Lecture 3i

Complex Systems of Equations

(pages 407-408)

Now that we have covered the fundamentals of the complex numbers, we want to move on to study the vector space properties of the complex numbers. As with vector spaces over $\mathbb{R}$, we will frequently find ourselves wanting to solve a system of linear equations over $\mathbb{C}$. That is to say, a system whose coefficients and variables are both complex numbers. Since we can add, subtract, multiply and divide complex numbers, just as we can real numbers, we find that the technique we used to solve systems of linear equations over $\mathbb{R}$ can also be used to solve systems of linear equations over $\mathbb{C}$. That is to say, we will row reduce a matrix. The elementary row operations remain the same, except that when we now say "multiply a row by a constant", we are now multiplying by a complex number, and when we "add a multiple of one row to another", we again are multiplying by a complex number. The definitions of row echelon form and reduced row echelon form remain the same, and the notion of a "bad row" (where we have set "$0 = a$" for any "$a \neq 0$ ") is still the same. When necessary, we still replace variables with parameters to write the general solution, again keeping in mind that our parameters will now be from $\mathbb{C}$, not just $\mathbb{R}$.

So, basically, it's all the same, except that it's completely different. Since the complex numbers are still new to you, even a simple "multiply a row by a constant" operation may prove difficult, so you are urged to take your time and write our all your work as you become acquainted with row reductions in the complex numbers.

Here are some examples of solving systems of equations over the complex numbers, designed to give you an idea of the different situations you may encounter.

Example: Solve the following system of linear equations:

$$\begin{aligned} z_1 + iz_2 + (-3+i)z_3 &= -1-i \\ 2z_1 + (1+3i)z_2 + (-4+2i)z_3 &= 2i \\ 2iz_1 - 2z_2 + (-2-3i)z_3 &= -1+i \end{aligned}$$

To solve this system, we will row reduce its augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & i & -3+i & -1-i \\ 2 & 1+3i & -4+2i & 2i \\ 2i & -2 & -2-3i & -1+i \end{array}\right]$$

Since our first row already has a 1 in the first column, we can establish this as a leading 1 by subtracting the correct multiples of row 1 from rows 2 and 3 to leave them with zeros in their first columns. So our first row operations (which can be performed simultaneously) are to replace row 2 with row 2 plus (-2) times row 1, and to replace row 3 with (-2i) times row 1. (Note that I prefer to to add a negative multiple rather than subtract a positive multiple, because I find it too easy to make a mistake when you end up combining negative signs and subtraction. It's better to simply deal with the matter when you are multiplying by your constant.)

To replace row 2 with row 2 plus (-2) times row 1, we first need to calculate (-2) times row 1. Multiplying by a real number is a simple matter of distributing, so we quickly find that

$$-2R_1 = \quad -2 \quad -2i \quad 6-2i \;\big|\; 2+2i$$

And so we find $R_2 + (-2)R_1$ as follows:

$$\begin{array}{r|rrr|r} R_2: & 2 & 1+3i & -4+2i & 2i \\ -2R_1: & -2 & -2i & 6-2i & 2+2i \\ \hline R_2+(-2)R_1: & 0 & 1+i & 2 & 2+4i \end{array}$$

Next, to replace row 3 with row 3 plus (-2i) times row 1, we first need to calculate (-2i) times row 1. Multiplying by a complex number does not become a simple case of distribution, because i^2 becomes -1, but with practice it can become easy to calculate.

$$\begin{array}{rrrr|r} -2iR_1: & -2i & -2i^2 & 6i-2i^2 & 2i+2i^2 \\ = & -2i & 2 & 2+6i & -2+2i \end{array}$$

And so we find $R_3 + (-2i)R_1$ as follows:

$$\begin{array}{rrrr|r} R_3: & 2i & -2 & -2-3i & -1+i \\ -2iR_1: & -2i & 2 & 2+6i & -2+2i \\ \hline R_2+(-2)R_1: & 0 & 0 & 3i & -3+3i \end{array}$$

Now we can replace R_2 with $R_2 + (-2)R_1$ and R_3 with $R_3 + (-2i)R_1$ to get the following matrix:

$$\left[\begin{array}{rrr|r} 1 & i & -3+i & -1-i \\ 0 & 1+i & 2 & 2+4i \\ 0 & 0 & 3i & -3+3i \end{array}\right]$$

This matrix is already in row echelon form, and from it we can see that our system will have a unique solution. To find this solution, we will continue to row reduce until we reach reduced row echelon form. The first step in that process will be to multiply rows 2 and 3 by constants to make the first non-zero entry a 1. We need to multiply row 2 by $1/1+i$, and we need to multiply row 3 by $1/3i$. But what are these? Remember that for any complex number z, we have $1/z = \overline{z}/|z|^2$, so when we multiply by $1/1+i$, we are multiplying by $(1-i)/(1^2+1^2) = (1-i)/2$, and when we multiply by $1/3i$, we are multiplying by $-3i/(0^2+3^2) = -i/3$. So rows 2 and 3 become:

$$
\begin{array}{rccr|r}
((1-i)/2)R_2: & 0 & 1 & 2((1-i)/2) & (2+4i)((1-i)/2) \\
= & 0 & 1 & 1-i & (1+2i)(1-i) \\
= & 0 & 1 & 1-i & 1-i+2i-2i^2 \\
= & 0 & 1 & 1-i & 3+i
\end{array}
$$

$$
\begin{array}{rccc|r}
(-i/3)R_3: & 0 & 0 & 1 & (-3+3i)(-i/3) \\
= & 0 & 0 & 1 & i-i^2 \\
= & 0 & 0 & 1 & 1+i
\end{array}
$$

So if we replace R_2 with $((1-i)/2)R_2$ and R_3 with $(-i/3)R_3$, we get the following matrix:

$$
\left[\begin{array}{ccr|r}
1 & i & -3+i & -1-i \\
0 & 1 & 1-i & 3+i \\
0 & 0 & 1 & 1+i
\end{array}\right]
$$

Now, we begin our back-substitution steps, by creating zeros above the leading 1 in row 3. That means we need to replace R_1 with $R_1 + (3-i)R_3$ and we need to replace R_2 with $R_2 + (-1+i)R_3$. Now, since the first two columns in row 3 are zero, these operations will not change the values in the first two columns of rows 1 and 2. Additionally, our row operation has been specifically chosen to make the third columns of rows 1 and 2 equal to zero. So the only calculation that actually needs to be performed is in the fourth column. So, instead of looking at these calculations on the entire rows, I will simply focus on the calculation needed for the fourth column. Which are:

$R_1 + (3-i)R_3 : (-1-i) + (3-i)(1+i) = -1-i+3+3i-i-i^2 = (-1+3+1) + i(-1+3-1) = 3+i$

$R_2 + (-1+i)R_3 : (3+i) + (-1+i)(1+i) = 3+i-1-i+i+i^2 = (3-1-1) + i(1-1+1) = 1+i$

and our new matrix is:

$$
\left[\begin{array}{ccc|c}
1 & i & 0 & 3+i \\
0 & 1 & 0 & 1+i \\
0 & 0 & 1 & 1+i
\end{array}\right]
$$

The final step in the row reduction is to replace R_1 with $R_1 + (-i)R_2$. Again we will have that the only calculations needed are in the fourth column, since the first and third columns of row 2 have a zero in them, and thus these columns of row 1 will be unchanged, and we have chosen our row operation to turn the entry in the second column of row 1 into a zero. In the fourth column we get:

$R_1 + (-i)R_2 : (3+i) + (-i)(1+i) = 3+i-i-i^2 = (3+1) + i(1-1) = 4$

And so we see that our reduced row echelon form matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1+i \\ 0 & 0 & 1 & 1+i \end{array}\right]$$

From this we see that the solution to our system is $z_1 = 4$, $z_2 = 1+i$, $z_3 = 1+i$.

Example: Solve the following system of linear equations:

$$\begin{aligned} z_1 + (1+i)z_2 - 2iz_3 - (1+i)z_4 &= 2+i \\ (1-i)z_1 + 2z_2 - (2+i)z_3 + (-3+i)z_4 &= 1-2i \\ iz_1 + (-1+i)z_2 + iz_3 - (2+2i)z_4 &= -1-3i \end{aligned}$$

To solve this system, we will row reduce its augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 1+i & -2i & -1-i & 2+i \\ 1-i & 2 & -2-i & -3+i & 1-2i \\ i & -1+i & i & -2-2i & -1-3i \end{array}\right]$$

Notice that I went ahead and distributed any subtractions when I created this matrix. For example $-(1+i)z_4$ in the first equation becomes $-1-i$. Our first row operations are to replace R_2 with $R_2 + (-1+i)R_1$, and to replace R_3 with $R_3 + (-i)R_1$. First we need to calculate $(-1+i)R_1$:

$$(-1+i)R_1: \quad 1(-1+i) \quad (1+i)(-1+i) \quad (-2i)(-1+i) \quad (-1-i)(-1+i) \;\big|\; (2+i)(-1+i)$$

Let's calculate all of these entries individually. Obviously, $1(-1+i) = -1+i$. Next, we find that $(1+i)(-1+i) = -1+i-i+i^2 = (-1-1)+i(1-1) = -2$. In the third column, we get $(-2i)(-1+i) = 2i-2i^2 = 2+2i$. In the fourth column we get $(-1-i)(-1+i) = 1-i+i-i^2 = (1+1)+i(-1+1) = 2$. And lastly, in the fifth column, we get $(2+i)(-1+i) = -2+2i-i+i^2 = (-2-1)+i(2-1) = -3+i$. So we have

$$(-1+i)R_1: \quad -1+i \quad -2 \quad 2+2i \quad 2 \;\big|\; -3+i$$

And now we can compute $R_2 + (-1+i)R_1$:

$$\begin{array}{rrrrr|r} R_2: & 1-i & 2 & -2-i & -3+i & 1-2i \\ (-1+i)R_1: & -1+i & -2 & 2+2i & 2 & -3+i \\ \hline R_2+(-1+i)R_1: & 0 & 0 & i & -1+i & -2-i \end{array}$$

Next, to replace R_3 with $R_3 + (-i)R_1$, we need to calculate $-iR_1$.

$$\begin{array}{rrrrr|r} -iR_1: & 1(-i) & (1+i)(-i) & (-2i)(-i) & (-1-i)(-i) & (2+i)(-i) \\ = & -i & -i-i^2 & 2i^2 & i+i^2 & -2i-i^2 \\ = & -i & 1-i & -2 & -1+i & 1-2i \end{array}$$

And now we can compute $R_3 + (-i)R_1$:

$$\begin{array}{r|rrrr|r} R_3: & i & -1+i & i & -2-2i & -1-3i \\ -iR_1: & -i & 1-i & -2 & -1+i & 1-2i \\ \hline R_3+(-i)R_1: & 0 & 0 & -2+i & -3-i & -5i \end{array}$$

Now we can replace R_2 with $R_2 + (-1+i)R_1$ and R_3 with $R_3 + (-i)R_1$ to get the following matrix:

$$\left[\begin{array}{rrrr|r} 1 & 1+i & -2i & -1-i & 2+i \\ 0 & 0 & i & -1+i & -2-i \\ 0 & 0 & -2+i & -3-i & -5i \end{array}\right]$$

The next thing we want to do is to establish a leading 1 in the third column. But do we want to do so in row 2 or row 3? Row 2 is by far the easiest, since it happens that dividing by i is the same thing as multiplying by $-i$. So, we can multiply row 2 by $-i$ and get

$$\begin{array}{rrrrr|r} -iR_2: & 0 & 0 & i(-i) & (-1+i)(-i) & (-2-i)(-i) \\ = & 0 & 0 & -i^2 & i-i^2 & 2i+i^2 \\ = & 0 & 0 & 1 & 1+i & -1+2i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{rrrr|r} 1 & 1+i & -2i & -1-i & 2+i \\ 0 & 0 & 1 & 1+i & -1+2i \\ 0 & 0 & -2+i & -3-i & -5i \end{array}\right]$$

Now, we want to replace R_3 with $R_3 + (2-i)R_2$. To do this, let's first compute $(2-i)R_2$:

$$\begin{array}{rrrlr|r} (2-i)R_2: & 0 & 0 & 2-i & (1+i)(2-i) & (-1+2i)(2-i) \\ = & 0 & 0 & 2-i & 2-i+2i-i^2 & -2+i+4i-2i^2 \\ = & 0 & 0 & 2-i & (2+1)+i(-1+2) & (-2+2)+i(1+4) \\ = & 0 & 0 & 2-i & 3+i & 5i \end{array}$$

And now we can compute $R_3 + (2-i)R_2$:

$$\begin{array}{r|rrrr|r} R_3: & 0 & 0 & -2+i & -3-i & -5i \\ (2-i)R_2: & 0 & 0 & 2-i & 3+i & 5i \\ \hline R_3+(2-i)R_2: & 0 & 0 & 0 & 0 & 0 \end{array}$$

and our matrix becomes

$$\left[\begin{array}{rrrr|r} 1 & 1+i & -2i & -1-i & 2+i \\ 0 & 0 & 1 & 1+i & -1+2i \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row echelon form, and since there are no bad rows, we know that it has a solution. Moreover, we already can tell that our general solution will have two parameters. But let's do one last row reduction step to put our matrix into reduced row echelon form before we attempt to find the general solution. And this operation will be to replace R_1 with $R_1 + (2i)R_2$. Since the first two columns of row 2 contain zero, this calculation will not change the first two columns of row 1. Moreover, this operation was chosen to change the entry in the third column of row 1 into a zero, so no calculation is needed here either. All that remains is to do the calculation for columns 4 and 5:

Column 4: $(-1-i)+(2i)(1+i) = -1-i+2i+2i^2 = (-1-2)+i(-1+2) = -3+i$

Column 5: $(2+i)+(2i)(-1+2i) = 2+i-2i+4i^2 = (2-4)+i(1-2) = -2-i$

So, if we replace R_1 with $R_1 + (2i)R_2$, our matrix becomes

$$\left[\begin{array}{rrrr|r} 1 & 1+i & 0 & -3+i & -2-i \\ 0 & 0 & 1 & 1+i & -1+2i \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row echelon form, and is equivalent to the following system:

$$\begin{aligned} z_1 + (1+i)z_2 + (-3+i)z_4 &= -2-i \\ z_3 + (1+i)z_4 &= -1+2i \end{aligned}$$

If we replace the variable z_2 with the parameter s and the variable z_4 with the parameter t, we see that the general solution to our system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} (-2-i)-(1+i)s-(-3+i)t \\ s \\ (-1+2i)-(1+i)t \\ t \end{bmatrix} = \begin{bmatrix} -2-i \\ 0 \\ -1+2i \\ 0 \end{bmatrix} + s\begin{bmatrix} -1-i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3-i \\ 0 \\ -1-i \\ 1 \end{bmatrix}$$

Example: Solve the following system of linear equations:

$$\begin{aligned} 2iz_1 + (4-2i)z_2 + 4z_3 &= 6-4i \\ 2z_1 + (-2-4i)z_2 + (1-2i)z_3 &= 1-i \\ (1+i)z_1 + (1-3i)z_2 + (3+i)z_3 &= 3-2i \end{aligned}$$

To solve this system, we will row reduce its augmented matrix:

$$\left[\begin{array}{rrr|r} 2i & 4-2i & 4 & 6-4i \\ 2 & -2-4i & 1-2i & 1-i \\ 1+i & 1-3i & 3+i & 3-2i \end{array}\right]$$

Our first goal is to create a 1 in the first column. At first glance, it may not seem obvious where to do this, and you may even be tempted to do this in the second row, even if it does create some fractions. But if you remember that dividing by i is the same as multiplying by $-i$, you see that it is easy to create a 1 in the first column of the first row by multiplying it by $-i/2$, making the matrix

$$\left[\begin{array}{ccc|c} 1 & -1-2i & -2i & -2-3i \\ 2 & -2-4i & 1-2i & 1-i \\ 1+i & 1-3i & 3+i & 3-2i \end{array}\right]$$

Now we need to replace R_2 with $R_2 + (-2)R_1$, and R_3 with $R_3 + (-1-i)R_1$. Since it is easy to calculate $-2R_1$, we quickly compute $R_2 + (-2)R_1$ as follows:

$$\begin{array}{rrrr|r} R_2: & 2 & -2-4i & 1-2i & 1-i \\ -2R_1: & -2 & 2+4i & 4i & 4+6i \\ \hline R_2+(-2)R_1: & 0 & 0 & 1+2i & 5+5i \end{array}$$

To compute $R_3 + (-1-i)R_1$, we will first want to compute $(-1-i)R_1$:

$$(-1-i)R_1: \quad 1(-1-i) \quad (-1-2i)(-1-i) \quad (-2i)(-1-i) \;\big|\; (-2-3i)(-1-i)$$

Let's calculate all of these entries individually. Obviously, $1(-1-i) = -1-i$. Next we find that $(-1-2i)(-1-i) = 1+i+2i+2i^2 = (1-2)+i(1+2) = -1+3i$. In the third column, we get $(-2i)(-1-i) = 2i+2i^2 = -2+2i$. And lastly, we find that $(-2-3i)(-1-i) = 2+2i+3i+3i^2 = (2-3)+i(2+3) = -1+5i$. So we now have $(-1-i)R_1$:

$$(-1-i)R_1: \quad -1-i \quad -1+3i \quad -2+2i \;\big|\; -1+5i$$

and this lets us compute $R_3 + (-1-i)R_1$:

$$\begin{array}{rrrr|r} R_3: & 1+i & 1-3i & 3+i & 3-2i \\ (-1-i)R_1: & -1-i & -1+3i & -2+2i & -1+5i \\ \hline R_3+(-1-i)R_1: & 0 & 0 & 1+3i & 2+3i \end{array}$$

So, if we replace R_2 with $R_2 + (-2)R_1$ and R_3 with $R_3 + (-1-i)R_1$ our matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -1-2i & -2i & -2-3i \\ 0 & 0 & 1+2i & 5+5i \\ 0 & 0 & 1+3i & 2+3i \end{array}\right]$$

The next step is to create a leading 1 in the third column. We can do this in row 2 by multiplying it by $1/(1+2i) = (1-2i)/(1^2+2^2) = (1-2i)/5$. We get

$$\begin{array}{rccr|r}
((1-2i)/5)R_2: & 0 & 0 & (1+2i)/(1+2i) & (5+5i)((1-2i)/5) \\
= & 0 & 0 & 1 & (1+i)(1-2i) \\
= & 0 & 0 & 1 & 1-2i+i-2i^2 \\
= & 0 & 0 & 1 & (1+2)+i(-2+1) \\
= & 0 & 0 & 1 & 3-i
\end{array}$$

and our new matrix is

$$\left[\begin{array}{rrr|r}
1 & -1-2i & -2i & -2-3i \\
0 & 0 & 1 & 3-i \\
0 & 0 & 1+3i & 2+3i
\end{array}\right]$$

Now we want to replace R_3 with $R_3 + (-1-3i)R_2$. This will make the first, second, and third entries in row 3 all equal to zero, so the only calculation we need to perform is in the fourth column:

Column 4: $(2+3i)+(-1-3i)(3-i) = 2+3i-3+i-9i+3i^2 = (2-3-3)+i(3+1-9) = -4-5i$

So, if we replace R_3 with $R_3 + (-1-3i)R_2$, our matrix becomes

$$\left[\begin{array}{rrr|r}
1 & -1-2i & -2i & -2-3i \\
0 & 0 & 1 & 3-i \\
0 & 0 & 0 & -4-5i
\end{array}\right]$$

Since the last row is a bad row, we see that our system is inconsistent, and has no solutions.