Automorphisms of varieties and critical density

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Let X be a topological space and let $\mathcal{S}$ be an infinite subset of X.

We say that $\mathcal{S}$ is *critically dense* if every infinite subset of $\mathcal{S}$ is dense in X.

Equivalently, $\mathcal{S} \cap Y$ is a finite set for every proper closed subset Y of X.

This notion was introduced by Dan Rogalski in his thesis in constructing counter-examples to old conjectures from non-commutative algebra.

Rogalski showed that the critical density property was closely linked with the Noetherian property for twisted homogeneous coordinate rings.

What do we mean by twisted homogeneous coordinate rings?

E.g.,

Let σ be an automorphism of $\mathbb{P}^d$. Notice that σ^* induces an automorphism of the Picard group of $\mathbb{P}^d$ and we can form the *twisted homogeneous coordinate ring* of $\mathbb{P}^d$ with respect to σ

$$\bigoplus_{i=0}^{\infty} \mathsf{H}^0(\mathbb{P}^d, \mathcal{O}(1) \otimes \mathcal{O}(1)^{\sigma} \otimes \cdots \otimes \mathcal{O}(1)^{\sigma^{i-1}}).$$

To think about what this ring looks like, notice that it is a graded ring whose n'th homogeneous part can be identified with the space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_d]$ of degree n.

Multiplication, however, is twisted in this ring by the formula

$$f \star g \;=\; f\phi^m(g),$$

where f is homogeneous of degree m and ϕ is the automorphism of $\mathbb{C}[x_0, \ldots, x_d]$ induced by σ.

Notice that to a point $\mathbf{c} \in \mathbb{P}^d$, we can associate a codimension 1 subspace of $\mathbf{H}^0(\mathbb{P}^d, \mathcal{O}(1))$.

Define $R(\sigma, \mathbf{c})$ to be the subring of the twisted homogeneous coordinate ring generated by this codimension 1 subspace.

THEOREM: $R(\sigma, \mathbf{c})$ is Noetherian if and only if

$$\{\sigma^n(\mathbf{c}) \mid n \in \mathbb{Z}\}$$

is critically dense in $\mathbb{P}^d$.

ROGALSKI'S QUESION: Can "critically dense" be replaced by "dense" in the theorem; i.e., is density equivalent to critical density for sets of the form

$$\{\sigma^n(\mathbf{c}) \mid n \in \mathbb{Z}\},$$

where $\mathbf{c}$ is a point in a quasi-projective variety?

The answer to Rogalski's question is 'no' if we are working in positive characteristic, as the following example due to Hochster shows.

Let $X = \mathbb{A}^2$ over the field $\mathbb{F}_p(t)$. Let σ be the automorphism given by

$$\sigma(x, y) \;=\; (tx, (1+t)y).$$

Take $\mathbf{c} = (1, 1)$.

Notice that

$$\sigma^{p^n}(1, 1) \;=\; (t^{p^n}, (1+t)^{p^n}) = (t^{p^n}, 1 + t^{p^n}).$$

Hence the orbit of $(1, 1)$ under σ is in the variety $Z(y - x - 1)$ infinitely often and so the orbit is not critically dense. It is not hard to show, however, that the orbit must be dense.

What happens in characteristic 0?

The characteristic 0 case is still open, but can be shown for many classes of varieties using p-adic methods.

Here is an over-simplified picture of how the proof works for affine d-space.

- Show that it is no loss of generality to assume that you are working over a p-adic field $\mathbb{Q}_p$.

- Show that the automorphism $\sigma : \mathbb{A}^d \to \mathbb{A}^d$ has the property that there exist analytic maps $f_1, \ldots, f_d : \mathbb{Z}_p \to \mathbb{Z}_p$ such that

$$\sigma^{an+b}(\mathbf{c}) = (f_1(n), \ldots, f_d(n))$$

for some a and b

- Argue that if $\sigma^{an+b}(\mathbf{c})$ is in $Z(P(x_1, \ldots, x_d))$ for infinitely many n, then $P(f_1(x), \ldots, f_d(x))$ is an analytic function with infinitely many zeros in the unit disc.

- Conclude that it is identically zero and hence $\sigma^{an+b}(\mathbf{c})$ must be in $Z(P)$ for all n.

Embedding in $\mathbb{Z}_p$.

The Lefschetz principle is well-known, saying that in some sense classical algebraic geometry over a field of characteristic 0 is equivalent to classical algebraic geometry over the complex numbers.

Less well-known is that any finitely generated extension of $\mathbb{Q}$ can be embedded in $\mathbb{Q}_p$ for some prime; moreover, one can assume that a given generating set is sent to elements of $\mathbb{Z}_p$.

The main idea is as follows. Write the f.g. extension of $\mathbb{Q}$ as

$$\mathbb{Q}(t_1, \ldots, t_d)(\theta),$$

where $t_1, \ldots, t_d$ are algebraically independent over $\mathbb{Q}$ and θ satisfies a polynomial

$$f \in \mathbb{Z}[t_1, \ldots, t_d][x].$$

Now pick integers $a_1, \ldots, a_d$ such that

$$f(a_1, \ldots, a_d; x)$$

is non-zero and the resultant of f with respect to x does not vanish at $a_1, \ldots, a_d$.

Next pick a prime p such that the resultant evaluated at $a_1, \ldots, a_d$ does not vanish and

$$f(a_1, \ldots, a_d; x)$$

has a root mod p.

Finally, pick d algebraically independent elements of $\mathbb{Z}_p$, $u_1, \ldots, u_d$. Send $t_i \mapsto a_i + pu_i$ and send θ to a root of $f(a_1 + u_1 p, \ldots, a_d + pu_d, x)$.

That is the basic idea about how to work in $\mathbb{Z}_p$.

But why is it advantageous to work over $\mathbb{Z}_p$? Can't we just work over $\mathbb{C}$? After all, I like $\mathbb{C}$ a lot more.

The problem lies in the fact that the integers are not compact in $\mathbb{C}$.

Now that we have translated the problem into one over a p-adic field, what do we do now?

E.g., Consider the automorphism of $\mathbb{A}^2(\mathbb{C})$ given by

$$\sigma(x, y) \;=\; (2x, 3y).$$

Notice that $\sigma^n(1, 1) = (f(n), g(n))$, where $f(x) = 2^x$, $g(x) = 3^x$. Thus we can embed the orbit of the point $(1, 1)$ under σ in a complex arc.

In general, automorphisms of $\mathbb{A}^d$ are quite complicated. Nevertheless, using Hensel's lemma it can be shown that:

If $\sigma : \mathbb{A}^d(\mathbb{Z}_p) \to \mathbb{A}^d(\mathbb{Z}_p)$ then there is some $a > 0$ such that for each b, $0 \le b < a$, there exist analytic functions $f_1, \ldots, f_d$ such that

$$\sigma^{an+b}(\mathbf{c}) = (f_1(n), \ldots, f_d(n)).$$

Thus we can embed arithmetic progressions into a p-adic arc.

Finally, we take a polynomial $P(x_1, \ldots, x_d)$ and create the analytic function

$$P(f_1, \ldots, f_d).$$

If this function has infinitely many zeros, then by Strassman's theorem it must be identically zero. Thus either $\{n \mid \sigma^n(\mathbf{c}) \in Z(P)\}$ is finite or it contains an arithmetic progression. If it contains an arithmetic progression then the orbit of $\mathbf{c}$ cannot be dense.

Thus we have the following results

THEOREM: Let σ be an automorphism of $\mathbb{A}^d$ over a field of characteristic 0. Then $\{\sigma^n(p) \mid n \in \mathbb{Z}\}$ is dense if and only if it is critically dense.

THEOREM: Let σ be an automorphism of $\mathbb{A}^d$ over a field of characteristic 0 and let Y be a subvariety of $\mathbb{A}^d$. Then $\{n \mid \sigma^n(p) \in Y\}$ is a finite union of 2-way arithmetic progressions possibly augmented by a finite set.

This can be seen as a generalization of the Skolem-Mahler-Lech therorem.

THEOREM (S-M-L) Let $f(z)$ be the power series expansion of a rational function. Then the set of n such that the coefficient of z^n in $f(z)$ is 0 is a finite union of one-way arithmetic progressions possibly augmented by a finite set.

The reason for this is that any rational function $f(z)$ can be realized as

$$\sum_{i=0}^{\infty} \mathbf{w}^{\top} A^n \mathbf{v} z^n.$$

Up to a finite number of coefficients, this series will agree with one in which A is an invertible matrix. Thus it is no loss of generality to assume that A is invertible. Notice that A can be thought of as giving a "linear" automorphism of $\mathbb{A}^d$, $\mathbf{v}$ can be thought of as a point in $\mathbb{A}^d$. Then saying that $\mathbf{w}^T A^n \mathbf{v} = 0$ is the same as saying that $A^n \mathbf{v}$ is in the hyperplane given by $\mathbf{w}^T \mathbf{x} = 0$.

FUTURE DIRECTIONS

Analogous results can be proven for:

- $\mathbb{P}^d$ and more generally any projective variety in which σ fixes an ample invertible sheaf (Rogalski)
- Fano varieties
- Abelian varieties (Rogalski)
- many surfaces (but still not all projective surfaces) Rogalski
- curves (of course)
- Toric varieties in which σ is a toric automorphism

Nevertheless, there are still many varieties for which the analogues are yet to be shown. The problem is particularly tantalizing for projective surfaces.

Look at the Kodaira dimension of a surface. In the case that the Kodaira dimension is 2, the automorphism group is finite. In the case that the Kodaira dimension is 0 or 1, our surface will have a unique minimal model and it is therefore sufficient to work with this surface. Thus nonnegative Kodaira dimension seems like is should be doable. The rational/ruled case might be harder, but there is also a lot of nice structure that can be exploited.

What about in characteristic p?

CONJECTURE: Let X be a q.p.v. over a field of characteristic p. Let σ be an automorphism of X, let $\mathbf{c} \in X$, and let Y be a subvariety of X. Then

$$\{n \mid \sigma^n(\mathbf{c}) \in Y\}$$

is sandwiched in between two sets T and $T \cup S_k$ where T is a finite union of two-way arithmetic progressions and S_k consists of all numbers whose base p expansion consists of at most k nonzero digits.