Solution to Practice 2j

B2(a): This is not an inner product, because $\langle x-x^3, x-x^3\rangle = (-1-(-1)^3)^2 + (0-0^2)^2 + (1-1^3)^2 = 0$ even though $x - x^3$ is not the zero polynomial. Thus, the function is not positive definite.

B2(b): This is an inner product. Let's look at the three properties.

(1) $\langle p,p\rangle = (p(0))^2 + (p(1))^2 + (p(3))^2 + (p(4))^2$, which is clearly greater than or equal to zero. If $\langle p,p\rangle = 0$, then we must have $p(0) = 0$, $p(1) = 0$, $p(3) = 0$, and $p(4) = 0$. This means that

$$\begin{array}{llll l} p_0 & & & & = 0 \\ p_0 & +p_1 & +p_2 & +p_3 & = 0 \\ p_0 & +3p_1 & +9p_2 & +27p_3 & = 0 \\ p_0 & +4p_1 & +16p_3 & +64p_3 & = 0 \end{array}$$

If we go ahead and plug in $p_0 = 0$, we are left looking for solutions to the following system of equations:

$$\begin{array}{lll l} p_1 & +p_2 & +p_3 & = 0 \\ 3p_1 & +9p_2 & +27p_3 & = 0 \\ 4p_1 & +16p_3 & +64p_3 & = 0 \end{array}$$

We solve this system by row reducing its coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 24 \\ 0 & 12 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 12 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 48 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the last matrix, we see that the only solution to our system is $p_1 = 0$, $p_2 = 0$, $p_3 = 0$. Combined with the fact that $p_0 = 0$, we have shown that if $\langle p,p\rangle = 0$, then p is the zero polynomial. And thus we have shown that $\langle\ ,\ \rangle$ is positive definite.

(2) $\langle p,q\rangle = p(0)q(0) + p(1)q(1) + p(3)q(3) + p(4)q(4) = q(0)p(0) + q(1)p(1) + q(3)p(3) + q(4)p(4) = \langle q,p\rangle$, so $\langle\ ,\ \rangle$ is symmetric.

(3)
$$\begin{aligned} \langle p, sq+tr\rangle &= p(0)(sq+tr)(0) + p(1)(sq+tr)(1) + p(3)(sq+tr)(3) + p(4)(sq+tr)(4) \\ &= p(0)(sq(0)+tr(0)) + p(1)(sq(1)+tr(1)) \\ &\quad +p(3)(sq(4)+tr(4)) + p(4)(sq(4)+tr(4)) \\ &= sp(0)q(0) + sp(1)q(1) + sp(3)q(3) + sp(4)q(4) \\ &\quad +tp(0)r(0) + tp(1)r(1) + tp(3)r(3) + tp(4)r(4) \\ &= s\langle p,q\rangle + t\langle p,r\rangle \end{aligned}$$

And so we see that $\langle\ ,\ \rangle$ is bilinear.

B2(c) This is not an inner product, because it is not symmetric. To see this, consider $p(x) = 1+2x$ and $q(x) = 2$. Then $\langle p, q\rangle = (1-2)(2)+(1+2)(2)+(1+4)(2)+(1+6)(2) = 28$, but $\langle q, p\rangle = (2)(1+0)+(2)(1+2)+(2)(1+4)+(2)(1+6) = 32$. So $\langle p, q\rangle \neq \langle q, p\rangle$.

D2(a) Let $\vec{x}, \vec{y} \in \mathbb{R}^2$. Then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ and $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2$. And this means that

$$\begin{aligned}\langle\vec{x}, \vec{y}\rangle &= \langle x_1\vec{e}_1 + x_2\vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2\rangle \\ &= x_1y_1\langle\vec{e}_1, \vec{e}_1\rangle + x_1y_2\langle\vec{e}_1, \vec{e}_2\rangle + x_2y_1\langle\vec{e}_2, \vec{e}_1\rangle + x_2y_2\langle\vec{e}_2, \vec{e}_2\rangle\end{aligned}$$

(1) To show that $\langle\vec{x}, \vec{y}\rangle = x_1y_1 + 3x_2y_2 + 2x_3y_3$ defines an inner product on $\mathbb{R}^3$, we need to verify the three defining properties:

(1) $\langle\vec{x}, \vec{x}\rangle = (x_1)^2 + 3(x_2)^2 + 2(x_3)^2$, which is clearly greater than or equal to zero since it is the sum of non-negative numbers. Moreover, if $\langle\vec{x}, \vec{x}\rangle = 0$, then we must have $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, which means that $\vec{x} = \vec{0}$. So $\langle\ ,\ \rangle$ is positive definite.

(2) $\langle\vec{x}, \vec{y}\rangle = x_1y_1 + 3x_2y_2 + 2x_3y_3 = y_1x_1 + 3y_2x_2 + 2y_3x_3 = \langle\vec{y}, \vec{x}\rangle$, so $\langle\ ,\ \rangle$ is symmetric.

(3) $$\begin{aligned}\langle\vec{x}, s\vec{y} + t\vec{z}\rangle &= x_1(sy_1 + tz_1) + 3x_2(sy_2 + tz_2) + 2x_3(sy_3 + tz_3) \\ &= sx_1y_1 + 3sx_2y_2 + 2sx_3y_3 + tx_1z_1 + 3tx_2z_2 + 2tx_3z_3 \\ & s\langle\vec{x}, \vec{y}\rangle + t\langle\vec{x}, \vec{z}\rangle\end{aligned}$$

So $\langle\ ,\ \rangle$ is bilinear.