Lecture 3h

n-th Roots

(pages 404-405)

De Moivre's theorem can just as easily be used to find the roots of a number instead of the powers of a number, since it is easy to see that

$$(r^{1/n}e^{i\theta/n})^n = (r^{1/n})^n e^{n(i\theta/n)} = re^{i\theta}$$

so $r^{1/n}e^{i\theta/n}$ is an n-th root of $re^{i\theta}$. I say "an" n-th root, because there could be more than one. At first glance, it may not look like the formula for the n-th root could yield more than one result, but the secret is in the fact that θ is not unique. We can replace θ with $\theta + 2\pi k$ for any integer k, and we get $re^{i\theta} = re^{i(\theta+2\pi k)}$, which means that $r^{1/n}e^{i(\theta+2\pi k)/n}$ is also an n-th root of $re^{i\theta}$. Let's look at this process in an example:

Example: Find the fourth roots of $8 - 8\sqrt{3}i$.

To solve this, we first need to find a polar form for $8 - 8\sqrt{3}i$. We start by calculating $r = \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{64 + 192} = \sqrt{256} = 16$. Next, we need to find θ such that $\cos\theta = 8/16 = 1/2$ and $\sin\theta = -8\sqrt{3}/16 = -\sqrt{3}/2$. Since the cosine is positive and the sine is negative, we want θ to be in the fourth quadrant, so we can use $\theta = 5\pi/3$. In fact, we can use $\theta = (5\pi/3) + 2\pi k$ for any integer k. Let's look at the roots we get by using various values for θ:

θ	$r^{1/4}e^{i\theta/4}$
$5\pi/3$	$2e^{i5\pi/12}$
$5\pi/3 + 2\pi = 11\pi/3$	$2e^{i11\pi/12}$
$5\pi/3 + 4\pi = 17\pi/3$	$2e^{i17\pi/12}$
$5\pi/3 + 6\pi = 23\pi/3$	$2e^{i23\pi/12}$
$5\pi/3 + 8\pi = 29\pi/3$	$2e^{i29\pi/12}$

Let's take a closer look at the last value we calculated. Since $29\pi/12$ is more that 2π, we can subtract 2π from it, getting $5\pi/12$. This is the same argument found in the first root we calculated. And, from this point on, if we were to continue adding multiples of 2π to θ, we would end up repeating the roots, in order, that we have already found.

θ	$r^{1/4}e^{i\theta/4}$
$5\pi/3 + 10\pi = 35\pi/3$	$2e^{i35\pi/12} = 2e^{i11\pi/12}$
$5\pi/3 + 12\pi = 41\pi/3$	$2e^{i41\pi/12} = 2e^{i17\pi/12}$
$5\pi/3 + 14\pi = 47\pi/3$	$2e^{i47\pi/12} = 2e^{i23\pi/12}$
$5\pi/3 + 16\pi = 53\pi/3$	$2e^{i53\pi/12} = 2e^{i29\pi/12} = 2e^{i5\pi/12}$

At this point, we see that the roots start cycling through again. Let's take a closer look at what is happening. We are starting with $\theta = 5\pi/3$, and then we are adding $2\pi k$, or $6\pi k/3$, to get $\theta_k = (5 + 6k)\pi/3$. Then, when we find the fourth root, divide θ_k by 4, giving us $(5 + 6k)\pi/12$. We can rewrite this as $(5\pi/12) + (k\pi/2)$. From this description, it is easy to see that when k is a multiple of four, say $k = 4d$, then the argument for our root is $(5\pi/12) + (4d\pi/2) = (5\pi/12) + (2\pi)d$, which is equivalent to $(5\pi/12)$. If k is one more than a multiple of four, say $k = 4d + 1$, then the argument for our root is $(5\pi/12)+((4d+1)\pi/2) = (5\pi/12)+(2\pi)d+\pi/2$, which is equivalent to $(11\pi/12)$. If k is two more than a multiple of four, say $k = 4d + 2$, then the argument for our root is $(5\pi/12)+((4d+2)\pi/2) = (5\pi/12)+(2\pi)d+\pi$, which is equivalent to $(17\pi/12)$. Lastly, if k is three more than a multiple of four, say $k = 4d+3$, then the argument for our root is $(5\pi/12) + ((4d+3)\pi/2) = (5\pi/12) + (2\pi)d + 3\pi/2$, which is equivalent to $(23\pi/12)$. We have now looked at all possible integer values k, and seen that the fourth roots of $2e^{i5\pi/3}$ are the following

$$2e^{i5\pi/12} \quad 2e^{i11\pi/12} \quad 2e^{i17\pi/12} \quad 2e^{i23\pi/12}$$

As such, we see that there are 4 different fourth roots of $8 - 8\sqrt{3}i$.

In general, we find that any non-zero complex number will have n distinct n-th roots. We see this by going through a similar argument to the one pursued in this example. For, if m is an integer, then we can write m as k more than a multiple of n, say $m = dn + k$. Then, when we add $2\pi m$ to our argument θ, we get $\theta + 2\pi m = \theta + 2(dn + k)\pi$. Then, when we divide θ by n to find the argument for the n-th root, we get $\theta/n+2d\pi+2k\pi/n$. Since $2d\pi$ is a multiple of 2π, we can remove it from the argument without changing the complex number, getting an argument of $\theta/n+2k\pi/n$. As such, it does not matter what m is, per se, it simply matters how much much m differs from a multiple of n. Since we need only consider k values of 0 to $n-1$ to cycle through all possible integers m, we find that no matter what multiple of 2π we add to θ, our list of arguments for the n-th root of $re^{i\theta}$ will be $\theta/n + 2k\pi/n$ for $k = 0, \ldots, n - 1$. And this means we have proven the following:

Theorem 9.1.6 Let z be a non-zero complex number. Then the n distinct n-th roots of $z = re^{i\theta}$ are

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \qquad k = 0, 1, \ldots, n-1$$

Example: Find all the cube roots of 27.

In the real numbers, 27 has only one cube root, namely 3. But as a complex number, 27 must have three distinct roots, one of which is still 3. Let's put the number 27 into polar form, and use Theorem 9.1.6 to find all cube roots. Since 27 is a positive real number, we know that $r = 27$ and $\theta = 0$ is an argument. This means that the cube roots of 27 are

$$27^{1/3} e^{i0/3} = 3e^{i0} \qquad 27^{1/3} e^{i(0+2\pi(1))/3} = 3e^{i2\pi/3} \qquad 27^{1/3} e^{i(0+2\pi(2))/3} = 3e^{i4\pi/3}$$

and we note that $3e^{i0} = 3$.

Example: Find all the fifth roots of $1 + i$ (i.e. find $(1+i)^{1/5}$).

We already know that $\sqrt{2}e^{i\pi/4}$ is a polar form for $1 + i$. This means that the fifth roots of $1 + i$ are

$k = 0 :\quad \sqrt{2}^{1/5} e^{i(\pi/4)/5} = 2^{1/10} e^{i\pi/20}$

$k = 1 :\quad \sqrt{2}^{1/5} e^{i(\pi/4+2\pi(1))/5} = 2^{1/10} e^{i9\pi/20}$

$k = 2 :\quad \sqrt{2}^{1/5} e^{i(\pi/4+2\pi(2))/5} = 2^{1/10} e^{i17\pi/20}$

$k = 3 :\quad \sqrt{2}^{1/5} e^{i(\pi/4+2\pi(3))/5} = 2^{1/10} e^{i25\pi/20}$

$k = 4 :\quad \sqrt{2}^{1/5} e^{i(\pi/4+2\pi(4))/5} = 2^{1/10} e^{i33\pi/20}$