Solution to Practice 2m

D2 Suppose that A is orthogonally diagonalizable. Then there is an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$. Now, $P^T = P^{-1}$, so we can rewrite this as $P^{-1}AP = D$. Multiplying both sides on the left by P, we get $AP = PD$. And then multiplying both sides on the right by $P^T = P^{-1}$, we get $A = PDP^T$. From this we see the following:

$$\begin{aligned}
A^T &= (PDP^T)^T \\
&= (DP^T)^T P^T \\
&= (P^T)^T D^T P^T \\
&= PDP^T \\
&= A
\end{aligned}$$

(Note that $D^T = D$, since D is a diagonal matrix.) So, we see that $A^T = A$, which means that A is symmetric.

D3 Suppose that A is an invertible symmetric matrix. Since A is symmetric, it is orthogonally diagonalizable. Let P be an orthogonal matrix and D be a diagonal matrix such that $P^T AP = D$. Since D is diagonal, D is clearly row equivalent to the identity matrix, and as such D is invertible. Moreover, D^{-1} is also a diagonal matrix, since we can get D^{-1} from D by replacing the diagonal entries of D with their inverse (in $\mathbb{R}$). That is, if $D = \text{diag}\{d_1, \ldots, d_n\}$, then $D^{-1} = \text{diag}\{1/d_1, \ldots, 1/d_n\}$.

Since $P^T AP = D$, we know that $P^T AP$ is invertible, with $(P^T AP)^{-1} = P^{-1}A^{-1}(P^T)^{-1}$. Of course, since P is orthogonal, we know that $P^{-1} = P^T$, and that $(P^T)^{-1} = (P^{-1})^{-1} = P$. Now, since P is orthogonal, P^T is also orthogonal. And since $P = (P^T)^T$, we have shown that P^T is an orthogonal matrix and D^{-1} is a diagonal matrix such that $(P^T)^T A^{-1} P^T = D^{-1}$. Which means that A^{-1} is orthogonally diagonalizable.