Network Bargaining: Using Approximate Blocking Sets to Stabilize Unstable Instances

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Abstract. We study a network extension to the Nash bargaining game, as introduced by Kleinberg and Tardos [6], where the set of players corresponds to vertices in a graph $G = (V, E)$ and each edge $ij \in E$ represents a possible deal between players i and j. We reformulate the problem as a cooperative game and study the following question: *Given a game with an empty core (i.e. an unstable game) is it possible, through minimal changes in the underlying network, to stabilize the game?* We show that by removing edges in the network that belong to a *blocking set* we can find a stable solution in polynomial time. This motivates the problem of finding small blocking sets. While it has been previously shown that finding the smallest blocking set is NP-hard [2], we show that it is possible to efficiently find approximate blocking sets in sparse graphs.

1 Introduction

In the classical *Nash bargaining* game [9], two players seek a mutually acceptable agreement on how to split a dollar. If no such agreement can be found, each player i receives her *alternative* α_i. Nash's solution postulates, that in an equilibrium, each player i receives her alternative α_i plus half of the surplus $1 - \alpha_1 - \alpha_2$ (if $\alpha_1 + \alpha_2 > 1$ then no mutually acceptable agreement can be reached, and both players settle for their alternatives).

In this paper, we consider a natural *network* extension of this game that was recently introduced by Kleinberg and Tardos [6]. Here, the set of players corresponds to the vertices of an undirected graph $G = (V, E)$; each edge $ij \in E$ represents a potential deal between players i and j of unit value. In Kleinberg and Tardos' model, players are restricted to bargain with at most one of their neighbours. Outcomes of the *network bargaining* game (NB) are therefore given by a matching $M \subseteq E$, and an *allocation* $x \in \mathbb{R}^V_+$ such that $x_i + x_j = 1$ for all $ij \in M$, and $x_i = 0$ if i is M-exposed; i.e., if it is not incident to an edge of M.

Unlike in the non-network bargaining game, the alternative α_i of player is not a given parameter but rather implicitly determined by the network neighbourhood of i. Specifically, in an outcome (M, x), player i's alternative is defined as

$$\alpha_i = \max\{1 - x_j \, : \, ij \in \delta(i) \setminus M\}, \tag{1}$$

where $\delta(i)$ is the set of edges incident to i. Intuitively, a neighbour j of i receives x_j in her current deal, and i may coerce her into a joint deal, yielding i a payoff of $1 - x_j$.

An outcome (M, x) of NB is called *stable* if $x_i + x_j \geq 1$ for all edges $ij \in E$, and it is *balanced* if in addition, the value of the edges in M is split according to Nash's bargaining solution; i.e., for a matching edge ij, $x_i - \alpha_i = x_j - \alpha_j$.

Kleinberg and Tardos gave an efficient algorithm to compute balanced outcomes in a graph (if these exist). Moreover, the authors characterize the class of graphs that admit such outcomes. In the following main theorem of [6], a vertex $i \in V$ is called *inessential* if there is a maximum matching in G that exposes i.

Theorem 1 ([6]). *An instance of NB has a balanced outcome iff it has a stable one. Moreover, it has a stable outcome iff no two inessential vertices are connected by an edge.*

The theory of *cooperative games* offers another useful angle for NB. In a cooperative game (with transferable utility) we are given a player set N, and a *valuation function* $v : 2^N \to \mathbb{R}_+$; $v(S)$ can be thought of as the value that the players in S can jointly create. The *matching game* [4, 12] is a specific cooperative game that will be of interest for us. Here, the set of players is the set of vertices V of a given undirected graph. The matching game has valuation function ν where $\nu(S)$ is the size of a maximum matching in the graph $G[S]$ induced by the vertices in S.

One goal in a cooperative game is to allocate the value $v(N)$ of the so called *grand coalition* fairly among the players. The *core* is in some sense the gold-standard among the solution concepts that prescribe such a fair allocation: a vector $x \in \mathbb{R}^N_+$ is in the core if (a) $x(N) = v(N)$, and (b) $x(S) \geq v(S)$ for all $S \subseteq N$, where we use $x(S)$ as a short-hand for $\sum_{i \in S} x_i$. In the special case of the matching game, this is seen to be equivalent to the following:

$$\mathcal{C}(G) = \{x \in \mathbb{R}^V_+ \, : \, x(V) = \nu(V) \text{ and } x_u + x_v \geq 1, \, \forall uv \in E\}. \tag{2}$$

Thus, the core of the matching game consists precisely of the set of stable outcomes of the corresponding NB game. This was recently also observed by Bateni et al. [1] who remarked that the set of balanced outcomes of an instance of NB corresponds to the elements in the intersection of core and *prekernel* (e.g., see [3, 10] for a definition),of the associated matching game instance.

1.1 Dealing with unstable instances

Using the language of cooperative game theory and the work of Bateni et al. [1], we can rephrase the main results of [6] as follows: *Given an instance of NB, if the core of the underlying matching game is non-empty then there is an efficient algorithm to compute a point in the intersection of core and prekernel.* Such an algorithm had previously been given by Faigle et al. in [5]. It is not hard to see that the core of an instance of the matching game is non-empty if and only if the fractional matching LP for this instance has an integral optimum solution. We state this LP and its dual below; we let $\delta(i)$ denote the set of edges incident to vertex i in the underlying graph, and use $y(\delta(i))$ as a shorthand for the sum of y_e over all $e \in \delta(i)$.

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e && \text{(P)} \\ \text{s.t.} \quad & y(\delta(i)) \leq 1 && \forall i \in V \\ & y \geq \mathbb{0} \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \sum_{i \in V} x_i && \text{(D)} \\ \text{s.t.} \quad & x_i + x_j \geq 1 && \forall ij \in E \\ & x \geq \mathbb{0}, \end{aligned} \tag{3}$$

LP (P) does of course typically have a fractional optimal solution, and in this case the core of the corresponding matching game instances is empty. Core assignments are highly desirable for their properties, but may simply not be available for many instances. For this reason, a number of more forgiving alternative solution concepts like *bargaining sets, kernel, nucleolus*, etc. have been proposed in the cooperative game theory literature (e.g., see [3, 10]).

This paper addresses network bargaining instances that are *unstable*; i.e., for which the associated matching game has an empty core. From the above discussion, we know that there is no solution x to (D) that also satisfies $\mathbb{1}^T x \leq \nu(V)$. We therefore propose to find an allocation x of $\nu(V)$ that violates the stability condition in the smallest number of places. Formally, we call a set B of edges a *blocking set* if there is $x \in \mathbb{R}^V_+$ such that $\mathbb{1}^T x \leq \nu(V)$, and $x_i + x_j \geq 1$ for all $ij \in E \setminus B$.

Blocking sets were previously discussed by Biró et al. [2]. The authors showed that finding a smallest such set is NP-hard (via a reduction from *maximum independent set*). In this paper, we complement this result by showing that approximate blocking sets can be computed in *sparse* graphs. A graph $G = (V, E)$ is ω*-sparse* for some $\omega \geq 1$ if for all $S \subseteq V$, the number of edges in the induced graph $G[S]$ is bounded by $\omega\,|S|$. For example, if G is planar, then we may choose $\omega = 3$ by Euler's formula.

Theorem 2. *Given an* ω*-sparse graph* $G = (V, E)$*, there is an efficient algorithm for computing blocking sets of size at most* $8\omega + 2$ *times the optimum.*

The main idea in our algorithm is a natural one: formulate the blocking set problem as a linear program, and extract a blocking set from one of its optimal fractional solutions via an application of the powerful technique of *iterative rounding* (e.g., see [8]). We first show that the proposed LP has an unbounded integrality gap in general graphs, and is therefore not useful for the design of approximation algorithms for such instances. We turn to the class of sparse graphs, and observe that, even here, extreme points of the LP can be highly fractional, ruling out the direct use of standard techniques. We carefully characterize problem extreme-points, and develop a direct rounding method for them. Our approach exploits problem-specific structure as well as the sparsity of the underlying graph.

Given a blocking set B, let $E' = E \setminus B$ be the non-blocking set edges, and let $G' = (V, E')$ be the induced graph. Notice that the matching game induced by G' may *still* have an empty core, and that the maximum matching in G' may even be smaller than that in G. We are however able to show that we can find a balanced allocation of $\nu(V)$ as follows: let M' be a maximum matching in G', and define the alternative of player i as

$$\alpha'_i = \max\{1 - x_i \,:\, ij \in \delta_{G'}(i) \setminus M'\},$$

for all $i \in V$. Call an assignment x is balanced if it satisfies the stability condition (3) for all edges $ij \in M'$, and

$$x_i - \alpha'_i = x_j - \alpha'_j,$$

for all $ij \in M'$. A straight-forward application of an algorithm of Faigle et al. [5] yields a polynomial-time method to compute such an allocation. Details are omitted from this extended abstract.

2 Finding small blocking sets in sparse graphs

We attack the problem of finding a small blocking set via iterative linear programming rounding. In order to do this, it is convenient to introduce a slight generalization of the blocking set problem. In an instance of the *generalized blocking set* problem (GBS), we are given a graph $G = (V, E)$, a partition $E_1 \cup E_2$ of E, and a parameter $\nu \geq 0$. The goal is to find a blocking set $B \subseteq E_1$, and an allocation $x \in \mathbb{R}^V_+$ such that $\mathbb{1}^T x \leq \nu$ and $x_u + x_v \geq 1$ for all $uv \in E \setminus B$, where $\mathbb{1}$ is a vector of 1s of appropriate dimension. The problem is readily formulated as an integer program. We give its relaxation below on the left.

$$
\begin{aligned}
\min \quad & \mathbb{1}^T z && (\mathrm{P}_B)\\
\text{s.t.} \quad & x_u + x_v + z_{uv} \geq 1 && \forall uv \in E_1 \quad (4)\\
& x_u + x_v \geq 1 && \forall uv \in E_2 \quad (5)\\
& \mathbb{1}^T x \leq \nu && (6)\\
& x, z \geq \mathbb{0}
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \mathbb{1}^T a + \mathbb{1}^T b - \gamma\, \nu && (\mathrm{D}_B)\\
\text{s.t.} \quad & a(\delta_{E_1}(u)) + b(\delta_{E_2}(u)) \leq \gamma && \forall u \in V \quad (7)\\
& a \leq \mathbb{1}\\
& a, b \geq \mathbb{0}
\end{aligned}
$$

The LP on the right is the dual of (P_B). It has a variable a_e for all $e \in E_1$, a variable b_e for all $e \in E_2$, and variable γ corresponds to the primal constraint limiting $\mathbb{1}^T x$. We can show the LP is weak and hence not useful for approximating the generalized blocking set problem in general graphs (for details, see [7]).

Lemma 1. *The integrality gap of* (P_B) *is* $\Omega(n)$*, where* n *is the number of vertices in the given instance of the blocking set problem.*

Given this negative result, we will focus on sparse instances (G, ν) and prove Theorem 2. We first characterize the extreme points of (P_B).

2.1 Extreme points of (P_B)

In the following, we assume that the underlying graph G is bipartite; this assumption will greatly simplify our presentation, and will turn out to be w.l.o.g. Let (x, z) be a feasible solution of LP (P_B), and let $A^{=}(x, z)^T = b^{=}$ be the set of tight constraints of the LP. It is well known (e.g., see [11] and also [8]) that (x, z) is an extreme point of the feasible region if $A^{=}$

has full column-rank. In particular, (x, z) is uniquely determined by any full-rank sub-system $A'(x, z)^T = b'$ of $A^{=}(x, z)^T = b^{=}$. If constraint (6) is not part of this system of equations, then

$$A' = [A'', I],$$

where A'' is a submatrix of the edge-vertex incidence matrix of a bipartite graph, and I is an identity matrix of appropriate dimension. Such matrices A' are well-known to be *totally unimodular* (e.g., see [11]), and (x, z) is therefore integral in this case. From now on, we therefore assume that constraint (6) is tight, and that (x, z) is the unique solution to

$$\begin{bmatrix} A'' & I \\ \mathbb{1}^T & \mathbb{0}^T \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ \nu \end{pmatrix}, \tag{8}$$

where A'' is a submatrix of the edge, vertex incidence matrix of bipartite graph G, I is an identity matrix, and $\mathbb{1}^T$ and $\mathbb{0}^T$ are row vectors of 1's and 0's, respectively. We obtain the following useful lemma.

Lemma 2. *Let (x, z) be a non-integral extreme point solution to* (P_B) *satisfying* (8)*. Then there is an $\alpha \in (0, 1)$ such that $x_u, z_{uv} \in \{0, \alpha, 1 - \alpha, 1\}$ for all $u \in V$, and $uv \in E_1$.*

Proof. Standard linear algebra implies that the solution space to the the system $[A''\, I](\bar{x}, \bar{z})^T$ is a line; i.e., it has dimension 1. Hence, there are two extreme points (x^1, z^1) and (x^2, z^2) of the integral polyhedron defined by constraints (4), (5), and the non-negativity constraints, and some $\alpha \in [0, 1]$ such that

$$\begin{pmatrix} x \\ z \end{pmatrix} = \alpha \begin{pmatrix} x^1 \\ z^1 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} x^2 \\ z^2 \end{pmatrix}.$$

In fact, α must be in $(0, 1)$ as (x, z) is assumed to be fractional. This implies the lemma. □

We call an extreme point *good* if there is a vertex u with $x_u = 1$, or an edge $uv \in E_1$ with $z_{uv} \in \{0\} \cup [1/3, 1]$. Let us call an extreme point *bad* otherwise. We will now characterize the structure of a bad extreme point (x, z). Let $G = (V, E_1 \cup E_2)$ be the bipartite graph for a given GBS instance. Let $\mathcal{T}_1 \subseteq E_1$ and $\mathcal{T}_2 \subseteq E_2$ be E_1 and E_2 edges corresponding to tight inequalities of (P_B) that are part of the defining system (8) for (x, z). Let α be as in Lemma 2. Since (x, z) is bad, it must be that either

α or $1-\alpha$ is larger than $2/3$; w.l.o.g., assume that $\alpha > 2/3$. We define the following useful sets:

$$X = \{u \in V \,:\, x_u = 1-\alpha\}$$
$$Y = \{u \in V \,:\, x_u = \alpha\}$$
$$O = \{u \in V \,:\, x_u = 0\}.$$

Lemma 3. *Let (x, z) be a bad extreme point. Using the notation defined above, we have*

(a) $z_{uv} = (1-\alpha)$ for all $uv \in E_1$,
(b) $O \cup X$ is an independent set in G
(c) Each $\mathcal{T}_1$ edge is incident to exactly one O and one Y vertex, and the edges of $\mathcal{T}_2$ form a tree spanning $X \cup Y$. Each edge in E is incident to exactly one Y vertex.

Proof. We know from Lemma 2 that $z_{uv} \in \{0, 1-\alpha, \alpha, 1\}$ for all $uv \in E_1$; (a) follows now directly from the fact that (x, z) is bad.

No two vertices $u, v \in O$ can be connected by an edge, as such an edge uv must then have $z_{uv} = 1$. Similarly, no two vertices $u, v \in X$ can be connected by an edge as otherwise $z_{uv} \geq 1-2(1-\alpha) > 1/3$. Finally, for an edge uv between O and X, we would have to have $z_{uv} \geq 1-(1-\alpha) > 2/3$, which once again can not be the case. This shows (b).

To see (c), consider first an edge uv in $\mathcal{T}_1$; we must have $x_u + x_v = \alpha$, and this is only possible if uv is incident to one O and one Y vertex. Similarly, $x_u + x_v = 1$ for all $uv \in \mathcal{T}_2$, and therefore one of u and v must be in X, and one must be in Y. It remains to show that the edges in $\mathcal{T}_2$ induce a tree. Let us first show acyclicity: suppose for the sake of contradiction that $u_1v_1, \ldots, u_pv_p \in \mathcal{T}_2$ form a cycle (i.e., $u_1 = v_p$). Then since G is bipartite, this cycle contains an even number of edges. Let $\chi_1, \ldots, \chi_p$ be the $0,1$-coefficient vector of the left-hand sides of the constraints belonging to these edges. We see that

$$\sum_{i=1}^{p} (-1)^i \chi_i = \mathbb{0},$$

contradicting the fact that the system in (8) has full (row) rank. Note that the size of the support of (x, z) is

$$|\mathcal{T}_1| + |X| + |Y| \tag{9}$$

by definition. On the other hand, the rank of the system in (8) is

$$|\mathcal{T}_1| + |\mathcal{T}_2| + 1 \leq |\mathcal{T}_1| + (|X| + |Y| - k) + 1,$$

where k is the number of components formed by the edges in $\mathcal{T}_2$. The rank of (8) must be at least the size of the support, and this is only the case when $k = 1$; i.e., when $\mathcal{T}_2$ forms a tree spanning $X \cup Y$. Since G is bipartite, X must be fully contained in one side of the bipartition of V, and Y must be fully contained in the other. Since Y is a vertex cover in G by (b), every edge in E must have exactly one endpoint in Y. □

2.2 Blocking sets in sparse graphs via iterative rounding

In this section we propose an iterative rounding (IR) type algorithm to compute a blocking set in a given sparse graph $G = (V, E)$. Recall that this means that there is a fixed parameter $\omega > 0$ such that the graph induced by any set S of vertices has at most $\omega|S|$ edges. Recall that we also initially assume that the underlying graph G is bipartite.

The algorithm we propose follows the standard IR paradigm (e.g., see [8]) in many ways: given some instance of the blocking set problem, we first solve LP (P_B) and obtain an extreme point solution (x, z). We now generate a *smaller* sub-instance of GBS such that (a) the *projection* of (x, z) onto the sub-instance is feasible, and (b) any integral solution to the sub-instance can cheaply be extended to a solution of the original GBS instance. In particular, the reader will see the standard steps familiar from other IR algorithms: if there is an edge $uv \in E_1$ with $z_{uv} = 0$ then we may simply drop the edge, if $z_{uv} \geq 1/3$ then we include the edge into the blocking set, and if $x_u = 1$ for some vertex, then we may install one unit of x-value at u permanently and delete u and all incident edges.

The problem is that the feasible region of (P_B) has bad extreme points, even if the underlying graph is sparse and bipartite. We will exploit the structural properties documented in Lemma 3 and show that a small number of edges can be added to our blocking set even in this case. Crucially, these edges will have to come from both E_1 and E_2.

In an iteration of the algorithm, we are given a sub-instance of GBS. We first solve (P_B) for this instance, and obtain an optimal basic solution (x, z). Inductively we maintain the following: The algorithm computes a set $\hat{B} \subseteq E$ of edges, and vector $\hat{x} \in \mathbb{R}^V$ such that

[I1] $\hat{x}_u + \hat{x}_v \geq 1$ for all $uv \in E \setminus \hat{B}$,
[I2] $\mathbb{1}^T \hat{x} \leq \nu$, and
[I3] $|\hat{B}| \leq (2\omega + 1) \cdot \mathbb{1}^T z$,

where ω is the sparsity parameter introduced above. Let us first assume that the extreme point solution (x, z) is good. In this case we proceed according to one of the following cases:

Case 1. ($\exists u \in V$ with $x_u = 1$) In this case, all edges incident to u are covered. We obtain a subinstance of GBS by removing u and all incident edges from G, and by reducing ν by 1.

Case 2. ($\exists uv \in E$ with $z_{uv} = 0$) In this case, obtain a new instance of GBS by removing uv from E_1, and adding it to E_2.

Case 3. ($\exists uv \in E_1$ with $z_{uv} \geq 1/3$) In this case add uv to the approximate blocking set B, and remove uv from E_1.

In each of these three cases, we inductively solve the generated subinstance of GBS. If this subinstance is the empty graph, then we can clearly return the empty set.

Let us now consider the case where (x, z) is a bad extreme point. This case will constitute a leaf of the recursion tree, and we will show that we can directly find a small blocking set. In the following lemma, we define the sets $X, Y, O \subseteq V$ as in Lemma 3. Its proof is deferred to [7].

Lemma 4. *Let (x, z) be a bad extreme point, and let ν be the current bound on $\mathbb{1}^T x$. Then $(|X| + |Y|)/2 < \nu < |Y|$.*

We can use this bound on ν to prove that we can find small blocking sets given a bad extreme point for (P_B).

Lemma 5. *Given a bad extreme point (x, z) to (P_B), we can find a blocking set $\hat{B} \subseteq E$, and corresponding $\hat{x}$ such that $\mathbb{1}^T \hat{x} \leq \nu$, and $|\hat{B}| \leq (2\omega + 1) \cdot \mathbb{1}^T z$.*

Proof. We will construct a blocking set $\hat{B}$ as follows: let $\hat{x}_u = 1$ for a carefully chosen set $\hat{Y}$ of ν vertices from the set Y, and let $\hat{x}_u = 0$ for all other vertices in V. Recall once more from Lemma 3 (b) that Y is a vertex cover in G, and hence it suffices to choose

$$\hat{B} = \bigcup_{u \in Y \setminus \hat{Y}} \delta(u) = \bigcup_{u \in Y \setminus \hat{Y}} \big(\delta_{E_1}(u) + \delta_{E_2}(u)\big) \tag{10}$$

as our blocking set, where $\delta_{E_i}(u)$ denotes the set of E_i edges incident to vertex u. Let (a, b, γ) be the optimal dual solution of (D_B) corresponding to extreme point (x, z). Then note that complementary slackness together with the fact that $z_{uv} > 0$ for all $uv \in E_1$ implies that $a_{uv} = 1$ for these edges as well. Thus γ is an upper bound on the number E_1-edges incident to a vertex u by dual feasibility. With (10) we therefore obtain

$$|\hat{B}| \leq \sum_{u \in Y \setminus \hat{Y}} (\gamma + |\delta_{E_2}(u)|) \leq (|Y| - \nu)\gamma + \sum_{u \in Y \setminus \hat{Y}} |\delta_{E_2}(u)|. \tag{11}$$

Lemma 3 (c) shows that each E_2 edge is incident to one X, and one Y vertex. As the subgraph induced by X and Y is sparse, there therefore must be a vertex $u_1 \in Y$ of degree at most $\omega(|X|+|Y|)/|Y|$. Removing this vertex from G leaves a sparse graph, and we can therefore find a vertex u_2 of degree at most $\omega(|X|+|Y|-1)/(|Y|-1)$. Repeating this $|Y|-\nu$ times we pick a set $u_1,\ldots,u_{|Y|-\nu}$ of vertices such that

$$\sum_{i=1}^{|Y|-\nu} |\delta_{E_2}(u_i)| \leq \sum_{i=1}^{|Y|-\nu} \frac{\omega(|X|+|Y|-i)}{|Y|-i} \leq (|Y|-\nu)\cdot\frac{\omega(|X|+|Y|)}{\nu} \leq 2\omega(|Y|-\nu), \quad (12)$$

where the last inequality follows from Lemma 4. We now let $\hat{Y} = Y \setminus \{u_1,\ldots,u_{|Y|-\nu}\}$, and hence let $\hat{x}_u = 1$ for $u \in \hat{Y}$, and $\hat{x}_u = 0$ for all other vertices $u \in V$; (11) and (12) together imply that

$$|\hat{B}| \leq (|Y|-\nu)(\gamma+2\omega) \leq (2\omega+1)\gamma(Y-\nu),$$

where the last inequality follows from the fact that $\gamma \geq 1$. Lemma 3(c) shows that each edge $e \in E$ has exactly one endpoint in Y. Applying complementary slackness together with the fact that $x_u > 0$ for all $u \in Y$, we can therefore rewrite the objective function of (D_B) as

$$\mathbb{1}^T a + \mathbb{1}^T b - \gamma\,\nu = \gamma(|Y|-\nu).$$

The lemma follows. □

We can now put things together.

Lemma 6. *Given an instance of GBS, the above procedure terminates with a set $\hat{B} \subseteq E$, and $\hat{x} \in \mathbb{R}^V$ such that $\mathbb{1}^T\hat{x} \leq \nu$, and $\hat{x}_u + \hat{x}_v \geq 1$ for all $uv \in E \setminus \hat{B}$. The set $\hat{B}$ has size at most $(2\omega+1)\mathbb{1}^T z$, where (x,z) is an optimal solution to (P_B) for the given GBS instance.*

Proof. The proof uses the usual induction on the recursion depth. Let us first consider the case where the current instance is a leaf of the recursion tree. The lemma follows vacuously if the graph in the given GBS instance is empty. Otherwise it follows immediately from Lemma 5.

Any internal node of recursion tree corresponds to an instance of GBS where (x,z) is a good extreme point. We claim that, no matter which one of the above cases we are in, we have that (a) a suitable projection of (x,z)

yields a feasible solution for the created GBS sub-instance, and (b) we can *augment* an approximate blocking set for this sub-instance to obtain a *good* blocking set for the instance given in this iteration. We proceed by looking at the three cases discussed above.

Case 1. Let (x', z') be the natural projection of (x, z) onto the GBS sub-instance; i.e., x'_v is set to x_v for all vertices in $V - u$, and $z'_{vw} = z_{vw}$ for the remaining edges $vw \in E_1 \setminus \delta(u)$. This solution is easily verified to be feasible. Inductively, we therefore know that we obtain a blocking set $\bar{B}$ and corresponding vector $\bar{x}$ such that $\bar{B}$ has no more than $(2\omega+1)\,\mathbb{1}^T\bar{z} \leq (2\omega+1)\,\mathbb{1}^T z$ elements, and $\mathbb{1}^T\bar{x} \leq \nu - 1$. Thus, letting $\hat{x}_v = \bar{x}_v$ for all $v \in V - u$, and $\hat{x}_u = 1$ together with $\hat{B} = \bar{B}$ gives a feasible solution for the original GBS instance.

Case 2. The argument for this case is virtually identical to that of Case 1, and we omit the details.

Case 3. Once again we project the current solution (x, z) onto the GBS subinstance; i.e., let $x' = x$, and $z'_{qr} = z_{qr}$ for all $qr \in E_1 - uv$. Clearly (x', z') is feasible for the GBS subinstance, and inductively we therefore obtain a vector $\bar{x}$ and corresponding feasible blocking set $\bar{B}$ of size at most $(2\omega + 1) \cdot \mathbb{1}^T z'$. Adding uv to $\bar{B}$ yields a feasible blocking set $\hat{B}$ for the original instance together with $\hat{x} = \bar{x}$. Its size is at most $(2\omega+1)\,\mathbb{1}^T z'+1 \leq (2\omega + 1)\, 1^T z$ as $\omega \geq 1$. □

Suppose now that we are given a non-bipartite, sparse instance of the blocking set problem: $G = (V, E)$ is a general sparse graph, and $\nu > 0$ is a parameter. We create a bipartite graph H in the usual way: for each vertex $u \in V$ create two copies u_1 and u_2 and add them to H. For each edge $uv \in E$, add two edges u_1v_2 and u_2v_1 to H. The new blocking set instance is given by (H, ν') where $\nu' = 2\nu$.

Given a feasible solution (x, z) to (P_B) for the instance (G, ν) , we let $x'_{u_i} = x_u$ for all $u \in V$ and $i \in \{1, 2\}$, and $z'_{u_iv_j} = z_{uv}$ for all edges u_iv_j. For any edge u_iv_j in H, we now have

$$x'_{u_i} + x'_{v_j} + z_{u_iv_j} = x_u + x_v + z_{uv} \geq 1,$$

and $\mathbb{1}^T x' \leq 2\mathbb{1}^T x \leq 2\nu$. Thus, (x', z') is feasible to (P_B) for instance (H, ν'), and its value is at most twice that of $\mathbb{1}^T z$. Let $\hat{x}, \hat{B}$ be a feasible solution to the instance on graph H. Then let

$$B = \{uv \in E \,:\, u_1v_2 \text{ or } u_2v_1 \text{ are in } \hat{B}\},$$

and note that B has size at most that of $\hat{B}$. Also let $x_u = (\hat{x}_{u_1} + \hat{x}_{u_2})/2$ for all $u \in V$. Clearly, $\mathbb{1}^T x \leq \nu$, and for any edge $uv \in E$, we have

$$x_u + x_v \geq \frac{\hat{x}_{u_1} + \hat{x}_{u_2} + \hat{x}_{v_1} + \hat{x}_{v_2}}{2},$$

and the right-hand side is at least 1 if none of the two edges u_1v_2, u_2v_1 is in $\hat{B}$. This shows feasibility of the pair x, B. In order to prove Theorem 2 it now remains to show that graph H is sparse. Pick any set S of vertices in H, and let

$$S' = \{v \in V \,:\, \text{at least one of } v_1 \text{ and } v_2 \text{ are in } S\}.$$

Then $|S'| \leq |S|$, and the number of edges of $H[S]$ is at most twice the number of edges in $G[S']$, and hence bounded by $2\omega\,|S|$; we let $\omega' = 2\omega$ be the sparsity parameter of H. Let (x, z) and (x', z') be optimal basic solutions to (P_B) for instances (G, ν), and (H, ν'), respectively. The blocking set B for G has size no more than

$$(2\omega' + 1)\mathbb{1}^T z' \leq 2(4\omega + 1)\mathbb{1}^T z.$$

Thus, we have proven Theorem 2.

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