Solution to Practice 2b

B2(a) Let $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. Then we have

$$a_1 = \vec{w} \cdot \vec{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = (6+0+1)/\sqrt{2} = 7/\sqrt{2}, \text{ and}$$

$$a_2 = \vec{w} \cdot \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = (-6+2+1)/\sqrt{3} = -3/\sqrt{3}, \text{ and}$$

$$a_3 = \vec{w} \cdot \vec{v}_3 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = (6+4-1)/\sqrt{6} = 9/\sqrt{6},$$

so $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 7/\sqrt{2} \\ -3/\sqrt{3} \\ 9/\sqrt{6} \end{bmatrix}$.

B2(b) Let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Then we have

$$b_1 = \vec{x} \cdot \vec{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = (-4+0+3)/\sqrt{2} = -1/\sqrt{2}, \text{ and}$$

$$b_2 = \vec{x} \cdot \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = (4+2+3)/\sqrt{3} = 9/\sqrt{3}, \text{ and}$$

$$b_3 = \vec{x} \cdot \vec{v}_3 = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = (-4+4-3)/\sqrt{6} = -3/\sqrt{6},$$

so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1/\sqrt{2} \\ 9/\sqrt{3} \\ -3/\sqrt{6} \end{bmatrix}$.

B2(c) Let $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Then we have

$$c_1 = \vec{y} \cdot \vec{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = (3 + 0 - 5)/\sqrt{2} = -2/\sqrt{2}, \text{ and}$$

$$c_2 = \vec{y} \cdot \vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = (-3 + 3 - 5)/\sqrt{3} = -5/\sqrt{3}, \text{ and}$$

$$c_3 = \vec{y} \cdot \vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = (3 - 6 + 5)/\sqrt{6} = 2/\sqrt{6},$$

so $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -2/\sqrt{2} \\ -5/\sqrt{3} \\ 2/\sqrt{6} \end{bmatrix}$.

B2(d) Let $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. Then we have

$$d_1 = \vec{z} \cdot \vec{v}_1 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = (-1 + 0 + 2)/\sqrt{2} = 1/\sqrt{2}, \text{ and}$$

$$d_2 = \vec{z} \cdot \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = (1 + 4 + 2)/\sqrt{3} = 7/\sqrt{3}, \text{ and}$$

$$d_3 = \vec{z} \cdot \vec{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} = (-1 + 8 - 2)/\sqrt{6} = 5/\sqrt{6},$$

so $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 1/\sqrt{2} \\ 7/\sqrt{3} \\ 5/\sqrt{6} \end{bmatrix}$.

B3(a) Let $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$. Then we have

$$a_1 = \vec{w} \cdot \vec{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = (3 - 2 + 6 + 1)/2 = 4, \text{ and}$$

$$a_2 = \vec{w} \cdot \vec{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = (3+2+6-1)/2 = 5, \text{ and}$$

$$a_3 = \vec{w} \cdot \vec{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = (-3+0+6+0)/\sqrt{2} = 3/\sqrt{2}, \text{ and}$$

$$a_4 = \vec{w} \cdot \vec{v}_4 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = (0-2+0-1)/\sqrt{2} = -3/\sqrt{2},$$

so $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \\ 3/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix}$.

B3(b) Let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b4 \end{bmatrix}$. Then we have

$$b_1 = \vec{x} \cdot \vec{v}_1 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = (2-4+0+4)/2 = 1, \text{ and}$$

$$b_2 = \vec{x} \cdot \vec{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = (2+4+0-4)/2 = 1, \text{ and}$$

$$b_3 = \vec{x} \cdot \vec{v}_3 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = (-2+0+0+0)/\sqrt{2} = -2/\sqrt{2}, \text{ and}$$

$$b_4 = \vec{x} \cdot \vec{v}_4 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = (0-4+0-4)/\sqrt{2} = -8/\sqrt{2},$$

so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -2/\sqrt{2} \\ -8/\sqrt{2} \end{bmatrix}$.

B3(c) Let $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$. Then we have

$$c_1 = \vec{y} \cdot \vec{v}_1 = \begin{bmatrix} 5 \\ 0 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = (5 + 0 - 2 + 2)/2 = 5/2, \text{ and}$$

$$c_2 = \vec{y} \cdot \vec{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = (5 + 0 - 2 - 2)/2 = 1/2, \text{ and}$$

$$c_3 = \vec{y} \cdot \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = (-5 + 0 - 2 + 0)/\sqrt{2} = -7/\sqrt{2}, \text{ and}$$

$$c_4 = \vec{y} \cdot \vec{v}_4 = \begin{bmatrix} 5 \\ 0 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = (0 + 0 + 0 - 2)/\sqrt{2} = -2/\sqrt{2},$$

so $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ 1/2 \\ -7/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix}$.

B3(d) Let $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$. Then we have

$$d_1 = \vec{z} \cdot \vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = (4 + 2 - 2 + 3)/2 = 7/2, \text{ and}$$

$$d_2 = \vec{z} \cdot \vec{v}_2 = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = (4 - 2 - 2 - 3)/2 = -3/2, \text{ and}$$

$$d_3 = \vec{z} \cdot \vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = (-4 + 0 - 2 + 0)/\sqrt{2} = -6/\sqrt{2}, \text{ and}$$

$$d_4 = \vec{z} \cdot \vec{v}_4 = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = (0 + 2 + 0 - 3)/\sqrt{2} = -1/\sqrt{2},$$

so $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 7/2 \\ -3/2 \\ -6/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.