Solution to Practice 1s

B1(a) We define $L : P_4 \to \mathbb{R}^5$ by $L(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$.

To prove that it is an isomorphism, we must prove that it is linear, one-to-one, and onto.

Linear: Let $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4$ and $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4$ be elements of P_4, and let $t \in \mathbb{R}$. Then we have

$$\begin{aligned} L(tp(x) + q(x)) &= L(t(p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4) + (q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4)) \\ &= L((tp_0 + q_0) + (tp_1 + q_1)x + (tp_2 + q_2)x^2 + (tp_3 + q_3)x^3 + (tp_4 + q_4)x^4) \\ &= \begin{bmatrix} tp_0 + q_0 \\ tp_1 + q_1 \\ tp_2 + q_2 \\ tp_3 + q_3 \\ tp_4 + q_4 \end{bmatrix} \\ &= t\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \\ &= tL(p(x)) + L(q(x)) \end{aligned}$$

Therefore, L is linear.

One-to-one: Let $p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4 \in \text{Null}(L)$. Then $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L(p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4) = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$. Hence $p_0 = 0$, $p_1 = 0$, $p_2 = 0$, $p_3 = 0$, and $p_4 = 0$, which means that $p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4$ is the zero polynomial. Since the only vector in the nullspace is the zero vector, by Lemma 4.7.1 we have that L is one-to-one.

Onto: For any $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{R}^5$, we have $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in P_4$ such that $L(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$. Hence, L is onto.

B1(b) We define $L : M(2,3) \to \mathbb{R}^6$ by $L\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$. To prove that it is an isomorphism, we must prove that it is linear, one-to-one, and onto.

Linear: Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}$ be elements of $M(2,3)$, and let $t \in \mathbb{R}$. Then we have

$$\begin{aligned} L(tA + B) &= L\left(t\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{bmatrix}\right) \\ &= L\left(\begin{bmatrix} ta_1 + b_1 & ta_2 + b_2 & ta_3 + b_3 \\ ta_4 + b_4 & ta_5 + b_5 & ta_6 + b_6 \end{bmatrix}\right) \\ &= \begin{bmatrix} ta_1 + b_1 \\ ta_2 + b_2 \\ ta_3 + b_3 \\ ta_4 + b_4 \\ ta_5 + b_5 \\ ta_6 + b_6 \end{bmatrix} \\ &= t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \\ &= tL(A) + L(B) \end{aligned}$$

Therefore, L is linear.

One-to-one: Let $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \in \text{Null}(L)$. Then $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$. Hence $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 =$, and $a_6 = 0$, which means that $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ is the zero matrix. Since the only vector in the nullspace is the zero vector, by Lemma 4.7.1 we have that L is one-to-one.

Onto: For any $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \in \mathbb{R}^6$, we have $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \in M(2,3)$ such that $L\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$. Hence, L is onto.

B1(c) We define $L : \mathbb{R}^2 \to \mathcal{S}$ by $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. To prove that it is an isomorphism, we must prove that it is linear, one-to-one, and onto.

Linear: Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be elements of $\mathbb{R}^2$, and let $t \in \mathbb{R}$. Then we have

$$
\begin{aligned}
L(t\vec{a}+\vec{b}) &= L\left(t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) \\
&= L\left(\begin{bmatrix} ta_1+b_1 \\ ta_2+b_2 \end{bmatrix}\right) \\
&= (ta_1+b_1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (ta_2+b_2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
&= ta_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + ta_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
&= tL(\vec{a}) + L(\vec{b})
\end{aligned}
$$

Therefore, L is linear.

One-to-one: Let $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \text{Null}(L)$. Then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = a_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1+a_2 \\ a_2 \\ a_1+a_2 \end{bmatrix}$. Setting the components equal, we see that this means that $a_1+a_2=0$ and $a_2=0$. Plugging $a_2=0$ into $a_1+a_2=0$ gives us $a_1=0$. As such $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since the only vector in the nullspace is the zero vector, by Lemma 4.7.1 we have that L is one-to-one.

Onto: For any $\vec{s} \in \mathcal{S}$, there are scalars s_1 and s_2 such that $\vec{s} = s_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Which means that $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \in \mathbb{R}^2$ is such that $L\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) = s_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \vec{s}$. Hence, L is onto.

B1(d) Before I define my isomorphism, I first want to find a basis for $\mathbb{P}$. Elements of $\mathbb{P}$ are polynomials $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$ such that $0 = p(1) = p_0 + p_1(1) + p_2(1) + p_3(1) = p_0 + p_1 + p_2 + p_3$. Since $p_0+p_1+p_2+p_3 = 0$, we know $p_3 = -p_0 - p_1 - p_2$. This leads me to think that $\mathcal{B} = \{1-x^3, x-x^3, x^2-x^3\}$ is a basis for $\mathbb{P}$. To see this, let's first show that $\mathcal{B}$ is linearly independent. To that end, suppose that $t_1, t_2, t_3 \in \mathbb{R}$ are such that

$$
t_1(1-x^3) + t_2(x-x^3) + t_3(x^2-x^3) = 0 + 0x + 0x^2 + 0x^3
$$

Then we have $t_1 + t_2x + t_3x^2 + (-t_1 - t_2 - t_3)x^3 = 0 + 0x + 0x^2 + 0x^3$. Setting the coefficients equal, we get that $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, and $-t_1 - t_2 - t_3 = 0$. Which means that $t_1 = t_2 = t_3 = 0$, so $\mathcal{B}$ is linearly independent. Now we need to show that $\mathcal{B}$ is a spanning set for $\mathbb{P}$. To that end, suppose that $p_0 + p_1x + p_2x^2 + p_3x^3 \in \mathbb{P}$. As noted above, this means that $p_3 = -p_0 - p_1 - p_2$. And this means that

$$p(x) = p_0 + p_1x + p_2x^2 + (-p_0 - p_1 - p_2)x^3 = p_0(1 - x^3) + p_1(x - x^3) + p_2(x^2 - x^3)$$

and so we see that $p(x) \in \text{Span } \mathcal{B}$, which means that $\mathcal{B}$ is a spanning set for $\mathbb{P}$. And since $\mathcal{B}$ is a linearly independent spanning set for $\mathbb{P}$, it is a basis for $\mathbb{P}$

And we can use this basis to define our linear mapping. For, given any $p(x) \in \mathbb{P}$, there are unique $a, b, c \in \mathbb{R}$ such that $p(x) = a(1 - x^3) + b(x - x^3) + c(x^2 - x^3)$. And so we define $L(p(x)) = L(a(1 - x^3) + b(x - x^3) + c(x^2 - x^3)) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. To prove that L is an isomorphism, we must prove that it is linear, one-to-one, and onto.

Linear: Let $p(x) = a_1(1 - x^3) + b_1(x - x^3) + c_1(x^2 - x^3)$ and $q(x) = a_2(1 - x^3) + b_2(x - x^3) + c_2(x^2 - x^3)$ be elements of $\mathbb{P}$, and let $t \in \mathbb{R}$. Then we have

$$\begin{aligned} L(tp(x) + q(x)) &= L(t(a_1(1 - x^3) + b_1(x - x^3) + c_1(x^2 - x^3)) + (a_2(1 - x^3) + b_2(x - x^3) + c_2(x^2 - x^3))) \\ &= L((ta_1 + a_2)(1 - x^3) + (tb_1 + b_2)(x - x^3) + (tc_1 + c_2)(x^2 - x^3)) \\ &= \begin{bmatrix} ta_1 + a_2 & tb_1 + b_2 \\ 0 & tc_1 + c_2 \end{bmatrix} \\ &= t\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_2 \\ 0 & c_2 \end{bmatrix} \\ &= tL(p(x)) + L(q(x)) \end{aligned}$$

Therefore, L is linear.

One-to-one: Let $a(1 - x^3) + b(x - x^3) + c(x^2 - x^3) \in \text{Null}(L)$. Then $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L(a(1 - x^3) + b(x - x^3) + c(x^2 - x^3)) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Hence $a = 0$, $b = 0$, and $c = 0$, which means that $a(1 - x^3) + b(x - x^3) + c(x^2 - x^3) = 0(1 - x^3) + 0(x - x^3) + 0(x^2 - x^3) = 0 + 0x + 0x^2 + 0x^3$. Since the only vector in the nullspace is the zero vector, by Lemma 4.7.1 we have that L is one-to-one.

Onto: For any $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{T}$, we have $a(1 - x^3) + b(x - x^3) + c(x^2 - x^3) \in \mathbb{P}$ such that $L(a(1 - x^3) + b(x - x^3) + c(x^2 - x^3)) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Hence, L is onto.

D2(a) Suppose that M and L are one-to-one, and suppose $M \circ L(\mathbf{u}_1) = M \circ$

$L(\mathbf{u}_2)$. Then we have $M(L(\mathbf{u}_1) = M(L(\mathbf{u}_2))$, and since M is one-to-one this means that $L(\mathbf{u}_1) = L(\mathbf{u}_2)$. And since L is one-to-one, this means that $\mathbf{u}_1 = \mathbf{u}_2$. And so we see that $M \circ L$ is one-to-one.

D2(b) Let $M : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $M(a,b,c,d) = (a,b)$ and $L : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $L(a,b) = (a,b,0,0)$. We first note that M is not one-to-one, since $M(1,2,1,2) = M(1,2,3,4)$. But $M \circ L$ is one-to-one, since $M \circ L(a,b) = M(L(a,b)) = M(a,b,0,0) = (a,b)$, so we see that $M \circ L$ is in fact the identity map.

D2(c) No, this is not possible. Suppose that $M \circ L$ is one-to-one, and suppose that $L(\mathbf{u}_1) = L(\mathbf{u}_2)$. Then $M(L(\mathbf{u}_1) = M(L(\mathbf{u}_2))$, which means that $M \circ L(\mathbf{u}_1) = M \circ L(\mathbf{u}_2)$. Since $M \circ L$ is one-to-one, we have $\mathbf{u}_1 = \mathbf{u}_2$. And so we see that L is one-to-one.

D3 Suppose the L and M are onto, and that $\mathbf{w} \in \mathbb{W}$. Then, since M is onto, there is some $\mathbf{v} \in \mathbb{V}$ such that $M(\mathbf{v}) = \mathbf{w}$. And since L is onto, there is some $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. This means that there is $\mathbf{u} \in \mathbb{U}$ such that $M \circ L(\mathbf{u}) = M(L(\mathbf{u})) = M(\mathbf{v}) = \mathbf{w}$. Thus, $M \circ L$ is onto.