# Introduction to Quantum Information Processing

## QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

## Lecture 11 (2017)

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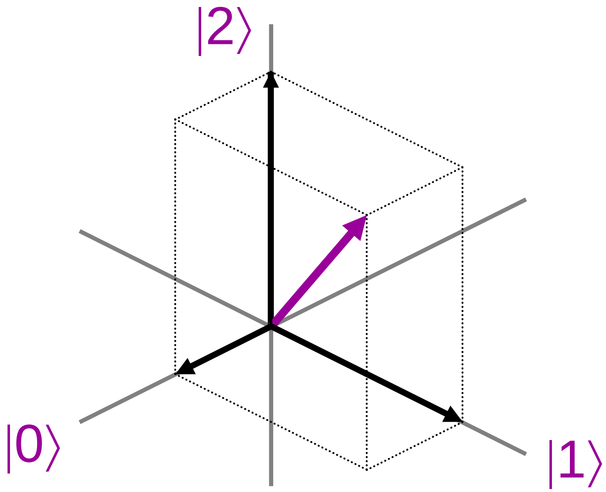
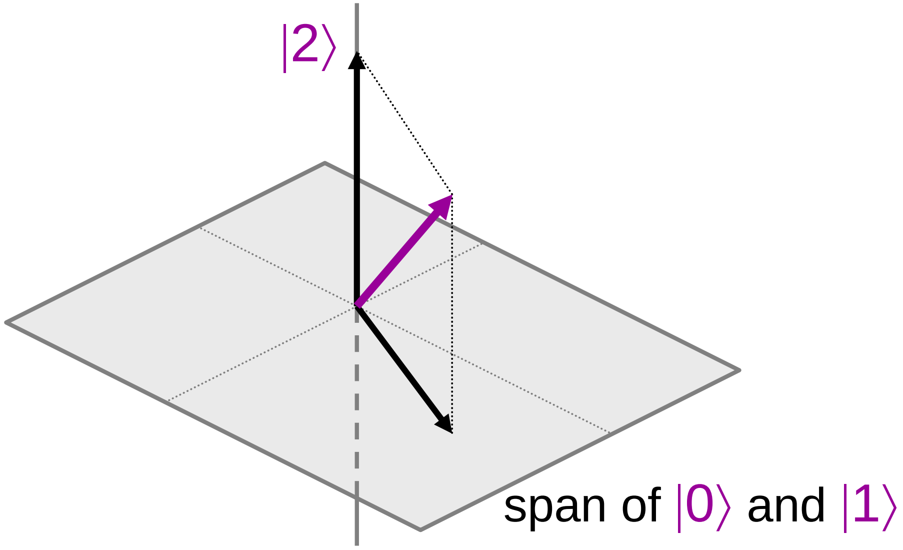
# General measurements and POVMS

(POVM = Positive Operator Valued Measure)

# Prelude: projective measurements




$$P_0 = |0\rangle\langle 0|$$
$$P_1 = |1\rangle\langle 1|$$
$$P_2 = |2\rangle\langle 2|$$




$$P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$$
$$P_2 = |2\rangle\langle 2|$$

In both cases, there is a complete set of mutually orthogonal projectors:

$\sum_j P_j = I$ and $P_i P_j = 0$

The probability of outcome $j$ is $\langle\psi|P_j^\dagger P_j|\psi\rangle = \mathrm{Tr}\big(|\psi\rangle\langle\psi|P_j^\dagger P_j\big)$ using $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$

The collapsed state is the projected vector, but normalized

# General measurements (1)

Let $A_1, A_2, \ldots, A_m$ be *any* matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Corresponding ***measurement*** is a stochastic operation on $\rho$ that, with probability $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)$, produces outcome:

$$\begin{cases} \boldsymbol{j} & \text{(classical information)} \\ \dfrac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} & \text{(the collapsed quantum state)} \end{cases}$$

**Example 1:** ($A_j = |\phi_j\rangle\langle\phi_j|$) (rank-1 orthogonal projectors)

Consistent with our first definition of measurements

**Question:** what if we do the above but don't look at $\boldsymbol{j}$ ?

# General measurements (2)

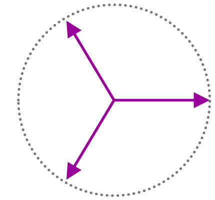
When $A_j = \left|\phi_j\right\rangle\left\langle\phi_j\right|$ are orthogonal projectors and $\rho = |\psi\rangle\langle\psi|$,

$$\begin{aligned}\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) &= \mathrm{Tr}\left|\phi_j\right\rangle\left\langle\phi_j|\psi\right\rangle\left\langle\psi|\phi_j\right\rangle\left\langle\phi_j\right| \\ &= \left\langle\phi_j|\psi\right\rangle\left\langle\psi|\phi_j\right\rangle\left\langle\phi_j|\phi_j\right\rangle \\ &= \left|\left\langle\phi_j|\psi\right\rangle\right|^2\end{aligned}$$

Moreover, $\dfrac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} = \dfrac{\left|\phi_j\right\rangle\left\langle\phi_j|\psi\right\rangle\left\langle\psi|\phi_j\right\rangle\left\langle\phi_j\right|}{\left|\left\langle\phi_j|\psi\right\rangle\right|^2} = \left|\phi_j\right\rangle\left\langle\phi_j\right|$

# General measurements (3)

**Example 3 (trine state measurement):**



Let $|\phi_0\rangle = |0\rangle$, $\quad |\phi_1\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$, $\quad |\phi_2\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

Define $A_0 = \sqrt{\frac{2}{3}}|\phi_0\rangle\langle\phi_0| = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

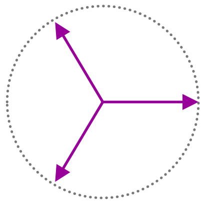
$$A_1 = \sqrt{\frac{2}{3}}|\phi_1\rangle\langle\phi_1| = \frac{1}{4}\begin{pmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{pmatrix}$$

$$A_2 = \sqrt{\frac{2}{3}}|\phi_2\rangle\langle\phi_2| = \frac{1}{4}\begin{pmatrix} \sqrt{2/3} & \sqrt{2} \\ \sqrt{2} & \sqrt{6} \end{pmatrix}$$

Then $A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I$.

If the input itself is an unknown trine state $|\phi_k\rangle\langle\phi_k|$, then the probability that classical outcome is $k$ is 2/3 = 0.6666…

# General measurements (3)



**Question:** Are there states the trine measurement can't distinguish?

$$|\phi_0\rangle = |0\rangle, \quad |\phi_1\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \quad |\phi_2\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

$$A_0 = \sqrt{\frac{2}{3}}|\phi_0\rangle\langle\phi_0| = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_1 = \sqrt{\frac{2}{3}}|\phi_1\rangle\langle\phi_1| = \frac{1}{4}\begin{pmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{pmatrix}$$

$$A_2 = \sqrt{\frac{2}{3}}|\phi_2\rangle\langle\phi_2| = \frac{1}{4}\begin{pmatrix} \sqrt{2/3} & \sqrt{2} \\ \sqrt{2} & \sqrt{6} \end{pmatrix}$$

**Answer:** $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$ ($Y$-eigenstates)

The trine measurement is **not informationally complete**!

# General measurements (4)

Often measurements arise in contexts where we only care about the classical part of the outcome (not the residual quantum state)

The probability of outcome ***j*** is $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) = \mathrm{Tr}\left(\rho A_j^\dagger A_j\right)$

***Simplified definition of such measurements***

Let $E_1, E_2, \dots, E_m$ be positive semidefinite and with $\sum_{j=1}^{m} E_j = I$

The probability of outcome ***j*** is $\mathrm{Tr}(\rho E_j)$.

Called a ***POVM (Positive Operator-Valued Measure)***

**It is a measure valued in positive (-semidefinite) operators.**

# Informationally-complete POVMs

A POVM $E_1, E_2, \dots, E_m$ is **informationally complete** if $\text{span}_{\mathbb{R}}(E_1, E_2, \dots, E_m) =$ all Hermitian $d \times d$ matrices.

Such POVMs can distinguish **any** states.

**Example:** Informationally complete POVMs such that $\text{rank}(E_j) = 1$ for each $j$, i.e. such that

$$E_1 = \alpha_1 |\phi_1\rangle\langle\phi_1|, E_2 = \alpha_2 |\phi_2\rangle\langle\phi_2|, \dots, E_m = \alpha_m |\phi_m\rangle\langle\phi_m|,$$

are sometimes called **tight frames.**

**(Very hard) question:** Do informationally-complete POVMs exist with $\text{rank}(E_j) = 1$ for every $j$

and $\text{Tr}(E_i E_j) =$ constant for $i \neq j$?

**Answer:** Apparently yes (no proof yet), known as SIC-POVMs (Symmetric Informationally Complete POVMs)

# "Mother of all operations"

Let $A_{1,1}, A_{1,2}, \ldots, A_{1,k_1}$ satisfy
$A_{2,1}, A_{2,2}, \ldots, A_{2,k_2}$
$A_{m,1}, A_{m,2}, \ldots, A_{m,k_m}$

$$\sum_{j=1}^{m} \sum_{i=1}^{k_m} A_{j,i}^{\dagger} A_{j,i} = I$$

Then there is a quantum operation that, on input $\rho$, produces with probability $\sum_{i=1}^{k_m} A_{j,i} \rho A_{j,i}^{\dagger}$ the state:

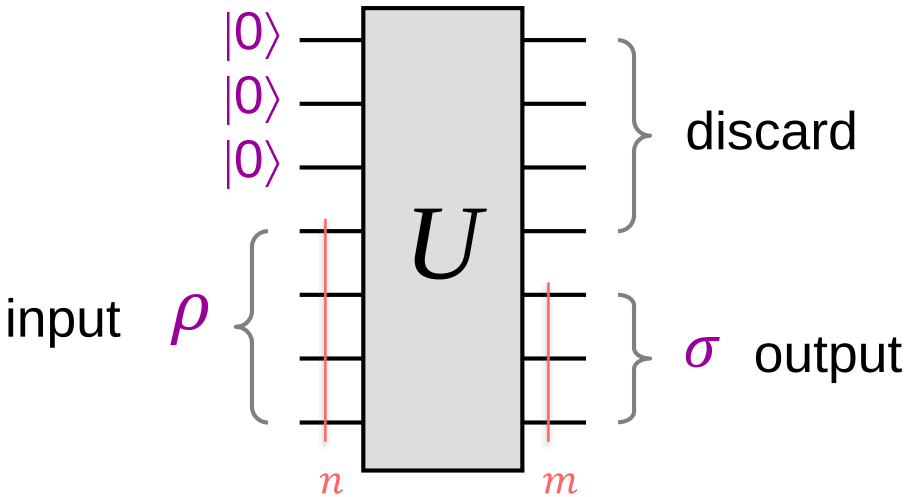
$$\begin{cases} j & \text{(classical information)} \\ \dfrac{\sum_{i=1}^{k_m} A_{j,i} \rho A_{j,i}^{\dagger}}{\sum_{i=1}^{k_m} \mathrm{Tr}\left(A_{j,i} \rho A_{j,i}^{\dagger}\right)} & \text{(the collapsed quantum state)} \end{cases}$$

Also known as **quantum instrument** or **heralded quantum channel**

# Simulations among operations

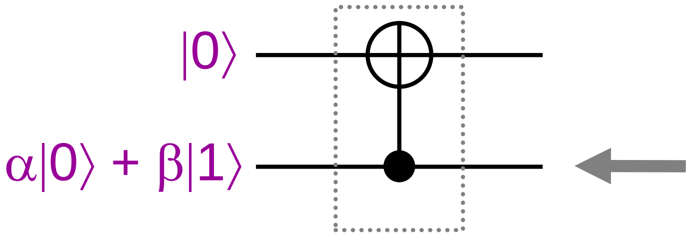
# Simulations among operations (1)

**Theorem 1:** any ***quantum operation*** can be simulated by applying a unitary operation on a larger quantum system:




This specification of a quantum operation is called the ***Stinespring*** form, or **isometric extension**

**Example:** decoherence




$$\rho = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

# Simulations among operations (2)

**Proof of Theorem 1:**

Let $A_1, A_2, \dots, A_{2^k}$ be any $2^m \times 2^n$ matrices such that

$$\sum_{j=1}^{2^k} A_j^\dagger A_j = I$$

This defines a mapping from $m$ qubits to $n$ qubits:

$$\rho \mapsto \sum_{j=1}^{2^k} A_j \rho A_j^\dagger$$

This specification of the quantum operation is called the ***Kraus*** form

# Simulations among operations (3)

Set $V = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2^k} \end{bmatrix}$

Since $V^\dagger V =$

$$\begin{bmatrix} A_1^\dagger & A_2^\dagger & \cdots & A_{2^k}^\dagger \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2^k} \end{bmatrix} = I$$

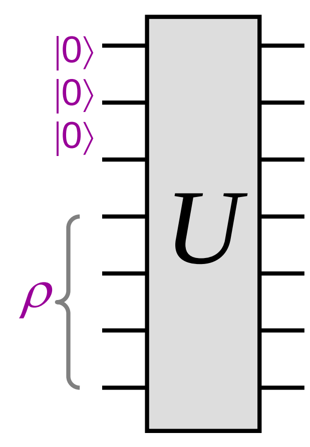
the columns of $V$ are ***orthonormal***

Let $U$ be any unitary matrix with first $2^n$ columns from $V$

$$U = [\, V \,\vdots\, W \,]$$

$U$ is a $2^{m+k} \times 2^{m+k}$ matrix (and its columns partition into $2^{m-n+k}$ blocks of size $2^n$)

Now, consider the circuit:



# Simulations among operations (4)

The output state of the circuit is $U(|00\cdots 0\rangle\langle 00\cdots 0| \otimes \rho)U^\dagger$

$$= \left[\begin{array}{c:c} A_1 & \\ A_2 & \\ \vdots & W \\ A_{2^k} & \end{array}\right] \begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \left[\begin{array}{cccc} A_1^\dagger & A_1^\dagger & \cdots & A_{2^k}^\dagger \\ \hdashline \multicolumn{4}{c}{W^\dagger} \end{array}\right]$$

$$= \begin{bmatrix} A_1\rho & 0 & \cdots & 0 \\ A_2\rho & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{2^k}\rho & 0 & \cdots & 0 \end{bmatrix} \left[\begin{array}{cccc} A_1^\dagger & A_1^\dagger & \cdots & A_{2^k}^\dagger \\ \hdashline \multicolumn{4}{c}{W^\dagger} \end{array}\right]$$

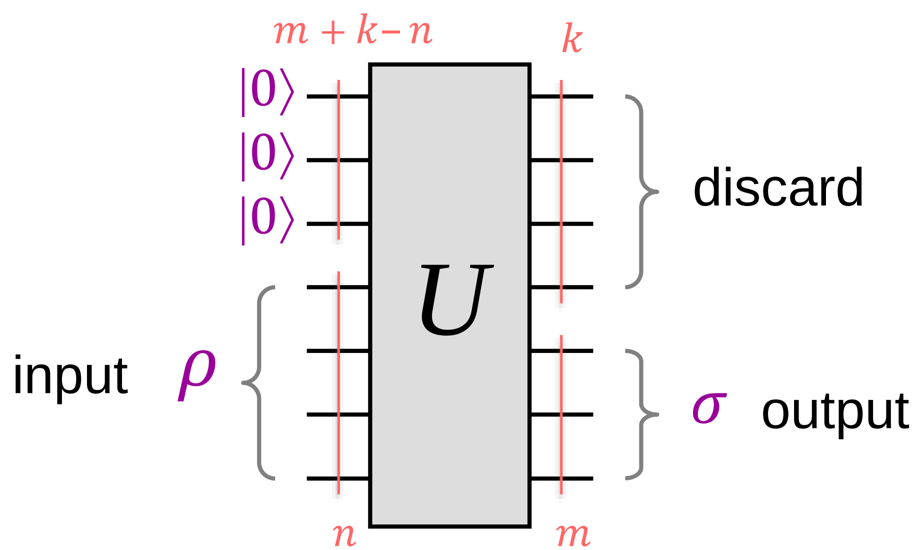
$$= \begin{bmatrix} A_1\rho A_1^\dagger & A_1\rho A_2^\dagger & \cdots & A_1\rho A_{2^k}^\dagger \\ A_2\rho A_1^\dagger & A_2\rho A_2^\dagger & \cdots & A_2\rho A_{2^k}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ A_{2^k}\rho A_1^\dagger & A_{2^k}\rho A_2^\dagger & \cdots & A_{2^k}\rho A_{2^k}^\dagger \end{bmatrix}$$

# Simulations among operations (5)

Tracing out the high-order $k$ qubits of this state yields

$$A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger + \cdots + A_{2^k} \rho A_{2^k}^\dagger$$

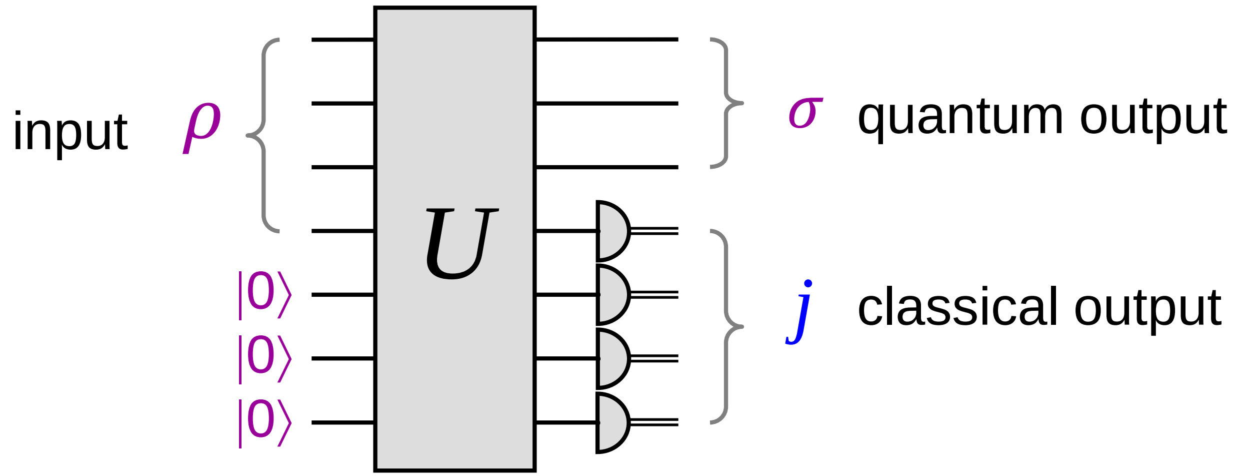
exactly the output of mapping that we want to simulate




**Note:** this approach is ***not always optimal*** in the number of ancilliary qubits used—there are more efficient methods

# Simulations among operations (6)

**Theorem 2:** ***any measurement*** can also be simulated by applying a unitary operation on a larger quantum system and then measuring:




This is the same diagram as for Theorem 1 (drawn with the extra qubits at the bottom) but where the “discarded” qubits are measured and part of the output