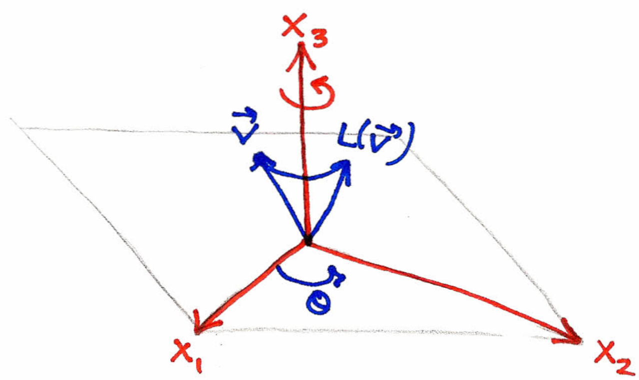
Lecture 2d

A Note on Rotation Transformations

(pages 329-330)

Back in Chapter 3, the textbook briefly discusses the matrix of a rotation around a coordinate axis in $\mathbb{R}^3$. At that time, the text book also noted that we would not be able to find the matrix of a rotation around a general vector in $\mathbb{R}^3$ until Chapter 7. Well, here we are in Chapter 7! Before we jump to the general case, let's take a closer look at how we find the matrix of a rotation around a coordinate axis in $\mathbb{R}^3$.

The easiest of all the cases is if we are rotating around the x_3-axis. Looking at the diagram below, you'll see that a rotation around the x_3-axis will leave our x_3 component the same (as the "height" of the point remains the same), but the x_1x_2-plane is rotated by θ.



rotation around X3

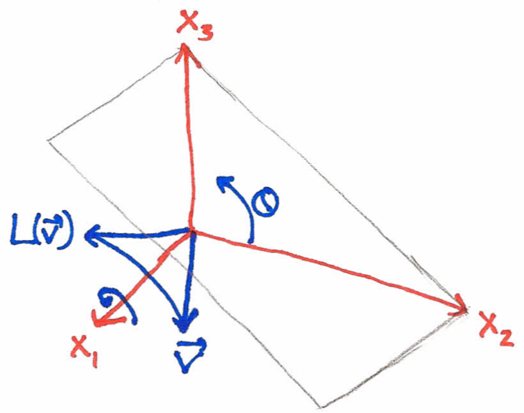
So the matrix for the rotation needs to leave our x_3 component unchanged, and it needs to treat our x_1 and x_2 components as if they are in $\mathbb{R}^2$, being rotated by θ. The matrix for a rotation by θ in $\mathbb{R}^2$ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

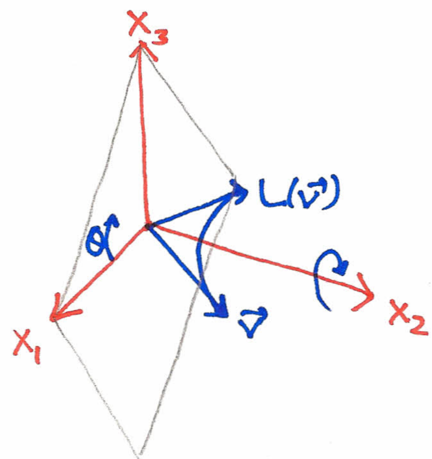
So we see that the matrix for the rotation about the x_3-axis through θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What about rotating about another axis? Again, the idea will be that one component will remain unchanged, while the plane containing the other two will be rotated by θ.



rotation around x_1



rotation around x_2

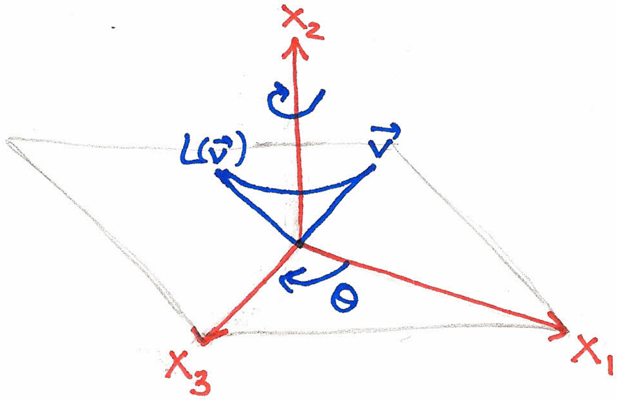
And so we see that the matrix for a rotation about the x_1-axis through θ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

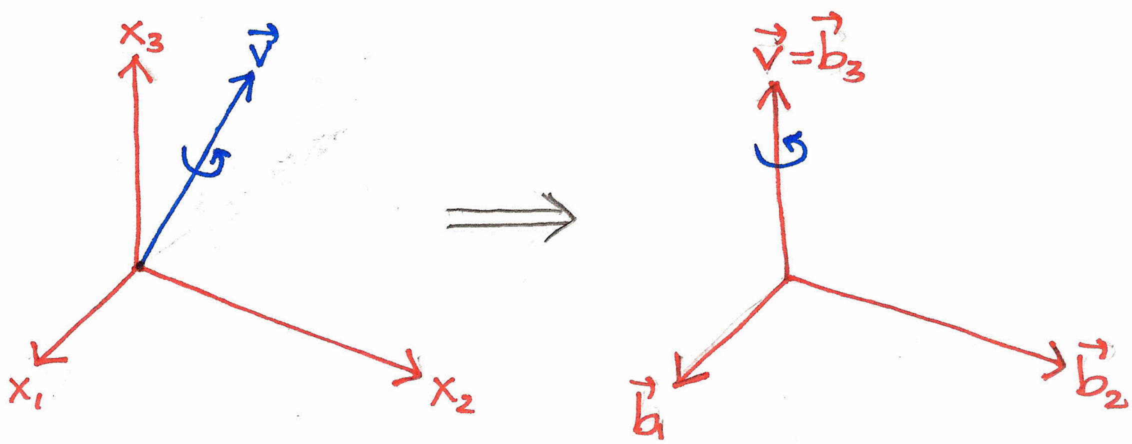
and the matrix for a rotation about the x_2-axis through θ is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

The matrix for x_2 looks a bit different, because in order to maintain the necessary "right hand system", the rotation in the x_1x_3 plane is reversed. This is easier to see if we re-orient our matrices so that x_2 becomes "up".



And reorienting "up" is how we will find a matrix around any vector. That is to say, if we want to find the matrix of a rotation about the vector $\vec{v}$ through θ, the easiest thing to do is to make $\vec{v}$ "up", and if we write our coordinates with respect to a basis that has $\vec{v}$ as its third member, then the value of our third coordinate stays the same, and the values of the first two coordinates are subject to a rotation in their plane.



We will need our new basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ to be orthonormal, to ensure that the rotation by θ is preserved (instead of being stretched, etc.). We also need to make sure that it forms a right hand system, so that the direction of our rotation is preserved. Lastly, we want $\vec{v}$ to be pointing "up", so we want $\vec{b}_3$ to be in the same direction as $\vec{v}$. If we meet these goals, then the matrix for a rotation about the x_3-axis will be our $[L]_\mathcal{B}$ (the matrix of the transformation with respect to $\mathcal{B}$ coordinates). But since our goal is to find the matrix of the transformation with respect to the usual coordinates, that is with respect to the standard basis, we will use the change of coordinates matrix P from $\mathcal{B}$ to $\mathcal{S}$, and the fact that $[L]_\mathcal{B} = P^{-1}[L]_\mathcal{S}P$, to get

$$[L]_\mathcal{S} = P[L]_\mathcal{B}P^{-1}$$

With this list in mind, we are left with the question of how to find $\mathcal{B}$? Well, this is more easily seen through an example:

Example: Find the standard matrix of the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates vectors about the axis defined by the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ counter-clockwise through an angle $\pi/6$.

Our first goal is to find an orthonormal basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ such that $\mathcal{B}$ is a right-handed system, and $\vec{b}_3$ is in the same direction as $\vec{v}$. Well, $\vec{b}_3$ is easy to find, since we have $\vec{b}_3 = \vec{v}/||\vec{v}|| = (1/\sqrt{1^2+2^2+1^2})\vec{v} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$. For $\vec{b}_1$, we will simply hunt for some vector that is orthogonal to $\vec{v}$, and then normalize it. It isn't all that difficult to find one. I choose to use $\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ (note that $\vec{v} \cdot \vec{y} = 0$), so $\vec{b}_1 = \vec{y}/||\vec{y}|| = (1/\sqrt{2^2+(-1)^2+0^2})\vec{y} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$. Lastly, we need to find a vector $\vec{z}$ that is orthogonal to both $\vec{v}$ and $\vec{y}$. Thankfully, that's what a cross product gives us. Even better, the cross product $\vec{v} \times \vec{y}$ will give us a vector $\vec{z}$ such that $\{\vec{y}, \vec{z}, \vec{v}\}$ is a right hand system. (But, for example, $\vec{y} \times \vec{v}$ would give us a vector in the opposite direction, and we would have similar problems if we made $\vec{y}$ our second basis vector instead of our first. If you are familiar with right-hand systems, have fun twisting your hand into the various directions to see that $\vec{v} \times \vec{y}$ is in fact the vector we want. If you are not familiar with right-hand systems, just trust me on this one!) And so we calculate that

$$\vec{z} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 2-0 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$$

And so we have $\vec{b}_2 = \vec{z}/||\vec{z}|| = (1/\sqrt{1^2+2^2+(-5)^2})\vec{z} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$.

With this, we have satisfied our first goal: $\mathcal{B} = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$ that forms a right-handed system, and whose third basis vector is in the same direction as the vector we are rotating around. This means that the matrix for the our rotation, RELATIVE TO $\mathcal{B}$, is the matrix for a rotation of $\theta = \pi/6$ about the x_3-axis:

$$[L]_{\mathcal{B}} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ \sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

From the fact that $\mathcal{B} = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$, we see that the change of basis matrix from $\mathcal{B}$ to $\mathcal{S}$ coordinates is the matrix whose columns are the vectors in $\mathcal{B}$

$$P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \\ 0 & -5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} 2\sqrt{6} & 1 & \sqrt{5} \\ -\sqrt{6} & 2 & 2\sqrt{5} \\ 0 & -5 & \sqrt{5} \end{bmatrix}$$

Since $\mathcal{B}$ is an orthonormal basis $P^{-1} = P^T$, so we have

$$P^{-1} = \frac{1}{\sqrt{30}}\begin{bmatrix} 2\sqrt{6} & -\sqrt{6} & 0 \\ 1 & 2 & -5 \\ \sqrt{5} & 2\sqrt{5} & \sqrt{5} \end{bmatrix}$$

All that's left now is to actually calculate $[L]_{\mathcal{S}} = P[L]_{\mathcal{B}}P^{-1}$:

$$
\begin{aligned}
[L]_\mathcal{S} &= P[L]_\mathcal{B}P^{-1} \\
&= \tfrac{1}{\sqrt{30}} \begin{bmatrix} 2\sqrt{6} & 1 & \sqrt{5} \\ -\sqrt{6} & 2 & 2\sqrt{5} \\ 0 & -5 & \sqrt{5} \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \tfrac{1}{\sqrt{30}} \begin{bmatrix} 2\sqrt{6} & -\sqrt{6} & 0 \\ 1 & 2 & -5 \\ \sqrt{5} & 2\sqrt{5} & \sqrt{5} \end{bmatrix} \\
&= \tfrac{1}{60} \begin{bmatrix} 2\sqrt{6} & 1 & \sqrt{5} \\ -\sqrt{6} & 2 & 2\sqrt{5} \\ 0 & -5 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2\sqrt{6} & -\sqrt{6} & 0 \\ 1 & 2 & -5 \\ \sqrt{5} & 2\sqrt{5} & \sqrt{5} \end{bmatrix} \\
&= \tfrac{1}{60} \begin{bmatrix} 2\sqrt{6} & 1 & \sqrt{5} \\ -\sqrt{6} & 2 & 2\sqrt{5} \\ 0 & -5 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 2\sqrt{18}-1 & -\sqrt{18}-2 & 5 \\ 2\sqrt{6}+\sqrt{3} & -\sqrt{6}+2\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{5} & 4\sqrt{5} & 2\sqrt{5} \end{bmatrix} \\
&= \tfrac{1}{60} \begin{bmatrix} 25\sqrt{3}+10 & -10\sqrt{3}-5\sqrt{6}+20 & -5\sqrt{3}+10\sqrt{6}+10 \\ -10\sqrt{3}+5\sqrt{6}+20 & 10\sqrt{3}+40 & -10\sqrt{3}-5\sqrt{6}+20 \\ -5\sqrt{3}-10\sqrt{6}+10 & -10\sqrt{3}+5\sqrt{6}+20 & 25\sqrt{3}+10 \end{bmatrix}
\end{aligned}
$$

I never said the matrix would be nice–I just said we could now find it!

The textbook ends this chapter with a discussion about the similarity between the matrix for a rotation R by θ in $\mathbb{R}^2$:

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and the change of basis matrix P from the orthonormal basis $\mathcal{B} = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$ to the standard basis. This is not surprising, considering they are the same matrix! What is surprising, however, is that they are the same matrix. That is, you might expect that the matrix for the linear transformation would actually take $\mathcal{S}$ coordinates to $\mathcal{B}$ coordinates. After all, we get that $R(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \vec{b}_1$ and $R(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \vec{b}_2$. But if $[R]$ was sending $\mathcal{S}$ coordinates to $\mathcal{B}$ coordinates, then we would get $R(\vec{e}_1) = [\vec{e}_1]_\mathcal{B}$, which is not $\vec{b}_1$. What is $[\vec{e}_1]_\mathcal{B}$? Well, we can use P^{-1} (the $\mathcal{S}$ to $\mathcal{B}$ change of coordinates matrix) to find out. And since $\mathcal{B}$ is an orthonormal basis, we know that $P^{-1} = P^T$, so we see that

$$[\vec{e}_1]_\mathcal{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \quad \text{and} \quad [\vec{e}_2]_\mathcal{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$

So, we see that $[\vec{e}_1]_\mathcal{B}$ is not the same as $R(\vec{e}_1)$. It is the inverse of it, though.

That is to say, if we instead looked at $R_{-\theta}$, the linear transformation of a rotation by $-\theta$, then we would have

$$[R_{-\theta}] = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = P^{-1}$$

And so we have shown that the change of basis matrix from $\mathcal{S}$ to $\mathcal{B}$ coordinates is the same as the matrix for a rotation by $-\theta$ (not a rotation by θ). While this may not seem right on the surface, you simply need to keep track of what you are doing. Say, for example, that you wanted to find the $\mathcal{S}$ coordinates for $\vec{b}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Sure, we get $\vec{b}_1$ from $\vec{e}_1$ by rotating by θ, but we aren't trying to get from $\vec{e}_1$ to $\vec{b}_1$. We are actually trying to find which vector gets sent TO $\vec{e}_1$, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the $\mathcal{B}$-coordinates for $\vec{b}_1$. To find out which vector is sent TO $\vec{e}_1$ by a rotation of θ, we rotate backwards by θ, i.e. we rotate by $-\theta$. So, the $\mathcal{S}$ coordinates of the vector that gets sent to $\vec{e}_1$ by R_θ is $R_{-\theta}(\vec{e}_1)$.

The textbook provides some drawings that may help you understand the process here. But, if nothing else, you should simply remember to be careful when looking at a change of basis inspired by a rotation, since your intuition may not lead you to the correct matrices.

ASSIGNMENT 2d: p.331 A5

Solution to Assignment 2d

A5(a) $\vec{g}_3 \cdot \vec{g}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -2 + 4 - 2 = 0$

$$\vec{g}_2 = \vec{g}_3 \times \vec{g}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+2 \\ 1-4 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix}$$

A5(b)

$$\vec{f}_1 = \frac{\vec{g}_1}{||\vec{g}_1||} = \frac{1}{\sqrt{(-1)^2 + 2^2 + 2^2}} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{f}_2 = \frac{\vec{g}_2}{||\vec{g}_2||} = \frac{1}{\sqrt{(6)^2 + (-3)^2 + 6^2}} \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{f}_3 = \frac{\vec{g}_3}{||\vec{g}_3||} = \frac{1}{\sqrt{2^2 + 2^2 + (-1)^2}} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

So $P = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

A5(c) $[L]_{\mathcal{B}} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

A5(d)

$$
\begin{aligned}
[L]_{\mathcal{S}} &= P[L]_{\mathcal{B}}P^{-1} \\
&= \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \\
&= \frac{1}{9\sqrt{2}}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 3 & 0 \\ 1 & 1 & 4 \\ 2\sqrt{2} & 2\sqrt{2} & -\sqrt{2} \end{bmatrix} \\
&= \frac{1}{9\sqrt{2}}\begin{bmatrix} 5+4\sqrt{2} & -1+4\sqrt{2} & 8-2\sqrt{2} \\ -7+4\sqrt{2} & 5+4\sqrt{2} & -4-2\sqrt{2} \\ -4-2\sqrt{2} & 8-2\sqrt{2} & 8+\sqrt{2} \end{bmatrix}
\end{aligned}
$$