Solution to Practice 1q

D4(a) $D(1) = 0$, so $[D(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $D(x) = 1$, so $[D(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $D(x^2) = 2x$, so $[D(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Thus,

$$_{\mathcal{C}}[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

D4(b) $L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = x^2$, so $\left[L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. And $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 3 + x^2$, but $[3 + x^2]_{\mathcal{C}}$ is not immediately obvious. To find $[3 + x^2]_{\mathcal{C}}$, we need to find $a, b, c \in \mathbb{R}$ such that

$$a(1 + x^2) + b(1 + x) + c(-1 - x + x^2) = 3 + x^2$$

This is the same as

$$(a + b - c) + (b - c)x + (a + c)x^2 = 3 + x^2$$

and this is equivalent to the system

$$\begin{array}{rrrl} a & +b & -c & = 3 \\ & b & -c & = 0 \\ a & & +c & = 1 \end{array}$$

To solve this system, we row reduce its augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -2 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] R_2 + R_3 \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

And so we see that $a = 3$, $b = -2$, and $c = -2$, which means that $[L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)]_{\mathcal{C}} = [3 + x^2]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$. Which tells us that

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 3 \\ 1 & -2 \\ 1 & -2 \end{bmatrix} $$

D4(c) First we note that $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. To find $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}_{\mathcal{C}}$, we need to find $a_1, b_1, c_1, d_1 \in \mathbb{R}$ such that

$$ \begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} &= a_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + d_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + b_1 + c_1 & a_1 + c_1 \\ d_1 & b_1 + c_1 \end{bmatrix} \end{aligned} $$

which is equivalent to the system

$$ \begin{array}{ccccl} a_1 & +b_1 & +c_1 & & = 1 \\ a_1 & & +c_1 & & = 0 \\ & & & d_1 & = 0 \\ & b_1 & +c_1 & & = 3 \end{array} $$

We also need to find $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}_{\mathcal{C}}$, which means we need to find $a_2, b_2, c_2, d_2 \in \mathbb{R}$ such that

$$ \begin{aligned} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} &= a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_2 + b_2 + c_2 & a_2 + c_2 \\ d_2 & b_2 + c_2 \end{bmatrix} \end{aligned} $$

which is equivalent to the system

$$ \begin{array}{ccccl} a_2 & +b_2 & +c_2 & & = 3 \\ a_2 & & +c_2 & & = 0 \\ & & & d_2 & = 0 \\ & b_2 & +c_2 & & = -1 \end{array} $$

Since our two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following doubly augmented matrix:

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & -1 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 \updownarrow R_4 \end{array} \sim \left[\begin{array}{cccc|c|c} 1 & 1 & 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ R_3 + R_2 \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ (-1)R_2 \end{array} \sim \left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

This tells us that $a_1 = -2$, $b_1 = 1$, $c_1 = 2$, $d_1 = 0$, $a_2 = 4$, $b_2 = 3$, $c_2 = -4$, and $d_4 = 0$. Which in turn tells us that

$$\left[T\left(\left[\begin{array}{r} 2 \\ -1 \end{array}\right]\right)\right]_{\mathcal{C}} \left[\begin{array}{r} -2 \\ 1 \\ 2 \\ 0 \end{array}\right] \quad \text{and} \quad \left[T\left(\left[\begin{array}{r} 1 \\ 2 \end{array}\right]\right)\right]_{\mathcal{C}} = \left[\begin{array}{r} 4 \\ 3 \\ -4 \\ 0 \end{array}\right]$$

and from this we get

$$_{\mathcal{C}}[T]_{\mathcal{B}} = \left[\begin{array}{rr} -2 & 4 \\ 1 & 3 \\ 2 & -4 \\ 0 & 0 \end{array}\right]$$

D4(d) $L(1+x^2) = \left[\begin{array}{r} 2 \\ -1 \end{array}\right] = 3\left[\begin{array}{r} 1 \\ 0 \end{array}\right] - \left[\begin{array}{r} 1 \\ 1 \end{array}\right]$, so $[L(1+x^2)]_{\mathcal{C}} = \left[\begin{array}{r} 3 \\ -1 \end{array}\right]$. $L(1+x) = \left[\begin{array}{r} 1 \\ 0 \end{array}\right] = 1\left[\begin{array}{r} 1 \\ 0 \end{array}\right] + 0\left[\begin{array}{r} 1 \\ 1 \end{array}\right]$, so $[L(1+x)]_{\mathcal{C}} = \left[\begin{array}{r} 1 \\ 0 \end{array}\right]$. $L(-1+x+x^2) = \left[\begin{array}{r} 0 \\ 2 \end{array}\right] = -2\left[\begin{array}{r} 1 \\ 0 \end{array}\right] + 2\left[\begin{array}{r} 1 \\ 1 \end{array}\right]$, so $[L(-1+x+x^2)]_{\mathcal{C}} = \left[\begin{array}{r} -2 \\ 2 \end{array}\right]$.

From this we get

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \left[\begin{array}{rrr} 3 & 1 & -2 \\ -1 & 0 & 2 \end{array}\right]$$

B1(a) $L(\vec{v}_1) = \vec{v}_1 + 3\vec{v}_2$, so $[L(\vec{v}_1)]_{\mathcal{B}} = \left[\begin{array}{r} 1 \\ 3 \end{array}\right]$, while $L(\vec{v}_2) = 5\vec{v}_1 - 7\vec{v}_2$, so $[L(\vec{v}_2)]_{\mathcal{B}} = \left[\begin{array}{r} 5 \\ -7 \end{array}\right]$. This means that $[L]_{\mathcal{B}} = \left[\begin{array}{rr} 1 & 5 \\ 3 & -7 \end{array}\right]$. We use this to find that

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 5 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 26 \end{bmatrix}$$

B1(b) $L(\vec{v}_1) = 2\vec{v}_1 - 3\vec{v}_2$, so $[L(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. $L(\vec{v}_2) = 3\vec{v}_1 + 4\vec{v}_2 - v_3$, so $[L(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$. And $L(\vec{v}_3) = -\vec{v}_1 + 2\vec{v}_2 + 6v_3$, so $[L(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$. This means that $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 3 & -1 \\ -3 & 4 & 2 \\ 0 & -1 & 6 \end{bmatrix}$. We use this to find that

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 3 & -1 \\ -3 & 4 & 2 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -25 \\ 9 \end{bmatrix}$$

B5(a) We need to find $a, b, c \in \mathbb{R}$ such that

$$\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ a+b \\ b+c \end{bmatrix}$$

This is equivalent to the system

$$\begin{array}{cccc} a & & +c & = 5 \\ a & +b & & = 2 \\ & b & +c & = 1 \end{array}$$

To solve this system, we row reduce its augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{c} \\ \\ (1/2)R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This means that $a = 3$, $b = -1$, and $c = 2$. And so we have that $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

B5(b) $L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, so $\left[L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$. $L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, so $\left[L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. And $L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$, so our result from B5(a) tells us that $\left[L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. We use these to find that

$$[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

B5(c) First, we know that

$$\begin{aligned} \left[L\left(\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}\right)\right]_{\mathcal{B}} &= [L]_{\mathcal{B}} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}} \\ &= \begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 13 \\ 2 \end{bmatrix} \end{aligned}$$

Since $\left[L\left(\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 13 \\ 2 \end{bmatrix}$, we have

$$L\left(\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}\right) = 6\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ 15 \end{bmatrix}$$