Solution to Practice 2l

B1 A and E are symmetric.

B2(a) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \\ &= (5-\lambda)(2-\lambda) - (2)(2) \\ &= 6 - 7\lambda + \lambda^2 \\ &= (6-\lambda)(1-\lambda) \end{aligned}$$

So, the eigenvalues of A are $\lambda = 1, 6$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = 1$, we have

$$A - \lambda_1 I = \begin{bmatrix} 5-1 & 2 \\ 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

So we need $2x_1 + x_2 = 0$, so if we replace the variable x_1 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -2s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\sqrt{5}$, $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 6$, we have

$$A - \lambda_2 I = \begin{bmatrix} 5-6 & 2 \\ 2 & 2-6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So we need $x_1 - 2x_2 = 0$, so if we replace the variable x_2 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} 2s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $\sqrt{5}$, $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

Noting that $\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = (2/5) + (-2/5) = 0$, we see that the matrix $P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is an orthonormal matrix. And we know that

A is diagonalized by P to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

(You can multiply out $P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ to double check this.)

B2(b) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (4-\lambda)(1-\lambda) - (2)(2) \\ &= -5\lambda + \lambda^2 \\ &= -\lambda(5-\lambda) \end{aligned}$$

So, the eigenvalues of A are $\lambda = 0, 5$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = 0$, we have

$$A - \lambda_1 I = \begin{bmatrix} 4-0 & 2 \\ 2 & 1-0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

So we need $2x_1 + x_2 = 0$, so if we replace the variable x_1 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -2s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\sqrt{5}$, $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 5$, we have

$$A - \lambda_2 I = \begin{bmatrix} 4-5 & 2 \\ 2 & \text{‘}-5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So we need $x_1 - 2x_2 = 0$, so if we replace the variable x_2 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} 2s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $\sqrt{5}$, $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

Noting that $\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = (2/5) + (-2/5) = 0$, we see that the matrix $P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is an orthonormal matrix. And we know that A is diagonalized by P to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$.

(You can multiply out $P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ to double check this.)

B2(c) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix} \\ &= (7-\lambda)(-1-\lambda) - (3)(3) \\ &== 16 - 6\lambda + \lambda^2 \\ &= (8-\lambda)(-2-\lambda) \end{aligned}$$

So, the eigenvalues of A are $\lambda = -2, 8$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = -2$, we have

$$A - \lambda_1 I = \begin{bmatrix} 7+2 & 3 \\ 3 & -1+2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

So we need $3x_1 + x_2 = 0$, so if we replace the variable x_1 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -3s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is $\sqrt{10}$, $\left\{ \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 8$, we have

$$A - \lambda_2 I = \begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

So we need $x_1 - 3x_2 = 0$, so if we replace the variable x_2 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} 3s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is $\sqrt{10}$, $\left\{ \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

Noting that $\begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \cdot \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} = (3/10) + (-3/10) = 0$, we see that the matrix $P = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ is an orthonormal matrix. And we know that A is diagonalized by P to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$.

(You can multiply out $P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ to double check this.)

B2(d) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= (1-\lambda)(1 - 3\lambda + \lambda^2) - 2(3 - 2\lambda) + (1 + \lambda) \\ &= -4 + \lambda + 4\lambda^2 - \lambda^3 \end{aligned}$$

We can quickly see that $\lambda = 1$ is a solution to the characteristic polynomial. So we can factor out $1 - \lambda$, leaving us with $-4 - 3\lambda + \lambda^2$, which factors as $(4 - \lambda)(-1 - \lambda)$. So we see that $\det\,(A - \lambda I) = (1 - \lambda)(4 - \lambda)(-1 - \lambda)$, which means that the eigenvalues of A are $\lambda = -1, 1, 4$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = -1$, we have

$$
\begin{aligned}
A - \lambda_1 I &= \begin{bmatrix} 1+1 & 2 & 1 \\ 2 & 1+1 & 1 \\ 1 & 1 & 2+1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

This means we need $x_1 + x_2 = 0$ and $x_3 = 0$. If we replace the variable x_2 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is $\sqrt{2}$, $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 1$, we have

$$
\begin{aligned}
A - \lambda_1 I &= \begin{bmatrix} 1-1 & 2 & 1 \\ 2 & 1-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}
$$

This means we need $x_1 + (1/2)x_3 = 0$ and $x_2 + (1/2)x_3 = 0$. If we replace the variable x_3 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} -s/2 \\ -s/2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ is $\sqrt{6}$, $\left\{ \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_3 = 4$, we have

$$A - \lambda_1 I = \begin{bmatrix} 1-4 & 2 & 1 \\ 2 & 1-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means we need $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. If we replace the variable x_3 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\sqrt{3}$, $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

So we know that the matrix $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ diagonalizes A to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, and a quick look verifies that P is an orthogonal matrix.

B2(e) First, we need to find the eigenvalues of A, which are the solutions to

$\det(A - \lambda I) = 0$.

$$\begin{aligned}
\det\,(A-\lambda I) &= \det \begin{bmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix} \\
&= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & -\lambda \\ -1 & 1 \end{vmatrix} \\
&= -\lambda(-1+\lambda^2) - (1-\lambda) - (1-\lambda) \\
&= -2 + 3\lambda - \lambda^3 \\
&= -(1-\lambda)(-2+\lambda+\lambda^2) \\
&= (1-\lambda)^2(-2-\lambda)
\end{aligned}$$

So we see that the eigenvalues of A are $\lambda = 1, -2$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = 1$, we have

$$A - \lambda_1 I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means we need $x_1 - x_2 + x_3 = 0$. If we replace the variable x_2 with the parameter s and x_3 with the parameter t, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace. To find an orthogonal basis for the eigenspace, we use the Gram-Schmidt Procedure. First, we set $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then we have

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{(-1)(1)+(0)(1)+(1)(0)}{1^2+1^2+1^2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \tfrac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}
\end{aligned}$$

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for the eigenspace, which means that $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = -2$, we have

$$\begin{aligned} A - \lambda_1 I &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This means we need $x_1 - x_3 = 0$ and $x_2 + x_3 = 0$. If we replace the variable x_3 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $\sqrt{3}$, $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

So we know that the matrix $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ diagonalizes A to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, and a quick look verifies that P is an orthogonal matrix.

B2(f) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned}
\det\,(A-\lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{bmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} + 0 - \begin{vmatrix} 0 & 1-\lambda \\ -1 & 1 \end{vmatrix} \\
&= (1-\lambda)(1-3\lambda+\lambda^2) - (1-\lambda) \\
&= (1-\lambda)(1-3\lambda+\lambda^2-1) \\
&= (1-\lambda)(-3\lambda+\lambda^2) \\
&= (1-\lambda)(-\lambda)(3-\lambda)
\end{aligned}$$

So we see that the eigenvalues of A are $\lambda = 0, 1, 3$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = 0$, we have

$$\begin{aligned}
A - \lambda_1 I &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ \\ 0 & 1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

This means we need $x_1 - x_3 = 0$ and $x_2 + x_3 = 0$. If we replace the variable x_3 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$.

Clearly, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $\sqrt{3}$, $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 1$, we have

$$
\begin{aligned}
A-\lambda_1 I &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

This means we need $x_1 - x_2 = 0$ and $x_3 = 0$. If we replace the variable x_2 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is $\sqrt{2}$, $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_3 = 4$, we have

$$
\begin{aligned}
A-\lambda_1 I &= \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & -2 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}
$$

This means we need $x_1 + (1/2)x_3 = 0$ and $x_2 - (1/2)x_3 = 0$. If we replace the variable x_3 with the parameter s, we see that the eigenspace is the set

$\left\{ \begin{bmatrix} -s/2 \\ s/2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is $\sqrt{6}$, $\left\{ \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

So we know that the matrix $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ diagonalizes A to $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and a quick look verifies that P is an orthogonal matrix.

B2(g) First, we need to find the eigenvalues of A, which are the solutions to $\det(A - \lambda I) = 0$.

$$\begin{aligned}
\det\,(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 2 & -4 \\ 2 & -2-\lambda & -2 \\ -4 & -2 & 1-\lambda \end{bmatrix} \\
&= (1-\lambda) \begin{vmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -4 & 1-\lambda \end{vmatrix} - 4 \begin{vmatrix} 2 & -2-\lambda \\ -4 & -2 \end{vmatrix} \\
&= (1-\lambda)(-6+\lambda+\lambda^2) - 2(-6-2\lambda) - 4(-12-4\lambda) \\
&= 54 + 27\lambda - \lambda^3 \\
&= (-3-\lambda)^2(6-\lambda)
\end{aligned}$$

So we see that the eigenvalues of A are $\lambda = -3, 6$. Now we need to find a basis for the eigenspaces.

For $\lambda_1 = -3$, we have

$$A - \lambda_1 I = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means we need $2x_1 + x_2 - 2x_3 = 0$. If we replace the variable x_1 with the parameter s and x_3 with the parameter t, we see that the eigenspace is the set $\left\{ \begin{bmatrix} s \\ -2s+2t \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$. Clearly, $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for this eigenspace. To find an orthogonal basis for the eigenspace, we use the

Gram-Schmidt Procedure. First, we set $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Then we have

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \tfrac{(1)(0)+(2)(-2)+(1)(0)}{1^2+(-2)^2+0^2} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \tfrac{4}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \\
&= \tfrac{1}{5} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}
\end{aligned}$$

So $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \right\}$ is an orthogonal basis for the eigenspace, which means that $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

For $\lambda_2 = 6$, we have

$$\begin{aligned}
A - \lambda_1 I &= \begin{bmatrix} -5 & 2 & -4 \\ 2 & -8 & -2 \\ -4 & -2 & -5 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -4 & -1 \\ -5 & 2 & 4 \\ -4 & -2 & -5 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -4 & -1 \\ 0 & -18 & -9 \\ 0 & -18 & -9 \end{bmatrix}
\end{aligned}$$

$$\sim \begin{bmatrix} 1 & -4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

This means we need $x_1 + x_3 = 0$ and $x_2 + (1/2)x_3 = 0$. If we replace the variable

x_3 with the parameter s, we see that the eigenspace is the set $\left\{ \begin{bmatrix} -s \\ -s/2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$.

Clearly, $\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for this eigenspace, and since the norm of $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is 3, $\left\{ \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace.

So we know that the matrix $P = \begin{bmatrix} 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$ diagonalizes A to $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, and a quick look verifies that P is an orthogonal matrix.

D1(a) Since $(A+B)_{ij} = A_{ij} + B_{ij} = A_{ji} + B_{ji} = (A+B)_{ji}$, we see that $A+B$ is symmetric.

D1(b) Note that if A is symmetric, then $A^T = A$, so this is the same as question D2(d). (See solution there.)

D1(c) No, it is not always the case that AB is symmetric. Consider, for example $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$. Then $AB = \begin{bmatrix} 3 & -3 \\ 4 & -2 \end{bmatrix}$, which is not symmetric.

D1(d) Let $C = A^2 = AA$. Then, using summation notation for matrix products, we see that $c_{ij} = \sum_{k=1}^n a_{ik}a_{kj}$. But, since A is symmetric, we know that $a_{ik} = a_{ki}$ and $a_{kj} = a_{jk}$, so we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}a_{ki} = c_{ji}$$

And since $c_{ij} = c_{ji}$, we know that $C = A^2$ is symmetric.