QIC 710 / CO 681 / AMATH 871 / CS 768 / PHYS 767 Fall 2018

Homework 3

Due date: Tuesday, November 13

1. **Distinguishing qubit states (20%)** Let

$$|\psi_{00}\rangle = |0\rangle, \ \ |\psi_{01}\rangle = |1\rangle, \ \ |\psi_{10}\rangle = |+\rangle, \ \ |\psi_{11}\rangle = |-\rangle.$$

Consider the general measurement $\{N_{00}, N_{01}, N_{10}, N_{11}\}$ defined by $N_j = \frac{1}{\sqrt{2}}|\psi_j\rangle\langle\psi_j|$.

(a) Suppose you perform this measurement on the state $|\psi_{00}\rangle$. What are the probabilities of the 4 measurement outcomes? What is the density matrix that results if you ignore the measurement outcome?

(b) Find a quantum circuit on 2 qubits, using single-qubit unitaries and controlled single-qubit unitaries, that implements the POVM $\{M_j\}$, where

$$M_j = N_j^\dagger N_j = \frac{1}{2}|\psi_j\rangle\langle\psi_j|,$$

via measurement in the computational basis. Your circuit should implement a unitary U such that

$$\langle j_1|\langle j_2|U(|0\rangle\langle 0| \otimes \rho)U^\dagger|j_1\rangle|j_2\rangle = \operatorname{Tr} M_j\rho$$

for every density matrix ρ.

2. **Measuring part of an entangled state (20%).** In this problem, we will be measuring the first qubit of the state $|\psi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$, which is a purification of the density matrix $\left(\begin{smallmatrix} 2/3 & 0 \\ 0 & 1/3 \end{smallmatrix}\right)$.

   (a) If you measure the first qubit in the computational basis, what are the probabilities and post-measurement states of the second qubit for each outcome? What is the average density matrix of the second qubit?

   (b) Now suppose you measure the first qubit using the trine measurement given in class. What are the measurement probabilities and post-measurement states of the second qubit for each of the three possible outcomes $j = 0, 1, 2$? What is the average density matrix of the second qubit?

3. **Qubit channels (30%).** Recall that every qubit density matrix can be written as

$$\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$$

for some $r \in \mathbb{R}^3$ satisfying $\|r\|^2 = r_x^2 + r_y^2 + r_z^2 \leq 1$. Any linear, trace-preserving map on $\mathbb{C}^{2\times 2}$ therefore acts as

$$\frac{1}{2}(I + r_x X + r_y Y + r_z Z) \mapsto \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)$$

for some affine linear map $r \mapsto r' = Mr + r_0$, where $M \in \mathbb{R}^{3\times 3}$ and $r_0 \in \mathbb{R}^3$. For such a map to take density matrices to density matrices, the corresponding map $r \mapsto r'$ must take the Bloch sphere into itself, i.e. $|r| \leq 1 \Rightarrow |r'| \leq 1$.

(a) Let $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$. What is the transformation $r \mapsto r'$ corresponding to the map $\rho \mapsto U\rho U^{-1}$?

(b) For $0 \leq p \leq 1$, the p-amplitude damping channel, which models noise caused by spontaneous emission, acts as $\rho \mapsto N_0 \rho N_0^\dagger + N_1 \rho N_1^\dagger$, where

$$N_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad 0 \leq p \leq 1.$$

What is the corresponding affine linear transformation $r \mapsto r'$?

(c) A universal NOT gate on a single qubit is a hypothetical gate modeled by a linear map $\mathcal{N} \colon \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ that takes each pure state to its orthogonal complement. Find the affine map $r \mapsto r'$ corresponding to such an operation and show $\mathcal{N}$ is not completely positive, hence it is not a physical operation.

4. **Distance measures (30%).**

   (a) Let $r_0, r_x, r_y, r_z \in \mathbb{R}$. Show that the eigenvalues of $r_0 + r_x X + r_y Y + r_z Z$ are $r_0 \pm \|r\|_2$, where $\|r\|_2 = \sqrt{r_x^2 + r_y^2 + r_z^2}$ is the Euclidean norm of the vector $r = (r_x, r_y, r_z)$.

   (b) Let $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and $\sigma = \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)$. Show that $\|\rho - \sigma\|_1 = \|r - r'\|_2$.

   (c) Let $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ be arbitrary pure states. Show that

$$\Big\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \Big\|_1 = 2\sqrt{1 - |\langle\psi|\phi\rangle|^2},$$

   i.e. that the upper bound on trace distance in terms of fidelity $F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = |\langle\psi|\phi\rangle|$ is saturated for pure states.