Solution to Practice 1r

B7(a) Our first step is to find P, the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$. So

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

Our next step is to find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] R_2 - 4R_1 \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array}\right] R_1 - R_2$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & -4 & 1 \end{array}\right].$$

So we see that $P^{-1} = \begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix}$.

Then we can compute

$$\begin{aligned} [L]_{\mathcal{B}} &= P^{-1}[L]_{\mathcal{S}}P \\ &= \begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -16 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

B7(a) Our first step is to find P, the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$. So

$$P = \begin{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

Our next step is to find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 5 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 \end{array}\right] R_2 - 5R_1$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -4 & 1 & -5 \end{array}\right] (-1/4)R_2 \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/4 & 5/4 \end{array}\right] R_1 - R_2$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1/4 & -1/4 \\ 0 & 1 & -1/4 & 5/4 \end{array}\right]$$

So we see that $P^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix}$.

Then we can compute

$$\begin{aligned} [L]_{\mathcal{B}} &= P^{-1}[L]_{\mathcal{S}}P \\ &= \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix} \begin{bmatrix} 6 & -10 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix} \begin{bmatrix} 20 & -4 \\ 4 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \end{aligned}$$

B8(a) Our first step is to find P, the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$. So

$$P = \begin{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

Our next step is to find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] R_1 - 2R_2 \sim \left[\begin{array}{cc|cc} -1 & -1 & 1 & -2 \\ 3 & 2 & 0 & 1 \end{array}\right] (-1)R_1$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{array}\right] R_2 - 3R_1 \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & -5 \end{array}\right] (-1)R_2$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 2 \\ 0 & 1 & -3 & 5 \end{array}\right] R_1 - R_2 \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

So we see that $P^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$.

Then we can compute

$$\begin{aligned} [L]_{\mathcal{B}} &= P^{-1}[L]_{\mathcal{S}}P \\ &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 12 & -15 \\ -16 & -7 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 15 & 6 \\ -101 & -62 \end{bmatrix} \\ &= \begin{bmatrix} 333 & 198 \\ -550 & -328 \end{bmatrix} \end{aligned}$$

And now we see that

$$\begin{aligned}[L(\vec{x})]_{\mathcal{B}} &= [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \\ &= \begin{bmatrix} 333 & 198 \\ -550 & -328 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 729 \\ -1206 \end{bmatrix}\end{aligned}$$

B8(b) Our first step is to find P, the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$. So

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}$$

Our next step is to find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right] \; R_2 - 5R_1 \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -5 & 1 \end{array}\right] \; (-1/3)R_2$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 5/3 & -1/3 \end{array}\right] \; R_1 - R_2 \; \left[\begin{array}{cc|cc} 1 & 0 & -2/3 & 1/3 \\ 0 & 1 & 5/3 & -1/3 \end{array}\right]$$

So we see that $P^{-1} = \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix}$.

Then we can compute

$$\begin{aligned}[L]_{\mathcal{B}} &= P^{-1}[L]_{\mathcal{S}}P \\ &= \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ 36 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 22 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ -7 & -4 \end{bmatrix}\end{aligned}$$

And now we see that

$$
\begin{aligned}
[L(\vec{x})]_{\mathcal{B}} &= [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \\
&= \begin{bmatrix} 3 & 6 \\ -7 & -4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 10 \end{bmatrix}
\end{aligned}
$$