Lecture 1q

The Matrix of a Linear Mapping

(pages 235-239)

When we first studied linear mappings in Math 106, they were only between $\mathbb{R}^n$ and $\mathbb{R}^m$. In this setting, we were always able to find a matrix $A$ such that our linear mapping $L(\vec{x})$ was the same as $A\vec{x}$. That's because, while $\vec{x} \in \mathbb{R}^n$ isn't technically a matrix, we could temporarily think of $\vec{x}$ like a matrix, and the matrix product $A\vec{x}$ had the properties we wanted. In general, we made great use of the matrix $A$, and would like to be able to do this for any linear mapping, not just ones between $\mathbb{R}^n$ and $\mathbb{R}^m$. But the product $A(2 + 3x - x^2)$ (for example) doesn't make any sense at all! So before we can find a matrix for a linear mapping whose domain is not $\mathbb{R}^n$, we first need to find some way to make our random vector look like a vector from $\mathbb{R}^n$. But wait–we've already done that! Once we fix a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for a vector space $\mathbb{V}$, we can look at the $\mathcal{B}$-coordinates for $\mathbf{x}$. That's only the first step in finding the matrix of a general linear mapping $L : \mathbb{V} \to \mathbb{W}$ though. Because our product $A[\mathbf{x}]_\mathcal{B}$ will be a vector from $\mathbb{R}^m$, not $\mathbb{W}$. So we will also need to fix a basis $\mathcal{C} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ for $\mathbb{W}$, and we will have our output be in $\mathcal{C}$-coordinates. So while we can't find a matrix for $L$ directly, we can find a matrix for $L$ relative to the $\mathcal{B}$ and $\mathcal{C}$ coordinates. As such, instead of simply denoting the matrix for the linear mapping as $[L]$, we denote it as ${}_\mathcal{C}[L]_\mathcal{B}$, so that we remember what coordinates we are using. And this matrix ${}_\mathcal{C}[L]_\mathcal{B}$ will be such that

$$[L(\mathbf{x})]_\mathcal{C} = {}_\mathcal{C}[L]_\mathcal{B}[\mathbf{x}]_\mathcal{B}$$

We know that the matrix ${}_\mathcal{C}[L]_\mathcal{B}$ must exist, since the equation above defines a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Now our only problem is to find it. We go about this process the same way we went about finding $[L]$ in Math 106. The first thing we want to note is that there is a vector $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ such that $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$. Then we have that

$$\begin{aligned} L(\mathbf{x}) &= L(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) \\ &= x_1L(\mathbf{v}_1) + \cdots + x_nL(\mathbf{v}_n) \end{aligned}$$

But, recalling Theorem 4.4.1, we then have that

$$\begin{aligned}
[L(\mathbf{x})]_{\mathcal{C}} &= [x_1 L(\mathbf{v}_1) + \cdots + x_n L(\mathbf{v}_n)]_{\mathcal{C}} \\
&= x_1 [L(\mathbf{v}_1)]_{\mathcal{C}} + \cdots + x_n [L(\mathbf{v}_n)]_{\mathcal{C}} \\
&= \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}
\end{aligned}$$

From this, we see that

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

<u>Definition</u>: Let $\mathbb{V}$ be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, let $\mathbb{W}$ be a vector space with basis $\mathcal{C}$, and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the **matrix of $L$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$** to be the matrix

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

**Example**: Let $L : M(2,2) \to P_2$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$, and let $\mathcal{B}$ be the standard basis for $M(2,2)$ (so $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$) and $\mathcal{C}$ be the standard basis for $P_2$ (so $\mathcal{C} = \{1, x, x^2\}$). To find ${}_{\mathcal{C}}[L]_{\mathcal{B}}$, we need to compute the $\mathcal{C}$-coordinates of the image under $L$ of the $\mathcal{B}$ basis vectors.

$$\begin{aligned}
L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= 1 + x + x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= 1, \text{ so} \\
\left[L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= x, \text{ so} \\
\left[L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

And thus we have that

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Most of the time we will be using the standard bases, but let's go ahead and look at the same linear mapping under different bases.

**Example**: Let $L : M(2,2) \to P_2$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$, and let $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ and $\mathcal{C} = \{1, 1+x, 1+x+x^2\}$. To find $_{\mathcal{C}}[L]_{\mathcal{B}}$, we need to compute the $\mathcal{C}$-coordinates of the image under $L$ of the $\mathcal{B}$ basis vectors.

$$
\begin{aligned}
L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= 1 + x + x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= 2 + x + x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
L\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) &= 2 + 2x + x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
L\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) &= 2 + 2x + 2x^2, \text{ so} \\
\left[L\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)\right]_{\mathcal{C}} &= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}
\end{aligned}
$$

And thus we have that

$$
{}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}
$$

If our domain and codomain are the same vector space, then we might use the same basis $\mathcal{B}$ for both. In these situations, we simply write ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ as $[L]_{\mathcal{B}}$.

**Example**: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a, b, c) = (2a, a + b, 4b + c)$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. To find $[L]_{\mathcal{B}}$, we need to compute the $\mathcal{B}$-coordinates of the image under $L$ of the $\mathcal{B}$ basis vectors. First, let's just find the image under $L$ of the $\mathcal{B}$ basis vectors.

$$
\begin{aligned}
L(1, 0, 1) &= (2, 1, 1) \\
L(2, 1, 1) &= (4, 3, 5) \\
L(-1, 1, 0) &= (-2, 0, 4)
\end{aligned}
$$

Now let's try to find the $\mathcal{B}$-coordinates for our results. We can quickly see that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. But the other two are harder to see. So, let's solve for them. That is, we need to find $a_1, b_1, c_1 \in \mathbb{R}$ such that

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

and $a_2, b_2, c_2 \in \mathbb{R}$ such that

$$a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

Our first equations is equivalent to the system

$$\begin{array}{cccl} a_1 & +2b_1 & -c_1 & = 4 \\ & b_1 & +c_1 & = 3 \\ a_1 & +b_1 & & = 5 \end{array}$$

while our second equation is equivalent to the system

$$\begin{array}{cccl} a_2 & +2b_2 & -c_2 & = -2 \\ & b_2 & +c_2 & = 0 \\ a_2 & +b_2 & & = 4 \end{array}$$

Since these systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following doubly augmented matrix:

$$\left[\begin{array}{ccc|c|c} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 5 & 4 \end{array}\right] R_3 - R_1 \sim \left[\begin{array}{ccc|c|c} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 6 \end{array}\right] R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & 4 & 6 \end{array}\right] (1/2)R_3 \sim \left[\begin{array}{ccc|c|c} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 2 & 0 & 6 & 1 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] R_1 - 2R_2 \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 4 & 7 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

And so we see that $a_1 = 4$, $b_1 = 1$, $c_1 = 2$, $a_2 = 7$, $b_2 = -3$, and $c_2 = 3$. This means that $\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ 3 \end{bmatrix}$. And all of this means that

$$[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 4 & 7 \\ 1 & 1 & -3 \\ 0 & 2 & 3 \end{bmatrix}$$

We've spent a lot of time looking at how to find ${}_{\mathcal{C}}[L]_{\mathcal{B}}$, but what do we do with it when we have it? We use it to compute $[L(\mathbf{x})]_{\mathcal{C}}$ from $[\mathbf{x}]_{\mathcal{B}}$ of course!

**Example**: Let $L : M(2,2) \to P_2$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)+(a+c)x+(a+d)x^2$, and let $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ and $\mathcal{C} = \{1, 1+x, 1+x+x^2\}$, as in our earlier example. We already found that

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Now, if we have $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$, then we get that

$$\begin{aligned} [L(\mathbf{x})]_{\mathcal{C}} &= {}_{\mathcal{C}}[L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ 10 \end{bmatrix} \end{aligned}$$

This means that $L(\mathbf{x}) = 2(1) - 1(1+x) + 10(1+x+x^2) = 11 + 9x + 10x^2$.

We can also compute $L(\mathbf{x})$ by first finding $\mathbf{x}$: since $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$, we have $\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$. And we can then compute that $L(\mathbf{x}) = L\left(\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}\right) = (6+5)+(6+3)x+(6+4)x^2 = 11 + 9x + 10x^2$, same as before.

**Course Author's Comments**: I have presented the material in this lecture rather differently than the text. For one thing, I start by defining the matrix for a linear mapping with respect to bases $\mathcal{B}$ and $\mathcal{C}$, whereas the text relegates this discussion to the assignment. (That's why your first practice question is actually a "D" question!) And then I work back towards the idea of a linear operator (where the domain and codomain are the same vector space), and while there is an example where the domain and codomain are both $\mathbb{R}^n$, this situation is the starting point in the text. Note that you will be expected to know the definition I stated in this lecture as well as the techniques presented in this lecture, and not just the material in the main part of the text.