = -\frac{1}{2i\zeta}\sum_{n=0}^\infty\left(\frac{\mathcal{L}^*}{\zeta}\right)^n v = \sum_{n=0}^\infty \frac{1}{(2i\zeta))^{n+1}}\frac{\delta\alpha_n}{\delta u}$$

Equating powers of ζ we see that

$$\frac{\delta\alpha_n}{\delta u} = -(2i)^n\mathcal{L}^{*n}v.$$

Now we substitute this result into the functional derivative of H:

$$\begin{aligned}\frac{\delta H}{\delta u} &= i\sum_{n=0}^{\infty}\omega_n i^n \frac{\delta\alpha_n}{\delta u}\\ &= \frac{2}{2i}\sum_{n=0}^{\infty}\omega_n(-2\mathcal{L}^*)^n v\\ &= 2\Omega(\mathcal{L}^*)v = -v_t\end{aligned}$$

where we have used the definitions of α_n and ω_n as well as the form of the general AKNS evolution equation. The other Hamiltonian equation is proved similarly. □

Lemma 7.5. *The infinite set of constants α_n are* all *in involution with the Hamiltonian H for the general AKNS evolution equation.*

Proof. we use the results of the previous theorem directly, and the definition of the Poisson bracket in equation (7.12):

$$\begin{aligned}0 = \frac{d\alpha_n}{dt} &= \int_{-\infty}^{\infty}\frac{\delta\alpha_n}{\delta u}\frac{du}{dt} + \frac{\delta\alpha_n}{\delta v}\frac{dv}{dt}dx\\ &= \int_{-\infty}^{\infty}\frac{\delta\alpha_n}{\delta u}\frac{\delta H}{\delta v} - \frac{\delta\alpha_n}{\delta v}\frac{\delta H}{\delta u}dx\\ &\{\alpha_n, H\}\end{aligned}$$

□

Thus we have an infinite set of conserved quantities in involution with H and this suggests that such a system might be completely integrable. In fact, the following is true (we could also somewhat weaken the hypotheses— see [12] and [11].):

Theorem 7.6. *If all the scattering data $(a, \tilde{a}, b, \tilde{b})$ are entire functions and $a(\zeta)$, $\tilde{a}(\zeta)$ have only simple zeroes and do not vanish on the real axis, then the following constitute a set of action-angle variables for the system:*

$$\begin{aligned}P(\zeta) &= \log\left(a(\zeta)\tilde{a}(\zeta)\right) & Q(\zeta) &= -\frac{1}{\pi}\log b(\zeta)\\ P_j &= \zeta_j & Q_j &= -2i\log(c_j)\\ \tilde{P}_j &= \tilde{\zeta}_j & \tilde{Q}_j &= -2i\log(\tilde{c}_j)\end{aligned}$$

In particular, we have a completely integrable system. The mapping given by the inverse scattering transform of the potential to its scattering data is a canonical transformation to action-angle variables.

Proof. See [12] for a proof. □

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