Lecture 3d

Roots of Polynomial Equations

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Another use of the complex conjugate is seen in the following theorem.

Theorem 9.1.2 Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_j \in \mathbb{R}$ for $1 \le j \le n$. If z is a root of $p(x)$, then $\overline{z}$ is also a root of $p(x)$.

Note that our polynomial has coefficients from the real numbers, not the complex numbers. This theorem is most certainly not true of polynomials with complex coefficients!

Proof of Theorem 9.1.2: Suppose that z is a root of $p(x)$. Then

$$a_n z^n + \cdots + a_1 z + a_0 = 0$$

Keeping in mind that our a_j are also complex numbers, we can use properties of the complex conjugate to see that

$$\begin{aligned} p(\overline{z}) &= a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \\ &= a_n \overline{z^n} + \cdots + a_1 \overline{z} + a_0 && \text{property (6)} \\ &= \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} && \text{property (5)} \\ &= \overline{a_n z^n + \cdots + a_1 z + a_0} && \text{property (4)} \\ &= \overline{0} \\ &= 0 \end{aligned}$$

Thus, $\overline{z}$ is a root of $p(x)$.

We can use this fact to help us factor polynomials with real coefficients, and find all their roots.

Example: Factor the polynomial $p(x) = x^2 + 9$, given that $3i$ is one of the roots.

From our theorem, we know that if $3i$ is a root, then $\overline{3i} = -3i$ is also a root. That means that both $(x - 3i)$ and $(x + 3i)$ are factors of $x^2 + 9$. And since $(x - 3i)(x + 3i) = x^2 + 3ix - 3ix - 9i^2 = x^2 + 9$, we see that $x^2 + 9$ factors as $(x - 3i)(x + 3i)$.

Example: Factor $p(x) = x^3 - 9x^2 + 36x - 54$, given that $3 + 3i$ is one of the roots.

From our theorem, we know that if $3+3i$ is one of the roots, then $\overline{3 + 3i} = 3-3i$ is also a root. This means that both $(x - 3 - 3i)$ and $(x - 3 + 3i)$ are factors of p, so we know that their product is also a factor of p. We see that

$(x-3-3i)(x-3+3i) = x^2-3x+3ix-3x+9-9i-3ix+9i-9i^2 = x^2-6x+18$

Dividing $p(x)$ by $x^2-6x+18$ we get $(x-3)$. So we have that

$$p(x) = (x-3-3i)(x-3+3i)(x-3)$$

Could we have started by dividing $p(x)$ by $(x-3-3i)$? Yes, and we would have found that $p(x) = (x-3-3i)(x^2+(-6+3i)x+(9-9i))$. We can even use the quadratic formula to factor the remaining degree 2 polynomial.

$$\begin{aligned}
\tfrac{-b\pm\sqrt{b^2-4ac}}{2a} &= \tfrac{6-3i\pm\sqrt{(-6+3i)^2-4(1)(9-9i)}}{2(1)} \\
&= \tfrac{6-3i\pm\sqrt{27-36i-36+36i}}{2} \\
&= \tfrac{6-3i\pm\sqrt{-9}}{2} \\
&= \tfrac{6-3i\pm\sqrt{-1}\sqrt{9}}{2} \\
&= \tfrac{6-3i\pm 3i}{2} \\
&= \tfrac{6-3i+3i}{2} \text{ and } \tfrac{6-3i-3i}{2} \\
&= 3 \text{ and } 3-3i
\end{aligned}$$

This, of course, confirms our earlier result. I, for one, preferred the first method, since it involves fewer calculations involving complex numbers. But I did want to take this chance to show that the quadratic formula can find the complex roots of a quadratic equation as well. Here's another example, that combines both techniques.

Example: Factor $p(x) = x^4+4x^3+6x^2+4x+5$, given that i is one of the roots.

Well, since i is one of the roots, $-i$ is also one of the roots, and so we know that both $(x-i)$ and $(x+i)$ are factors of this polynomial. As such, their product $(x-i)(x+i) = x^2+ix-ix-i^2 = x^2+1$ is a factor of p. If we divide $p(x)$ by x^2+1, we get x^2+4x+5. Let's plug this into the quadratic formula to find its roots:

$$\begin{aligned}
\tfrac{-b\pm\sqrt{b^2-4ac}}{2a} &= \tfrac{-4\pm\sqrt{4^2-4(1)(5)}}{2(1)} \\
&= \tfrac{-4\pm\sqrt{16-20}}{2} \\
&= \tfrac{-4\pm\sqrt{-4}}{2} \\
&= \tfrac{-4\pm\sqrt{-1}\sqrt{4}}{2} \\
&= \tfrac{-4\pm 2i}{2} \\
&= -2+i \text{ and } -2-i
\end{aligned}$$

Note that, of course, these two complex roots are conjugates of each other!

So we see that $p(x) = (x-i)(x+i)(x+2-i)(x+2+i)$.