Solution to Practice 1p

B4(b) The range of L is polynomials of the form $b + 2cx$. But we can write any polynomial in P_1 in this form, so the range of L is P_1, which has as a basis $\{1, x\}$. As for the nullspace of L, it will be all polynomials $a + bx + cx^2$ where a, b, and c satisfy $b = 0$ and $2c = 0$ (so that $b + 2cx = 0 + 0x$). This means that we need $b = 0$ and $c = 0$, but there are no restrictions on a. As such, the nullspace of L is all polynomials of the form $a + 0x + 0x^2$. We can write this as Span $\{1\}$. So $\{1\}$ is a spanning set for the nullspace of L, and this set is clearly linearly independent, so it is a basis for Null(L).

Since we found that $\{1, x\}$ is a basis for the range of L, the rank of L is 2. We then found that $\{1\}$ is a basis for the nullspace of L, and thus the nullity of L is 1. So we have $\text{rank}(L) + \text{nullity}(L) = 2 + 1 = 3 = \dim P_2$, and thus our results are consistent with the Rank-Nullity Theorem.

B4(d) The range of L is polynomials of the form $a+(a+b)x+bx^2$. This can be rewritten as $a(1+x)+b(x+x^2)$. Thus, we have $\text{Range}(L) = \text{Span}\ \{1+x, x+x^2\}$. So $\{1 + x, x + x^2\}$ is a spanning set for the range of L, and this set is clearly linearly independent, so $\{1 + x, x + x^2\}$ is a basis for Range(L). As for the nullspace, it will be all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ such that $a+(a+b)x+bx^2 = 0+0x+0x^2$. Setting the coefficients equal, we see that we need $a = 0$, $a+b = 0$, and $b = 0$. Which means that $a = 0$, $b = 0$ is the only possibility. And so, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the only element of the nullspace, which means that the empty set is a basis for Null(L).

Since we found that$\{1 + x, x + x^2\}$ is a basis for the range of L, the rank of L is 2. We then found that the empty set is our basis for the nullspace of L, and thus the nullity of L is 0. So we have $\text{rank}(L)+\text{nullity}(L) = 2+0 = 2 = \dim \mathbb{D}$, and thus our results are consistent with the Rank-Nullity Theorem.

B4(e) The range of L is matrices of the form $\begin{bmatrix} a-b & b-c \\ c-d & d-a \end{bmatrix}$. This can be rewritten as

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

And so we see that $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\right\}$ is a spanning set for Range(L). But is it linearly independent? Let's look for $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculation on the left, we get

$$\begin{bmatrix} x_1 - x_2 & x_2 - x_3 \\ x_3 - x_4 & x_4 - x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Setting the entries equal to each other, we get the following system:

$$\begin{array}{rrrrl} x_1 & -x_2 & & & = 0 \\ & x_2 & -x_3 & & = 0 \\ & & x_3 & -x_4 & = 0 \\ -x_1 & & & +x_4 & = 0 \end{array}$$

We solve this system by row reducing its coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} R_4 + R_1 \quad \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} R_4 + R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} R_4 + R_3 \quad \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 + R_2 \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see from the reduced row echelon form of the coefficient matrix that there is one parameter in the solution to our system, which means that our set is not linearly independent. However, using our technique for obtaining a basis from a spanning set, we simply need to remove any dependent members of our set. To that end, we note that our system is equivalent to

$$x_1 - x_4 = 0, \quad x_2 - x_4 = 0, \quad x_3 - x_4 = 0$$

Replacing the variable x_4 with the parameter s, we see that the general solution to our system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \\ s \end{bmatrix}$$

To write one vector as a linear combination of the others, we set $s = -1$, which gives us

$$-\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and so we have

$$-\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

(In fact, we can pull any of the matrices to the right...) And so we see that $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ is a dependent member of our spanning set. So, let $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ Then $\mathcal{B}_1$ is still a spanning set for Range(L). But is it linearly independent? To find out, we need to look for solutions to the equation

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculation on the left, we get

$$\begin{bmatrix} x_1 - x_2 & x_2 - x_3 \\ x_3 & -x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} x_1 & -x & 2 & = 0 \\ & x_2 & -x_3 & = 0 \\ & & x_3 & = 0 \\ -x_1 & & & = 0 \end{array}$$

The third equation tells us that $x_3 = 0$. Plugging this into the second equation, we have that $x_2 - 0 = 0$, so $x_2 = 0$. And the fourth equation tells us that $x_1 = 0$. As such, we must have that $x_1 = x_2 = x_3 = 0$. And this means that $\mathcal{B}_1$ is linearly independent.

At long last, we have shown that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for the range of L.

As for the nullspace, it will consist of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose entries a, b, c, d satisfy $\begin{bmatrix} a-b & b-c \\ c-d & d-a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Setting the entries equal, this means that

$$a - b = 0 \quad b - c = 0 \quad c - d = 0 \quad d - a = 0$$

The first equation tells us that $b = a$. Plugging this into the second equation, we now have that $a - c = 0$, which means that $c = a$. Plugging this into the third equation, we now have $a - d = 0$, which means that $d = a$. This is consistent with the fourth equation, which also says that $d = a$. And so we see that $a = b = c = d$. That is, all the entries of our matrix are the same! So we write $\text{Null}(L) = \text{Span}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$. And so $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ is a spanning set for the nullspace of L, and it is clearly linearly independent, so it is a basis for the nullspace of L.

Since we found that $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for the range of L, the rank of L is 3. We then found that $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ is a basis for the nullspace of L, and thus the nullity of L is 1. So we have $\text{rank}(L) + \text{nullity}(L) = 3 + 1 = 4 = \dim M(2,2)$, and thus our results are consistent with the Rank-Nullity Theorem.

D2(a) We have

$$\begin{aligned} L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= L\left(a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= aL\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + bL\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + cL\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= ax^2 + b(2x) + c(1 + x + x^2) \\ &= c + (2b + c)x + (a + c)x^2 \end{aligned}$$

And so the only linear mapping satisfying the given conditions is $L : \mathbb{R}^3 \to P_2$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = c + (2b + c)x + (a + c)x^2$.

D2(b) This question can have multiple answers, but the most obvious one is to have $L(a + bx + cx^2) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

First, we note that L is linear, since $L(t(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) = L((ta_1+a_2)+(tb_1+b_2)x+(tc_1+c_2)x^2) = \begin{bmatrix} ta_1 + a_2 & tb_1 + b_2 \\ 0 & tc_1 + c_2 \end{bmatrix} = t\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} +$

$\begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = tL(a_1 + b_1x + c_1x^2) + L(a_2 + b_2x + c_2x^2).$

Second we see that the nullspace of L is all polynomials $a + bx + cx^2$ whose coefficients a, b, and c have the property that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Setting the entries equal to each other, we immediately see that the only such a, b, and c are $a = 0$, $b = 0$, and $c = 0$. As such, the only element of the nullspace is the zero polynomial, i.e. we have shown that $\text{Null}(L) = \{\mathbf{0}\}$, as required.

Lastly, we note that the range of L is all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is clearly a spanning set for the range of L, as required.

D2(c) This question can have multiple answers, but the most obvious one is to have $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a \\ d \\ 0 \end{bmatrix}$.

First, we note that L is linear, since

$$\begin{aligned} L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} ta_1 + a_2 & tb_1 + b_2 \\ tc_1 + c_2 & td_1 + d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ ta_1 + a_2 \\ td_1 + d_2 \\ 0 \end{bmatrix} \\ &= t\begin{bmatrix} 0 \\ a_1 \\ d_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \\ b_2 \\ 0 \end{bmatrix} \\ &= tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

Second we see that the nullspace of L is the set of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose entries a, b, c, and d satisfy $\begin{bmatrix} 0 \\ a \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. So we need $a = 0$ and $d = 0$, but there are not restrictions on b and c. As such, the nullspace of L is $\left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \mid b, c \in \mathbb{R}\right\}$. We can write such matrices as $b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so we see that $\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a spanning set for Null(L). Moreover, since $\mathcal{B}$ is a subset of a linearly independent set (specifically, the standard basis for $M(2,2)$), we know that $\mathcal{B}$ is also linearly independent. Thus, $\mathcal{B}$ is a basis for Null(L), and so we have shown that the nullity of L is 2, as required.

Third, we note that the range of L is vectors of the form $\begin{bmatrix} 0 \\ a \\ d \\ 0 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. As such, we have that $\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a spanning set for Range(L). Moreover, since $\mathcal{C}$ is a subset of a linearly independent set (namely the standard basis for $\mathbb{R}^4$), we know that $\mathcal{C}$ is also linearly independent. As such, we have that $\mathcal{C}$ is a basis for Range(L), and thus that the rank of L is 2, as required.

Lastly, we note that $L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, as required.

D4 Note that the key to this proof is the fact that $\dim \mathbb{V} = \dim \mathbb{W}$. The statement is not true otherwise.

And so, let us assume that $\mathbb{V}$ and $\mathbb{W}$ are n-dimensional vector spaces, and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. To show that the Range(L) $= \mathbb{W}$ if and only if Null(L) $= \{\mathbf{0}\}$, we need to prove both directions of this statement: ($\Rightarrow$) if Range(L) $= \mathbb{W}$ then Null(L) $= \{\mathbf{0}\}$, and ($\Leftarrow$) if Null(L) $= \{\mathbf{0}\}$ then Range(L) $= \mathbb{W}$.

($\Rightarrow$): Suppose that Range(L) $= \mathbb{W}$. Then we have that $\text{rank}(L) = \dim(\text{Range}(L)) = \dim W = n$. Plugging this fact into the Rank-Nullity Theorem, we have that $n + \text{nullity}(L) = n$. This means that $\text{nullity}(L) = 0$, and so we see that the dimension of the nullspace is 0. But the only subspace of $\mathbb{V}$ with dimension 0 is $\{\mathbf{0}\}$. And so we see that, if Range(L) $= \mathbb{W}$, then Null(L) $= \{\mathbf{0}\}$.

($\Leftarrow$): Suppose that Null(L) $= 0$. Plugging this fact into the Rank-Nullity Theorem, we have that $\text{rank}(L) + 0 = n$, so we see that the rank of L is n. That is, we have that the dimension of the range of L is n. Then there must be a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for Range(L). But this means that $\mathcal{B}$ is a set of n vectors from $\mathbb{W}$ (since Range(L) is a subset of $\mathbb{W}$) that are linearly independent (since if they are linearly independent in Range(L) they will still be linearly independent in $\mathbb{W}$). By the two-out-of-three rule (Theorem 4.3.4 (3)), we also have that $\mathcal{B}$ is a spanning set for $\mathbb{W}$. And so we have that Range(L) $= \text{Span}\,\mathcal{B} = \mathbb{W}$. And so we see that, if $\mathrm{Null}(L) = \{\mathbf{0}\}$, then $\mathrm{Range}(L) = \mathbb{W}$.