Solution to Practice 2h

B1(a) For this question we want $X = \begin{bmatrix} \vec{1} & \vec{t} \end{bmatrix}$, with $\vec{t} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 9 \\ 8 \\ 5 \\ 3 \\ 1 \end{bmatrix}$. Then we get

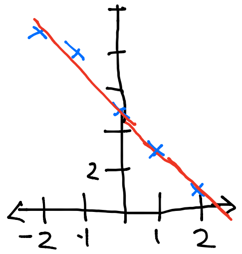
$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

So $(X^T X)^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix}$ This means that

$$(X^T X)^{-1} X^T = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ -2/10 & -1/10 & 0 & 1/10 & 2/10 \end{bmatrix}.$$

$$\vec{a} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ -2/10 & -1/10 & 0 & 1/10 & 2/10 \end{bmatrix} \begin{bmatrix} 9 \\ 8 \\ 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 26/5 \\ -21/10 \end{bmatrix}$$

So $y = (26/5) - (21/10)t$ is the equation of the form $y = a + bt$ that best fits the data.



B1(b) For this question we want $X = \begin{bmatrix} \vec{1} & \vec{t} \end{bmatrix}$, with $\vec{t} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 5 \\ 5 \end{bmatrix}$.

Then we get

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

Next, we use the matrix inverse algorithm to find $(X^T X)^{-1}$:

$$\left[\begin{array}{cc|cc} 5 & 15 & 1 & 0 \\ 15 & 55 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 5 & 15 & 1 & 0 \\ 0 & 10 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & 1/5 & 0 \\ 0 & 1 & -3/10 & 1/10 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 11/10 & -3/10 \\ 0 & 1 & -3/10 & 1/10 \end{array}\right]$$

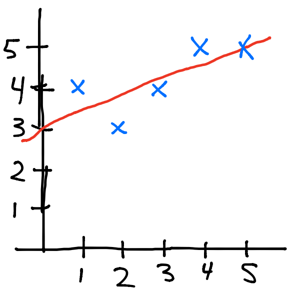
So $(X^T X)^{-1} = \begin{bmatrix} 11/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$ This means that

$$(X^T X)^{-1} X^T = \begin{bmatrix} 11/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 8/10 & 5/10 & 2/10 & -1/10 & -4/10 \\ -2/10 & -1/10 & 0 & 1/10 & 2/10 \end{bmatrix}.$$

$$\vec{a} \;=\; (X^T X)^{-1} X^T \vec{y} \;=\; \begin{bmatrix} 8/10 & 5/10 & 2/10 & -1/10 & -4/10 \\ -2/10 & -1/10 & 0 & 1/10 & 2/10 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 4 \\ 5 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \\ 2/5 \end{bmatrix}$$

So $y = 3 + (2/5)t$ is the equation of the form $y = a + bt$ that best fits the data.



B2 For this question we want $X = \begin{bmatrix} \vec{1} & \vec{t} & \vec{t^2} \end{bmatrix}$, with $\vec{t} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{t^2} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}$,

and $\vec{y} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 8 \end{bmatrix}$. Then we get

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

Next, we use the matrix inverse algorithm to find $(X^T X)^{-1}$:

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 10 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 1 & 0 \\ 10 & 0 & 34 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & 10 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 1 & 0 \\ 0 & 0 & 14 & -2 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/10 & 0 \\ 0 & 0 & 1 & -1/7 & 0 & 1/14 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 17/35 & 0 & -2/14 \\ 0 & 1 & 0 & 0 & 1/10 & 0 \\ 0 & 0 & 1 & -1/7 & 0 & 1/14 \end{array}\right]$$

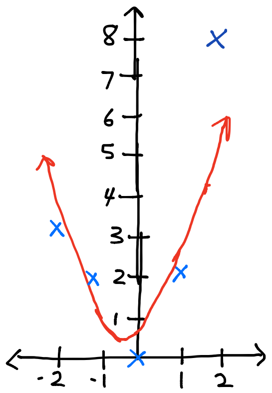
So $(X^T X)^{-1} = \begin{bmatrix} 17/35 & 0 & -2/14 \\ 0 & 1/10 & 0 \\ -1/7 & 0 & 1/14 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 34 & 0 & -10 \\ 0 & 7 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ This means that

$$(X^T X)^{-1} X^T = \frac{1}{70} \begin{bmatrix} 34 & 0 & -10 \\ 0 & 7 & 0 \\ -10 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} -6 & 24 & 34 & 24 & -6 \\ -14 & -7 & 0 & 7 & 14 \\ 10 & -5 & -10 & -5 & 10 \end{bmatrix}.$$

$$\vec{a} = (X^T X)^{-1} X^T \vec{y} = \frac{1}{70} \begin{bmatrix} -6 & 24 & 34 & 24 & -6 \\ -14 & -7 & 0 & 7 & 14 \\ 10 & -5 & -10 & -5 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 8 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 30 \\ 70 \\ 90 \end{bmatrix} =$$

$$\begin{bmatrix} 3/7 \\ 1 \\ 9/7 \end{bmatrix}$$

So $y = (3/7) + t + (9/7)t^2$ is the equation of the form $y = a + bt + ct^2$ that best fits the data.



B3(a) For this question we want $X = \begin{bmatrix} \vec{t} & \vec{t^2} \end{bmatrix}$, with $\vec{t} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{t^2} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Then we get

$$X^T X = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 9 & 17 \end{bmatrix}.$$

Next, we use the matrix inverse algorithm to find $(X^T X)^{-1}$:

$$\left[\begin{array}{cc|cc} 5 & 9 & 1 & 0 \\ 9 & 17 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 9/5 & 1/5 & 0 \\ 9 & 17 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 9/5 & 1/5 & 0 \\ 0 & 4/5 & -9/5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 9/5 & 1/5 & 0 \\ 0 & 1 & -9/4 & 5/4 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 17/4 & -9/4 \\ 0 & 1 & -9/4 & 5/4 \end{array}\right]$$

So $(X^T X)^{-1} = \begin{bmatrix} 17/4 & -9/4 \\ -9/4 & 5/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 17 & -9 \\ -9 & 5 \end{bmatrix}$. This means that

$$(X^TX)^{-1}X^T = \frac{1}{4}\begin{bmatrix} 17 & -9 \\ -9 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 8 & -2 \\ 0 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1/2 \\ 0 & -1 & 1/2 \end{bmatrix}.$$

$$\vec{a} = (X^TX)^{-1}X^T\vec{y} = \begin{bmatrix} 0 & 2 & -1/2 \\ 0 & -1 & 1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -3/2 \end{bmatrix}.$$

So $y = (5/2)t - (3/2)t^2$ is the equation of the form $y = at + bt^2$ that best fits the data.

B3(b) For this question we want $X = \begin{bmatrix} \vec{1} & \vec{t} & \vec{t^3} \end{bmatrix}$, with $\vec{t} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{t^3} = \begin{bmatrix} -8 \\ -1 \\ 0 \\ 1 \\ 8 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} -5 \\ -2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$. Then we get

$$X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & -2 & -8 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 34 \\ 0 & 34 & 128 \end{bmatrix}$$

Next, we use the matrix inverse algorithm to find $(X^TX)^{-1}$:

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 10 & 34 & 0 & 1 & 0 \\ 0 & 34 & 128 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 34/10 & 0 & 1/10 & 0 \\ 0 & 34 & 128 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 34/10 & 0 & 1/10 & 0 \\ 0 & 0 & 124/10 & 0 & -34/10 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 34/10 & 0 & 1/10 & 0 \\ 0 & 0 & 1 & 0 & -34/124 & 10/124 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 128/124 & -34/124 \\ 0 & 0 & 1 & 0 & -34/124 & 10/124 \end{array}\right]$$

So $(X^TX)^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 128/124 & -34/124 \\ 0 & -34/124 & 10/124 \end{bmatrix} = \frac{1}{310}\begin{bmatrix} 62 & 0 & 0 \\ 0 & 320 & -85 \\ 0 & -85 & 25 \end{bmatrix}$. This means that

$$(X^TX)^{-1}X^T = \frac{1}{310}\begin{bmatrix} 62 & 0 & 0 \\ 0 & 320 & -85 \\ 0 & -85 & 25 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix}$$

$$= \frac{1}{310}\begin{bmatrix} 62 & 62 & 62 & 62 & 62 \\ 40 & -235 & 0 & 235 & -40 \\ -30 & 60 & 0 & -60 & 30 \end{bmatrix}$$

$$\vec{a} = (X^TX)^{-1}X^T\vec{y} = \frac{1}{310}\begin{bmatrix} 62 & 62 & 62 & 62 & 62 \\ 40 & -235 & 0 & 235 & -40 \\ -30 & 60 & 0 & -60 & 30 \end{bmatrix}\begin{bmatrix} -5 \\ -2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{310}\begin{bmatrix} -434 \\ -45 \\ 150 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 9/62 \\ 15/31 \end{bmatrix}.$$

So $y = (7/5) + (9/62)t + (15/31)t^3$ is the equation of the form $y = a + bt + ct^3$ that best fits the data.