QIC 710 / CO 681 / AMATH 871 / CS 768 / PHYS 767 Fall 2017

Homework 1

Due Friday, September 28

Completed homeworks will be submitted online in Crowdmark. You will receive an email with instructions closer to the due date. A requirement for this is that each of your solutions *must begin on a new page.* Typesetting solutions in LaTeX is recommended but not required. If you do write your solutions by hand, please ensure that your scans are legible before you upload them.

1. **Single-qubit quantum circuits (20%)** Suppose you have access to a single-qubit quantum computer that can perform, with essentially perfect accuracy, any of the following three gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_8 \end{pmatrix},$$

where $\zeta_8 = e^{2\pi i/8}$. Then the computer can also perfectly perform any sequence of these gates, for instance

$$-\boxed{U_1}-\boxed{U_2}-\boxed{U_3}-\boxed{U_4}- \quad = U_4 U_3 U_2 U_1,$$

where $U_i \in \{X, H, T\}$. Note that the order is *reversed* between the circuit notation and matrix multiplication, i.e. matrices act on vectors from the left, whereas time moves to the right in the circuit.

Write each matrix below as a product of some number of gates from the set $\{X, H, T\}$. You do not have to draw the circuit.

(a) Z

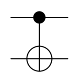
(b) $\begin{pmatrix} \zeta_8 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta_8 & \zeta_8 \\ 1 & 1 \end{pmatrix}$

(d) $\frac{1}{2} \begin{pmatrix} 1 + \zeta_8 & \zeta_8 - i \\ 1 - \zeta_8 & \zeta_8 + i \end{pmatrix}$ (note that $i^2 = -1$ and $\zeta_8^2 = i$)

2. **CNOT and the Bell basis (20%)**

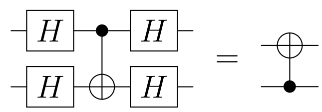
Recall from class that the controlled-NOT gate, or CNOT, implements the following unitary in the computational basis:



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

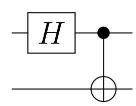
(a) Give a set of eigenvectors and corresponding eigenvalues of CNOT.

(b) Show that



and write down the corresponding 4×4 unitary matrix. This shows that in the $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ basis, a CNOT acts as if the second qubit controls the first.

(c) Write down the 4×4 unitary matrix implemented by the following circuit:



You should find that this unitary transforms the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to the Bell basis:

$$|\Psi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, |\Psi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi_+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, |\Phi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Which Bell state does each computational basis state get mapped to?

3. **Measuring individual qubits (20%)** Consider the following pure state of two qubits:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle.$$

(a) Suppose the first qubit is measured in the computational basis. Compute the probability of each outcome and the corresponding "collapsed" post-measurement states by finding probabilities $p_0 + p_1 = 1$ and normalized states $|\psi_0\rangle$ and $|\psi_1\rangle$ such that

$$|\psi\rangle = \sqrt{p_0}|0\rangle|\psi_0\rangle + \sqrt{p_1}|1\rangle|\psi_1\rangle.$$

(b) Compute the probabilities and post-measurement states associated with a measurement in the $|\pm\rangle$ basis by finding probabilities $p_+ + p_- = 1$ and normalized states $|\psi_+\rangle$ and $|\psi_-\rangle$ such that

$$|\psi\rangle = \sqrt{p_+}|+\rangle|\psi_+\rangle + \sqrt{p_-}|-\rangle|\psi_-\rangle.$$

4. **Distinguishing nondistinguishable states (20%)**

   Recall from class that two states $|\psi_0\rangle$ and $|\psi_1\rangle$ are perfectly distinguishable iff they are orthogonal. When they are orthogonal, they are distinguished by a measurement in the basis $|\psi_0\rangle$, $|\psi_1\rangle$. Otherwise, we may try to distinguish them as well as possible by performing a measurement in some orthonormal basis $|\phi_0\rangle, |\phi_1\rangle$. For each i, this measurement successfully identifies the state $|\psi_i\rangle$ with probability $|\langle\phi_i|\psi_i\rangle|^2$.

   Suppose further that with probability p_0, $|\psi_0\rangle$ is prepared and, with probability p_1, $|\psi_1\rangle$ is prepared. Then measuring in the $|\phi_0\rangle, |\phi_1\rangle$ basis successfully identifies i with probability

   $$\Pr(\text{correct}) = |\langle\phi_0|\psi_0\rangle|^2 p_0 + |\langle\phi_1|\psi_1\rangle|^2 p_1.$$

   Maximizing Pr(correct) over orthonormal measurement bases $|\phi_0\rangle, |\phi_1\rangle$ gives a measure of the distinguishability of the probabilistic preparation $(p_0, |\psi_0\rangle), (p_1, |\psi_1\rangle)$.

   (a) Suppose that

   $$|\psi_0\rangle = |0\rangle, |\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

   are prepared with equal probabilities $p_0 = p_1 = \frac{1}{2}$. Find the maximum value of Pr(correct) over all measurement bases $|\phi_0\rangle, |\phi_1\rangle$ and give the optimal basis.

   Note that because the $|\langle\phi_i|\psi_i\rangle|^2$ are invariant under multiplying $|\phi_i\rangle$ by a phase, it is sufficient to optimize over bases of the form

   $$|\phi_0\rangle = \alpha x|0\rangle - \sqrt{1-x^2}|1\rangle, \;\; |\phi_1\rangle = \sqrt{1-x^2}|0\rangle + \bar{\alpha}x|1\rangle,$$

   where $0 \le x \le 1$ and $|\alpha|^2 = 1$.

   (b) Same question, only with $p_0 = \frac{3}{4}, p_1 = \frac{1}{4}$.

5. **Entanglement swapping (20%).** Consider four qubits, labeled 1,2,3,4, where qubits 1 and 2 are in the Bell state $|\Psi^+\rangle$, and qubits 3 and 4 also in the Bell state $|\Psi^+\rangle$. The state of all four qubits is then given by the tensor product $|\Psi^+\rangle|\Psi^+\rangle$. Here we will see what happens if we perform a Bell measurement on qubits 2 and 3.

   (a) Let U be the unitary that moves the 4th qubit into the second position, defined by $U|abcd\rangle = |adbc\rangle$. Show that

   $$U|\Psi^+\rangle|\Psi^+\rangle = \frac{1}{2}|\Psi^+\rangle|\Psi^+\rangle + \frac{1}{2}|\Psi^-\rangle|\Psi^-\rangle + \frac{1}{2}|\Phi^+\rangle|\Phi^+\rangle + \frac{1}{2}|\Phi^-\rangle|\Phi^-\rangle.$$

   (b) If one does a Bell measurement on qubits 2 and 3 of the state $|\Psi^+\rangle|\Psi^+\rangle$, what are the probabilities and corresponding post-measurement states of qubits 1 and 4 for each possible outcome of the Bell measurement?